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· Examples of Hopf algebras ·

$$H = \underbrace{(H, m, u)}_{\text{algebra}} + \underbrace{(H, \Delta, \epsilon)}_{\text{coalgebra}} + \underbrace{S}_{\text{antipode}}$$

Let $\boxed{G}$ be a finite group, $\boxed{\mathfrak{g}}$ be a Lie algebra, $\boxed{q} \in \mathbb{C}^\times$, and $\boxed{\zeta}$ be a primitive n-th root of unity.

$\mathbb{C}G$ group algebra

algebra structure ✓

$\Delta(g) = g \otimes g$
$\epsilon(g) = 1$
$S(g) = g^{-1}$ $\quad \forall g \in G$

semisimple | finite dimensional | pointed | cocommutative

$U(\mathfrak{g})$ universal enveloping algebra

algebra structure ✓

$\Delta(x) = 1 \otimes x + x \otimes 1$
$\epsilon(x) = 0$
$S(x) = -x$ $\quad \forall x \in \mathfrak{g}$

non-semisimple | infinite dimensional | pointed | cocommutative

H_8 Kac-Paljutkin algebra

algebra generated by x, y, z with
$x^2 = y^2 = 1, \quad xy = yx, \quad zx = yz,$
$zy = xz, \quad z^2 = \frac{1}{2}(1 + x + y - xy)$

$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y$
$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z)$
$\epsilon(x) = \epsilon(y) = \epsilon(z) = 1$
$S(x) = x, \quad S(y) = y, \quad S(z) = z$

semisimple | 8-dimensional | non-pointed | non(co)commutative

$(\mathbb{C}G)^*$ dual of a group algebra

algebra generated by $\{e_g\}_{g \in G}$ with:
$e_g e_h = 0$ for $g \neq h, \quad e_g^2 = e_g, \quad \sum_{g \in G} e_g = 1_{(\mathbb{C}G)^*}$

$\Delta(e_g) = \sum_{h \in G} e_{gh^{-1}} \otimes e_h$
$\epsilon(e_g) = \delta_{1,g}$
$S(e_g) = e_{g^{-1}}$

semisimple | finite dimensional | non-pointed | commutative

$\mathcal{O}_q(SL_2)$ quantum special linear group

algebra generated by $e_{11}, e_{12}, e_{21}, e_{22}$ with:
$e_{k2}e_{k1} = q e_{k1} e_{k2}, \quad e_{2k}e_{1k} = q e_{1k} e_{k2}, \quad$ for $k = 1, 2,$
$e_{21}e_{12} = e_{12}e_{21}, \quad e_{22}e_{11} - e_{11}e_{22} = (q - q^{-1})e_{12}e_{21}$
$e_{11}e_{22} - q^{-1}e_{12}e_{21} = 1$

$\Delta(e_{ij}) = \sum_{k=1}^{2} e_{ik} \otimes e_{kj}$
$\epsilon(e_{ij}) = \delta_{i,j}$
$S(e_{11}) = e_{22}, \quad S(e_{12}) = -q e_{21},$
$S(e_{21}) = -q^{-1} e_{12}, \quad S(e_{22}) = e_{11}$

non-semisimple | infinite dimensional | non-pointed | non(co)mmutative

$T(n)$ Taft algebra

H_{Sw} for $n=2$: Sweedler algebra

algebra generated by g, x with:
$g^n = 1, \quad x^n = 0, \quad gx = \zeta x g$

$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g$
$\epsilon(g) = 1, \quad \epsilon(x) = 0$
$S(g) = g^{-1} = g^{n-1}, \quad S(x) = -x g^{n-1}$

non-semisimple | n^2- dimensional | pointed | non(co)commutative

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· Some Noetherian (filtered) Artin-Schelter (AS) regular algebras ·

$A = \bigoplus_{i\geq 0} A_i$ with $A_0 = \mathbb{C}$; gl.dim $n < \infty$; and AS Gorenstein condition: $\mathrm{Ext}^i_A(\mathbb{C}, A) \cong \delta_{i,n}\mathbb{C}$

(or filtered algebra A so that $\mathrm{gr}(A)$ satisfy properties above)

Global dimension 2

$\boxed{\mathbb{C}_q[u,v]}$ **(skew or) q-polynomial ring**

generated by u, v with:
$vu - quv = 0$ for $q \in \mathbb{C}^\times$

Noetherian domain | commutative for $q = 1$ | PI for q a root of unity | non-PI for q a not a root of unity

$\boxed{\mathbb{C}_J[u,v]}$ **Jordan plane**

generated by u, v with:
$vu - uv - u^2 = 0$

Noetherian domain | non-PI

Global dimension 3

$\boxed{S(a,b,c)}$ **Sklyanin algebra**

generated by v_0, v_1, v_2 with:

$$av_iv_{i+1} + bv_{i+1}v_i + cv_{i+2}^2 = 0$$

for $i = 0, 1, 2$, indices modulo 3

$a, b, c \in \mathbb{C}$ nonzero and generic

Noetherian domain | generally non-PI, PI in special cases

$\boxed{A(\alpha,\beta,0)}$ **Down-Up algebra**

generated by u, d with:

$$d^2u = \alpha dud + \beta ud^2$$

$$du^2 = \alpha udu + \beta u^2d$$

$\alpha, \beta \in \mathbb{C}$

Noetherian domain iff $\beta \neq 0$ | generally non-PI, PI in special cases

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Global dimension n

$\boxed{\mathbb{C}_{q_{ij}}[v_1, v_2, \dots, v_n]}$ **skew polynomial ring** or **quantum affine space**

generated by $v_1, v_2, \dots, v_n$ with $v_jv_i = q_{ij}v_iv_j$ for $q_{ij} \in \mathbb{C}^\times$

Noetherian domain | commutative for $q_{ij} = 1$ | PI iff all q_{ij} are roots of unity

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Filtered

$\boxed{A_n(\mathbb{C})}$ **n-Weyl algebra**

generated by $u_1, \dots, u_n$
and $v_1, \dots, v_n$
with:
$[u_i, u_j] = [v_i, v_j] = 0$,
$[v_i, u_j] = \delta_{i,j}$

$\boxed{A_n^q(\mathbb{C})}$ **quantum n-Weyl algebra**

generated by $u_1, \dots, u_n$
and $v_1, \dots, v_n$
with:
$u_ju_i = qu_iu_j$, $v_jv_i = qv_iv_j$
$v_ju_i - qu_iv_j = \delta_{i,j}$

$\boxed{U(\mathfrak{sl}_2)}$

generated by X, Y, H
with:

$[H, X] = 2X$, $[H, Y] = -2Y$
$[X, Y] = H$