C. Walton
Oct. 25, 2024
TAMU
Alg. & Comb. Seminar

On Extended Frobenius Structures

Based on joint work with: ≡ on ArXiv ≡

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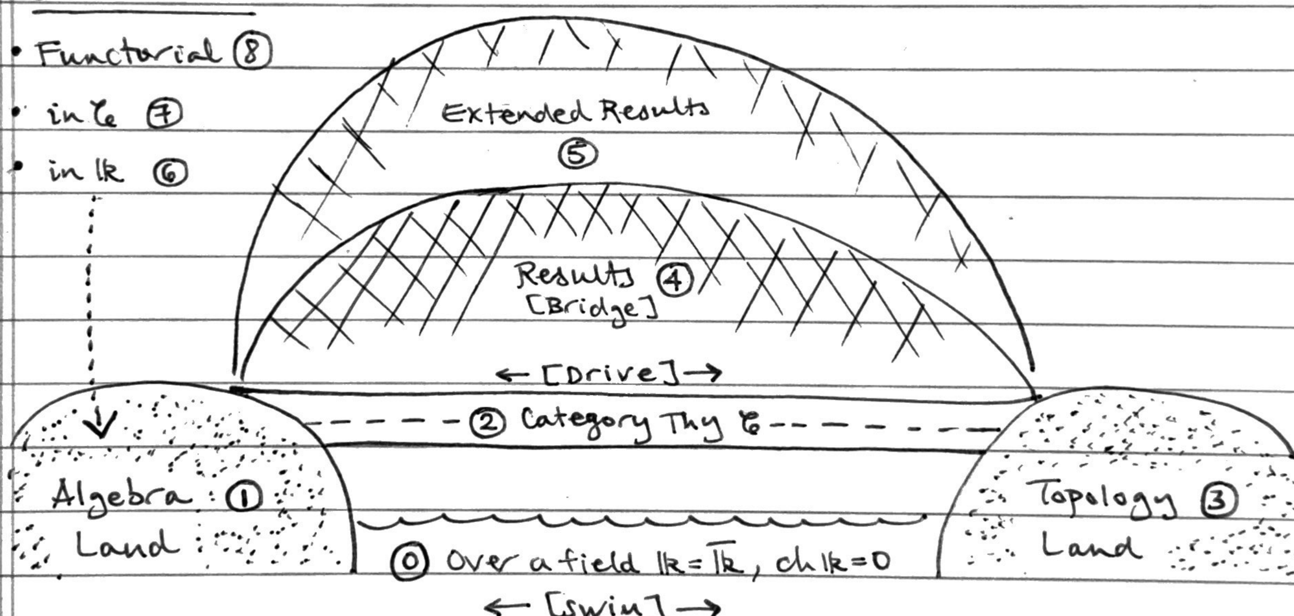
Story setting: Quantum Algebra / Quantum Topology

<u>Outline</u>: ①–⑧ on using algebraic structures to get top. invariants.

<u>Contributions</u>



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① <u>Algebraic Structures</u>

The algebraic structures of interest here, to start, are "Frobenius algebras" over $\Bbbk$: $\otimes := \otimes_{\Bbbk}$

$$(\underset{\Bbbk\text{-vs}}{A},\ m: A\otimes A \to A,\ u: \Bbbk \to A,\ \Delta: A \to A\otimes A,\ \varepsilon: A \to \Bbbk)$$

$\Bbbk$-vs

$\Bbbk$-linear maps

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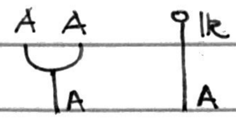
[drawing from top downward]

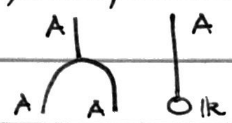
where (A, m, u) is a $\Bbbk$-algebra, that is, we have

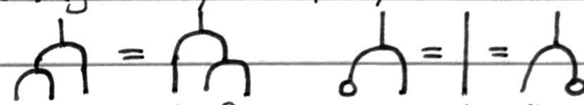
graphical diagram:



associativity unitality

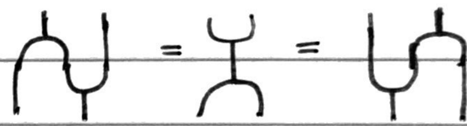
Where (A, Δ, ε) is a $\Bbbk$-coalgebra, that is, we have





coassociativity counitality

& where Frobenius law holds:
(a certain order of operations btw m, Δ)



<u>Example</u> Take G a finite group.

$\Bbbk G$ with basis = elts of G

$m(g \otimes h) = gh$
$u(1_{\Bbbk}) := e$
$\Delta(g) := \sum_{h \in G} gh^{-1} \otimes h$
$\varepsilon(g) := \delta_{g,e} 1_{\Bbbk}$

} extended linearly ... forms a Frob. alg/$\Bbbk$ $\boxed{\Bbbk G}$

② Categorical Structures

We transport btw Algebraland & Topology Land by packaging data into [certain types of] categories & then using [certain types of] functors as the vehicle.

[Ref: CW books "Symmetries of Algebras"]

A "monoidal category" is $(\mathcal{C}, \otimes, \mathbb{1})$ [modeled on monoids]

category — $\mathcal{C}$; bifunctor — $\otimes$: $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$; object in $\mathcal{C}$ — $\mathbb{1}$ ∋. $\mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1}$

+ axioms...

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$(\mathcal{C}, \otimes, \mathbb{1})$ is "symmetric" if $\exists$ nat'l isom
$c = \{c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X\}$ + axioms.

<u>Examples of symmetric monoidal categories:</u>

(1) $(\mathrm{Vec}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk, \text{flip})$

(2) $(\mathrm{FrobAlg}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk, \text{flip})$ & full subcat. of <u>com</u>. Frob. alg/$\Bbbk$.

(3) For a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, c)$

$$\left(\mathrm{ComFrobAlg}(\mathcal{C}), \otimes, \mathbb{1}, c\right)$$

objects are tuples $(A, m, u, \Delta, \varepsilon)$ — A: object in $\mathcal{C}$; $m, u, \Delta, \varepsilon$: morphisms in $\mathcal{C}$; defined like $\mathrm{ComFrobAlg}_{\Bbbk}$

③ <u>Topological structures</u>

One symmetric monoidal category of interest here is $\boxed{\text{2-Cob}}$

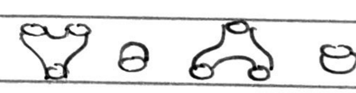
objects: oriented closed 1-manifolds  S^1

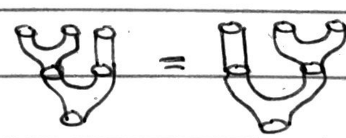
morphisms: orientation preserving cobordisms (2-mflds) 

$\otimes$: disjoint union

$\mathbb{1}$: $\emptyset$

c : flip.

<u>Generators</u> of 2-Cob : S^1 & 

<u>Relations</u> of 2-Cob :  + other Frobenius diagrams.
+ commutativity

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If we want (linear algebraic) invariants of 2-mflds in a sym. mon. cat. $\mathcal{C}$ ($\mathrm{Vec}_{\Bbbk}$)

Build $\boxed{\text{2-TQFT}}$ $\equiv$ "symmetric monoidal functor" (defined later)

$$Z : \text{2-Cob} \longrightarrow \mathcal{C}$$

Here, $\quad S^1 \longmapsto Z(S^1) \in \mathrm{ComFrobAlg}(\mathcal{C})$.

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Ref: Koch's 2-TQFT book

④ Classical Bridge

2-TQFTs$/_{\mathcal{C}}$ form a symmetric monoidal category &

$$\boxed{\text{2-TQFT}_{\mathcal{C}} \overset{\otimes}{\simeq} \mathrm{ComFrobAlg}(\mathcal{C})}$$

So, commutative Frobenius algebras in $\mathcal{C}$
yield 2-TQFTs, which in turn,
yield invariants of 2-mflds

could amp up dimension

could add/remove top. conditions.

... This is a small part of a large story on {n-TQFTs*}
The "toy" example in this programme!

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Ref: Turaev-Turner (2006)

⑤ Extended Bridge

* stick with $n=2$, & remove "oriented" above

↪ Work with $\boxed{\text{2-uCob}}$ $\qquad \circlearrowleft \neq \circlearrowright$

↪ can define $\boxed{\text{2-uTQFT}_{\mathcal{C}}}$ objects: sym ⊗ func: $Z : \text{2-uCob} \longrightarrow \mathcal{C}$

... pick up more data in $\mathcal{C}$...

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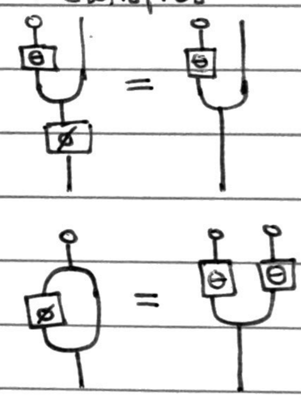
$$\boxed{\begin{array}{ccc} \text{2-UTQFTs} & \overset{\otimes}{\simeq} & \text{ComExtFrobAlg}(\mathcal{C}) \\ Z & \longmapsto & Z(S^1) \end{array}}$$

An "extended Frobenius algebra" in $\mathcal{C}$ is a tuple in $(\mathcal{C}, \otimes, \mathbb{1}, c)$

$$(\underbrace{A, m, u, \Delta, \varepsilon}_{\in \text{ComFrobAlg}(\mathcal{C})},\ \phi : A \to A,\ \theta : \mathbb{1} \to A)$$

ϕ: involution of Frob alg in $\mathcal{C}$

θ: satisfies



[Ref. You Qi's TQFT coursenotes]

Here, $\boxed{\phi}$ needed to handle reversing orientation

$\boxed{\theta}$ needed to handle gluing with a moebius band

We refer to (ϕ, θ) as the "extended structure" of $(A, m, u, \Delta, \varepsilon)$
& call it "ϕ-trivial" if $\phi = id_A$
"θ-trivial" if $\theta = 0$ (if zero morphisms exist)

We study, classify, and construct extended Frob Algebras...

(6) Results over a field $\Bbbk$

<u>Theorem</u> [CKQW] The extended structures on the following well-known Frobenius algebras are classified.

(a) $\Bbbk$ [ϕ-triv]

(b) $\mathbb{C}/\mathbb{R}$ [ϕ-triv or θ-triv]

(c) $\Bbbk[x]/(x^n)$ [ϕ-triv if n odd, not extendable if n even]

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(d) $\Bbbk C_2$ [ϕ-triv or θ-triv]
(e) $\Bbbk C_3$ [ϕ not nec. triv]
(f) $\Bbbk C_4$ [" ——— "
(g) $\Bbbk C_2 \times C_2$ [" ——— "]
(h) $T_2(-1)$ Sweedler alg [ϕ-triv]

[Also: Have conjecture for $\Bbbk C_n$]

Note: Most of the Frobenius algebras above are fin. dim. Hopf algebras (with different comultip'n, counit). Indeed f.d Hopf algebras always Frobenius.

7 Results in $(\mathcal{C}, \otimes, \mathbb{1}, c)$

Likewise Hopf algebras in symmetric monoidal cats $\mathcal{C}$, equipped with a normalized integral, are Frobenius algebras in $\mathcal{C}$
[Known fact, but we give a careful graphical argument in Appendix A]

$\boxed{\Psi : \mathrm{IntHopfAlg}(\mathcal{C}) \longrightarrow \mathrm{FrobAlg}(\mathcal{C})}$ is a well-defined functor

$$(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S^{\pm 1}, \quad \longmapsto \quad (H, m, u,$$

integral $\Lambda : \mathbb{1} \to H$, $\quad$ $\Delta := (m \otimes S)(\mathrm{id} \otimes \underline{\Delta}\Lambda) : H \to H \otimes H$

cointegral $\lambda : H \to \mathbb{1}$) $\quad$ $\varepsilon := \lambda : H \to \mathbb{1}$)

Definition–Proposition [CKOW]

(a) We introduce the notion of an "extended Hopf algebra" in $\mathcal{C}$

$$(\underbrace{H, m, u, \underline{\Delta}, \varepsilon, S^{\pm 1}, \Lambda, \lambda}_{\in \mathrm{IntHopfAlg}(\mathcal{C})}, \phi : H \to H, \theta : \mathbb{1} \to H)$$

certain morphisms in $\mathcal{C}$.

such that $\exists$ forgetful functor

$$\mathcal{U}: \mathrm{ExtHopfAlg}(\mathcal{C}) \longrightarrow \mathrm{IntHopfAlg}(\mathcal{C})$$

(b) If $(H, \phi, \theta) \in \mathrm{ExtHopfAlg}(\mathcal{C})$, then

$\Psi\mathcal{U}(H) \in \mathrm{FrobAlg}(\mathcal{C})$ &

$(\Psi\mathcal{U}(H), \phi, \theta) \in \mathrm{ExtFrobAlg}(\mathcal{C})$.

Extend them $\rightsquigarrow$ to yield extended Frob Algs

Certain Hopf Algs $\rightsquigarrow$ Frob Algs

<u>Example</u>: The converse is not true:

$\exists$ extended structure (ϕ, θ) on Frobenius $\Bbbk G$,

that is not an extended structure on Hopf $\Bbbk G$.

Namely, take $\phi(g) = -g$ and $\theta = 0$.

not comultiplicative with resp. to $\Delta(g) = g \otimes g$.

⑧ Functorial Results

[Ref
(shameless plug)
CW's book
"Sym. of Algs"]

Back to "~~symmetric~~ monoidal functors"...

This is a functor $F: (\mathcal{C}, \otimes, \mathbb{1}) \longrightarrow (\mathcal{C}', \otimes', \mathbb{1}')$ btw mon. cats

equipped with a natural transformation & a morphism in $\mathcal{C}'$

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, bend left, "\otimes \circ (F \times F)"] \arrow[r, bend right, "F \circ \otimes"'] \arrow[r, phantom, "\Downarrow F^{(2)}"] & \mathcal{C}' \end{tikzcd} \qquad F^{(0)}: \mathbb{1}' \longrightarrow F(\mathbb{1})$$

+ axioms.

$\hookrightarrow$ It preserves algebras. Given $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$,

$$(F(A),\ m_{F(A)} = F(m_A) F^{(2)}_{A,A},\ u_{F(A)} = F(u_A) F^{(0)}) \in \mathrm{Alg}(\mathcal{C}').$$

$\hookrightarrow$ Composition of monoidal functors is monoidal as well.

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↪ Get higher cat. structures:

$\boxed{\text{MON}}$ obj = monoidal categories
1-morph = monoidal functors
2-morph = monoidal nat'l transfs.

[Ref
Day–Pastro
(~2009)]

Also have "Frobenius monoidal functors"

$(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}) : (\mathcal{C}, \otimes, \mathbb{1}) \longrightarrow (\mathcal{C}', \otimes', \mathbb{1}')$

↪ inducing functor $\text{FrobAlg}(\mathcal{C}) \longrightarrow \text{FrobAlg}(\mathcal{C}')$

↪ preserved under composition

↪ yielding 2-category $\boxed{\text{FROBMON}}$

Definition–Theorem [CKQW]

(a) We introduce the notion of an "extended Frob. monoidal func."

$$(\underbrace{F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}}_{\in\,\text{FROBMON}};\ \underbrace{\hat{F} : F \Rightarrow F}_{\text{nat'l trans}},\ \underbrace{\check{F} : \mathbb{1}' \to F(\mathbb{1})}_{\text{morphism in } \mathcal{C}'}) : (\mathcal{C}, \otimes, \mathbb{1}) \to (\mathcal{C}', \otimes', \mathbb{1}')$$

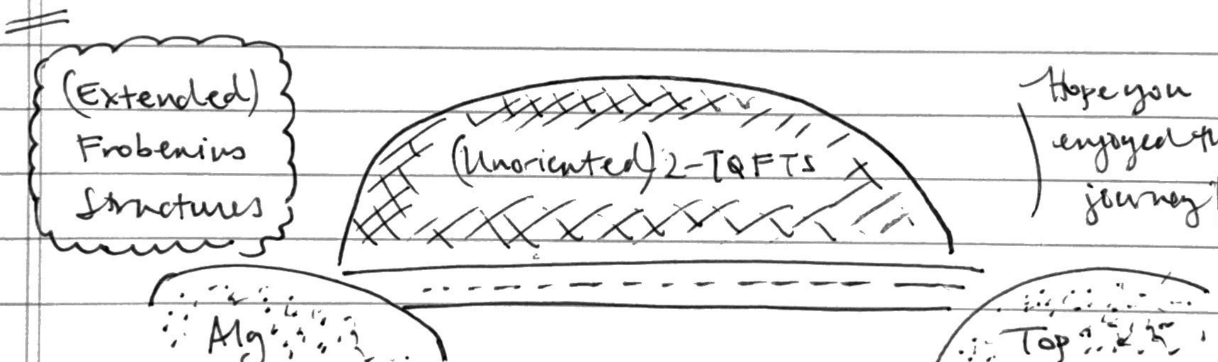
+ axioms.

(b) It induces a functor $\text{ExtFrobAlg}(\mathcal{C}) \longrightarrow \text{ExtFrobAlg}(\mathcal{C}')$

(c) preserved under composition

(d) yielding 2-category $\boxed{\text{EXTFROBMON}}$

Pf/ involves lengthy commutative diagram arguments
(reserved for preprint version only).



Hope you
enjoyed the
journey!