A Counterexample to Noether's Bound for Noncommutative Noetherian Monomial Algebras

A study of invariant rings in the noncommutative context...

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Abstract

For a field K and a finite group G that acts faithfully on a polynomial ring $K[V] = K[x_1, x_2, ..., x_n]$, the ring of invariants $K[V]^G$ satisfies Noether's bound: if d is the maximum degree of a generator in a minimal generating set of $K[V]^G$, then $d \leq |G|$. However, it has been hitherto unknown whether the bound generalizes for noncommutative algebras over a field K with group representations or Hopf algebra representations. We describe a noncommutative Noetherian monomial algebra with a permutation group representation that does not satisfy Noether's bound. We also propose a conjecture on the degree bound on invariant rings for faithful permutation representations of Noetherian monomial algebras.

Introduction

Invariant theory is a branch of abstract algebra that studies symmetries of objects by studying their "symmetry groups" of transformations of objects, and invariants which are parts of objects do not change under the symmetries.

Example: Consider an arrangement of four identical points, with four different positions each occupied by one point.

(The set of all symmetries of this object is the 4th symmetric group Σ_4, which has $4! = 24$ symmetries since there are $4!$ ways to permute four objects.)

For example, under the symmetry that swaps points 1 and 2, the "invariants" are 3 and 4.

We can summarize this information as:

Object: an arrangement of four points.
Symmetry: a way of rearranging these four points.
Invariant: points that remain fixed under the symmetries.

The present area of study:

Algebraic structures such as algebras can also be studied this way.

Roughly speaking, an "algebra" is a set where all elements are sums and products of generators, and can be multiplied by a number; expressions called relations are considered to equal 0.

This time,

Object: An algebra
Symmetry: Group action
Invariants: elements of the algebra that remain fixed under the group action

More precisely, let A be an algebra and let G be a group. Then a group action of G on A is a function $\rho : G \to Sym(A)$; it is a rule specifying how each group element rearranges elements in A. We write an element of G acting on an element of A as $\rho(g).a$, or simply $g.a$ when ρ is understood from context.

For the remainder we will assume G is a finite group, and G acts faithfully on A ("faithful" means that no two elements of G act on A the same way).

Invariant theory concerns itself with the ring of invariants of a given action of a group G on an algebra A. Invariants of A under G are elements that do not change under the action of G (i.e. elements $x \in A$ such that for all $g \in G$, $g.x = x$). The invariants of A under G form an invariant ring which is denoted A^G.

Question: Given a G-action on an algebra A, what generators and relations does A^G have? A priori, A^G need not have finitely many generators or finitely many relations.

To understand A^G it is useful to compute $\beta(A, G)$, which is the degree of the highest degree generator in a minimal generating set of A^G.

For commutative algebras, which are synonymous with polynomial rings (such as $\mathbb{R}[x]$, the set of all polynomials of one variable with real coefficients), a result called Noether's bound holds.

Noether's bound For a polynomial ring $K[V]$ over a field K, and G a finite group acting faithfully on $K[V]$, $\beta(K[V], G) \leq |G|$.

Goal: We investigate Noether's bound for special, not necessarily commutative algebras.

Our algebra: A "monomial algebra" is generated as a ring (is built by addition and multiplication which is not necessarily commutative) by n variables and has relations of a fixed degree N which specify the expressions that are equal to 0. A monomial algebra is also closed under scalar multiplication by elements of $\mathbb{R}$.

To make our examples easier to study and calculate we assume that A is "Noetherian". An important property of Noetherian monomial algebras is that they are finitely generated (have finitely many generators).

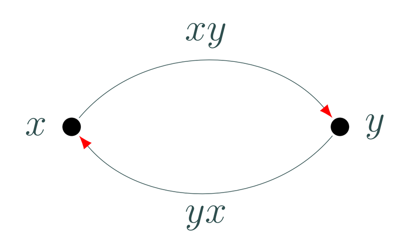
Working Definition. A is Noetherian if and only if no arrows go into or out of any cycles in its "Ufnarovskii graph" $U(A)$.

Definition. Given a monomial algebra A with fixed n, N, the ***Ufnarovskii graph*** of A, denoted $U(A)$, is equal to a directed graph (V, E) where:

1) V the vertex set is the set of all words of n variables of length $N - 1$;

2) each edge in E corresponds to a "valid word" $x_{i_1}x_{i_2}...x_{i_N}$ which is not in the relation space R, and goes from the vertex $x_{i_1}...x_{i_{N-1}}$ to the vertex $x_{i_2}...x_{i_N}$.

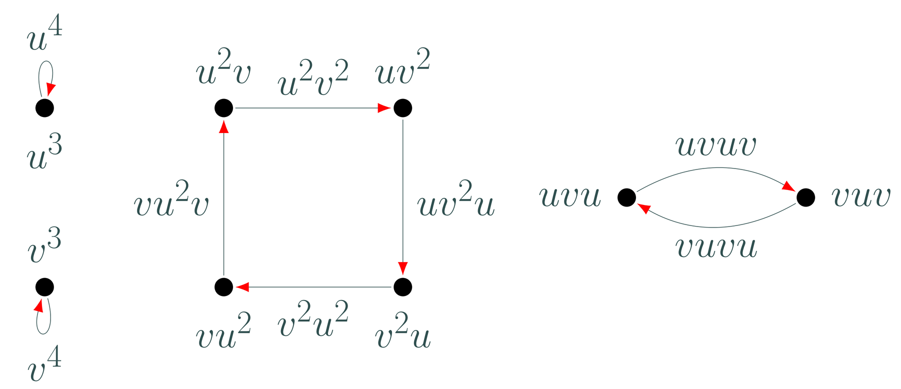
Example: The algebra $A = \frac{\mathbb{R}\langle x,y \rangle}{(x^2, y^2)}$ under the action of $G = \langle \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$ is a Noetherian algebra and has invariant ring $A^G = \mathbb{R}\langle x + y \rangle$ (a free algebra generated by $x+y$ as a ring). For instance, $x+y$ is an invariant; $g.(x + y) = y + x = x + y$.



Our finding:

We have found that Noether's bound fails for noncommutative algebras.

The Counterexample. Let $A = \frac{K\langle u,v \rangle}{(u^3v, v^3u, u^2vu, v^2uv, uvu^2, vuv^2, uv^3, vu^3)}$, and let $G = \langle g \rangle = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$, acting via $g.u = v$ and $g.v = u$.



Note that for this A, $n = 2$ and $N = 4$ (Recall n is the number of basis elements of A, and N is degree of relations in A).

Proof sketch: The "Hilbert series" of an algebra A is defined by:

$$\mathcal{H}_A(t) = \sum_{i=0}^{\infty} \dim(A_i) t^i.$$

The ith coefficient of the Hilbert series of A tells us how many linear basis elements are in A_i, the ith degree homogeneous subspace of A.

We first calculate the Hilbert series of A^G by using Molien's Theorem:

$$\mathcal{H}_{A^G}(t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I - tg)} = \frac{1}{|G|} \sum_{g \in G} \sum_{i=0}^{\infty} \text{tr}(g|_{A^i}) = \frac{1}{|G|} \sum_{i=0}^{\infty} \sum_{g \in G} \text{tr}(g|_{A^i}).$$

To compute $\text{tr}(g|_{A^i})$ we look at what linear basis elements are in A_i, $i \geq 0$. Looking at $U(A)$ helps with this. Then we use the action of g on each element to find out how many elements are mapped to themselves.

Then we make a guess for A^G and check that its Hilbert series is equal to the Hilbert series we have found above.

More specifically, by using Molien's Theorem and U(A) we get the following traces:

degree-d subspace of A	$\text{tr}(1_G\|_{A_i})$	$\text{tr}(g\|_{A_i})$
$A_0 = \langle 1 \rangle$	1	1
$A_1 = \langle u, v \rangle$	2	0
$A_2 = \langle u^2, v^2, uv, vu \rangle$	4	0
$A_3 = \langle u^3, v^3, u^2v, v^2u, uv^2, vu^2, uvu, vuv \rangle$	8	0
$A_4 = \langle u^4, v^4, u^2v^2, v^2u^2, uv^2u, vu^2v, uvuv, vuvu \rangle$	8	0
...	...	...

By adding the traces for each degree we find that the Hilbert series of A^G is equal to

$$\mathcal{H}_{A^G}(t) = \frac{1}{2}((1+1)+2t+4t^2+8t^3+8t^4+...) = 1+t+2t^2+4t^3 \sum_{i \geq 0} t^i.$$

We use the guess $L = \frac{K\langle a,b,c \rangle}{(b^2-a^2b, a^2b-ba^2, ac-ca, bab-bc, bc-cb, bab-aca, abc-c^2)}$, where $a = u + v$, $b = u^2 + v^2$, and $c = u^3 + v^3$. (It can be shown that a, b, c are algebraically independent.)

The Hilbert series of L is also $1+t+2t^2+4t^3 \sum_{i \geq 0} t^i$. So we conclude $A^G = L$.

This shows that $\beta(A, G) = 3$. But $|G| = 2$. Hence Noether's bound fails to hold for A.

Interestingly, we looked for an algebra with $|G| = 2$ and $N = 4$ in order to get this answer. This suggests a pattern for permutation group actions:

Conjecture. Let $A = \frac{K\langle u_1, u_2, ..., u_n \rangle}{R}$ be a Noetherian monomial algebra and let $\phi : G \to Aut(A)$ be a linear faithful group action acting on A by permutation of basis elements. Let N be the degree of generators of R. Then $\beta(A, G) = N - 1$.