Algebraic Properties of Weak Quantum Symmetries

Fabio Alejandro Calderón Mateus

Universidad Nacional de Colombia
Facultad de Ciencias
Departamento de Matemáticas
Bogotá, D.C., Colombia
2023

Algebraic Properties of Weak Quantum Symmetries

Fabio Alejandro Calderón Mateus
Mathematician, M.Sc.

Thesis Work to Obtain the Degree of
Doctor in Science-Mathematics

Advisor
Chelsea Walton, Ph.D.
Rice University, TX, USA

Coadvisor
Armando Reyes, Ph.D.
Universidad Nacional de Colombia, Bogotá, D.C., Colombia

Universidad Nacional de Colombia
Facultad de Ciencias
Departamento de Matemáticas
Bogotá, D.C., Colombia
2023

Title

Algebraic Properties of Weak Quantum Symmetries

Título

Propiedades algebraicas de las simetrías cuánticas débiles

Abstract: This thesis investigates the properties of weak bialgebras and weak Hopf algebras, their (co)representations, and applications in groupoids, path algebras, and Lie algebroids. The research employs algebraic and categorical techniques to explore the foundational properties of these structures, establishing connections between algebraic and categorical frameworks, and addressing open problems related to their actions on noncommutative graded algebras. By combining theoretical findings and practical examples, this work enhances our understanding of weak Hopf algebras as symmetry generators and their broader implications in various mathematical contexts. Our results contribute to the field of noncommutative algebra and Hopf algebras, paving the way for future research in these areas.

Resumen: Esta tesis investiga las propiedades de las biálgebras débiles (weak bialgebras) y las álgebras de Hopf débiles (weak Hopf algebras), sus (co)representaciones y aplicaciones en groupoides, álgebras de caminos y álgebroides de Lie. La investigación emplea técnicas algebraicas y categóricas para explorar las propiedades fundamentales de estas estructuras, estableciendo conexiones entre los marcos algebraicos y categóricos, y abordando problemas abiertos relacionados con sus acciones en álgebras graduadas no conmutativas. Combinando hallazgos teóricos y ejemplos prácticos, este trabajo mejora nuestra comprensión de las álgebras de Hopf débiles como generadores de simetrías y sus implicaciones más amplias en diversos contextos matemáticos. Nuestros resultados contribuyen al campo del álgebra no conmutativa y las álgebras de Hopf, allanando el camino para futuras investigaciones en estas áreas.

Keywords: *Weak Hopf algebra, monoidal category, representation theory, groupoid, Lie algebroid, path algebra, quiver.*

Palabras clave: *Álgebra de Hopf débil, categoría monoidal, teoría de la representación, grupoide, algebroide de Lie, álgebra de caminos, quiver.*

Dedicatory

To my mother, Gladys, who has tirelessly guided me to become the best version of myself.

To Juan, my beloved family, and dear friends, whose unwavering support and encouragement have been invaluable throughout this journey.

To my esteemed professors, O. Lezama, A. Reyes, and C. Walton, whose exceptional mentorship and teachings have illuminated the path toward becoming a skilled mathematician.

To my cherished pets, Luna, Laika, Sol, and Coco, for their companionship that has brought warmth and joy to every moment.

Finally, to patience and perseverance, steadfast companions that have become the guiding principles in pursuing and achieving every goal.

Acknowledgments

This journey has revealed to me the joy of pursuing Mathematics in the company of others. Throughout the past four years, and even before, I have been fortunate to encounter numerous individuals who have supported, celebrated, and collaborated with me in various ways. Therefore, this list, though extensive, is by no means exhaustive.

My deepest gratitude goes to my mother and Juan, who have been unwavering companions throughout this time, even when distance kept us apart.

I extend heartfelt thanks to the professors who have guided my growth as a professional and researcher. Professor Chelsea Walton, thank you for your unwavering support and pragmatic guidance. Professor Armando Reyes, I am grateful for your constant mentorship, friendship and trust. Professor Oswaldo Lezama, thank you for entrusting the future of your legacy to us young individuals. Professor Carolina Neira, thank you for the warm conversations and supportive hugs.

I must also express my gratitude to acquaintances, friends, and those who hold a special place in my life, whether they are no longer part of my journey or continue to be. Each of them has contributed in some way to both my personal growth and the achievements of this thesis. Special thanks to Haliaphne Acosta for her unwavering support and love from afar. Special thanks to Nahum Shemi for the extensive and cathartic conversations we have shared on the streets of Houston. Special thanks to Andrés Rubiano for our enduring friendship and mutual growth.

I would also like to acknowledge the mathematicians, researchers, and collaborators who have provided me with opportunities to work together on projects, events, and presentations in recent years. Special thanks to Hongdi Huang, Robert Won, and Elizabeth Wicks.

Lastly, I wish to express my gratitude to all the individuals and institutions who have supported me throughout these years. I am thankful to Universidad Nacional de Colombia for being my alma mater and for providing the foundation that has shaped my academic journey. My sincere thanks to the Dirección Académica, Facultad de Ciencias, and Departamento de Matemáticas for their financial and administrative support. I also extend my gratitude to Comisión Fulbright Colombia and Rice University for opening the doors to an extraordinary experience that has been instrumental in shaping this thesis.

Thank you all very much.

Contents

Introduction

This thesis investigates the properties of weak bialgebras and weak Hopf algebras, along with their (co)representations, utilizing a combination of algebraic and categorical techniques. The research explores examples from groupoids, path algebras, and Lie algebroids, highlighting the versatility of weak actions in studying symmetries in (non-commutative) positively graded algebras over a field, particularly in the non-connected setting.

Weak bialgebras and weak Hopf algebras have emerged as powerful tools for capturing algebraic structures with flexible properties. These structures provide a broader framework beyond traditional bialgebras and Hopf algebras, accommodating algebraic objects with less rigid axioms and enabling a more nuanced understanding of various phenomena in mathematics. Several applications in diverse areas of mathematics have been found, demonstrating their significance and potential impact (see, e.g., [BCJ11, BNS99, BS97, HWWW23, Nik02, NV02, WWW22]).

Let $\Bbbk$ be a field. Symmetries of noncommutative $\Bbbk$-algebras have been extensively studied in the context of Hopf algebra (co)actions, with notable success for connected graded $\Bbbk$-algebras (see, e.g., [AS10, CK98, Mon93, Vár03]). However, extending these results to not-necessarily-connected $\Bbbk$-algebras is not a straightforward task. In recent years, the study of (co)actions of weak Hopf algebras has emerged as a promising avenue for investigating symmetries of such $\Bbbk$-algebras, with several attempts made in this direction (see, e.g., [AFSS00, BNS99, NSW98, Nik02, HWWW23]). In this work, we contribute to this ongoing effort by providing further insights and results for weak Hopf algebra actions on noncommutative, not necessarily connected $\Bbbk$-algebras. Specifically, we address three open problems:

1. Monoidal categories and (weak) Hopf algebras are closely related, as the (co)module category over these structures have a canonical monoidal product making them monoidal categories. For traditional bialgebras and Hopf algebras, the monoidal product in the categories of (co)modules is given the classical tensor product of $\Bbbk$-vector spaces, $\otimes_{\Bbbk}$, so there is an immediate correspondence between categorical objects in these categories (e.g., monoids) and classical algebraic structures over $\Bbbk$-algebras (e.g., (co)module algebras) [BCJ11, NTV03]. However, in the weak case the monoidal product depends on certain substructures, making harder to match the categorical setup with its algebraic counterpart. Hence, we discuss recent

advances in this area while providing some correspondences that we expect to give a categorical framework to understand partial (co)actions of Hopf-like structures (see, e.g. [ABV15, BFP10, CPQS15, Dok18, FMF20]) for future research.

2. If A is a $\Bbbk$-algebra, then the group $\mathrm{Aut}_{\mathrm{Alg}}(A)$ of $\Bbbk$-algebra automorphisms of A can be viewed as the algebraic object which captures the symmetries of A [CM84, Proposition 1.2]. Likewise, the Lie algebra $\mathrm{Der}(A)$ of derivations of A can also be viewed as capturing the symmetries of A. We unite and generalize these notions by defining an object $\mathrm{Sym}_{\mathcal{C}}(A)$, which captures the symmetries of A by actions of objects in a category $\mathcal{C}$ whose objects resemble cocommutative (weak) Hopf algebras.

3. A line of research was prompted by Manin's inquiry in the work of Artin-Schelter-Tate [AST91] on whether the bialgebras that coact universally on a well-behaved class of connected graded algebras, Artin-Schelter regular algebras, also enjoy nice ring-theoretic and homological properties. In that work, the question was addressed for key examples of connected graded comodule algebras, skew polynomial rings [AST91], and has been addressed later for other Artin-Schelter regular algebras, especially for the Noetherian, domain, prime, growth conditions and homological dimensions (see, e.g., [BG02, Sections I.2 and II.9] and [WW16]). But Manin's question is unresolved, in general. Hence, we address this inquiry for a class of graded algebras that are usually non-connected: path algebras of finite quivers.

By addressing these three problems, we aim to deepen our understanding of weak Hopf algebras and their representations, uncovering their foundational properties, and exploring the relationships between their operations and substructures. Categorical techniques, particularly those based on monoidal categories, are employed to study the (co)representations of weak Hopf algebras, enhancing our understanding of their structure and behavior in a broader mathematical context. Examples drawn from groupoids, path algebras, and Lie algebroids are provided to demonstrate the practical significance of the theoretical findings, showcasing the versatility of weak Hopf algebra actions in modeling complex algebraic structures arising in diverse mathematical domains.

Our research contributes to the field of algebra by offering a comprehensive investigation of weak Hopf algebras, their representations, and their applications in groupoids, path algebras, and Lie algebroids. The research outcomes deepen our understanding of these structures, their role as symmetry generators, and their broader impact in various mathematical contexts, thereby advancing both theoretical and applied algebraic mathematics.

This thesis is structured as follows. Chapter 1 delves into the basic algebraic and categorical structures and properties studied throughout, including the notion and examples of weak bialgebras and weak Hopf algebras. Then, we delve into their (co)representations, giving those categories a canonical monoidal structure. We prove a natural connection between certain objects in these categories and their algebraic counterparts. Chapter 2 constructs a distinguished object in remarkable categories of Hopf-like structures, providing a bijective correspondence between actions of these structures on a given algebra and morphisms in the corresponding category. Chapter 3 examines the ring-theoretic and homological properties shared between path algebras and the only weak bialgebra acting universally on them, focusing on graph-theoretic properties of quivers. Throughout this work, new results, examples, and open questions are presented, which may serve as future research directions for researchers in the field.

Notations and conventions

Throughout this document, by $\Bbbk$ we denote any field that, if needed, is assumed to be algebraically closed and of characteristic 0. Unless stated otherwise, all rings and $\Bbbk$-algebras are associative and unital. Ring and $\Bbbk$-algebra morphisms are supposed to be unitary. If not explicitly specified, modules (resp. comodules) are considered left-sided (resp. right-sided). Unadorned tensor products are over $\Bbbk$.

Let f, g, h be functions. If defined, we denote the composition of f with g by fg and the composition of h with itself n-times as h^n. $\mathrm{Id}_X : X \to X$ will always denote the identity map of X. Arrow diagrams will be constantly used; they represent composition of functions as concatenation of arrows. A diagram is said to be *commutative* if, no matter what path one follows, the composition of arrows gives always the same result.

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the usual numerical systems, assuming that $0 \in \mathbb{N}$.

Statement of contributions

Chapters 2 and 3 in this thesis correspond to the following publications and preprints containing original results.

- **Chapter 2**: **Calderón, F.**, Huang, H., Wicks, E., and Won, R. Symmetries of algebras captured by actions of weak Hopf algebras. Submitted for publication. Available online at `arXiv:2209.11903`, 2023. [CHWW23]
- **Chapter 3**: **Calderón, F.**, and Walton, C. Algebraic properties of face algebras. *Journal of Algebra and Its Applications*, 22 (03) 2350076, 2023. [CW23]

Part of Chapter 1 is extracted from the following upcoming preprint, in which partial and global (co)actions of weak Hopf algebras are studied.

- **Section 1.3.1**: **Calderón, F.**, and Reyes, A. On the (partial) representation category of weak Hopf algebras. In preparation, 2023. [CR23]

Furthermore, our research findings have been presented in the following events and seminars:

1. *On the study of quantum groupoids and their actions*. June 2023. XXIII Congreso Colombiano de Matemáticas. Universidad Pedagógica y Tecnológica de Colombia - UPTC, Tunja, Colombia.

2. *Hopf-like actions on algebras: an invitation to quantum groups*. November 2022. Seminario de Matemática de Postgrado. Universidad de Santiago de Chile, Santiago, Chile.

3. Poster: *Hopf-like actions on algebras: an invitation to quantum groups*. November 2022. Encuentro Nacional de Estudiantes de Matemática. Pontificia Universidad Católica de Chile, Santiago, Chile.

4. *Hopf-like actions on algebras: an invitation to quantum groups*. October 2022. Seminario de estudiantes de posgrado en Matemáticas. Pontificia Universidad Católica de Chile, Santiago, Chile.

5. *Hopf-like actions on algebras: an invitation to quantum groups*. October 2022. Seminario del Departamento Matemáticas. Universidad Técnica Federico Santa María, Valparaiso, Chile.

6. *Hopf-like actions on algebras*. October 2022. MiniSimposio Álgebra y Teoría de Categorías. Pontificia Universidad Javeriana, Bogotá, Colombia.

7. *Cocommutative Hopf-like actions on algebras*. September 2022. Noncommutative Geometry and Noncommutative Invariant Theory. Banff International Research Station, Banff, Alberta, Canada [Virtual].

8. *Representations of (weak) Hopf algebras from a categorical point of view*. September 2022. Categories and Companions Symposium. MATRIX, Creswick, Victoria, Australia [Virtual].

9. *An invitation to the fantastic world of groupoids*. August 2022. Seminario de Álgebra Constructiva – SAC2. Universidad Nacional de Colombia, Bogotá, Colombia.

10. *On the study of weak Hopf algebras and their (co)actions*. July 2022. Hopf algebras, monoidal categories and related topics. Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania [Virtual].

11. *Una invitación a los grupos cuánticos y sus acciones*. April 2022. Coloquio de Matemáticas por Estudiantes. Universidad Nacional de Colombia, Bogotá, Colombia [Virtual].

12. *On the study of algebraic properties of universal quantum semigroupoids*. April 2022. AMS Joint Mathematical Meetings - Special Session on Hopf Algebras and Tensor Categories. Seattle, WA, USA [Virtual].

13. *Una invitación a los grupos cuánticos.* January 2022. CONVImaTE 2022. Universidad del Valle, Cali, Colombia [Virtual]

14. *On the study of quantum symmetries for non-connected graded algebras.* October 2021. Seminario de Álgebra Constructiva – SAC2. Universidad Nacional de Colombia, Bogotá, Colombia [Virtual].

15. *Algebraic properties of universal quantum semigroupoids*. July 2021. Mathematical Congress of the Americas. Universidad de Buenos Aires, Buenos Aires, Argentina [Virtual].

16. *Algebraic properties of universal quantum semigroupoids*. June 2021. UN Encuentro de Matemáticas 2021. Universidad Nacional de Colombia, Bogotá, Colombia [Virtual].

17. *Some algebraic properties of universal quantum semigroupoids*. April 2021. Quantum Symmetries Student Seminar. The Ohio State University, Columbus, OH, USA [Virtual].

18. *Some algebraic and homological properties of face algebras*. April 2021. AMS Spring Central Sectional Meeting – Special Session on Interactions between Representation Theory, Poisson Geometry, and Noncommutative Algebra. University of Cincinnati, Cincinnati, OH, USA [Virtual].

19. *Algebraic properties of face algebras*. March 2021. AMS Spring Eastern Sectional Meeting – Special Session on Hopf Algebras, Tensor Categories, and Related Homological Methods. Brown University, Providence, RI, USA [Virtual].

20. *Algebraic properties of face algebras*. March 2021. Seminario de Álgebra Constructiva – SAC2. Universidad Nacional de Colombia, Bogotá, Colombia.

21. *Algebraic properties of face algebras*. December 2020. SNAD - Seattle Noncommutative Algebra Day. University of Washington, Seattle, WA, USA [Virtual].

CHAPTER 1

Preliminaries

This chapter lays the groundwork for our study of weak quantum symmetries, which extend classical actions of Hopf algebras on algebras. Section 1.1 introduces the concept of weak bialgebras and weak Hopf algebras, exploring their structure and relevant properties. In Section 1.2, we delve into monoidal categories as a framework for understanding classical and weak quantum symmetries. Section 1.3 investigate (co)actions of weak bialgebras and weak Hopf algebras on algebras. Section 1.4 explores the connections between groupoids and weak Hopf algebras, while Section 1.5 examines quivers and their path algebras in relation to our study. Finally, in Section 1.6, we provide a concise overview of the algebraic and homological properties that will be studied throughout.

1.1 Structure and properties of weak Hopf algebras

First, we set our notation for basic algebraic structures; see e.g. [DNR01, EGNO15, Mon93] for details and examples. Recall that a triple (A, m, u) is a $\Bbbk$-*algebra* if A is a $\Bbbk$-vector space, and the maps $m : A \otimes A \to A$ (*multiplication*) and $u : \Bbbk \to A$ (*unit*) satisfy the following properties:

$$m(m \otimes \mathrm{Id}) = m(\mathrm{Id} \otimes m) \qquad (\textit{associativity}),$$
$$m(u \otimes \mathrm{Id}) = \mathrm{Id} = m(\mathrm{Id} \otimes u) \qquad (\textit{unit property}).$$

We denote $1_A := u(1_{\Bbbk})$. A $\Bbbk$-*algebra morphism* between two algebras (A, m_A, u_A) and (B, m_B, u_B) is a $\Bbbk$-linear map $f : A \to B$ that preserves the multiplication and unit maps, that is, satisfies the following properties:

$$m_B(f \otimes f) = f m_A \qquad (\textit{multiplicative}),$$
$$u_B = f u_A \qquad (\textit{unital}).$$

A *left module* over a $\Bbbk$-algebra (A, m, u) is a pair (M, γ) where M is a $\Bbbk$-vector space and

$\gamma : A \otimes M \longrightarrow M$ (left *action*) is a $\Bbbk$-linear map satisfying the following conditions:

$$\gamma(m \otimes \mathrm{Id}_M) = \gamma(\mathrm{Id}_A \otimes \gamma),$$
$$\gamma(u \otimes \mathrm{Id}_M) = \mathrm{Id}_M .$$

In these definitions, we make use of the well-known $\Bbbk$-isomorphisms $\Bbbk \otimes A \cong A$ and $\Bbbk \otimes M \cong M$. We write $a \cdot m := \gamma(a \otimes m)$, for all $a \in A$ and $m \in M$. A morphism between two left A-modules (M, γ_M) and (N, γ_N) is a $\Bbbk$-linear map $f : M \longrightarrow N$ such that $\gamma_N(\mathrm{Id}_A \otimes f) = f\gamma_M$. The definitions for right modules and their morphisms are similar.

Let (A, m_A, u_A) and (B, m_B, u_B) be two $\Bbbk$-algebras. A $\Bbbk$-vector space M that is both a left A-module via left action map $\gamma : A \otimes M \longrightarrow M$ and a right B-module via right action map $\tau : M \otimes B \longrightarrow M$ is called an (A, B)-*bimodule* if $\gamma(\mathrm{Id}_A \otimes \tau) = \tau(\gamma \otimes \mathrm{Id}_B)$.

Dually, (C, Δ, ε) is a $\Bbbk$-*coalgebra* if C is a $\Bbbk$-vector space, and the maps $\Delta : C \longrightarrow C \otimes C$ (*comultiplication*) and $\varepsilon : C \longrightarrow \Bbbk$ (*counit*) are such that:

$$(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta \qquad (\textit{coassociativity}),$$
$$(\varepsilon \otimes \mathrm{Id})\Delta = \mathrm{Id} = (\mathrm{Id} \otimes \varepsilon)\Delta \qquad (\textit{counit property}).$$

We use sumless notation, which means that for any $c \in C$ we write $\Delta(c) := c_1 \otimes c_2$. Using the coassociativity, we also denote $\Delta^2(c) = (\Delta \otimes \mathrm{Id})(c) = c_1 \otimes c_2 \otimes c_3$, etc. A $\Bbbk$-*coalgebra morphism* between two coalgebras $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ is a $\Bbbk$-linear map $g : C \longrightarrow D$ that preserves the coalgebraic structure, that is, satisfies the following properties:

$$(g \otimes g)\Delta_C = \Delta_D g \qquad (\textit{comultiplicative}),$$
$$\varepsilon_C = \varepsilon_D g \qquad (\textit{counital}).$$

A $\Bbbk$-subspace $I \subseteq C$ of a coalgebra (C, Δ, ε) is a *coideal* if $\Delta(I) \subseteq I \otimes C + C \otimes I$ and $\varepsilon(I) = 0$. A *right comodule* over (C, Δ, ε) is a pair (M, ρ) where M is a $\Bbbk$-vector space and $\rho : M \longrightarrow M \otimes C$ (right *coaction*) is a $\Bbbk$-linear map satisfying the following conditions:

$$(\mathrm{Id}_M \otimes \Delta)\rho = (\rho \otimes \mathrm{Id}_C)\rho,$$
$$(\mathrm{Id}_M \otimes \varepsilon)\rho = \mathrm{Id}_M .$$

There is also sumless notation for right comodules: for any $m \in M$ we write $\rho(m) = m_0 \otimes m_1$. A morphism between two right C-comodules (M, ρ_M) and (N, ρ_N) is a $\Bbbk$-linear map $f : M \longrightarrow N$ such that $\rho_N f = (f \otimes \mathrm{Id}_C)\rho_M$. The definitions for left comodules and their morphisms are similar. For a left coaction of the form $\eta : M \longrightarrow C \otimes M$, we write $\eta(m) = m_{-1} \otimes m_0$, for any $m \in M$, preserving the convention that $m_i \in C$ for $i \neq 0$.

Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be two $\Bbbk$-coalgebras. A $\Bbbk$-vector space M that is both a left D-comodule via left coaction map $\eta : M \longrightarrow D \otimes M$ and a right C-comodule via right action map $\rho : M \longrightarrow M \otimes C$ is called an (D, C)-*bicomodule* if $(\eta \otimes \mathrm{Id}_C)\rho = (\mathrm{Id}_D \otimes \rho)\eta$.

A $\Bbbk$-*bialgebra* is defined as a quintuple $(H, m, u, \Delta, \varepsilon)$, where (H, m, u) forms a $\Bbbk$-algebra and (H, Δ, ε) forms a $\Bbbk$-coalgebra, such that both Δ and ε are algebra maps, or equivalently,

m and u are coalgebra maps.

To simplify the notation, if no further mention of the operations is required, we will use the same letter to denote both the algebraic structure and the underlying $\Bbbk$-vector space. For example, a $\Bbbk$-algebra (A, m, u) will be denoted simply as A.

It is important to note that the mere presence of algebra and coalgebra structures on a vector space H does not guarantee that the comultiplication Δ and the counit ε are both multiplicative and unital. This realization serves as a driving force behind the development of more flexible concepts that accommodate structures where the complete multiplicativity and unitality requirements may not be satisfied. To this end, we recall the notion of a $\Bbbk$-*Frobenius algebra*, defined as a quintuple $(F, m, u, \Delta, \varepsilon)$ where (F, m, u) forms a $\Bbbk$-algebra and (F, Δ, ε) forms a $\Bbbk$-coalgebra, subject to the *Frobenius constraint*, that is,

$$(m \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta) = \Delta m = (\mathrm{Id} \otimes \mathrm{m})(\Delta \otimes \mathrm{Id}).$$

If additionally $m\Delta = \mathrm{Id}$, we call the Frobenius algebra *separable*. Every $\Bbbk$-Frobenius algebra is finite-dimensional [EN55, Section 3].

However, in the context of our investigation of weak quantum symmetries, we aim to define a concept that bears a closer resemblance to classical bialgebras. Thus we introduce weak bialgebras. It is worth mentioning that in earlier works (e.g., [BNS99, BS97, Nil98]) the vector space H was required to be finite-dimensional; we do not assume such a restriction here.

Definition 1.1 (Weak bialgebra). A quintuple $(H, m, u, \Delta, \varepsilon)$ is called a $\Bbbk$-*weak bialgebra* if the following conditions are satisfied:

(i) (H, m, u) is a $\Bbbk$-algebra;

(ii) (H, Δ, ε) is a $\Bbbk$-coalgebra;

(iii) Δ is *multiplicative*, that is, $\Delta(ab) = \Delta(a)\Delta(b)$, for all $a, b \in H$;

(iv) ε is *weak multiplicative*, that is, $\varepsilon(abc) = \varepsilon(ab_1)\varepsilon(b_2c) = \varepsilon(ab_2)\varepsilon(b_1c)$, for all $a, b, c \in H$;

(v) Δ is *weak comultiplicative*, that is, $\Delta^2(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$.

Remark 1.2. Several comultiplications of the unit could appear in the same formula, so we denote different copies of these using primed notation, that is,

$$\Delta(1) \otimes \Delta(1) = (1_1 \otimes 1_2) \otimes (1'_1 \otimes 1'_2) = 1_1 \otimes 1_2 \otimes 1'_1 \otimes 1'_2.$$

Hence condition (v) of Definition 1.1 may be expressed as

$$1_1 \otimes 1_2 \otimes 1_3 = 1_1 \otimes 1_2 1'_1 \otimes 1'_2 = 1_1 \otimes 1'_1 1_2 \otimes 1'_2.$$

As mentioned above, the transition from classical bialgebras to weak bialgebras consists on a weakening of the usual required compatibility between the algebra and coalgebra structures. More precisely, even thought in a weak bialgebra we still require the comultiplication Δ to respect products (condition (iii)), the counit ε is no longer multiplicative and we do not necessarily have $\Delta(1) = 1 \otimes 1$ or $\varepsilon(1) = 1$ (and instead we place conditions (iv)-(v)). In fact, a weak bialgebra is a bialgebra if and only if Δ is unital if and only if ε is multiplicative [BNS99, p. 5].

Naturally, morphisms between weak bialgebras are those preserving the operations, that is, those being morphism of both algebras and coalgebras. Finally, notice that the axioms in Definition 1.1 are self-dual, so in the finite-dimensional case the linear dual $H^* := \mathrm{Hom}_{\Bbbk}(H, \Bbbk)$ of a weak bialgebra is also a weak bialgebra, denoted by H^* [BNS99, p. 3]. However, even for traditional bialgebras, the infinite-dimensional case is not so-well behaved since the dual of the algebra structure is not necessarily a coalgebra (see e.g. [Mon93, Section 1.1]).

Within the framework of weak bialgebras, we encounter a notable pair of maps known as counital maps. These measure "how far" a weak bialgebra is from satisfying the classical compatibility requirements [Proposition 1.4(viii)].

Definition 1.3 (Counital maps, counital subalgebras). Let $(H, m, u, \Delta, \varepsilon)$ be a $\Bbbk$-weak bialgebra. The *source* and *target counital maps* of H are respectively defined as follows:

$$\begin{aligned} \varepsilon_s : H &\to H, \\ h &\mapsto 1_1 \varepsilon(h 1_2), \end{aligned} \qquad\qquad \begin{aligned} \varepsilon_t : H &\to H, \\ h &\mapsto \varepsilon(1_1 h) 1_2. \end{aligned}$$

We denote $H_s := \varepsilon_s(H)$ and $H_t := \varepsilon_t(H)$, which are called the *counital subalgebras* of H.

The presence of these counital maps and subalgebras enables us to establish several remarkable properties specific to weak bialgebras.

Proposition 1.4. *Let $(H, m, u, \Delta, \varepsilon)$ be a weak bialgebra.*

(i) *H_s and H_t are separable Frobenius $\Bbbk$-algebras, and hence finite-dimensional.*

(ii) *$y \in H_s$ if and only if $\Delta(y) = 1_1 \otimes 1_2 y = 1_1 \otimes y 1_2$. Similarly, $z \in H_t$ if and only if $\Delta(z) = 1_1 z \otimes 1_2 = z 1_1 \otimes 1_2$.*

(iii) *$\varepsilon_s(y) = y$ for every $y \in H_s$. Similarly, $\varepsilon_t(z) = z$, for every $z \in H_t$.*

(iv) *If $y \in H_s$ and $z \in H_t$, then $yz = zy$.*

(v) *H_s (resp., H_t) is a left (resp., right) coideal subalgebra of H. Moreover, if H is finite-dimensional, then*

$$H_s = \{(\mathrm{Id} \otimes \varphi)\Delta(1) : \varphi \in H^*\} \qquad \textit{and} \qquad H_t = \{(\varphi \otimes \mathrm{Id})\Delta(1) : \varphi \in H^*\}.$$

(vi) *H_s is a coalgebra with counit $\varepsilon|_{H_s}$ and comultiplication*

$$\begin{aligned} \Delta_s : H_s &\to H_s \otimes H_s \\ y &\mapsto 1_1 \otimes \varepsilon_s(y 1_2) = y_1 \otimes \varepsilon_s(y_2). \end{aligned}$$

Similarly, H_t is a coalgebra with counit $\varepsilon|_{H_t}$ and comultiplication

$$\begin{aligned}\Delta_t : H_t &\to H_t \otimes H_t \\ z &\mapsto \varepsilon_t(1_1 z) \otimes 1_2 = \varepsilon_t(z_1) \otimes z_2.\end{aligned}$$

(vii) *ε_t is an anti-isomorphism of algebras from H_s to H_t, that is, $H_s \cong H_t^{\mathrm{op}}$ as $\mathbb{k}$-algebras.*

(viii) *H is a bialgebra if and only if $\dim_{\mathbb{k}} H_s = 1$ if and only if $\dim_{\mathbb{k}} H_t = 1$.*

(ix) *Any nonzero weak bialgebra morphism $\alpha : H \to K$ preserves counital subalgebras, that is, $H_s \cong K_s$ and $H_t \cong K_t$ as $\mathbb{k}$-algebras.*

Proof. (i): the assertion follows from [BCJ11, Proposition 4.4] and [BNS99, Proposition 2.11].

(ii), (iii), (iv), (v): the claims are consequences of [BNS99, Section 2.2] and [NV02, Propositions 2.2.1 and 2.2.2].

(vi): this is the content of [BCJ11, Proposition 1.17].

(vii): this is [BCJ11, Propositions 1.15 and 1.18].

(viii): it follows from (vii) and [Nik02, Definition 3.1 and Remark 3.2].

(ix): this is [WWW22, Proposition 2.3(h)]. □

We recall some useful weak bialgebra identities that will be use onward. A detailed proof of these facts can be found in [NV02, Proposition 2.2.1].

Lemma 1.5. *Let $(H, m, u, \Delta, \varepsilon)$ be a weak bialgebra and $h \in H$. The following relations hold:*

$$\varepsilon_s(\varepsilon_s(h)) = \varepsilon_s(h), \qquad \varepsilon_t(\varepsilon_t(h)) = \varepsilon_t(h), \tag{1.1}$$

$$h_1\varepsilon_s(h_2) = h = \varepsilon_t(h_1)h_2, \tag{1.2}$$

$$\Delta(1) = 1_1 \otimes \varepsilon_t(1_2) = \varepsilon_s(1_1) \otimes 1_2 \in H_s \otimes H_t. \tag{1.3}$$

Recall that a classical $\mathbb{k}$-bialgebra H is called a $\mathbb{k}$-*Hopf algebra* if there is a $\mathbb{k}$-linear map $S : H \to H$ such that $S(h_1)h_2 = \varepsilon(h)1_H = h_1S(h_2)$. Naturally, there is a notion of antipode for weak bialgebras.

Definition 1.6 (Weak Hopf algebra). A quintuple $H = (H, m, u, \Delta, \varepsilon, S)$ is a $\mathbb{k}$-*weak Hopf algebra* if $(H, m, u, \Delta, \varepsilon)$ is a $\mathbb{k}$-weak bialgebra, and there exists a $\mathbb{k}$-linear map $S : H \to H$ (*antipode*) that satisfies the following properties for all $h \in H$:

(i) $S(h_1)h_2 = \varepsilon_s(h)$;

(ii) $h_1S(h_2) = \varepsilon_t(h)$;

(iii) $S(h_1)h_2S(h_3) = S(h)$.

Remark 1.7. Notice that the antipode axioms for a weak Hopf algebra use the maps ε_s and ε_t instead of ε as for ordinary Hopf algebras. In fact, a weak Hopf algebra H is a classical

Hopf algebra if and only if H is a bialgebra and $S(h_1)h_2 = \varepsilon(h)1$ (or $h_1 S(h_2) = \varepsilon(h)1$), for all $h \in H$ [BNS99, p. 5]. Moreover, it follows from Definition 1.6 that S is anti-multiplicative with respect to m, and anti-comultiplicative with respect to Δ.

As expected, a *morphism of weak Hopf algebras* between H and K (with respective antipodes S_H and S_K) is a $\Bbbk$-weak bialgebra map $f : H \to K$ such that $f S_H = S_K f$.

A $\Bbbk$-subspace I of a weak bialgebra H is called a *(weak) biideal* if it is a two-sided ideal of the underlying algebra structure of H and a coideal of the underlying coalgebra structure of H. Further, I is called a *(weak) Hopf ideal* if it is a biideal of a weak Hopf algebra H such that $S(I) \subseteq I$.

Now, we mention some remarkable examples of weak bialgebras and weak Hopf algebras over $\Bbbk$. More can be found in [BCJ11, BNS99, RWZ21, Szl01, NV02].

Example 1.8 (Groupoid algebras; e.g., [NV02, Example 2.5]). Let $\mathcal{G}$ be a *finite groupoid*, that is, a finite category in which every morphism has inverse. Consider the $\Bbbk$-vector space $\Bbbk\mathcal{G}$ with basis given by the morphisms g in $\mathcal{G}$. We define the product of two morphisms as their composition if it is defined and 0 otherwise. This extends linearly to define a multiplication on $\Bbbk\mathcal{G}$, so that the unit of $\Bbbk\mathcal{G}$ is $1_{\Bbbk\mathcal{G}} = \sum_{i=1}^{n} e_i$, where e_i denotes the identity morphism of the ith object of $\mathcal{G}$. The algebra $\Bbbk\mathcal{G}$ is known as the *groupoid algebra* of $\mathcal{G}$. Furthermore, $\Bbbk\mathcal{G}$ has structure of (finite-dimensional) weak Hopf algebra via

$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \qquad S(g) = g^{-1}, \qquad \forall g \in \mathcal{G}.$$

Moreover, for every $g \in \mathcal{G}$,

$$\varepsilon_s(g) = 1_1 \varepsilon(g 1_2) = \sum_{i=1}^{n} e_i \varepsilon(g e_i) = e_{t(g)},$$

where $t(g)$ denotes the target object of g and composition is written left-to-right. Hence $(\Bbbk\mathcal{G})_s = \bigoplus_{i=1}^{n} \Bbbk e_i = (\Bbbk\mathcal{G})_t$. Groupoids, their algebras and their (co)representations are studied in dept in Section 1.4. The groupoid algebra is a Hopf algebra if and only if $\mathcal{G} = G$ is a group (that is, a groupoid with only one object).

Notice that, for any integer $n > 0$, the matrix algebra $M_n(\Bbbk)$ can be seen as a groupoid algebra and thus these are a particular examples of weak Hopf algebras.

Example 1.9 (Path algebras; e.g., [WWW22, Example 4.9]). Let Q be a *finite quiver*, that is, a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 (resp., Q_1) is a finite collection of vertices (resp., arrows), and $s, t : Q_1 \to Q_0$ denote the source and target maps, respectively. We read paths of Q from left-to-right. For any quiver Q, its associative, unital *path algebra* $\Bbbk Q$ is the $\Bbbk$-algebra having as basis the paths of Q with ring structure determined by path concatenation when possible: $a * b = \delta_{t(a),s(b)} ab$, for all paths a, b of Q. The unit is $1_{\Bbbk Q} = \sum_{i \in Q_0} e_i$, where each e_i is the trivial path at vertex ith of Q. Furthermore, $\Bbbk Q$ has a weak bialgebra structure given by

$$\Delta(e_i) = e_i \otimes e_i, \qquad \varepsilon(e_i) = 1, \qquad 1 \leq i \leq n,$$
$$\Delta(p) = p \otimes p, \qquad \varepsilon(p) = 1, \qquad \forall p \in Q_1.$$

Also, $(\Bbbk Q)_s = \bigoplus_{i=1}^n \Bbbk e_i = (\Bbbk Q)_t$. Quivers, their algebras and their (co)representations are studied in dept in Section 1.5. The path algebra $\Bbbk Q$ is a bialgebra if and only if $|Q_0| = 1$. In general, $\Bbbk Q$ does not admit antipode. The path algebra $\Bbbk Q$ is $\mathbb{N}$-graded by path length, where $(\Bbbk Q)_k = \Bbbk(Q_k)$, for Q_k consisting of paths of length $k \in \mathbb{N}$.

Example 1.10 (Hayashi's face algebras; [Hay96, Example 1.1]). Let Q be a finite quiver, and denote by Q_k the set of all paths of length $k \in \mathbb{N}$ in Q. For a path a, let $s(a)$ and $t(a)$ denote the source and target vertex of a, respectively. By $\mathfrak{H}(Q)$ we denote the $\Bbbk$-vector space that has $\Bbbk$-basis given by elements $\{x_{a,b}\}_{a,b \in Q_k}$, for each $k \geq 0$. The ring structure of $\mathfrak{H}(Q)$ is determined by the following relations:

$$x_{i,j} x_{i',j'} = \delta_{i,i'} \delta_{j,j'} x_{i,j}, \qquad \forall i, j, i', j' \in Q_0,$$
$$x_{s(p),s(q)} x_{p,q} = x_{p,q} = x_{p,q} x_{t(p),t(q)}, \qquad \forall p, q \in Q_1,$$
$$x_{p,q} x_{p',q'} = \delta_{t(p),s(p')} \delta_{t(q),s(q')} x_{pp',qq'}, \qquad \forall p, p', q, q' \in Q_1.$$

The unit is $1_{\mathfrak{H}(Q)} = \sum_{i,j \in Q_0} x_{i,j}$. The algebra $\mathfrak{H}(Q)$ is known as the *Hayashi's face algebra attached to* Q, and it has weak bialgebra structure given by

$$\Delta(x_{a,b}) = \sum_{c \in Q_k} x_{a,c} \otimes x_{c,b} \qquad \varepsilon(x_{a,b}) = \delta_{a,b}, \qquad \forall a, b \in Q_k, k \geq 0.$$

For each $j \in Q_0$, the *face idempotents* of $\mathfrak{H}(Q)$ are $a_j := \sum_{i \in Q_0} x_{i,j}$ and $a'_j = \sum_{i \in Q_0} x_{j,i}$. Hence, for $a, b \in Q_k$, $\varepsilon_s(x_{a,b}) = \delta_{a,b} \sum_{i \in Q_0} x_{i,t(b)}$ and $\varepsilon_t(x_{a,b}) = \delta_{a,b} \sum_{j \in Q_0} x_{s(b),j}$. Thus, as $\Bbbk$-vector spaces, $\mathfrak{H}(Q)_s = \bigoplus_{j \in Q_0} \Bbbk a_j$ and $\mathfrak{H}(Q)_t = \bigoplus_{j \in Q_0} \Bbbk a'_j$. In general, $\mathfrak{H}(Q)$ does not admit antipode. Hayashi's face algebra has a $\mathbb{N}$-grading given by

$$(\mathfrak{H}(Q))_k = \bigoplus_{a,b \in Q_k} \Bbbk x_{a,b}, \qquad \text{for all } k \in \mathbb{N}.$$

More details on the construction of this weak bialgebra can be found in [Hay93].

Example 1.11 (Direct sums; e.g. [HWWW23, Lemma 2.24]). If H, K are weak bialgebras, then their direct sum $H \oplus K$ is again a weak bialgebra with the following structure for all $h, g \in H$ and $k, l \in K$:

multiplication:	$(h, k)(g, l) := (hg, kl)$,
unit:	$1_{H \oplus K} := (1_H, 1_K)$,
comultiplication:	$\Delta_{H \oplus K}(h, k) := (h_1, 0) \otimes (h_2, 0) + (0, k_1) \otimes (0, k_2)$,
counit	$\varepsilon_{H \oplus K}(h, k) := \varepsilon_H(h) + \varepsilon_K(k)$.

Also,

$$(\varepsilon_{H\oplus K})_s(h,k) = ((\varepsilon_H)_s(h), (\varepsilon_K)_s(k)), \qquad (H \oplus K)_s = H_s \oplus K_s,$$
$$(\varepsilon_{H\oplus K})_t(h,k) = ((\varepsilon_H)_t(h), (\varepsilon_K)_t(k)), \qquad (H \oplus K)_t = H_t \oplus K_t.$$

Furthermore, if both H and K have antipode, then $S_{H\otimes K} := (S_H(h), S_K(k))$ defines an antipode for $H \oplus K$. This construction covers three remarkable contexts:

(i) It is known that the direct sum of Hopf algebras is not necessarily a Hopf algebra, but via this example, it is always a weak Hopf algebra;

(ii) The n-space $\Bbbk^n$ and the matrix algebra $M_n(H)$ are weak Hopf algebra for any integer $n > 0$;

(iii) In [Nik01, Section 3.2] a *Lie algebroid* is defined as the weak Hopf algebra obtained from a direct sum of universal enveloping algebras of Lie algebras, and this is used to prove a generalized version of the Cartier-Gabriel-Kostant-Milnor-Moore Theorem for cocommutative weak Hopf algebras. These Lie algebroids will be studied in dept in Section 2.3.

Example 1.12 (e.g., [RWZ21, Example 5.4]). Let H be a weak Hopf algebra, σ a weak Hopf algebra automorphism of H, and $\mathbb{Z} = \langle a \rangle$ an infinite cyclic group. H is a $\Bbbk\mathbb{Z}$-module algebra (action induced by σ; see Definition 1.27 below). Then the *smash product* $H\#\Bbbk\mathbb{Z}$, which as vector space is $H \otimes \Bbbk\mathbb{Z}$, is a weak Hopf algebra via the following structure for all $h, k \in H$ and $m, n \in \mathbb{Z}$:

$$\begin{aligned}
\text{multiplication:} &\quad (h \otimes a^m)(k \otimes a^n) := h\sigma^m(k) \otimes a^{m+n},\\
\text{unit:} &\quad 1_{H\#\Bbbk\mathbb{Z}} := 1_H \otimes a^0,\\
\text{comultiplication:} &\quad \Delta_{H\#\Bbbk\mathbb{Z}}(h \otimes a^m) := (h_1 \otimes a^m) \otimes (h_2 \otimes a^m),\\
\text{counit} &\quad \varepsilon_{H\#\Bbbk\mathbb{Z}}(h \otimes a^m) := \varepsilon(h),\\
\text{antipode:} &\quad S_{H\#\Bbbk\mathbb{Z}}(h \otimes a^m) := \sigma^{-m}(S(h)) \otimes a^{-m}.
\end{aligned}$$

This construction is isomorphic, as algebra, to the skew Laurent polynomial ring $H[x^{\pm 1}; \sigma]$, via $h \mapsto h \otimes a^0$, for all $h \in H$, and $x \mapsto 1 \otimes a$.

1.2 Monoidal categories

Let $\mathcal{C}$ be any category. The collection of *objects* of $\mathcal{C}$ is denoted $\mathrm{Ob}(\mathcal{C})$, but we write $X \in \mathcal{C}$ as a shorthand for $X \in \mathrm{Ob}(\mathcal{C})$. For every pair of objects $X, Y \in \mathcal{C}$, we denote the set of *morphisms* from X to Y by $\mathrm{Hom}_{\mathcal{C}}(X,Y)$. If $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ we also write $f : X \to Y$. For all pairs of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$ the the *composition* of g and f is denoted $gf : X \to Z$. In this case, we call f and g *composable*. For each $X \in \mathcal{C}$, the *identity morphism* of X is denoted $\mathrm{Id}_X : X \to X$. Also, recall that an additive category is called $\Bbbk$*-linear* if the hom-sets are $\Bbbk$-vector spaces and the composition of morphisms is bilinear over $\Bbbk$.

Example 1.13. Let A, B be algebras and C, D be coalgebras over $\Bbbk$. We consider the following categories:

- Set, of sets together with set-theoretic functions,
- $\Bbbk$- Vec, of $\Bbbk$-vector spaces together with $\Bbbk$-linear maps,
- Alg, of $\Bbbk$-algebras together with $\Bbbk$-algebra morphisms,
- X-Alg, of $\Bbbk$-algebras with a complete set of nonzero orthogonal idempotents indexed by a nonempty finite set X, together with X-preserving $\Bbbk$-algebra morphisms (see Definition 2.26),
- Coalg, of $\Bbbk$-coalgebras together with $\Bbbk$-coalgebra morphisms,
- WBA, of weak bialgebras together with $\Bbbk$-weak bialgebra morphisms,
- WHA, of weak Hopf algebras together with $\Bbbk$-weak Hopf algebra morphisms.

We also denote the categories of left/right modules over A respectively by A-mod and mod-A. Similarly, the category of left/right comodules over C are respectively denoted by C-comod and comod-C. Furthermore, we denote the category of (A, B)-bimodules by A-bimod-B and the category of (D, C)-bicomodules by D-$\mathsf{bicomod}$-C.

Next, we introduce the concept of a category having a structure resembling a monoid. This concept plays a fundamental role in our subsequent work.

Definition 1.14 (Monoidal category, e.g. [EGNO15, Definition 2.2.8]). A *monoidal category* $(\mathcal{C}, \otimes, \mathbb{1})$ is a category $\mathcal{C}$ together with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (*monoidal product*), a natural isomorphism $\alpha_{\square,\triangle,\diamond} : (\square \otimes \triangle) \otimes \diamond \xrightarrow{\sim} \square \otimes (\triangle \otimes \diamond)$ (*associative constraint*), an object $\mathbb{1} \in \mathcal{C}$ (*monoidal unit*), and natural isomorphisms $l_\square : \mathbb{1} \otimes \square \xrightarrow{\sim} \square$ and $r_\square : \square \otimes \mathbb{1} \xrightarrow{\sim} \square$ (*unital constraints*), such that the pentagon and triangle axioms are satisfied:

- For all $W, X, Y, Z \in \mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "\alpha_{W,X,Y} \otimes \mathrm{Id}_Z"'] \arrow[dr, "\alpha_{W \otimes X, Y, Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "\alpha_{W, X \otimes Y, Z}"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\alpha_{W,X,Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{Id}_W \otimes \alpha_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

- For all $X, Y \in \mathcal{C}$ the following diagram commutes:

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\alpha_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{Id}_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{Id}_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

Some examples of monoidal categories are the following. More can be found in, e.g., [EGNO15, Section 2.3].

Example 1.15 (e.g., [EGNO15, Example 2.3.3]). The category $\mathsf{Vec}_{\Bbbk} := (\Bbbk\text{-}\mathrm{Vec}, \otimes_{\Bbbk}, \Bbbk)$ of $\Bbbk$-vector spaces, together with the canonical associativity and unit isomorphisms, is monoidal. An important subcategory that carries the monoidal structure is that of finite-dimensional vector spaces, $\mathrm{vec}_{\Bbbk}$.

It is important to note that in this chapter we will use different notations for both the category itself and the enriched monoidal structure. Furthermore, in the sequel, when discussing a monoidal category without explicitly mentioning the associative or unital constraints, it is understood that they are the same as those in the category $\mathsf{Vec}_{\Bbbk}$.

Example 1.16 (e.g., [BCJ11, Section 2]). Let A be a $\Bbbk$-algebra. Given two (A, A)-bimodules M and N, the *tensor product* of M and N over A is the (A, A)-bimodule given by

$$M \otimes_A N := (M \otimes N)/\mathrm{Img}(\nu_M \otimes \mathrm{Id}_N - \mathrm{Id}_M \otimes \mu_N).$$

Notice that $M \otimes_A N$ is a $\Bbbk$-quotient space of $M \otimes N$. The category of (A, A)-bimodules, ${}_A\mathcal{M}_A := (A\text{-}\mathsf{bimod}\text{-}A, \otimes_A, A)$, is monoidal.

Example 1.17 (e.g., [BCJ11, Section 3]). Let C be a $\Bbbk$-coalgebra. Given two (C, C)-bicomodules M and N, the *cotensor product* of M and N over C is the (C, C)-bimodule given by

$$M \otimes^C N := \ker(\rho_M \otimes \mathrm{Id}_N - \mathrm{Id}_M \otimes \lambda_N).$$

Notice that $M \otimes^C N$ is a $\Bbbk$-subspace of $M \otimes N$. The category of (C, C)-comodules, ${}^C\mathcal{M}^C := (C\text{-}\mathsf{bicomod}\text{-}C, \otimes^C, C)$, is monoidal.

Now, we proceed to define special functors between monoidal categories; see e.g. [DP08], [Str07, Chapter 13], [Szl05, Equations 6.46 and 6.47] or [WWW22, Definition 3.3] for further details.

Definition 1.18 ((Co)monoidal functor). Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ be two monoidal categories.

(i) A functor $F : \mathcal{C} \to \mathcal{D}$ is *monoidal* if it is equipped with a natural transformation $F_{\square,\triangle} : F(\square) \otimes_{\mathcal{D}} F(\triangle) \to F(\square \otimes_{\mathcal{C}} \triangle)$, and a morphism $F_0 : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$ in $\mathcal{D}$, such that associative and unital constraints are satisfied, that is, for all $X, Y, Z \in \mathcal{C}$,

$$\begin{aligned} F_{X,Y\otimes_{\mathcal{C}}Z}(\mathrm{Id}_{F(X)} \otimes_{\mathcal{D}} F_{Y,Z})\alpha_{F(X),F(Y),F(Z)} &= F(\alpha_{X,Y,Z})F_{X\otimes_{\mathcal{C}}Y,Z}(F_{X,Y} \otimes_{\mathcal{D}} \mathrm{Id}_{F(Z)}), \\ F(l_X)^{-1} l_{F(X)} &= F_{\mathbb{1}_{\mathcal{C}},X}(F_0 \otimes_{\mathcal{D}} \mathrm{Id}_{F(X)}), \\ F(r_X)^{-1} r_{F(X)} &= F_{X,\mathbb{1}_{\mathcal{C}}}(\mathrm{Id}_{F(X)} \otimes_{\mathcal{D}} F_0). \end{aligned}$$

Moreover, the functor is called *strong monoidal* if F_0 and $F_{X,Y}$ are isomorphisms, for all $X, Y \in \mathcal{C}$.

(ii) A functor $F : \mathcal{C} \to \mathcal{D}$ is *comonoidal* if it is equipped with a natural transformation $F^{\square,\triangle} : F(\square \otimes_{\mathcal{C}} \triangle) \to F(\square) \otimes_{\mathcal{D}} F(\triangle)$, and a morphism $F^0 : F(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$ in $\mathcal{D}$, such that coassociativity and counitality constraints are satisfied, that is, for all $X, Y, Z \in \mathcal{C}$,

$$\begin{aligned} \alpha_{F(X),F(Y),F(Z)}(F^{X,Y} \otimes_{\mathcal{D}} \mathrm{Id}_{F(Z)})F^{X\otimes_{\mathcal{C}}Y,Z} &= (\mathrm{Id}_{F(X)} \otimes_{\mathcal{D}} F_{Y,Z})F_{X,Y\otimes_{\mathcal{C}}Z}F(\alpha_{X,Y,Z}), \\ l^{-1}_{F(X)}F(l_X) &= (F^0 \otimes_{\mathcal{D}} \mathrm{Id}_{F(X)})F^{\mathbb{1}_{\mathcal{C}},X}, \\ r^{-1}_{F(X)}F(r_X) &= (\mathrm{Id}_{F(X)} \otimes_{\mathcal{D}} F^0)F^{X,\mathbb{1}_{\mathcal{C}}}. \end{aligned}$$

Moreover, the functor is called *strong comonoidal* if F^0 and $F^{X,Y}$ are isomorphisms for all $X, Y \in \mathcal{C}$.

Our notion of (co)monoidal functor is also referred to as a *lax (co)monoidal functor* in the literature. In those cases, a strong (co)monoidal functor is simply referred to as a (co)monoidal functor. It is worth noting that a strong monoidal functor is the same as a strong comonoidal functor.

Two monoidal categories are said to be *monoidally equivalent* if there exists a strong (co)monoidal functor between them that is an equivalence of ordinary categories. Similarly, two monoidal categories are said to be *monoidally isomorphic* if there exists a strong (co)monoidal functor between them that is an isomorphism of ordinary categories.

Definition 1.19 (Frobenius monoidal functor). A functor $F : \mathcal{C} \to \mathcal{D}$ is *Frobenius monoidal* if it is simultaneously monoidal and comonoidal, and such that for all $X, Y, Z \in \mathcal{C}$, the following identities hold:

$$\begin{aligned} (F_{X,Y} \otimes_{\mathcal{D}} \mathrm{Id}_{F(Z)})\alpha^{-1}_{F(X),F(Y),F(Z)}(\mathrm{Id}_{F(X)} \otimes_{\mathcal{D}} F^{Y,Z}) &= F^{X\otimes_{\mathcal{C}}Y,Z}F(\alpha^{-1}_{X,Y,Z})F_{X,Y\otimes_{\mathcal{C}}Z}, \\ (\mathrm{Id}_{F(X)} \otimes_{\mathcal{D}} F_{Y,Z})\alpha_{F(X),F(Y),F(Y)}(F^{X,Y} \otimes_{\mathcal{D}} \mathrm{Id}_{F(Z)}) &= F^{X,Y\otimes_{\mathcal{C}}Z}F(\alpha_{X,Y,Z})F_{X\otimes_{\mathcal{C}}Y,Z}. \end{aligned}$$

Any strong (co)monoidal functor is Frobenius monoidal [DP08, Proposition 3].

In monoidal categories, there exist distinguished objects that exhibit behavior reminiscent of (co)algebras. This leads to the following concepts; see e.g., [BÏ8, Definition 7.1] or [EGNO15, Definition 7.20.3] for further details.

Definition 1.20 (Monoid, comonoid, Frobenius monoid). Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ be a monoidal category.

(i) A triple (A, m, u) is a *monoid* in $\mathcal{C}$ if $A \in \mathcal{C}$, and the morphisms $m : A \otimes A \to A$ and

$u : \mathbb{1}_{\mathcal{C}} \to A$ in $\mathcal{C}$ satisfy:

$$m(m \otimes \mathrm{Id}) = m(\mathrm{Id} \otimes m)\alpha_{A,A,A} \qquad (\textit{associativity constraint}),$$
$$m(u \otimes \mathrm{Id}) = l_A, \qquad m(\mathrm{Id} \otimes u) = r_A \qquad (\textit{unitality constraints}).$$

Given two monoids (A, m_A, u_A) and (B, m_B, u_B) in $\mathcal{C}$, a *morphism of monoids* from A to B is a morphism $f : A \to B$ in $\mathcal{C}$ so that $f m_A = m_B(f \otimes f)$ and $f u_A = u_B$. Monoids in $\mathcal{C}$ together with their morphisms form a category, which we denote by $\mathrm{Mon}(\mathcal{C})$.

(ii) A triple (C, Δ, ε) is a *comonoid* in $\mathcal{C}$ if $C \in \mathcal{C}$, and the morphisms $\Delta : C \to C \otimes C$ and $\varepsilon : C \to 1$ in $\mathcal{C}$ satisfy:

$$\alpha_{C,C,C}(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta \qquad (\textit{coassociativity constraint}),$$
$$(\varepsilon \otimes \mathrm{Id})\Delta = l_C^{-1}, \qquad (\mathrm{Id} \otimes \varepsilon)\Delta = r_C^{-1}, \qquad (\textit{counitality constraints}).$$

Given two comonoids $(C, \Delta_C, \varepsilon_C)$, $(D, \Delta_D, \varepsilon_D)$ in $\mathcal{C}$, a *morphism of comonoids* from C to D is a morphism $g : C \to D$ in $\mathcal{C}$ so that $\Delta_D g = (g \otimes g)\Delta_C$ and $\varepsilon_D g = \varepsilon_C$. Comonoids in $\mathcal{C}$ together with their morphisms form a category, which we denote by $\mathrm{Comon}(\mathcal{C})$.

(iii) A quintuple $(A, m, u, \Delta, \varepsilon)$ is a *Frobenius monoid* in $\mathcal{C}$ if $(A, m, u) \in \mathrm{Mon}(\mathcal{C})$ and $(A, \Delta, \varepsilon) \in \mathrm{Comon}(\mathcal{C})$, so that

$$(m \otimes \mathrm{Id})\alpha^{-1}_{A,A,A}(\mathrm{Id} \otimes \Delta) = \Delta m = (\mathrm{Id} \otimes m)\alpha_{A,A,A}(\Delta \otimes \mathrm{Id}).$$

A *morphism of Frobenius monoids* in $\mathcal{C}$ is a morphism in $\mathcal{C}$ that lies in both $\mathrm{Mon}(\mathcal{C})$ and $\mathrm{Comon}(\mathcal{C})$. Frobenius monoids in $\mathcal{C}$ and their morphisms form a category, which we denote by $\mathrm{FrobMon}(\mathcal{C})$.

Our notion of monoid object (resp. comonoid object, Frobenius monoid object) is also referred to as an *algebra* (resp. *coalgebra*, *Frobenius algebra*) in $\mathcal{C}$. In fact, it is worth noting that monoids, comonoids, and Frobenius monoids in the category $\mathsf{Vec}_{\Bbbk}$ correspond to $\Bbbk$-algebras, $\Bbbk$-coalgebras, and Frobenius algebras over $\Bbbk$, respectively. Furthermore, if we restrict our attention to the category of finite-dimensional vector spaces $\mathsf{vec}_{\Bbbk}$, we recover the finite-dimensional versions of these concepts.

We end this section by recalling that these distinguished objects are preserved by the corresponding type of functor. This result follows from [Str07, p. 100-101], [Szl05, Lemma 2.1], [DP08, Corollary 5], and [KR09, Proposition 2.13].

Proposition 1.21. *Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ be two monoidal categories. We have the following assertions:*

(i) *If $F : \mathcal{C} \to \mathcal{D}$ is a monoidal functor and $(A, m, u) \in \mathrm{Mon}(\mathcal{C})$, then*

$$(F(A), F(m)F_{A,A}, F(u)F_0) \in \mathrm{Mon}(\mathcal{D}).$$

(ii) *If* $F : \mathcal{C} \to \mathcal{D}$ *is a comonoidal functor and* $(C, \Delta, \varepsilon) \in \mathrm{Comon}(\mathcal{C})$*, then*

$$(F(C), F^{C,C}F(\Delta), F^0F(\varepsilon)) \in \mathrm{Comon}(\mathcal{D}).$$

(iii) *If* $F : \mathcal{C} \to \mathcal{D}$ *is a Frobenius monoidal functor and* $(A, m, u, \Delta, \varepsilon) \in \mathrm{FrobMon}(\mathcal{C})$*, then*

$$(F(A), F(m)F_{A,A}, F(u)F_0, F^{A,A}F(\Delta), F^0F(\varepsilon)) \in \mathrm{FrobMon}(\mathcal{D}).$$

1.3 (Co)actions of weak bialebras

This section is dedicated to proving that both the category of modules and the category of comodules over a weak bialgebra are monoidal. Additionally, we will establish a correspondence between (co)monoids in these categories and the traditional notions of (co)module (co)algebras over a weak Hopf algebra. In this section H will denote a $\Bbbk$-weak bialgebra.

1.3.1 Weak actions

We start by recalling the monoidal structure of the left representation category H-mod; see e.g., [BCJ11, Section 2], [NTV03, Section 4], [NV02, Section 5] or [Nil98, Section 2] for further details. It suffices to work with left representations since H^{op} is also a weak bialgebra and $H\text{-mod} \cong \text{mod-}H^{\mathrm{op}}$ [NV02, Remark 2.4.1]. As stated in the Introduction, for the rest of this section, we assume that all modules are left-sided.

Definition 1.22 (*H*-monoidal product of modules)**.** Given two H-modules M and N, the *H-monoidal product* of M with N is the subspace

$$M\underline{\otimes}N := \langle x \otimes y \in M \otimes N \mid x \otimes y = 1_1 \cdot x \otimes 1_2 \cdot y \rangle \subseteq M \otimes N.$$

If $x \otimes y \in M\underline{\otimes}N$, we write $x\underline{\otimes}y$. Given two morphisms $f : M \to M'$ and $g : N \to N'$ in H-mod, the *H-monoidal product* of f with g is given by

$$\begin{aligned} f\underline{\otimes}g : M\underline{\otimes}N &\to M'\underline{\otimes}N' \\ x\underline{\otimes}y &\mapsto f(x)\underline{\otimes}g(y). \end{aligned}$$

Remark 1.23. Let M, N, f, g as in Definition 1.22.

(i) Notice that $M\underline{\otimes}N = \Delta(1)(M \otimes N)$, which corresponds to the unital submodule $1 \cdot (M \otimes N)$ of the non-unital left H-action on $M \otimes N$ given by

$$h \cdot (x \otimes y) := h_1 \cdot x \otimes h_2 \cdot y, \tag{1.4}$$

for all $h \in H$, $x \in M$ and $y \in N$. Hence, $M\underline{\otimes}N \in H\text{-mod}$. Furthermore, there are inclusion and projection maps given by

$$\begin{aligned} \iota_{M,N} : M\underline{\otimes}N &\longrightarrow M\otimes N \\ x\underline{\otimes}y &\mapsto 1_1\cdot x\otimes 1_2\cdot y, \end{aligned} \qquad\qquad \begin{aligned} \pi_{M,N} : M\otimes N &\longrightarrow M\underline{\otimes}N \\ x\otimes y &\mapsto 1_1\cdot x\underline{\otimes}1_2\cdot y. \end{aligned}$$

(ii) Since for every $x\underline{\otimes}y \in M\underline{\otimes}N$ we have $1_1\cdot f(x)\otimes 1_2\cdot g(x) = f(x)\otimes g(y)$, the H-monoidal product of H-module maps is well defined. Moreover, $f\underline{\otimes}g = (f\otimes g)|_{M\underline{\otimes}N}$.

The counital subalgebra H_t is an H-module via $h\cdot k = \varepsilon_t(hk)$, for all $h\in H$ and $k\in H_t$. This allows us to define the H-isomorphisms given by

$$\begin{aligned} l_M : H_t\underline{\otimes}M &\longrightarrow M \\ k\underline{\otimes}x &\mapsto k\cdot x, \end{aligned} \qquad\qquad \begin{aligned} r_M : M\underline{\otimes}H_t &\longrightarrow M \\ x\underline{\otimes}k &\mapsto k\cdot x, \end{aligned}$$

for every $M\in H$-mod. Using (1.3), the inverses of these morphisms are defined as

$$\begin{aligned} l_M^{-1} : M &\longrightarrow H_t\underline{\otimes}M \\ x &\mapsto 1_1\underline{\otimes}1_2\cdot x, \end{aligned} \qquad\qquad \begin{aligned} r_M^{-1} : M &\longrightarrow M\underline{\otimes}H_t \\ x &\mapsto 1_1\cdot x\otimes 1_2. \end{aligned}$$

These constructions guarantee the following result.

Proposition 1.24 ([BCJ11, Theorem 2.3]). *The category ${}_H\mathcal{M} := (H\text{-}\mathsf{mod}, \underline{\otimes}, H_t)$ is monoidal.*

Now, we present some functors of ${}_H\mathcal{M}$ used later on.

Proposition 1.25 ([BCJ11, Theorem 2.4], [Szl05, Section 6]). *Let $U : {}_H\mathcal{M} \longrightarrow \mathsf{Vec}_{\Bbbk}$ be the forgetful functor. Then U is a Frobenius monoidal functor with monoidal functor structure given by*

$$\begin{aligned} U_{M,N} = \pi_{M,N} : M\otimes N &\longrightarrow M\underline{\otimes}N \\ x\otimes y &\mapsto 1_1\cdot x\otimes 1_2\cdot y, \end{aligned} \qquad\qquad \begin{aligned} U_0 = u_{H_t} : \Bbbk &\longrightarrow H_t \\ 1_{\Bbbk} &\mapsto 1_H, \end{aligned} \tag{1.5}$$

and comonoidal functor structure given by

$$\begin{aligned} U^{M,N} = \iota_{M,N} : M\underline{\otimes}N &\longrightarrow M\otimes N \\ x\underline{\otimes}y &\mapsto 1_1\cdot x\otimes 1_2\cdot y, \end{aligned} \qquad\qquad \begin{aligned} U^0 = \varepsilon|_{H_t} : H_t &\longrightarrow \Bbbk \\ k &\mapsto \varepsilon(k), \end{aligned} \tag{1.6}$$

for all $M, N\in H$-mod.

Remark 1.26. Notice that

$$U_{M,N}U^{M,N} = \mathrm{Id}_{M\underline{\otimes}N}\,. \tag{1.7}$$

Moreover, as a consequence, any H-module has structure of H_t-bimodule with left action $\mu_M : H_t\otimes M \longrightarrow M$ and right H_t-action $\nu_M : M\otimes H_t \longrightarrow M$ given, respectively, by

$$\mu_M(k\otimes x) := k\cdot x, \quad \text{and} \quad \nu_M(x\otimes k) := \varepsilon(1_2k)1_1\cdot x, \tag{1.8}$$

for all $k\in H_t$ and $x\in M$. Furthermore, any H-module map is a morphism of H_t-bimodules.

Now, we establish an identification between monoid objects in ${}_H\mathcal{M}$ and $\Bbbk$-algebras that also possess a compatible H-module structure. This duality allows us to bridge the algebraic and categorical perspectives.

Definition 1.27 (H-module algebra, e.g. [CG00, Section 4.5]). Let A be a $\Bbbk$-algebra. We say that A is an *H-module algebra* if $A \in H\text{-}\mathsf{mod}$, the action is multiplicative, that is, satisfies

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b), \quad \text{for all } h \in H \text{ and } a, b \in A, \tag{1.9}$$

and the action is unital, that is, satisfies

$$\ker(\varepsilon_t) \cdot 1_A = 0. \tag{1.10}$$

Naturally, a morphism between H-module algebras is a $\Bbbk$-algebra map that lies in $H\text{-}\mathsf{mod}$. The category of H-module algebras together with their morphisms is denoted by ${}_H\mathcal{A}$.

Remark 1.28. Adapting [CG00, Proposition 4.15] to the left case, we get that (1.10) is equivalent to each of the following conditions, for all $h, k \in H$ and $a \in A$:

$$\begin{aligned}
(hk) \cdot 1_A &= \varepsilon(h_1 k) h_2 \cdot 1_A; && (1.11)\\
(hk) \cdot 1_A &= \varepsilon(h_2 k) h_1 \cdot 1_A; && (1.12)\\
\varepsilon(1_1 h) 1_2 \cdot a &= (h \cdot 1_A) a; && (1.13)\\
\varepsilon(1_2 h) 1_1 \cdot a &= a (h \cdot 1_A); && (1.14)\\
\varepsilon(1_1 h) 1_2 \cdot 1_A &= h \cdot 1_A; && (1.15)\\
\varepsilon(1_2 h) 1_1 \cdot 1_A &= h \cdot 1_A. && (1.16)
\end{aligned}$$

Recall the notation of Definition 1.20.

Theorem 1.29. *The categories ${}_H\mathcal{A}$ and $\mathrm{Mon}({}_H\mathcal{M})$ are isomorphic.*

Proof. We prove the statement by defining two functors $F : {}_H\mathcal{A} \to \mathrm{Mon}({}_H\mathcal{M})$ and $G : \mathrm{Mon}({}_H\mathcal{M}) \to {}_H\mathcal{A}$, and proving that they are inverse of each other. Firstly, given $A = (A, m_A, u_A) \in {}_H\mathcal{A}$, define

$$F(A) := (A, \underline{m}_A := m_A U^{A,A}, \underline{u}_A := \mu_A(\mathrm{Id}_{H_t} \otimes u_A)),$$

where $\mu_A : H_t \otimes A \to A$ denotes the induced left H_t-action on A of (1.8). By construction,

$$\begin{aligned}
\underline{m}_A : A \underline{\otimes} A &\to A & \qquad \underline{u}_A : H_t &\to A\\
a \underline{\otimes} b &\mapsto ab, & k &\mapsto k \cdot 1_A.
\end{aligned} \tag{1.17}$$

Similarly, given a morphism $f : (A, m_A, u_A) \to (B, m_B, u_B)$ in ${}_H\mathcal{A}$, we define the corresponding morphism $F(f) : (A, \underline{m}_A, \underline{u}_A) \to (B, \underline{m}_B, \underline{u}_B)$ simply as $F(f) := f$. To guarantee that F is in fact a functor, it suffices to show the following:

- $\underline{m}_A \in {}_H\mathcal{M}$: for all $h \in H$ and $a, b \in A$,

$$\underline{m}_A(h \cdot (a \underline{\otimes} b)) \overset{(1.4)}{=} \underline{m}_A(h_1 \cdot a \underline{\otimes} h_2 \cdot b) \overset{(1.17)}{=} (h_1 \cdot a)(h_2 \cdot b)$$
$$\overset{(1.9)}{=} h \cdot (ab) \overset{(1.17)}{=} h \cdot \underline{m}_A(a \underline{\otimes} b).$$

- $\underline{u}_A \in {}_H\mathcal{M}$: for all $h \in H$ and $k \in H_t$,

$$\underline{u}_A(h \cdot k) = \underline{u}_A(\varepsilon_t(hk)) \overset{(1.17)}{=} \varepsilon_t(hk) \cdot 1_A$$
$$\overset{(1.15)}{=} hk \cdot 1 = h \cdot (k \cdot 1_A) \overset{(1.17)}{=} h \cdot \underline{u}_A(k).$$

- $\underline{m}_A$ satisfies the associative constrain: this is immediate form the associativity of m_A.
- $\underline{m}_A$ and $\underline{u}_A$ satisfy the unitality constrains: recall that

$$l_A : H_t \underline{\otimes} A \longrightarrow A \qquad\qquad r_A : A \underline{\otimes} H_t \longrightarrow A$$
$$k \underline{\otimes} a \longmapsto k \cdot a, \qquad\qquad a \underline{\otimes} k \longmapsto k \cdot a,$$

  so for every $k \in H_t$ and $a \in A$,

$$\underline{m}_A(\underline{u}_A \underline{\otimes} \mathrm{Id}_A)(k \underline{\otimes} a) \overset{(1.17)}{=} (k \cdot 1_A)a \overset{(1.13)}{=} \varepsilon_t(k) \cdot a = k \cdot a,$$

  where in the last equality we have used Proposition 1.4(iii); the proof is similar for the right unitality constraint.

- $F(f) = f \in {}_H\mathcal{M}$: this follows from the definition.
- $F(f) = f$ is a morphism of monoids: for $a, b \in A$ and $k \in H_t$,

$$f\ \underline{m}_A\ (a \underline{\otimes} b) = f(ab) = f(a)f(b) = \underline{m}_B(f(a) \underline{\otimes} f(b)),$$
$$f\ \underline{u}_A\ (k) = f(k \cdot 1_A) = k \cdot f(1_A) = k \cdot 1_B = \underline{u}_B(k).$$

The functoriality of F is clear and hence, as claimed, $F : {}_H\mathcal{A} \longrightarrow \mathrm{Mon}({}_H\mathcal{M})$ is a functor.

Conversely, for $A = (A, \underline{m}_A, \underline{u}_A) \in \mathrm{Mon}({}_H\mathcal{M})$, we define

$$G(A) := (A, m_A := \underline{m}_A U_{A,A}, u_A := \underline{u}_A U_0).$$

By construction, $G(A) = (U(A), U(\underline{m}_A)U_{A,A}, U(\underline{u}_A)U_0)$. Since the functor U is monoidal, by Proposition 1.21 we have that $G(A)$ is indeed a $\Bbbk$-algebra. Similarly, for a morphism of monoids $g : (A, \underline{m}_A, \underline{u}_A) \longrightarrow (B, \underline{m}_B, \underline{u}_B)$ we define $G(g) := g$. To guarantee that G is indeed a functor, it suffices to prove the following:

- m_A satisfies (1.9): for $h \in H$ and $a, b \in A$,

$$\begin{aligned} h \cdot m_A(a \otimes b) &= h \cdot \underline{m}_A(1_1 \cdot a \underline{\otimes} 1_2 \cdot b) = \underline{m}_A(h_1 \cdot (1_1 \cdot a) \underline{\otimes} h_2 \cdot (1_2 \cdot b)) \\ &= \underline{m}_A(1_1 \cdot (h_1 \cdot a) \underline{\otimes} 1_2 \cdot (h_2 \cdot b)) = m_A(h_1 \cdot a \otimes h_2 \cdot b) \\ &= m_A(h \cdot (a \otimes b)). \end{aligned}$$

- u_A satisfies (1.10): for $h \in H$,

$$\begin{aligned} h \cdot u_A(1_{\Bbbk}) &= h \cdot \underline{u}_A(1_H) = \underline{u}_A(h \cdot 1_H) = \underline{u}_A(\varepsilon_t(h)) = \underline{u}_A(\varepsilon_t(h) \cdot 1_H) \\ &= \varepsilon_t(h) \cdot \underline{u}_A(1_H) = \varepsilon_t(h) \cdot u_A(1_{\Bbbk}), \end{aligned}$$

  which proves (1.15).

- $G(g) = g$ is a H-module map: by construction $G(g) = g \in {}_H\mathcal{M}$ and for $a, b \in A$,

$$\begin{aligned} g(m_A(a \otimes b)) &= g\, \underline{m}_A\,(1_1 \cdot a \underline{\otimes} 1_2 \cdot b) = \underline{m}_B(1_1 \cdot g(a) \underline{\otimes} 1_2 \cdot g(b)) \\ &= m_B(g \otimes g)(a \otimes b), \end{aligned}$$

  which shows that g is multiplicative, and

$$g(u_A(1_{\Bbbk})) = g(\underline{u}_A(1_H)) = \underline{u}_B(1_H) = u_B(1_{\Bbbk}),$$

  so g is an algebra map.

The functoriality of G is clear and hence, as claimed, $G : \mathrm{Mon}({}_H\mathcal{M}) \longrightarrow {}_H\mathcal{A}$ is a functor.

Finally, we prove that F and G are inverse to each other. If $(A, \underline{m}_A, \underline{u}_A) \in \mathrm{Mon}({}_H\mathcal{M})$, then

$$FG(A) = (A, \underline{m}_A U_{A,A} U^{A,A}, \mu_A(\mathrm{Id}_{H_t} \otimes \underline{u}_A\, U_0)).$$

It is clear from (1.7) that $\underline{m}_A U_{A,A} U^{A,A} = \underline{m}_A$, and

$$\mu_A(\mathrm{Id}_{H_t} \otimes \underline{u}_A\, U_0)(1_H) = \mu_A(1_H \otimes \underline{u}_A(1_H)) = 1_H \cdot \underline{u}_A(1_H) = \underline{u}_A(1_H),$$

so $FG(A) = A$. Clearly, for any morphism of monoids g in $\mathrm{Mon}({}_H\mathcal{M})$ we have $FG(g) = g$. Hence $FG = \mathrm{Id}_{\mathrm{Mon}({}_H\mathcal{M})}$.

Reciprocally, if $A = (A, m_A, u_A) \in {}_H\mathcal{A}$, then

$$GF(A) = (A, m_A U^{A,A} U_{A,A}, \mu_A(\mathrm{Id}_{H_t} \otimes u_A) U_0).$$

For $a, b \in A$ we have

$$\begin{aligned} m_A U^{A,A} U_{A,A}(a \otimes b) &= m_A U^{A,A}(1_1 \cdot a \underline{\otimes} 1_2 \cdot b) = m_A(1_1 \cdot a \otimes 1_2 \cdot b) \\ &= (1_1 \cdot a)(1_2 \cdot b) \overset{(1.9)}{=} 1 \cdot (ab) = ab, \end{aligned}$$

and thus $m_A U^{A,A} U_{A,A} = m_A$. Similarly,

$$[\mu_A(\mathrm{Id}_{H_t} \otimes u_A)U_0](1_{\Bbbk}) = \mu_A(\mathrm{Id}_{H_t} \otimes u_A)(1_H) = \mu_A(1_H \otimes 1_A) = 1_H \cdot 1_A = 1_A,$$

which means $\mu_A(\mathrm{Id}_{H_t} \otimes u_A)U_0 = u_A$. Clearly, for any morphism of H-module algebras $f \in {}_H\mathcal{A}$ we have $GF(f) = f$ and thus $GF = \mathrm{Id}_{{}_H\mathcal{A}}$. □

Example 1.30. Since the counital subalgebra H_t is clearly a left H-module algebra, as consequence of the previous result we have $H_t \in \mathrm{Mon}({}_H\mathcal{M})$, which is consistent with the fact that in a monoidal category the unit object is always a monoid.

The isomorphism of Theorem 1.29 allows us to make the following convention: if A is a $\Bbbk$-algebra and an H-module so that $A \in \mathrm{Mon}({}_H\mathcal{M})$ (by satisfying (1.9)-(1.10)), we call A an *H-module algebra*. This makes the proof of the following proposition straightforward.

Proposition 1.31. *Let $\{H_x\}_{x\in X}$ be a finite collection of a weak Hopf algebras.*

(i) *Then $H = \bigoplus_{x\in X} H_x$ is a weak Hopf algebra. The algebra operations and antipode are defined component-wise, while the coalgebra structure is given by the sum of the component-wise operations. Moreover, $H_s = \bigoplus_{x\in X}(H_x)_s$ and $H_t = \bigoplus_{x\in X}(H_x)_t$.*

(ii) *If, for each $x \in X$, A_x is a H_x-module algebra, then $\bigoplus_{x\in X} A_x$ is an H-module algebra with H-action defined component-wise.*

1.3.2 Weak coactions

Dually, we start by recalling the monoidal structure of the right corepresentation category comod-H; see e.g., [BCJ11, Section 4] or [WWW22, Section 3.2] for further details. It suffices to work with right corepresentations since H^{cop} is also a weak bialgebra and comod-$H^{\mathrm{cop}} \cong H$-comod [NV02, Remark 2.4.1]. As stated in the Introduction, for the rest of this section, we assume that all comodules are right-sided.

Definition 1.32 (H-monoidal product of comodules). Given two H-comodules M and N, the *H-monoidal product* of M with N is the subspace

$$M\overline{\otimes}N := \langle x \otimes y \in M \otimes N \mid x \otimes y = \varepsilon(x_1 y_1)x_0 \otimes y_0\rangle \subseteq M \otimes N.$$

If $x \otimes y \in M\overline{\otimes}N$, we write $x\overline{\otimes}y$. Given two morphisms $f : M \to M'$ and $g : N \to N'$ in comod-H, the *H-monoidal product* of f with g is given by

$$\begin{aligned} f\overline{\otimes}g : M\overline{\otimes}N &\to M'\overline{\otimes}N' \\ x\overline{\otimes}y &\mapsto f(x)\overline{\otimes}g(y). \end{aligned}$$

Remark 1.33. Let M, N, f, g as in Definition 1.32.

(i) The H-coaction on $M\overline{\otimes}N$ is given by

$$\begin{aligned} \rho_{M\overline{\otimes}N} : M\overline{\otimes}N &\longrightarrow M\overline{\otimes}N \otimes H \\ x\overline{\otimes}y &\longmapsto x_0\overline{\otimes}y_0 \otimes x_1y_1, \end{aligned} \tag{1.18}$$

for all $h \in H$, $x \in M$ and $y \in N$. Hence, $M\overline{\otimes}N \in \mathsf{comod}\text{-}H$. Furthermore, there are inclusion and projection maps given by

$$\begin{aligned} \iota_{M,N} : M\overline{\otimes}N &\longrightarrow M \otimes N \\ x\overline{\otimes}y &\longmapsto \varepsilon(x_1y_1)x_0 \otimes y_0, \end{aligned} \qquad \begin{aligned} \pi_{M,N} : M \otimes N &\longrightarrow M\overline{\otimes}N \\ x \otimes y &\longmapsto \varepsilon(x_1y_1)x_0\overline{\otimes}y_0. \end{aligned}$$

(ii) Since we have

$$\varepsilon(f(x)_1g(y)_1)f(x)_0\overline{\otimes}g(y)_0 = \varepsilon(x_1y_1)f(x_0)\overline{\otimes}g(y_0),$$

for every $x\overline{\otimes}y \in M\overline{\otimes}N$, the H-monoidal product of H-comodule maps is well defined. Moreover, $f\overline{\otimes}g = (f \otimes g)|_{M\overline{\otimes}N}$.

The counital subalgebra H_s is an H-comodule since the image of Δ_{H_s} is a subspace of $H_s \otimes H$ [Proposition 1.4(vi)], and so we can set $\Delta_{H_s} H_s \longrightarrow H_s \otimes H$. This allows us to define the H-isomorphisms given by

$$\begin{aligned} l_M : H_s\overline{\otimes}M &\longrightarrow M \\ k\overline{\otimes}x &\longmapsto \varepsilon(kx_1)x_0, \end{aligned} \qquad \begin{aligned} r_M : M\overline{\otimes}H_s &\longrightarrow M \\ x\overline{\otimes}k &\longmapsto \varepsilon(x_1k)x_0, \end{aligned}$$

for every $M \in \mathsf{comod}\text{-}H$. Using (1.3), the inverses of these morphisms are defined as

$$\begin{aligned} l_M^{-1} : M &\longrightarrow H_s\overline{\otimes}M \\ x &\longmapsto \varepsilon(1_2x_1)1_1\overline{\otimes}x_0, \end{aligned} \qquad \begin{aligned} r_M^{-1} : M &\longrightarrow M\overline{\otimes}H_t \\ x &\longmapsto \varepsilon(x_11_2)1_1. \end{aligned}$$

These constructions guarantee the following result.

Proposition 1.34 ([BCJ11, Theorem 3.1]). *The category $\mathcal{M}^H := (\mathsf{comod}\text{-}H, \overline{\otimes}, H_s)$ is monoidal.*

Now, we present some functors of $\mathcal{M}^H$ used later.

Proposition 1.35 ([BCJ11, Theorem 3.2], [Szl05, Section 6]). *Let $U : \mathcal{M}^H \longrightarrow \mathsf{Vec}_{\Bbbk}$ be the forgetful functor. Then U is a Frobenius monoidal functor with monoidal functor structure given by*

$$\begin{aligned} U_{M,N} = \pi_{M,N} : M \otimes N &\longrightarrow M\overline{\otimes}N \\ x \otimes y\, &\varepsilon(x_1y_1)x_0\overline{\otimes}y_0, \end{aligned} \qquad \begin{aligned} U_0 = u_{H_s} : \Bbbk &\longrightarrow H_s \\ 1_{\Bbbk} &\longmapsto 1_H, \end{aligned} \tag{1.19}$$

and comonoidal functor structure given by

$$\begin{aligned} U^{M,N} = \iota_{M,N} : M\overline{\otimes}N &\longrightarrow M \otimes N \\ x\overline{\otimes}y &\longmapsto \varepsilon(x_1y_1)x_0 \otimes y_0, \end{aligned} \qquad \begin{aligned} U^0 = \varepsilon|_{H_s} : H_s &\longrightarrow \Bbbk \\ k &\longmapsto \varepsilon(k), \end{aligned} \tag{1.20}$$

for all $M, N \in \mathsf{comod}\text{-}H$.

Remark 1.36. Notice that
$$U_{M,N}U^{M,N} = \mathrm{Id}_{M\overline{\otimes}N}\,. \tag{1.21}$$
Moreover, as a consequence, any H-comodule has structure of H_s-bimodule with left H_s-action $\mu_M : H_s \otimes M \to M$ and right H_s-action $\nu_M : M \otimes H_s \to M$ given, respectively, by
$$\mu_M(k \otimes x) := \varepsilon(kx_1)x_0, \quad \text{and} \quad \nu_M(x \otimes k) := \varepsilon(x_1 k)x_0, \tag{1.22}$$
for all $k \in H_s$ and $x \in M$. Furthermore, any H-comodule map is a morphism of H_s-bimodules.

Dual to Section 1.3.1, we establish an identification between monoid objects in $\mathcal{M}^H$ and $\Bbbk$-algebras that also possess a compatible H-comodule structure.

Definition 1.37 (H-comodule algebra, e.g. [CG00, Section 4.5]). Let A be a $\Bbbk$-algebra. We say that A is an *H-comodule algebra* if $A \in \mathsf{comod}\text{-}H$ via $\rho : A \to A \otimes H$, the coaction is multiplicative, that is, satisfies
$$(ab)_0 \otimes (ab)_1 = a_0 b_0 \otimes a_1 b_1, \quad \text{for all } a, b \in A, \tag{1.23}$$
and the coaction is unital, that is, satisfies
$$\rho(1_A) \in A \otimes H_t. \tag{1.24}$$
Naturally, a morphism between H-comodule algebras is a $\Bbbk$-algebra map that lies in $\mathsf{comod}\text{-}H$. The category of H-comodule algebras together with their morphisms is denoted by $\mathcal{A}^H$.

Recall the notation of Definition 1.20.

Theorem 1.38 ([BCM02, Proposition 3.9], [WWW22, Theorem 4.4]). *The categories $\mathcal{A}^H$ and* $\mathrm{Mon}(\mathcal{M}^H)$ *are isomorphic.*

Proof. Since the proof is somewhat dual to that of Theorem 1.29 and can be found in the referenced sources, we will only outline how to construct the mutually inverse functors $F : \mathcal{A}^H \to \mathrm{Mon}(\mathcal{M}^H)$ and $G : \mathrm{Mon}(\mathcal{M}^H) \to \mathcal{A}^H$.

Firstly, for a given $A = (A, m_A, u_A) \in \mathcal{A}^H$, define
$$F(A) := (A, \overline{m}_A := m_A U^{A,A}, \overline{u}_A := \nu_A(u_A \otimes \mathrm{Id}_{H_s})),$$
where $\nu_A : A \otimes H_s \to A$ denotes the induced right H_s-action on A of (1.22). By construction,
$$\begin{aligned} \overline{m}_A : A\overline{\otimes}A &\to A & \qquad \overline{u}_A : H_s &\to A \\ a\overline{\otimes}b &\mapsto ab, & k &\mapsto \varepsilon(1_1 k)1_0. \end{aligned}$$
Similarly, given $f : (A, m_A, u_A) \to (B, m_B, u_B)$ in $\mathcal{A}^H$, we define the corresponding morphism $F(f) : (A, \overline{m}_A, \overline{u}_A) \to (B, \overline{m}_B, \overline{u}_B)$ simply as $F(f) := f$.

Reciprocally, for $A = (A, \overline{m}_A, \overline{u}_A) \in \mathrm{Mon}(\mathcal{M}^H)$, we define

$$G(A) := (A, m_A := \overline{m}_A U_{A,A}, u_A := \overline{u}_A U_0).$$

By construction, $G(A) = (U(A), U(\overline{m}_A)U_{A,A}, U(\overline{u}_A)U_0)$ and since the functor U is monoidal, by Proposition 1.21 we have that $G(A)$ is indeed a $\Bbbk$-algebra. Moreover, for a morphism of monoids $g : (A, \overline{m}_A, \overline{u}_A) \to (B, \overline{m}_B, \overline{u}_B)$ we simply define $G(g) := g$. □

Example 1.39. Since the counital subalgebra H_s is clearly a right H-comodule algebra, as consequence of the previous result we have $H_s \in \mathrm{Mon}(\mathcal{M}^H)$, which is consistent with the fact that in a monoidal category the unit object is always a monoid.

Example 1.40 ([Hay99, Equation 2.15]). Let Q be a finite quiver, $\Bbbk Q$ its path algebra and $\mathfrak{H}(Q)$ its attached Hayashi's face algebra. Recall from Example 1.10 that a $\Bbbk$-basis for $\mathfrak{H}(Q)_t$ is given by $\{\sum_{i \in Q_0} x_{i,j}\}_{j \in Q_0}$. Let $k \in \mathbb{N}$ and define for any path $p \in Q_k$ the map

$$\begin{aligned} \rho : \Bbbk Q &\to \Bbbk Q \otimes \mathfrak{H}(Q) \\ p &\mapsto \sum_{q \in Q_k} q \otimes x_{q,p}. \end{aligned}$$

Hence ρ is a $\mathfrak{H}(Q)$-coaction on $\Bbbk Q$. Moreover, $\Bbbk Q$ is a $\mathfrak{H}(Q)$-comodule algebra. A complete proof of these facts can be found in [WWW22, Example 4.9].

The isomorphism of Theorem 1.38 allows us to make the following convention: if A is a $\Bbbk$-algebra and an H-comodule so that $A \in \mathrm{Mon}(\mathcal{M}^H)$ (by satisfying (1.23)-(1.24)), we call A an *H-comodule algebra*.

Finally, we give a notion that will be mentioned in Chapter 3.

Definition 1.41. Let H be a $\Bbbk$-weak bialgebra and A a $\Bbbk$-algebra. We say that H *coacts universally* on A if it satisfies the following properties:

(i) A is an H-comodule algebra via coaction $\rho : A \to A \otimes H$,

(ii) if H' is another $\Bbbk$-weak bialgebra so that A is also a H'-comodule algebra via coaction $\rho' : A \to A \otimes H'$, then there is a unique weak bialgebra map $\pi : H \to H'$ so that $(\pi \otimes \mathrm{Id}_A)\rho = \rho'$.

1.4 Groupoids and groupoid algebras

In this section, we introduce the fundamental terminology of groupoids, which will be used in Chapter 2. Although we have already mentioned it in Example 1.8, we provide the precise definition of a groupoid for clarity.

Definition 1.42. ($\mathcal{G}$, $\mathcal{G}_0$, $\mathcal{G}_1$, e_x) A *groupoid* $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ is a small category in which every morphism is an isomorphism. Here, $\mathcal{G}_0$ (resp., $\mathcal{G}_1$) is the set of objects (resp., morphisms) of $\mathcal{G}$. If $g \in \mathcal{G}_1$, then $s(g)$ (resp., $t(g)$) denotes the source (resp., target) object of g. By convention, we compose elements of a groupoid from right to left. For each $x \in \mathcal{G}_0$, the identity morphism of $\mathrm{Hom}_{\mathcal{G}}(x,x)$ is denoted by e_x. A *morphism of groupoids* is simply a functor between groupoids.

A group G can be viewed as a groupoid with only one object. Unless otherwise stated, we assume that $\mathcal{G}_0$ is a nonempty finite set. If $\mathcal{G}_1$ is also a finite set, then we call $\mathcal{G}$ *finite*. Throughout we will use the following notation.

Notation 1.43 (X). Henceforth, let X be a nonempty finite set.

Definition 1.44 (X-Grpd). Let X-Grpd be the category defined as follows:

- the objects are groupoids $\mathcal{G}$ such that $\mathcal{G}_0 = X$; we call these *X-groupoids*.
- the morphisms are groupoid morphisms leaving X fixed, that is, functors of the form $\pi : \mathcal{G} \to \mathcal{G}'$ such that $\pi(x) = x$ for all $x \in X$; we call these functors *X-groupoid morphisms*.

1.4.1 Modules over groupoids

Some of the concepts here are partially adapted from [BHS11, Definition 7.1.7] and [PT14, p. 85].

Definition 1.45 (X-decomposable vector space). A vector space V is *X-decomposable* if there exists a family $\{V_x\}_{x\in X}$ of subspaces of V such that $V = \bigoplus_{x\in X} V_x$. We call the $\{V_x\}_{x\in X}$ the *components* of V. If $V = \bigoplus_{x\in X} V_x$ and $W = \bigoplus_{x\in X} W_x$ are X-decomposable vector spaces, a $\Bbbk$-linear map $f : V \to W$ is said to be *X-decomposable* if there is a family $\{f_x : V_x \to W_x\}_{x\in X}$ of $\Bbbk$-linear maps such that $f|_{V_x} = f_x$ for all $x \in X$. In this case, we write $f = (f_x)_{x\in X}$.

For instance, every vector space V has a trivial X-decomposition if $|X| = 1$. Even if $|X| > 1$, V has an X-decomposition where $V_x = V$ for one $x \in X$, and $V_y = 0$ otherwise.

Definition 1.46 ($\mathcal{G}$-module). Let $\mathcal{G}$ be an X-groupoid. An X-decomposable vector space $V = \bigoplus_{x\in X} V_x$ is said to be a *left $\mathcal{G}$-module* if it is equipped with, for each $x, y \in X$, a $\Bbbk$-linear map $\mathrm{Hom}_{\mathcal{G}}(x,y) \times V_x \to V_y$, denoted $(g, v) \mapsto g \cdot v$, such that

- $(gh) \cdot v = g \cdot (h \cdot v)$, for all $g, h \in \mathcal{G}_1$ with $t(h) = s(g)$ and all $v \in V_{s(h)}$, and
- $e_x \cdot v = v$, for all $x \in X$ and $v \in V_x$.

Given two left $\mathcal{G}$-modules $V = \bigoplus_{x\in X} V_x$ and $W = \bigoplus_{x\in X} W_x$, a *$\mathcal{G}$-module morphism* $f = (f_x)_{x\in X} : V \to W$ is an X-decomposable $\Bbbk$-linear map such that

$$g \cdot f_{s(g)}(v) = f_{t(g)}(g \cdot v), \qquad \text{for all } g \in \mathcal{G}_1 \text{ and } v \in V_{s(g)}. \tag{1.25}$$

For two $\mathcal{G}$-module morphisms $f = (f_x)_{x\in X} : V \to W$ and $f' = (f'_x)_{x\in X} : W \to Z$, their *composition* is defined as the $\mathcal{G}$-module morphism $f'f = (f'_x f_x)_{x\in X} : V \to Z$.

Remark 1.47. Let $\mathcal{G}$ be an X-groupoid and let $V = \bigoplus_{x\in X} V_x$ be a left $\mathcal{G}$-module. Notice that the action is defined locally, so if $g \in \mathcal{G}_1$, then $g \cdot v$ makes sense only when $v \in V_{s(g)}$.

For this reason, in the literature, the linear maps associated to the module are called a *partial (groupoid) action* (see e.g., [BP12, Section 1]). Also, when $|X| = 1$ (that is, when $\mathcal{G}$ is a group), Definition 1.46 recovers the classical notion of a module over a group.

The following result is an adapted version of [IR19a, Proposition 9.3].

Lemma 1.48. *Let $\mathcal{G}$ be an X-groupoid and let $V = \bigoplus_{x\in X} V_x$ be a left $\mathcal{G}$-module. Then, for each $g \in \mathcal{G}_1$, the linear map $\nu_g : V_{s(g)} \to V_{t(g)}$ given by $v \mapsto g \cdot v$ is an isomorphism. In particular, $(\{V_x\}_{x\in X}, \{\nu_g\}_{g\in\mathcal{G}_1})$ is a groupoid.*

Proof. If $g : x \to y$ is a morphism in $\mathcal{G}_1$, then $\nu_g^{-1} = \nu_{g^{-1}}$. Indeed, for every $v \in V_x$ we have $\nu_{g^{-1}}\nu_g(v) = g^{-1} \cdot (g \cdot v) = (g^{-1}g) \cdot v = e_x \cdot v = v$. Similarly, for any $v' \in V_y$ we have $\nu_g\nu_{g^{-1}}(v') = e_y \cdot v' = v'$. In particular, $\nu_{e_x} = \mathrm{Id}_{V_x}$, for every $x \in X$. □

Notation 1.49 ($\nu_g, \omega_g, \alpha_g$)**.** The maps $\{\nu_g\}_{g\in\mathcal{G}_1}$ above are called the *structure isomorphisms* of the $\mathcal{G}$-module $V = \bigoplus_{x\in X} V_x$. When dealing with groupoids, we will use the Greek letters $\{\nu_g\}_{g\in\mathcal{G}_1}$, $\{\omega_g\}_{g\in\mathcal{G}_1}$, $\{\alpha_g\}_{g\in\mathcal{G}_1}$, etc., to denote the respective structure isomorphisms of given $\mathcal{G}$-modules $V = \bigoplus_{x\in X} V_x$, or $W = \bigoplus_{x\in X} W_x$, or $A = \bigoplus_{x\in X} A_x$, etc.

For instance, if $f = (f_x)_{x\in X} : V \to W$ is a $\mathcal{G}$-morphism, then condition (1.25) can be restated as $\omega_g f_{s(g)} = f_{t(g)}\nu_g$, for all $g \in \mathcal{G}_1$.

The category of left $\mathcal{G}$-modules can be endowed with a monoidal structure. The proof of the following result is straightforward.

Lemma 1.50 ($\mathcal{G}$-mod)**.** *Let $\mathcal{G}$ be an X-groupoid let $\mathcal{G}$-mod be the category of left $\mathcal{G}$-modules. This category admits a monoidal structure as follows.*

- *If $V = \bigoplus_{x\in X} V_x, W = \bigoplus_{x\in X} W_x \in \mathcal{G}\text{-}\mathsf{mod}$, then $V \otimes_{\mathcal{G}\text{-}\mathsf{mod}} W = \bigoplus_{x\in X} V_x \otimes W_x$; the structure isomorphisms are $\{\nu_g \otimes \omega_g\}_{g\in\mathcal{G}_1}$,*
- *$\mathbb{1}_{\mathcal{G}\text{-}\mathsf{mod}} = \bigoplus_{x\in X} \Bbbk$ with the structure isomorphisms $\{\kappa_g = \mathrm{Id}_{\Bbbk}\}_{g\in\mathcal{G}_1}$.*

If $f = (f_x)_{x\in X} : V \to W$ and $f' = (f'_x)_{x\in X} : V' \to W'$ are two $\mathcal{G}$-module morphisms, then $f \otimes_{\mathcal{G}\text{-}\mathsf{mod}} f' = (f_x \otimes f'_x)_{x\in X}$. The associativity constraint is that induced by the tensor product of components, and the left/right unital constraints

$$l_V : \mathbb{1}_{\mathcal{G}\text{-}\mathsf{mod}} \otimes_{\mathcal{G}\text{-}\mathsf{mod}} V \longrightarrow V, \qquad r_V : V \otimes_{\mathcal{G}\text{-}\mathsf{mod}} \mathbb{1}_{\mathcal{G}\text{-}\mathsf{mod}} \longrightarrow V,$$

are given by scalar multiplication, that is, for each $x \in X$,

$$(l_V)_x : \Bbbk \otimes V_x \to V_x, \;\; k \otimes v \mapsto kv \qquad (r_V)_x : V_x \otimes \Bbbk \to V_x, \;\; v \otimes k \mapsto kv.$$

For a group G, Lemma 1.50 implies the well-known result that G-mod is a monoidal category under the usual tensor product $\otimes_{\Bbbk}$. Now we provide specific examples of modules over a groupoid.

Example 1.51. Consider the following groupoid:

$$\mathcal{G} = \begin{tikzcd} \bullet_x \arrow[loop left, "e_x"] \arrow[r, bend left, "g"] & \bullet_y \arrow[loop right, "e_y"] \arrow[l, bend left, "g^{-1}"] \end{tikzcd} \tag{1.26}$$

(a) Let $V_x = V_y = \Bbbk^2$ and define for all $a, b \in \Bbbk$,

$$g \cdot (a,b) = g^{-1} \cdot (a,b) := (b,a), \qquad e_x \cdot (a,b) = e_y \cdot (a,b) := (a,b).$$

Then $V = V_x \oplus V_y$ is a left $\mathcal{G}$-module; $\nu_g, \nu_{g^{-1}} : \Bbbk^2 \to \Bbbk^2$ are given by $(a,b) \mapsto (b,a)$.

(b) Let $W_x = W_y = \Bbbk^3$ and define for all $a, b, c \in \Bbbk$,

$$g \cdot (a,b,c) = g^{-1} \cdot (a,b,c) := (-a,c,b), \qquad e_x \cdot (a,b,c) = e_y \cdot (a,b,c) := (a,b,c).$$

Then $W = W_x \oplus W_y$ is a left $\mathcal{G}$-module. Also, $\omega_g, \omega_{g^{-1}} : \Bbbk^3 \to \Bbbk^3$ are both given by $(a,b,c) \mapsto (-a,c,b)$. Furthermore, the maps $f_x, f_y : \Bbbk^2 \to \Bbbk^3$ both defined as $(a,b) \mapsto (0,a,b)$ make $f = (f_x, f_y)$ a $\mathcal{G}$-module morphism from V to W.

(c) Let $Z_x = Z_y = \Bbbk[t,t^{-1}]$ and define for every $r \in \Bbbk[t,t^{-1}]$,

$$g \cdot r := tr, \qquad g^{-1} \cdot r := t^{-1}r, \qquad e_x \cdot r = e_y \cdot r := r.$$

Then $Z = Z_x \oplus Z_y$ is an infinite-dimensional left $\mathcal{G}$-module. The structure isomorphisms $\xi_g, \xi_{g^{-1}} : \Bbbk[t,t^{-1}] \to \Bbbk[t,t^{-1}]$ are given by $\xi_g(r) = tr$ and $\xi_{g^{-1}}(r) = t^{-1}r$, for every $r \in \Bbbk[t,t^{-1}]$.

(d) Let $A_x = A_y = \Bbbk[t,t^{-1}]$ and let σ be the automorphism of $\Bbbk[t,t^{-1}]$ define by $\sigma(t) = t^{-1}$. For $r \in \Bbbk[t,t^{-1}]$, define

$$g \cdot r = \sigma(r) = g^{-1} \cdot r, \quad e_x \cdot r = e_y \cdot r = r.$$

Then $A = A_x \oplus A_y$ is an infinite-dimensional left $\mathcal{G}$-module with structure isomorphisms $\alpha_g = \alpha_{g^{-1}} = \sigma$.

1.4.2 Representations of groupoids

Now we focus our study on representations of groupoids, by adapting terminology present in [IR19a, Section 9.3], [IR19b, Section 3.1] and [PF13, Section 2.3]. First, we introduce a generalization of the general linear group, $\mathrm{GL}(V)$, over a vector space V.

Definition 1.52 ($\mathrm{GL}_X(V), \mathrm{GL}_{(d_1,\ldots,d_n)}(\Bbbk)$)**.** Let $V = \bigoplus_{x \in X} V_x$ be an X-decomposable vector space. We define the X-groupoid $\mathrm{GL}_X(V)$, which we call the X-*general linear groupoid* of V, as follows:

- the object set is X,

- for any $x, y \in X$, $\mathrm{Hom}_{\mathrm{GL}_X(V)}(x, y)$ is the space of linear isomorphisms between the vector spaces V_x and V_y.

If $X = \{1, \dots, n\}$ and V_i has dimension d_i, then we also denote $\mathrm{GL}_X(V)$ by $\mathrm{GL}_{(d_1,\dots,d_n)}(\Bbbk)$ for $d_1 \leq d_2 \leq \cdots \leq d_n$.

This generalizes the classical notation $\mathrm{GL}_d(\Bbbk) = \mathrm{GL}(V)$ when V has dimension d. Note that, in general, this X-groupoid is not finite (but always has finitely many objects).

Example 1.53. If $X = \{x, y\}$, then $\Bbbk^4$ is X-decomposable by taking $(\Bbbk^4)_x := (\Bbbk, \Bbbk, 0, 0)$ and $(\Bbbk^4)_y := (0, 0, \Bbbk, \Bbbk)$. Moreover, we have

$$\mathrm{GL}_X(\Bbbk^4) = \mathrm{GL}_{(2,2)}(\Bbbk) = \begin{tikzcd} x \arrow[loop left, dashed] \arrow[r, bend left, dashed] & y \arrow[l, bend left, dashed] \arrow[loop right, dashed] \end{tikzcd} \cong \begin{tikzcd} \Bbbk^2 \arrow[loop left, dashed] \arrow[r, bend left, dashed] & \Bbbk^2 \arrow[l, bend left, dashed] \arrow[loop right, dashed] \end{tikzcd},$$

where the dashed arrows can be identified with the spaces $\mathrm{GL}_2(\Bbbk)$ of linear isomorphisms.

Definition 1.54 (Representation of $\mathcal{G}$). Let $\mathcal{G}$ be an X-groupoid. A *representation of $\mathcal{G}$* is an X-decomposable vector space $V = \bigoplus_{x \in X} V_x$ equipped with a X-groupoid morphism $\pi : \mathcal{G} \to \mathrm{GL}_X(V)$, called the *representation map* of V. We often denote this as (V, π). Given two representations (V, π), (W, τ), a *morphism of $\mathcal{G}$-representations* is a $\Bbbk$-linear natural transformation $\varphi : \pi \Rightarrow \tau$, that is, a family of $\Bbbk$-linear maps $\varphi = \{\varphi_x : V_x \to W_x\}_{x \in X}$ such that $\varphi_{t(g)}\pi(g) = \tau(g)\varphi_{s(g)}$ for every $g \in \mathcal{G}_1$.

The $\mathcal{G}$-representations, together with their morphisms, form a category that possesses a monoidal structure.

Lemma 1.55 ($\mathsf{rep}(\mathcal{G})$). *Let $\mathcal{G}$ be an X-groupoid and let $\mathsf{rep}(\mathcal{G})$ be the category of $\mathcal{G}$-representations. This category admits a monoidal structure as follows.*

- *If $(V, \pi), (W, \tau) \in \mathsf{rep}(\mathcal{G})$, then $V \otimes_{\mathsf{rep}(\mathcal{G})} W = \bigoplus_{x \in X} V_x \otimes W_x$; the representation map $\mathcal{G} \to \mathrm{GL}_X(V \otimes_{\mathsf{rep}(\mathcal{G})} W)$ is given by $g \mapsto \pi(g) \otimes \tau(g)$, for all $g \in \mathcal{G}_1$,*
- *$\mathbb{1}_{\mathsf{rep}(\mathcal{G})} = \bigoplus_{x \in X} \Bbbk$; the representation map $\mathcal{G} \to \mathrm{GL}_X(\mathbb{1}_{\mathsf{rep}(\mathcal{G})})$ is given by $g \mapsto \mathrm{Id}_{\Bbbk}$, for all $g \in \mathcal{G}_1$.*

Given $\mathcal{G}$-representations $(V, \pi), (W, \tau), (V', \pi'), (W', \tau')$, if $\varphi = \{\varphi_x : V_x \to W_x\}_{x \in X}$ and $\varphi' = \{\varphi'_x : V'_x \to W'_x\}_{x \in X}$ are morphisms of $\mathcal{G}$-representations, then $\varphi \otimes_{\mathsf{rep}(\mathcal{G})} \varphi' = (\varphi_x \otimes \varphi'_x)_{x \in X}$. The associativity constraint is that induced by the tensor product of components, and the left/right unital constraints

$$l_V : \mathbb{1}_{\mathsf{rep}(\mathcal{G})} \otimes_{\mathsf{rep}(\mathcal{G})} V \longrightarrow V, \qquad r_V : V \otimes_{\mathsf{rep}(\mathcal{G})} \mathbb{1}_{\mathsf{rep}(\mathcal{G})} \longrightarrow V,$$

are given by scalar multiplication, that is, for each $x \in X$,

$$(l_V)_x : \Bbbk \otimes V_x \to V_x, \quad k \otimes v \mapsto kv \qquad (r_V)_x : V_x \otimes \Bbbk \to V_x, \quad v \otimes k \mapsto kv.$$

Remark 1.56. In [IR19b, Definition 2], a $\mathcal{G}$-representation is defined as a functor from $\mathcal{G}$ to $\mathsf{Vec}_{\Bbbk}$; our definition is a repackaging of the information carried by such a functor. One benefit of this repackaging is that in the case that $|X| = 1$ (that is, when $\mathcal{G}$ is a group), Definition 1.54 recovers the classical notion of a group representation, that is, a vector space V equipped with a group morphism $\pi : G \to \mathrm{GL}(V)$.

Example 1.57. Let $\mathcal{G}$ be as in (1.26). Then the $\{x, y\}$-decomposable vector space $\Bbbk^4$ of Example 1.53 is a $\mathcal{G}$-representation by taking $\pi : \mathcal{G} \to \mathrm{GL}_X(\Bbbk^4) = \mathrm{GL}_{(2,2)}(\Bbbk)$ as

$$\begin{aligned} \pi(g) : (\Bbbk, \Bbbk, 0, 0) &\longrightarrow (0, 0, \Bbbk, \Bbbk) \\ (a, b, 0, 0) &\mapsto (0, 0, b, a), \end{aligned} \qquad \begin{aligned} \pi(g^{-1}) : (0, 0, \Bbbk, \Bbbk) &\longrightarrow (\Bbbk, \Bbbk, 0, 0) \\ (0, 0, a, b) &\mapsto (b, a, 0, 0). \end{aligned}$$

The following result reconciles various notions of (linear) groupoid actions in the literature and reduces to a well-known result for groups when $\mathcal{G}$ has one object; see, e.g. [BHS11, Section 7.1.ii], [BP12, Section 1], [IR19a, Section 9.3], [IR19b, Section 3.1], [PF13, Section 2.3], or [PT14, Section 3].

Lemma 1.58. *Let $\mathcal{G}$ be an X-groupoid. Then, the categories $\mathcal{G}$-mod and $\mathsf{rep}(\mathcal{G})$ are monoidally isomorphic.*

Proof. Consider the functor $F : \mathcal{G}\text{-}\mathsf{mod} \to \mathsf{rep}(\mathcal{G})$ that sends a left $\mathcal{G}$-module $V = \bigoplus_{x\in X} V_x$ with associated structure isomorphisms $\{v_g\}_{g\in\mathcal{G}_1}$ to the representation $V = \bigoplus_{x\in X} V_x$ with representation map $\pi : \mathcal{G} \to \mathrm{GL}_X(V)$ defined as $\pi(g) := v_g$ for all $g \in \mathcal{G}_1$. Here, we must have $\pi(x) = x$, for all $x \in X$, by Definition 1.54. Moreover, if $f = (f_x)_{x\in X} : V \to W$ is a $\mathcal{G}$-module morphism, then $F(f) = f$. It is clear that the functor is an isomorphism of categories.

For any two left $\mathcal{G}$-modules V, W consider $F_{V,W} : F(V) \otimes_{\mathsf{rep}(\mathcal{G})} F(W) \to F(V \otimes_{\mathcal{G}\text{-}\mathsf{mod}} W)$ given by $(F_{V,W})_x = \mathrm{Id}_{V_x \otimes W_x}$ for all $x \in X$. Then $\{F_{V,W}\}_{V,W\in\mathcal{G}\text{-}\mathsf{mod}}$ is clearly a natural isomorphism. Also, consider the morphism $F_0 : \mathbb{1}_{\mathsf{rep}(\mathcal{G})} \to F(\mathbb{1}_{\mathcal{G}\text{-}\mathsf{mod}})$ in $\mathsf{rep}(\mathcal{G})$ given by $(F_0)_x = \mathrm{Id}_{\Bbbk} : \Bbbk \to \Bbbk$ for all $x \in X$, which is clearly invertible. By straightforward verification, we can confirm that F satisfies the associativity and unit constraints outlined in Definition 1.18. We can conclude that F is a strong monoidal functor and thus, F is a monoidal isomorphism. □

When $\mathcal{G} = G$ is a group, this results recovers the classical monoidal isomorphism between G-mod and $\mathsf{rep}(G)$, given by $V \mapsto V$ and $g \cdot v = \pi(g)(v)$, for all $g \in G$ and $v \in V$.

Example 1.59. The $\mathcal{G}$-module $\Bbbk^2 \oplus \Bbbk^2$ of Example 1.51(a) corresponds to the $\mathcal{G}$-representation $\Bbbk^4 \cong \Bbbk^2 \oplus \Bbbk^2$ of Example 1.57 via the correspondence of Lemma 1.58.

1.4.3 Module algebras over groupoids

Next, we study module algebras over groupoids.

Definition 1.60. Let A be a $\Bbbk$-algebra. We say that A is an *X-decomposable $\Bbbk$-algebra* if there exists a family $\{A_x\}_{x\in X}$ of unital $\Bbbk$-algebras (some of which may be 0) such that $A = \bigoplus_{x\in X} A_x$ as $\Bbbk$-algebras.

In other words, X-decomposable $\Bbbk$-algebras are simply direct sums of $|X|$ unital $\Bbbk$-algebras with the canonical (unital) $\Bbbk$-algebra structure. This is a stronger condition than the $\Bbbk$-algebra A being an X-decomposable vector space, in the sense of Definition 1.45, since we also require that the decomposition respects the $\Bbbk$-algebra structure of A.

Remark 1.61. Let $A = \bigoplus_{x\in X} A_x$ be an X-decomposable $\Bbbk$-algebra. If 1_x denotes the multiplicative identity element of A_x, for each $x \in X$, then $1_A = \sum_{x\in X} 1_x$ by definition. We refer to the set of elements $\{1_x \mid x \in X \text{ such that } 1_x \neq 0\}$ as the *local identities* of A. The local identities form a complete set of orthogonal central idempotents of A.

However, the local identities could be a sum of nonzero central orthogonal idempotents (that is, non-primitive), as the next example shows. Thus, an X-decomposition of a $\Bbbk$-algebra is not unique.

Example 1.62. The $\{x, y\}$-decomposable vector space $\Bbbk^4$ of Example 1.53 is, in fact, a $\{x, y\}$-decomposable $\Bbbk$-algebra when considering component-wise operations. Here, the local identities are $1_x = (1, 1, 0, 0)$ and $1_y = (0, 0, 1, 1)$, which are not primitive.

We point out an example of an X-decomposable $\Bbbk$-algebra which is used in [CL09] for their classification of weak Hopf algebra structures derived from $U_q(sl_2)$.

Example 1.63. Take $q \in \Bbbk^\times$. We follow [CL09] for the following construction. The *weak quantum group* $A := wsl_q2$ is the unital $\Bbbk$-algebra generated by indeterminates $E, F, K, \overline{K}$ subject to the following relations:

$$KE = q^2EK, \qquad \overline{K}E = q^{-2}E\overline{K}, \qquad KF = q^{-2}FK, \qquad \overline{K}F = q^2F\overline{K},$$
$$K\overline{K} = \overline{K}K, \qquad K\overline{K}K = K, \qquad \overline{K}K\overline{K} = \overline{K}, \qquad EF - FE = (K - \overline{K})/(q - q^{-1}).$$

For $X = \{x, y\}$, let $A_x = A1_x$, $A_y = A1_y$, with $1_x = K\overline{K}$, and $1_y = 1_A - K\overline{K}$. Thus, $A = A_x \oplus A_y$ is an X-decomposable $\Bbbk$-algebra; in fact, $A_x \cong U_q(\mathfrak{sl}_2)$ and $A_y \cong \Bbbk[t_1, t_2]$ as $\Bbbk$-algebras [CL09, Theorems 2.3, 2.5, 2.7].

Now, we define two remarkable X-decomposable linear maps that arise when an X-decomposable $\Bbbk$-algebra has the structure of module over an X-groupoid $\mathcal{G}$.

Remark 1.64. If $A = \bigoplus_{x\in X} A_x$ is an X-decomposable $\Bbbk$-algebra, the multiplication map $m_A : A \otimes A \to A$ and unit map $u_A : \Bbbk \to A$ immediately decompose into the respective multiplication map $m_x : A_x \otimes A_x \to A_x$ and unit map $u_x : \Bbbk \to A_x$ of each $\Bbbk$-algebra A_x, for all $x \in X$. If additionally A is a $\mathcal{G}$-module then, using the notation of Lemma 1.50, m_A and u_A induce X-decomposable linear maps $\underline{m}_A := (m_x)_{x\in X} : A \otimes_{\mathcal{G}\text{-mod}} A \to A$ and $\underline{u}_A := (u_x)_{x\in X} : \mathbb{1}_{\mathcal{G}\text{-mod}} \to A$, which we call the *monoidal multiplication* and *monoidal unit* of A, respectively. It is clear that these maps satisfy associativity and unital condition. However, in general, these maps are not necessarily $\mathcal{G}$-module morphisms.

By definition, it follows immediately that the monoidal multiplication and monoidal unit maps are $\mathcal{G}$-module morphisms precisely when they make A a monoid in the category of $\mathcal{G}$-modules. Conversely, if A is monoid in $\mathcal{G}$-mod, then it comes equipped with maps in $\mathcal{G}$-mod, $\underline{m}_A = (m_x)_{x\in X} A \otimes_{\mathcal{G}\text{-mod}} A \to A$ and $\underline{u}_A = (u_x)_{x\in X} : \mathbb{1}_{\mathcal{G}\text{-mod}} \to A$. These maps extend naturally to maps $m_A : A \otimes A \to A$ (where if $a \in A_x$ and $b \in A_y$ for $x \neq y$, we define $m_A(a \otimes b) = 0$) and $u_A : \Bbbk \to A$ (defined by $\Bbbk \to \mathbb{1}_{\mathcal{G}\text{-mod}} \to A$ where $\Bbbk \to \mathbb{1}_{\mathcal{G}\text{-mod}}$ maps $1_{\Bbbk}$ to $(1, 1, \dots, 1)$). Hence, we have proved the following result.

Lemma 1.65. *Let $A = \bigoplus_{x \in X} A_x$ be an X-decomposable $\Bbbk$-algebra and let $\mathcal{G}$ be an X-groupoid. Then the following statements are equivalent.*

(i) $A \in \mathrm{Mon}(\mathcal{G}\text{-mod})$*, via the monoidal product $\underline{m}_A : A \otimes_{\mathcal{G}\text{-mod}} A \to A$ and monoidal unit $\underline{u}_A : \mathbb{1}_{\mathcal{G}\text{-mod}} \to A$ of Remark 1.64.*

(ii) *A is a $\mathcal{G}$-module such that*

$$g \cdot (ab) = (g \cdot a)(g \cdot b), \tag{1.27}$$
$$g \cdot 1_{s(g)} = 1_{t(g)}, \tag{1.28}$$

for all $g \in \mathcal{G}_1$ and $a, b \in A_{s(g)}$.

As a consequence of this result, an X-decomposable $\Bbbk$-algebra satisfying the conditions in Lemma 1.65(ii) can be referred to as a $\mathcal{G}$-module algebra. This terminology emphasizes that the $\Bbbk$-algebra is equipped with a compatible action of the X-groupoid $\mathcal{G}$, making it a monoid object within the category of $\mathcal{G}$-modules (see also Remark 2.6).

1.4.4 Actions of groupoid algebras on algebras

In this subsection, we introduce important concepts related to groupoid algebra actions. We recall the notation used in Example 1.8, and emphasize that while a group algebra is a Hopf algebra, a groupoid algebra is generally a weak Hopf algebra.

Remark 1.66. Let $\mathcal{G}$ be an X-groupoid and $\Bbbk\mathcal{G}\text{-mod}$ be the category of *left $\Bbbk\mathcal{G}$-modules* in the sense of Example 1.13 and Proposition 1.24. It is straightforward to show that every $\Bbbk\mathcal{G}$-module V can be given the structure of a $\mathcal{G}$-module in the sense of Definition 1.46 by taking $V_x = e_x \cdot V$ for all $x \in X$. Conversely, every $\mathcal{G}$-module is a $\Bbbk\mathcal{G}$-module by linearizing the action of $\mathcal{G}$. Hence, the categories $\mathcal{G}\text{-mod}$ and $\Bbbk\mathcal{G}\text{-mod}$ are isomorphic.

For a vector space V, we let $\mathrm{End}(V) = \mathrm{Hom}_{\mathsf{Vec}_{\Bbbk}}(V, V)$ denote the usual endomorphism ring of V. If $V = \bigoplus_{x \in X} V_x$ is X-decomposable with each $V_x \neq 0$, then for each $x \in X$, we let $\pi_x : V \to V_x$ denote the canonical projection and $\iota_x : V_x \to V$ denote the canonical inclusion. Note that $\iota_x \pi_x$ is an idempotent of $\mathrm{End}(V)$ whose restriction to V_x is Id_{V_x}.

Definition 1.67 ($\mathsf{rep}(\Bbbk\mathcal{G})$)**.** Let $\mathcal{G}$ be an X-groupoid and $\mathsf{rep}(\Bbbk\mathcal{G})$ be the category of *representations of* $\Bbbk\mathcal{G}$, that is, X-decomposable vector spaces $V = \bigoplus_{x \in X} V_x$ equipped with a $\Bbbk$-algebra morphism $\widetilde{\pi} : \Bbbk\mathcal{G} \to \mathrm{End}(V)$, such that $\widetilde{\pi}(e_x) = \iota_x \pi_x$, for all $x \in X$. A morphism between representations $(V, \widetilde{\pi})$ and $(V', \widetilde{\pi}')$ is a linear map $\phi : V \to V'$ with $\phi(V_x) \subset V'_x$ for all $x \in X$, such that $\phi \circ \widetilde{\pi}(g) = \widetilde{\pi}'(g) \circ \phi$ for $g \in \mathcal{G}_1$.

Remark 1.68. Similar to the group case (where $|X| = 1$), we have an isomorphism of categories $\mathsf{rep}(\Bbbk\mathcal{G}) \cong \Bbbk\mathcal{G}\text{-mod}$. This will follow from Remark 2.32, which shows that

$\mathsf{rep}(\Bbbk\mathcal{G}) \cong \mathsf{rep}(\mathcal{G})$, along with the isomorphisms $\mathsf{rep}(\mathcal{G}) \cong \mathcal{G}\text{-}\mathsf{mod}$ [Lemma 1.58] and $\mathcal{G}\text{-}\mathsf{mod} \cong \Bbbk\mathcal{G}\text{-}\mathsf{mod}$ [Remark 1.66].

1.5 Quivers and path algebras

In this section we recall preliminary graph-theoretic concepts of quivers and corresponding algebraic properties of path algebras. These will be used in Chapter 3. Recall from Example 1.9 that a *quiver* is simply a directed graph, which is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 (resp., Q_1) is a collection of vertices (resp., arrows), and $s, t : Q_1 \to Q_0$ denote the source and target maps, respectively.

We say that Q is *finite* if both $|Q_0|$ and $|Q_1|$ are finite sets. We read paths of Q from left-to-right, and all cycles are assumed to be oriented here, so by "cycle" we mean "oriented cycle". A quiver is said to be *acyclic* if it contains no cycles.

Hypothesis 1.69. We assume throughout this work that quivers are finite.

For any quiver Q, its associative, unital *path algebra* $\Bbbk Q$ over $\Bbbk$ is the $\Bbbk$-algebra having $\Bbbk$-basis given by the paths of Q, with ring structure determined by path concatenation when possible: $a * b = \delta_{t(a),s(b)}ab$, for all paths a, b of Q. The unit is $1_{\Bbbk Q} = \sum_{i \in Q_0} e_i$, where each e_i is the trivial path at vertex i. The path algebra $\Bbbk Q$ is $\mathbb{N}$-graded by path length, where $(\Bbbk Q)_k = \Bbbk(Q_k)$, for Q_k consisting of paths of length $k \in \mathbb{N}$.

First, we introduce some basic terminology.

Definition 1.70. Let $Q = (Q_0, Q_1, s, t)$ be a quiver.

(i) A cycle $p_1 p_2 \cdots p_k \in Q_k$ is called a *simple* if $s(p_i) \neq t(p_j)$ for $2 \leq i, j \leq k$.

(ii) A cycle $c := p_1 p_2 \cdots p_k$ of Q is said to be a *source* (resp., *sink*) *cycle* if there exists an arrow leaving (resp., entering) c, that is, there exists an arrow $p \in Q_1$, not in c, with $s(p) = s(p_i)$ (resp., $t(p) = t(p_i)$) for some $i = 1, \ldots, k$.

(iii) A cycle of Q is called *isolated* if it is neither a source nor sink cycle.

(iv) A cycle of Q is called *exclusive* (or *cyclically simple*) if it is disjoint with every other cycle.

(v) Q is said to satisfy the *exclusive condition* if every cycle of Q is exclusive.

(vi) For two cycles c, d of Q, we write $c \Rightarrow d$ if there is a path that starts at a vertex in c and ends at a vertex in d. A sequence of distinct cycles $c_1, \ldots, c_n$ of Q is a *chain of cycles* of length n if $c_1 \Rightarrow c_2 \Rightarrow \cdots \Rightarrow c_n$.

In Figure 1.1 below, we present examples of (non-)isolated and (non-)exclusive cycles. Notice that every isolated cycle is an exclusive cycle.

Next, we turn our attention to the connected condition of quivers.

Definition 1.71. Let $Q = (Q_0, Q_1, s, t)$ be a quiver.

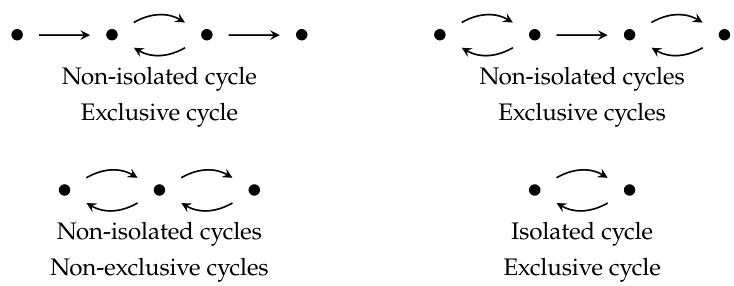


Figure 1.1: Non-/isolated and non-/exclusive cycles

(i) Q is said to be *connected* if for any given decomposition $Q_0 = Q_0' \cup Q_0''$, with $Q_0' \cap Q_0'' = \emptyset$ and both Q_0', Q_0'' non-empty, there exists at least one arrow $p \in Q_1$ such that either $s(p) \in Q_0'$, $t(p) \in Q_0''$ or $s(p) \in Q_0''$, $t(p) \in Q_0'$.

(ii) Q is said to be *strongly connected* (or *oriented connected*) if for every $i, j \in Q_0$ with $i \neq j$, there exists a path $p_1 \cdots p_k$ such that $s(p_1) = i$ and $t(p_k) = j$.

(iii) Q is said to be *pairwise strongly connected* if for every $i, j, i', j' \in Q_0$ with $i \neq j$ and $i' \neq j'$, there exist paths $p_1 \cdots p_k$ and $q_1 \cdots q_k$ of the same length such that $s(p_1) = i$, $t(p_k) = j$, $s(q_1) = i'$ and $t(q_k) = j'$.

(iv) Q is said to be *path reversible* if for every path $p_1 p_2 \cdots p_k \in Q_k$, there exists a path $q_1 q_2 \cdots q_l \in Q_l$ such that $s(p_1) = t(q_l)$ and $t(p_k) = s(q_1)$. Here, l need not equal k.

It is clear that pairwise strongly connected $\Rightarrow$ strongly connected $\Rightarrow$ connected. However, the converses do not hold as we see in Figure 1.2 below.

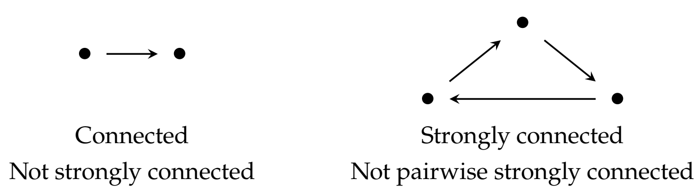


Figure 1.2: Various connected quivers

Now we recall results on algebraic properties of path algebras which depend on graph-theoretic properties of the underlying quiver; many of these results are standard.

Notation 1.72 **(C, C^k, $c_{i,j}^{(k)}$).** For a quiver Q, denote by $C = (c_{i,j})_{i,j \in Q_0}$ the adjacency matrix of (arrows in) Q, and by $C^k := (c_{i,j}^{(k)})_{i,j \in Q_0}$ the adjacency matrix of paths of length k in Q (which is equal to the k-th power of C).

Proposition 1.73. *Let $Q = (Q_0, Q_1, s, t)$ be a quiver.*

(i) (a) *[ASS06, Lemma II.1.4] $\Bbbk Q$ is finite dimensional if and only if Q is acyclic.*

(b) *When $\Bbbk Q$ is finite dimensional,* $\dim_{\Bbbk} \Bbbk Q = \sum_{i,j \in Q_0,\, k \geq 0} c_{i,j}^{(k)}$.

(c) *In general, the Hilbert series of* $\Bbbk Q$ *is given by*

$$H_{\Bbbk Q}(t) = (I - Ct)^{-1} = I + Ct + C^2t^2 + C^3t^3 + \cdots,$$

where I *denotes the* $|Q_0| \times |Q_0|$ *identity matrix.*

(ii) *[Ufn82]* (see also [MFSM18, Theorem 3.12]) $\Bbbk Q$ *has finite GK-dimension if and only if* Q *satisfies the exclusive condition. In this case,* $\mathrm{GKdim}(\Bbbk Q)$ *equals the maximal length of chains of cycles in* Q.

(iii) (see, e.g., [CL00, Theorems 2.2 and 2.3]) $\Bbbk Q$ *is (resp., right, left) Noetherian if and only if every cycle in* Q *is (resp., not a source cycle, not a sink cycle) isolated.*

(iv) (see, e.g., [CL00, Theorem 2.1]) $\Bbbk Q$ *is prime if and only if* Q *is strongly connected.*

(v) *[SM08, Proposition 2.1]* $\Bbbk Q$ *is semiprime if and only if* Q *is path reversible.*

(vi) (see, e.g., [Bri12, Section 1.4]) $\Bbbk Q$ *is hereditary, and* $\mathrm{gldim}(\Bbbk Q) = 0$ *if and only if* Q *is arrowless.* □

1.6 Overview of algebraic and homological properties

In this section, we present an overview of several ring-theoretic and homological properties that will be recalled throughout this work, with a particular focus on Chapter 3. Take R a ring, $\Bbbk$ an arbitrary field, and A a $\Bbbk$-algebra here. We refer the reader to [GW04, Section 1.1], [MR01, Sections 0.2, 7.1, 8.1], [Frö99, Section 1], and [EE07, Section 2.1] for further details of the material here.

A right R-module M_R is called *Noetherian* if every submodule of M is finitely generated. In particular, a ring R is said to be *right Noetherian* if R_R is Noetherian. Likewise, we can define *left Noetherian* rings, and R is *Noetherian* if it is both left and right Noetherian.

The *projective dimension* of a module M_R, written $\mathrm{pd}(M_R)$, is the shortest length n of a projective resolution of M, or equal to ∞ if no such n exists. The *right global dimension* of R is defined by $\mathrm{r.gldim}(R) := \sup\{\mathrm{pd}(M) \mid M \text{ any right } R\text{-module}\}$. Likewise, $\mathrm{l.gldim}(R)$ is defined; when $\mathrm{r.gldim}(R) = \mathrm{l.gldim}(R)$, we simply write $\mathrm{gldim}(R)$. A ring R is called *hereditary* if $\mathrm{gldim}(R) \le 1$.

An $\mathbb{N}$-graded $\Bbbk$-algebra A is said to be *Koszul* if it has a linear minimal graded free resolution, that is, there exists an exact sequence

$$\cdots \to A(-i)^{b_i} \to \cdots \to A(-2)^{b_2} \to A(-1)^{b_1} \to A \to \Bbbk \to 0,$$

where $A(-j)$ is the graded algebra A with grading shifted up by j, that is $A(-j)_i = A_{i-j}$, and the exponents b_i refer to the b_i-fold direct sum.

A ring R is called *prime* if for every pair of nonzero two-sided ideals I, J of R it follows $IJ \neq 0$, and is called *semiprime* if R has no nonzero nilpotent two-sided ideals.

Let A be a finitely generated $\Bbbk$-algebra. The *Gelfand-Kirillov dimension* of A is defined by $\mathrm{GKdim}(A) = \sup_V \overline{\lim}_{n\to\infty} \log_n(\dim_{\Bbbk} V^n)$, where the supremum is taken over all finite dimensional $\Bbbk$-subspaces V of A and V^n denotes the subspace spanned by all elements of the form $v_1 \cdots v_n$, where $v_i \in V$, $1 \leq i \leq n$. A well known result is that $\mathrm{GKdim}(A) = 0$ if and only if A is finite-dimensional as a $\Bbbk$-vector space.

Finally, recall that a positively graded bimodule $M = \bigoplus_{n\in\mathbb{N}} M_n$ over a graded $\Bbbk$-algebra $A = \bigoplus_{n\in\mathbb{N}} A_n$ is said to be *locally finite* if each M_n is finite-dimensional. Let J be a finite set and consider the $\Bbbk$-algebra $A := \Bbbk^{|J|}$. Note that a locally finite $\mathbb{N}$-graded A-bimodule M can be seen as an $J \times J$-graded vector space $M = \bigoplus_{i,j\in J} M_{i,j}$. We define the *(matrix) Hilbert series* $h_M(t)$ of M to be a matrix-valued series with entries given by

$$h_M(t)_{i,j} = \sum_{k\geq 0} \dim_{\Bbbk}((M_k)_{i,j})t^k.$$

CHAPTER 2

Weak quantum symmetries

The objective of this chapter is to investigate the symmetries of $\Bbbk$-algebras A through the use of actions by algebraic structures H that resemble cocommutative Hopf algebras. To accomplish this, we will extend and formalize the well-known correspondence between group modules and representations of groups. This will allow us to establish a connection between categorical and representation-theoretic frameworks for these H-actions on A. Throughout this chapter, we will focus on the following categories.

Notation 2.1. Let X be a nonempty set. We consider the following categories:

- Grp, of groups together with group morphisms,
- Lie, of Lie algebras together with Lie algebra morphisms,
- Hopf, of Hopf algebras together with Hopf algebra morphisms,
- Grpd, of groupoids together with groupoid morphisms.
- X-Grpd, the subcategory of Grpd whose objects are groupoids with object set X, together with X-preserving groupoid morphisms (see Definition 1.44),
- X-Lie, of X-Lie algebroids, together with X-Lie algebroid morphisms (see Definition 2.37),
- X-WHA, of weak Hopf algebras with a complete set of grouplike idempotents indexed by X, together with X-preserving weak Hopf algebra morphisms (see Definition 2.26).

We also consider the following full subcategories of Hopf:

- GrpAlg, of group algebras,
- EnvLie, of enveloping algebras of Lie algebras,
- $\mathsf{CocomHopf}$, of cocommutative Hopf algebras,

as well as the following full subcategories of X-WHA:

- X-$\mathsf{GrpdAlg}$, of X-groupoid algebras (see Definition 2.26),
- X-EnvLie, of X-enveloping algebras of X-Lie algebroids (see Definition 2.49),
- X-$\mathsf{CocomWHA}$, of cocommutative X-weak Hopf algebras.

Note that all the categories $\mathcal{C}$ mentioned above have a common property: for any object H in $\mathcal{C}$, there is a notion of a category of left H-modules, which we denote H-mod, and this category is a $\Bbbk$-linear monoidal category (although $\otimes_{H\text{-}\mathsf{mod}}$ is not necessarily given by $\otimes_{\Bbbk}$). We refer to any of the categories above as a *category of Hopf-like structures*.

We would like to emphasise that we do not provide a precise set of axioms that define a category $\mathcal{C}$ as consisting of "Hopf-like structures". However, the categories presented in Notation 2.1 share several common properties, which we discuss in Remark 2.2. It is our aspiration that the findings of our research can be extended to other categories, including non-cocommutative Hopf-like structures. We consider this an open question for future investigation and exploration.

Remark 2.2. Let $\mathcal{C}$ be any of the categories of Hopf-like structures introduced in Notation 2.1. We will see that each of these categories possesses the following properties:

(i) $\mathcal{C}$ is a concrete category.

(ii) For each H in $\mathcal{C}$, there exists a notion of a quotient object H/I in $\mathcal{C}$ for certain subsets I of H. It is important to note that in this context, I may not necessarily be a subobject of H in $\mathcal{C}$.

(iii) One of the following cases occurs:

(I) There exists a bifunctor $\boxtimes_{\mathcal{C}} : \mathcal{C} \times \mathsf{Vec}_{\Bbbk} \longrightarrow \mathsf{Set}$, so that the elements of $H \boxtimes_{\mathcal{C}} V$ have the form $h \boxtimes_{\mathcal{C}} v$, for certain $h \in H$ and $v \in V$; in this case we say $\mathcal{C}$ is of *type I*,

(II) There exists a bifunctor $\boxtimes_{\mathcal{C}} : \mathcal{C} \times \mathsf{Vec}_{\Bbbk} \longrightarrow \mathsf{Vec}_{\Bbbk}$, so that the elements of $H \boxtimes_{\mathcal{C}} V$ are sums of elements of the form $h \boxtimes_{\mathcal{C}} v$, for certain $h \in H$ and $v \in V$; in this case we say $\mathcal{C}$ is of *type II*.

The details of $\boxtimes_{\mathcal{C}}$ depend on the category $\mathcal{C}$ and will be clarify for each studied category (see Example 2.3).

(iv) For each object H in $\mathcal{C}$, there is a notion of endowing a $\Bbbk$-vector space V with a structure of H-module via a set-theoretic function $\ell_{H,V} : H \boxtimes_{\mathcal{C}} V \longrightarrow V$ (if $\mathcal{C}$ is of type I) or a $\Bbbk$-linear map $\ell_{H,V} : H \boxtimes_{\mathcal{C}} V \longrightarrow V$ (if $\mathcal{C}$ is of type II). In either case, we use the notation $h \cdot v$ to denote $\ell_{H,V}(h \boxtimes v)$ and say that H *acts* on V.

(v) For each object H in $\mathcal{C}$, there is also a notion of a morphism of H-modules, and thus there exists a category H-mod of H-modules and their morphisms. H-mod forms a $\Bbbk$-linear monoidal category $(H\text{-}\mathsf{mod}, \otimes_{H\text{-}\mathsf{mod}}, \mathbb{1}_{H\text{-}\mathsf{mod}})$.

In short, our categories of Hopf-like structures are categories in which it is possible to associate to each object a $\Bbbk$-linear monoidal category of modules.

EXAMPLE 2.3. (a) When $C = \mathsf{Grp}$, we may take $\boxtimes_{\mathsf{Grp}} = \times$, the Cartesian product of sets. In this case, if G is a group, $\ell_{G,V} : G \times V \to V$ is a $\Bbbk$-linear group action of G on a vector space V. Then $(G\text{-}\mathsf{mod}, \otimes_{\Bbbk}, \Bbbk)$ is a $\Bbbk$-linear monoidal category where if V, W are G-modules, then $V \otimes W$ is a G-module via the action $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$ for $g \in G$, $v \in V$, and $w \in W$.

(b) When $C = \mathsf{Lie}$, we may take $\boxtimes_{\mathsf{Lie}} = \times$, the Cartesian product. In this case, if $\mathfrak{g}$ is a Lie algebra, $\ell_{\mathfrak{g},V} : \mathfrak{g} \times V \to V$ is a $\Bbbk$-bilinear map such that $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$, for all $x, y \in \mathfrak{g}$ and $v \in V$. Then $(\mathfrak{g}\text{-}\mathsf{mod}, \otimes_{\Bbbk}, \Bbbk)$ is a $\Bbbk$-linear monoidal category where if V, W are $\mathfrak{g}$-modules, then $V \otimes W$ is a $\mathfrak{g}$-module via the action $x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w)$.

(c) When $C = \mathsf{Hopf}$, we may take $\boxtimes_{\mathsf{Hopf}} = \otimes_{\Bbbk}$. In this case, $\ell_{H,V} : H \otimes V \to V$ is $\Bbbk$-linear map that makes V a left module over the $\Bbbk$-algebra H. Then $(H\text{-}\mathsf{mod}, \otimes_{\Bbbk}, \Bbbk)$ is a $\Bbbk$-linear monoidal category where if V, W are H-modules, then $V \otimes W$ is an H-module via the action $h \cdot (v \otimes w) = \sum (h_1 \cdot v) \otimes (h_2 \cdot w)$. As consequence, GrpAlg, EnvLie and $\mathsf{CocomHopf}$ are also categories of Hopf-like structures.

(d) When $C = X\text{-}\mathsf{Grpd}$, for an X-groupoid $\mathcal{G}$ and an X-decomposable vector space $V = \bigoplus_{x \in X} V_x$, we may take

$$\mathcal{G} \boxtimes_{X\text{-}\mathsf{Grpd}} V = \{(f, v) \in \mathcal{G} \times V \mid f \in \mathcal{G}, v \in V_{s(f)}\}.$$

Then a left action of $\mathcal{G}$ on V can be viewed as a map $\mathcal{G} \boxtimes_{X\text{-}\mathsf{Grpd}} V \to V$. By the Lemma 1.50, $(\mathcal{G}\text{-}\mathsf{mod}, \otimes_{\mathcal{G}\text{-}\mathsf{mod}}, \Bbbk^{|X|})$ is a monoidal category. Note that since the morphisms in $X\text{-}\mathsf{Grpd}$ fix the set X, the subobjects of $\mathcal{G}$ in $X\text{-}\mathsf{Grpd}$ are all *wide* subgroupoids $\mathcal{H}$ (that is, subcategories with $\mathcal{H}_0 = X$). A wide subgroupoid is called *normal* if $ghg^{-1} \in \mathcal{H}$ for all $h \in \mathcal{H}_1$ and all $g \in \mathcal{G}_1$ such that $s(g) = t(h)$. If $\mathcal{H}$ is a normal subgroupoid, then the set of cosets of $\mathcal{H}$ in $\mathcal{G}$ forms a quotient groupoid $\mathcal{G}/\mathcal{H} \in X\text{-}\mathsf{Grpd}$.

Later we will show that the remaining categories $X\text{-}\mathsf{Lie}$, $X\text{-}\mathsf{WHA}$, $X\text{-}\mathsf{GrpdAlg}$, $X\text{-}\mathsf{EnvLie}$ and $X\text{-}\mathsf{CocomWHA}$ also satisfy the properties in Remark 2.2 (see Remarks 2.6 and 2.40).

Next we introduce the notion of inner-faithful action by a Hopf-like structure.

DEFINITION 2.4 (INNER-FAITHFUL ACTION). Let C be a category of Hopf-like structures with bifunctor $\boxtimes_C : C \times \mathsf{Vec}_{\Bbbk} \to \mathsf{Set}$ (if C is of type I) or a bifunctor $\boxtimes_C : C \times \mathsf{Vec}_{\Bbbk} \to \mathsf{Vec}_{\Bbbk}$ (if C is of type II). If H is an object in C and V is an H-module via $\ell_{H,V}$, we say that H acts *inner-faithfully* on V if there is no proper quotient H/I of H in C such that H/I acts on V

via $\ell_{H/I,V}$ and the following diagram commutes (in Set or in $\mathsf{Vec}_{\Bbbk}$):

$$\begin{tikzcd}
H \boxtimes_{C} V \arrow[d, "\pi \boxtimes_{C} \mathrm{Id}"'] \arrow[dr, "\ell_{H,V}"] & \\
\frac{H}{I} \boxtimes_{C} V \arrow[r, "\ell_{H/I,V}"'] & V,
\end{tikzcd}$$

where $\pi : H \to H/I$ is the natural projection.

Definition 2.5 (H-module algebra). Let H be an object in a category of Hopf-like structures C. We say that A is a *left H-module algebra* if A is a monoid object in the monoidal category H-mod.

Remark 2.6. While we require the objects of H-mod to be $\Bbbk$-vector spaces, it is important to note that the monoidal product $\otimes_{H\text{-}\mathsf{mod}}$ may not coincide with the tensor product $\otimes_{\Bbbk}$. Consequently, not every monoid object A in H-mod automatically possesses a $\Bbbk$-algebra structure. However, it is worth highlighting that in the categories of Notation 2.1, this correspondence between monoid objects and $\Bbbk$-algebras with compatible H-actions holds. Specifically:

(i) For Grp, Lie, and Hopf, where the monoidal product is given by $\otimes_{\Bbbk}$, it is immediate that every monoid object A in H-mod can be endowed with a $\Bbbk$-algebra structure.

(ii) In the case of X-Grpd, Lemma 1.65 establishes that monoid objects A in $\mathcal{G}$-mod precisely correspond to $\Bbbk$-algebras equipped with a compatible $\mathcal{G}$-action. This result will also hold for X-Lie [Lemma 2.45].

(iii) The main objective of Theorem 1.29 was to prove that for weak bialgebras (and hence, weak Hopf algebras), monoid objects in H-mod correspond to H-module algebras as defined in Definition 1.27. Consequently, this correspondence also extends to X-WHA and the full subcategories X-$\mathsf{GrpdAlg}$, X-EnvLie and X-$\mathsf{CocomWHA}$.

We are now ready to introduce the object $\mathrm{Sym}_{C}(A)$ which captures the symmetries of a $\Bbbk$-algebra A by all actions of the structures in C.

Definition 2.7 ($\mathrm{Sym}_{C}(A)$). Let A be a $\Bbbk$-algebra and let C be a category of Hopf-like structures, as in Notation 2.1. We denote by $\mathrm{Sym}_{C}(A)$ an object in C (if it exists) such that:

(i) A is $\mathrm{Sym}_{C}(A)$-module algebra via $\ell_{\mathrm{Sym}_{C}(A),A} : \mathrm{Sym}_{C}(A) \boxtimes_{C} A \to A$ (where if $f \boxtimes a \in \mathrm{Sym}_{C}(A) \boxtimes_{C} A$, we write $f \triangleright a$ to denote $\ell_{\mathrm{Sym}_{C}(A),A}(f \boxtimes a)$).

(ii) For each object H in C, if A is an H-module algebra via $\ell_{H,A} : H \boxtimes_{C} A \to A$, then there exists a unique morphism $\phi : H \to \mathrm{Sym}_{C}(A)$ in C such that the following diagram commutes

$$\begin{tikzcd} H \boxtimes_C A \arrow[d, dashed, "\phi \boxtimes_C \mathrm{Id}"'] \arrow[drr, "\ell_{H,A}"] & & \\ \mathrm{Sym}_C(A) \boxtimes_C A \arrow[rr, "\ell_{\mathrm{Sym}_C(A),A}"'] & & A, \end{tikzcd}$$

and conversely, every morphism $\phi : H \to \mathrm{Sym}_C(A)$ in C gives A the structure of an H-module algebra via $\ell_{H,A}(h \boxtimes a) = \phi(h) \triangleright a$.

Remark 2.8. (i) Note that, while the definition of an H-module algebra is "category-theoretic", if the object $\mathrm{Sym}_C(A)$ exists, then it provides a "representation-theoretic" framework for actions on $\Bbbk$-algebras: A is an H-module algebra if there is a morphism $H \to \mathrm{Sym}_C(A)$ in C.

(ii) It is not obvious that the object $\mathrm{Sym}_C(A)$ exists. For example, let $C = \mathsf{AbGrp}$ be the category of abelian groups and let $A = \Bbbk[x]$. Let $G = \langle g \rangle$ be the group of order 2. Then G acts on A via the action $\cdot$ defined by $g \cdot x = -x$ and also the action $*$ defined by $g * x = -x + 1$. If $\mathrm{Sym}_{\mathsf{AbGrp}}(A)$ existed, then we would obtain abelian group morphisms $\phi : G \to \mathrm{Sym}_{\mathsf{AbGrp}}(A)$ and $\psi : G \to \mathrm{Sym}_{\mathsf{AbGrp}}(A)$ so that $\phi(g) \triangleright x = -x$ and $\psi(g) \triangleright x = -x + 1$. But then the elements $\phi(g)$ and $\psi(g)$ would not commute in $\mathrm{Sym}_{\mathsf{AbGrp}}(A)$ (since they have different actions on x), contradicting the fact that $\mathrm{Sym}_{\mathsf{AbGrp}}(A)$ is an abelian group.

(iii) For the categories C in Notation 2.1, we conjecture that if $\mathrm{Sym}_C(A)$ exists, then it acts inner-faithfully on A and it is unique up to unique isomorphism in C. This remains as an open question that we expect to answer in future research.

In the remainder of this chapter, we will show that $\mathrm{Sym}_C(A)$ exists for the categories C appearing in Notation 2.1. We note that these categories are all categories of objects which are (closely related to) weak Hopf algebras. Hence, weak Hopf algebras can be viewed as capturing symmetries of $\Bbbk$-algebras.

2.1 Weak quantum symmetries captured by groupoids

The main objective of this section is to establish a correspondence between two action frameworks in the context of groupoids. Specifically we aim to generalize the following well-known correspondence: if G is a group, a G-module algebra structure on a $\Bbbk$-algebra A yields a group morphism from G to the algebra automorphism group $\mathrm{Aut}_{\mathrm{Alg}}(A)$; conversely any group morphism $G \to \mathrm{Aut}_{\mathrm{Alg}}(A)$ induces a G-module algebra structure on A [CM84, Proposition 1.2].

First, we introduce a generalization of the algebra automorphism group to accommodate groupoid actions. Recall the notation of Section 1.4 and Example 2.3.(d).

Definition 2.9 ($\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$)**.** Let $A = \bigoplus_{x \in X} A_x$ be an X-decomposable $\Bbbk$-algebra. We define $\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$, the *$X$-algebra automorphism groupoid* of A, as follows:

- the object set is X,
- for any $x, y \in X$, $\mathrm{Hom}_{\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)}(x, y)$ is the space of unital $\Bbbk$-algebra isomorphisms between the unital $\Bbbk$-algebras A_x and A_y. The composition of morphisms is determined by the composition of the corresponding $\Bbbk$-algebra morphisms.

Remark 2.10. Note that the X-algebra automorphism groupoid $\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$ is as a *subgroupoid* (that is, a subcategory closed under taking inverses) of $\mathrm{GL}_X(A)$ from Definition 1.52, and it is usually *proper* in the sense that it is not a full subcategory. Clearly $\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$ is an X-groupoid. Also, when $|X| = 1$ we recover the classical group of automorphisms $\mathrm{Aut}_{\mathsf{Alg}}(A)$ as a subgroup of $\mathrm{GL}(A)$.

Now, we explicitly calculate $\mathrm{Sym}_{X\text{-}\mathsf{Grpd}}(A)$ for any X-decomposable $\Bbbk$-algebra A (recall Definition 2.7).

Proposition 2.11. *Let $A = \bigoplus_{x \in X} A_x$ be an X-decomposable $\Bbbk$-algebra and $\mathcal{G}$ be an X-groupoid. Then:*

(i) *A is an $\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$-module algebra via*

$$\ell_{\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A),A}(f \boxtimes a) = f(a), \quad \textit{for all } f \boxtimes a \in \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A) \boxtimes_{X\text{-}\mathsf{Grpd}} A$$

(that is, for all $f \in \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)_1$ and $a \in A_{s(f)}$). We denote $\ell_{\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A),A}(f \boxtimes a)$ by $f \triangleright a$.

(ii) *Suppose that A is a $\mathcal{G}$-module algebra via $\ell_{\mathcal{G},A}$, and denote $g \cdot a := \ell_{\mathcal{G},A}(g \boxtimes a)$ for all $g \boxtimes a \in \mathcal{G} \boxtimes_{X\text{-}\mathsf{Grpd}} A$. Then there is a unique X-groupoid morphism $\pi : \mathcal{G} \to \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$ such that $g \cdot a = \pi(g) \triangleright a$ for all $g \boxtimes a \in \mathcal{G} \boxtimes_{X\text{-}\mathsf{Grpd}} A$.*

(iii) *Every X-groupoid morphism $\pi : \mathcal{G} \to \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$ gives A the structure of a $\mathcal{G}$-module algebra via $\ell_{\mathcal{G},A}(g \boxtimes a) = \ell_{\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A),A}(\pi(g) \boxtimes a)$ for all $g \boxtimes a \in \mathcal{G} \boxtimes_{X\text{-}\mathsf{Grpd}} A$.*

Hence $\mathrm{Sym}_{X\text{-}\mathsf{Grpd}}(A) = \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$.

Proof. (i): For all $f \in \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)_1$ and $a, b \in A_{s(f)}$ we have

$$f \triangleright (ab) = f(ab) = f(a)f(b) = (f \triangleright a)(f \triangleright b),$$
$$f \triangleright 1_{s(f)} = f(1_{s(f)}) = 1_{t(f)}$$

so by Lemma 1.65 it follows that A is an $\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$-module algebra.

(ii): If A is a $\mathcal{G}$-module via $\cdot$, then by Lemma 1.48, there is an associated subgroupoid of $\mathrm{GL}_X(A)$ with object set X and its structure isomorphisms α_g (see Notation 1.49). This induces a unique X-groupoid morphism $\pi : \mathcal{G} \to \mathrm{GL}_X(A)$ such that and $\pi(g) = \alpha_g$ for all $x \in X$ and $g \in \mathcal{G}_1$ (see Lemma 1.58). Moreover, if A is a $\mathcal{G}$-module algebra, then it satisfies (1.27) and (1.28), which are equivalent to map $\pi(g)$ being a unital $\Bbbk$-algebra map for all

$g \in G$. Hence, the image of the groupoid morphism π is contained in the subgroupoid $\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$ of $\mathrm{GL}_X(A)$. Corestricting π to a map $\mathcal{G} \to \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$ gives the result.

(iii): If $\pi : \mathcal{G} \to \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$ is any X-groupoid morphism, then we can define a map $\ell_{\mathcal{G},A} : \mathcal{G} \boxtimes_{X\text{-}\mathsf{Grpd}} A \to A$ by $\ell_{\mathcal{G},A}(g \boxtimes a) = \ell_{\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A),A}(\pi(g), a)$ for all $g \boxtimes a \in \mathcal{G} \boxtimes_{X\text{-}\mathsf{Grpd}} A$. Since π is an X-groupoid morphism, we see that if $g, h \in \mathcal{G}_1$ with $t(h) = s(g)$ and $a \in A_{s(h)}$, then

$$(gh) \cdot a = \pi(gh) \triangleright a = [\pi(g)\pi(h)] \triangleright a = \pi(g) \triangleright (\pi(h) \triangleright a) = g \cdot (h \cdot a),$$

and for all $a \in A_x$

$$e_x \cdot a = \pi(e_x) \triangleright a = a.$$

Hence, $\cdot$ makes A a $\mathcal{G}$-module. Further for all $g \in \mathcal{G}_1$ and $a, b \in A_{s(g)}$, we have

$$g \cdot (ab) = \pi(g) \triangleright (ab) = (\pi(g) \triangleright a)(\pi(g) \triangleright b) = (g \cdot a)(g \cdot b)$$

and

$$g \cdot 1_{s(g)} = \pi(g) \triangleright 1_{s(g)} = 1_{t(\pi(g))} = 1_{t(g)}$$

and so A is a $\mathcal{G}$-module algebra. □

Example 2.12. In Example 1.51(a), the $\mathcal{G}$-module $\Bbbk^4 \cong \Bbbk^2 \oplus \Bbbk^2$ is a $\mathcal{G}$-module algebra with functor $\pi : \mathcal{G} \to \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(\Bbbk^4)$ induced from π as defined in Example 1.57. In Example 1.51(d), the $\mathcal{G}$-module $A = \Bbbk[t, t^{-1}] \oplus \Bbbk[t, t^{-1}]$ is a $\mathcal{G}$-module algebra with the functor $\pi : \mathcal{G} \to \mathrm{Aut}_{\mathcal{G}_0\text{-}\mathsf{Alg}}(A)$. However, the $\mathcal{G}$-module in Example 1.51(b) is not an example of a $\mathcal{G}$-module algebra since the structure isomorphisms are not unital. Similarly, the induced X-decomposition of the $\mathcal{G}$-module of Example 1.51(c) does not make it into an $\mathcal{G}$-module algebra.

We end this section with a well-known result, which is the reinterpretation of Proposition 2.11 in the case that $|X| = 1$ (that is, when $\mathcal{G}$ is a group).

Corollary 2.13. *Let A be a $\Bbbk$-algebra. Then A is an $\mathrm{Aut}_{\mathrm{Alg}}(A)$-module algebra and for a group G, the following are equivalent.*

(i) *A is a G-module algebra.*

(ii) *There exists a group morphism $\pi : G \to \mathrm{Aut}_{\mathsf{Alg}}(A)$.*

Hence, $\mathrm{Sym}_{\mathsf{Grp}}(A) = \mathrm{Aut}_{\mathrm{Alg}}(A)$.

2.2 Weak quantum symmetries captured by groupoid algebras

In this section we explore a linearization approach for groupoid actions using a generalization of a well-known adjuntion for groups: the functor $\Bbbk() : \mathsf{Grp} \to \mathsf{Alg}$, which maps a group to its group algebra, is left adjoint to the functor $()^{\times} : \mathsf{Alg} \to \mathsf{Grp}$, which assigns the group of units of an algebra.

2.2.1 Local units of algebras

Next, we introduce the notion of local units of a $\Bbbk$-algebra. This generalizes the group of units functor to the groupoid setting; see Theorem 2.31(i) below.

Definition 2.14 ($\widehat{a}$). Let A be a $\Bbbk$-algebra and $\{e_x\}_{x\in X}$ be a set of nonzero orthogonal idempotents of A. For $x, y \in X$, an element $a \in e_y A e_x$ is called a *local unit* if there exists an element $b \in e_x A e_y$ such that $ab = e_y$ and $ba = e_x$. The element b is called a *local inverse* of a, which we denote by $\widehat{a} := b$.

The notions above are well-defined due to the following straightforward lemma.

Lemma 2.15. *Let A be a $\Bbbk$-algebra and let $\{e_x\}_{x\in X}$ be a set of nonzero orthogonal idempotents of A. Suppose that $a \in e_y A e_x$ be a local unit of A for some $x, y \in X$. Then:*

(i) *a is nonzero.*

(ii) *If $a \in e_{y'} A e_{x'}$ for $x', y' \in X$, then $x = x'$ and $y = y'$.*

(iii) *If there exists $b, b' \in e_x A e_y$ such that $ab = ab' = e_y$ and $ba = b'a = e_x$, then $b = b'$.*

(iv) *For $x, y, z \in X$, if $a_1 \in e_y A e_x$ and $a_2 \in e_z A e_y$ are local units of A, then the product $a_2 a_1 \in e_z A e_x$ is also a local unit of A.*

Proof. (i): Recall that $ab = e_y$ for some $y \in X$ and $b \in A$. It is clear that a is nonzero.

(ii): Since $a \in e_y A e_x$, we assume that $a = e_y a' e_x$ for some $a' \in A$. One can see that $e_y a e_x = e_y^2 a' e_x^2 = e_y a' e_x = a$. If $a \in e'_y A e'_x$ for some $x', y' \in X$, then we have $a = e'_y a e'_x$. So $a = e_y a e_x = e_y e_{y'} a e_{x'} e_x$. Since the idempotents are orthogonal and a is nonzero, we have that $x = x'$ and $y = y'$.

(iii): We have that $b = e_x b e_y = b' abab' = b' e_y$. So by multiplying on the left by e_x and on the right by e_y, we get that $b = b'$.

(iv): Since $a_1\widehat{a}_1 = e_y$, $\widehat{a}_1 a_1 = e_x$, $a_2\widehat{a}_2 = e_z$ and $\widehat{a}_2 a_2 = e_y$, we have that

$$a_2 a_1 \widehat{a}_1 \widehat{a}_2 \; = \; a_2 e_y \widehat{a}_2 \; = \; a_2 \widehat{a}_2 a_2 \widehat{a}_2 \; = \; e_z e_z \; = \; e_z.$$

Similarly, $\widehat{a}_1 \widehat{a}_2 a_2 a_1 = e_x$. ☐

Remark 2.16. In contrast with the notion of local identities introduced in Remark 1.61 for X-decomposable $\Bbbk$-algebras, the idempotents in Lemma 2.15 are not required to be central, nor do we require $1_A = \sum_{x\in X} e_x$. This allows us to work with a broader scope of $\Bbbk$-algebras towards our main result, Theorem 2.31. For example, for an X-groupoid $\mathcal{G}$, the groupoid algebra $\Bbbk\mathcal{G}$ has the set of nonzero orthogonal idempotents $\{e_x\}_{x\in X}$, and even though $1_{\Bbbk\mathcal{G}} = \sum_{x\in X} e_x$, the idempotents are e_x are not central, in general. Hence, unless $\mathcal{G}$ is totally disconnected, $\Bbbk\mathcal{G}$ is not a X-decomposable $\Bbbk$-algebra via these elements.

Definition 2.17 ($(-)^\times_X$). Let A be a $\Bbbk$-algebra and $\{e_x\}_{x\in X}$ be a set of nonzero orthogonal idempotents of A. The *groupoid of local units* of A, denoted by $A^\times_X$, is defined as follows.

- The objects are the elements $x \in X$
- For each $x, y \in X$, $\mathrm{Hom}_{A_X^\times}(x, y) = \{a \in e_y A e_x \mid a \text{ is a local unit of } A\}$. In this case we write $a : x \to y$.
- Given two morphisms $a : x \to y$ and $a' : y \to z$ we define their composition as the product $a'a : x \to z$.
- The inverse of a morphism $a : x \to y$ is given by its local inverse $\widehat{a} : y \to x$.

Remark 2.18. Lemma 2.15 guarantees that the construction above is well-defined, and it is indeed an X-groupoid.

The construction of $A_X^\times$ strongly depends on the choice of the family of idempotents, so a $\Bbbk$-algebra might have several different associated groupoids of local units.

Example 2.19. (a) If $|X| = 1$, then for any $\Bbbk$-algebra A we can take $\{1_A\}$ as the set of idempotents indexed by X. In this case, $A_X^\times$ is precisely the group of units $A^\times$.

(b) If $\mathcal{G}$ is an X-groupoid and $A = \Bbbk\mathcal{G}$, then the identity morphisms $\{e_x\}_{x\in X}$ of $\mathcal{G}$ form a set of orthogonal nonzero idempotents in $\Bbbk\mathcal{G}$. Hence, $\mathcal{G}$ is a subgroupoid of $(\Bbbk\mathcal{G})_X^\times$ since $gg^{-1} = e_{t(g)}$ and $g^{-1}g = e_{s(g)}$ for all $g \in \mathcal{G}_1$. If $|X| = 1$, then this reduces to the fact that $\mathcal{G}$ is a subgroup of the group $(\Bbbk\mathcal{G})^\times$.

(c) Let $A = \bigoplus_{x\in X} A_x$ be an X-decomposable $\Bbbk$-algebra. Then $A_X^\times$ is a disjoint collection of groups due to the fact that $\mathrm{Hom}_{A_X^\times}(x, x) \cong (A_x)^\times$ as multiplicative groups and $\mathrm{Hom}_{A_X^\times}(x, y) = 0$, for every $x, y \in X$ with $x \neq y$.

(d) Let $V = \bigoplus_{x\in X} V_x$ be an X-decomposable vector space with $V_x \neq 0$ for all $x \in X$. Let $A = \mathrm{End}(V)$. Recalling the notation introduced before Definition 1.67, for all $x \in X$, set $e_x = \iota_x \pi_x$. Then, $e_y A e_x \cong \mathrm{Hom}_{\mathsf{Vec}_\Bbbk}(V_x, V_y)$, and so $f \in e_y A e_x$ is a local unit if and only if it is an isomorphism of vector spaces between V_x and V_y. Hence, $A_X^\times \cong \mathrm{GL}_X(V)$. If $|X| = 1$, then we recover that $(\mathrm{End}(V))^\times = \mathrm{GL}(V)$.

2.2.2 Groupoids of grouplike elements

The groupoid of local units introduced in the previous section provides a groupoid $A_X^\times$ which is associated to a $\Bbbk$-algebra A with a set $\{e_x\}_{x\in X}$ of nonzero orthogonal idempotents. For the groupoid algebra $\Bbbk\mathcal{G}$ with its set of identity morphisms, $(\Bbbk\mathcal{G})_X^\times$ is not necessarily equal to $\Bbbk\mathcal{G}$. In this section, we study the groupoid of grouplike elements $\Gamma(H)$ of a weak Hopf algebra H and we will see that $\Gamma(\Bbbk\mathcal{G}) = \mathcal{G}$. Following [BGTLC14, Corollary 6.6, Proposition 6.8, Theorem 8.4], consider the construction below. Recall that the object set (resp. morphism set) of a groupoid $\mathcal{G}$ is denoted by $\mathcal{G}_0$ (resp. $\mathcal{G}_1$).

Definition 2.20 ($\Gamma(H)$, [BGTLC14]). Let H be a $\Bbbk$-weak Hopf algebra. The *groupoid of grouplike elements* of H, denoted $\Gamma(H)$, is defined as follows. The morphisms of $\Gamma(H)$ are the elements in

$$\Gamma(H)_1 = \{h \in H : \Delta(h) = h \otimes h,\ \varepsilon(h) = 1_\Bbbk\}.$$

The object set $\Gamma(H)_0$ is $\Gamma(H)_1 \cap H_t$. Each $h \in \Gamma(H)_1$ is an element in $\mathrm{Hom}_{\Gamma(H)}(\varepsilon_s(h), \varepsilon_t(h))$ with composition defined by restriction of the product in H. The inverse of the morphism $h \in \mathrm{Hom}_{\Gamma(H)}(\varepsilon_s(h), \varepsilon_t(h))$ is given by $S(h) \in \mathrm{Hom}_{\Gamma(H)}(\varepsilon_t(h), \varepsilon_s(h))$.

If H is a Hopf algebra, then this definition gives the classical group of grouplike elements of H, and so we use the same notation $\Gamma(H)$ whether H is a Hopf algebra or a weak Hopf algebra. We remark that $\Gamma(H)_0$ is a finite set since H_t is finite-dimensional and grouplike elements are linearly independent. However, $\Gamma(H)_0$ may be empty (see Example 2.23 below). We now seek to understand $\Gamma(H)_0$ in two special cases: when $H_s \cap H_t \cong \Bbbk$ and when $H_s = H_t$.

The next result shows that it is possible to understand the object set $\Gamma(H)_0$ in terms of its *minimal weak Hopf subalgebra* $H_{\min} := H_s H_t$, which was studied in [Nik02, Nik04].

Lemma 2.21. *Let H be a weak Hopf algebra. Then*

$$\Gamma(H)_0 = \{p \in H_s \cap H_t : p \textit{ is a nonzero idempotent and } \dim(H_{\min} p) = 1\}.$$

Proof. Suppose that $x \in \Gamma(H)_1$. Since S is anti-comultiplicative by Remark 1.7, we have $\Delta(\varepsilon_s(x)) = \Delta(S(x_1)x_2) = \Delta(S(x)x) = (S(x) \otimes S(x))(x \otimes x) = \varepsilon_s(x) \otimes \varepsilon_s(x)$, so $\varepsilon_s(x)$ is an object in $\Gamma(H)$. Similarly, $\varepsilon_t(x)$ is an object in $\Gamma(H)$. This forces $\Gamma(H)_0 \subseteq H_t \cap H_s$. Now let $p \in \Gamma(H)_0$. Since p is identified with the identity morphism at p, we also see that $p^2 = p$. Since $\varepsilon(p) = 1$, this shows that p is a nonzero idempotent. Now by [Nik04, p. 643], we have that $H_{\min} p = p H_{\min} p$ is a weak Hopf algebra with unit p. However, since $\Delta(p) = p \otimes p$, we get that $H_{\min} p$ is actually a Hopf algebra. Thus $\dim(H_{\min} p) = 1$ as $H_{\min} p = (H_{\min} p)_{\min} = \Bbbk p$.

Conversely, suppose that $p \in H_s \cap H_t$ is a nonzero idempotent such that $H_{\min} p$ has dimension 1. Since $H_{\min} p$ has a weak Hopf algebra structure, $\Delta(H_{\min} p) \subset H_{\min} p \otimes H_{\min} p$. Write $\Delta(p) = \alpha p \otimes p$ for $\alpha \in \Bbbk$. Since $\Delta(p) = \Delta(p^k) = \alpha^k p \otimes p$ for any positive integer k, $\alpha^k = \alpha$ and so $\alpha = 1$. So, $\Delta(p) = p \otimes p$. Next, $p = (\varepsilon \otimes \mathrm{Id})\Delta(p) = \varepsilon(p)p$; thus, $\varepsilon(p) = 1_{\Bbbk}$. Hence, $p \in \Gamma(H)_0$, as desired. □

Next, we consider the case when $H_s \cap H_t \cong \Bbbk$.

Proposition 2.22. *If H is a weak Hopf algebra with $H_s \cap H_t = \Bbbk$, then H is either a Hopf algebra or $\Gamma(H)_0 = \emptyset$.*

Proof. By Lemma 2.21, we have that $|\Gamma(H)_0| \leq 1$. If $|\Gamma(H)_0| = 1$, then there exists some scalar $\alpha \in \Bbbk$ such that $\Gamma(H)_0 = \alpha 1_H$. Since $\alpha 1_H$ is grouplike, $\Delta(1_H) = \alpha(1_H \otimes 1_H)$. By Proposition 1.4, $H_t = H_s = \Bbbk 1_H$ and hence H is a Hopf algebra. Otherwise, $|\Gamma(H)_0| = 0$, whence $\Gamma(H)_0 = \emptyset$. □

Example 2.23. Let $N \geq 2$ be an integer, let $\epsilon \in \{1, -1\}$, and consider the face algebra $H = \mathfrak{S}(A_{N-1}; t)_\epsilon$ introduced by Hayashi [Hay99, Example 2.1]. Then H is a weak Hopf algebra with $H_s \cap H_t = \Bbbk 1_H$. So by the result above, we have that $\Gamma(H)_0 = \emptyset$.

We now consider the case when $H_s = H_t$. In particular, this holds if H is a cocommutative weak Hopf algebra.

Proposition 2.24. *Let H be a weak Hopf algebra with $H_s = H_t$. Then the following statements hold.*

(i) $\Gamma(H)_0 = \{e_1, \dots, e_n\}$*, where* $1_H = \sum_{i=1}^n e_i$ *and* $\{e_1, \dots, e_n\}$ *is a complete set of primitive orthogonal idempotents of H. Moreover, each e_i is grouplike.*

(ii) *If A is an H-module algebra, then $A = \bigoplus_{i=1}^n A_i$ is an X-decomposable $\Bbbk$-algebra where $X = \{e_1, \dots, e_n\}$ is a complete set of primitive idempotents of H. The local identities of A are given by the family of orthogonal idempotents $\{e_i \cdot 1_A \mid 1 \le i \le n\}$.*

(iii) *Suppose, further, that $H = \bigoplus_{x \in X} H_x$ is a direct sum of weak Hopf algebras H_x. If A is an H-module algebra, then A is X-decomposable, and A_x is a H_x-module algebra obtained by restricting the action of H on A, for each $x \in X$. Moreover, $H_y \cdot A_x = 0$ if $x \neq y$.*

Proof. (i): If $H_s = H_t$, then $H_s = H_t = H_{\min}$ is a commutative semisimple $\Bbbk$-algebra by Proposition 1.4(i) and (vii). Hence, $H_t = \bigoplus_{i=1}^n H_t e_i$, where $\{e_1, \dots, e_n\}$ is a complete set of primitive orthogonal idempotents. Since $\Bbbk$ is algebraically closed, $H_t e_i \cong \Bbbk e_i$ for i. Now apply Lemma 2.21 to get the first statement.

Next, by Proposition 1.4(iii), $\Delta(e_i) = 1_1 e_i \otimes 1_2 = 1_1 \otimes e_i 1_2 = 1_1 \otimes 1_2 e_i$. Since each e_i is idempotent, we have

$$\Delta(e_i) = \Delta(e_i^2) = (1_1 e_i \otimes 1_2)(1'_1 \otimes 1'_2 e_i) = 1_1 e_i \otimes 1_2 e_i = \Delta(1_H)(e_i \otimes e_i).$$

Writing $\Delta(1_H) = \sum_{i,j=1}^n \alpha_{i,j} e_i \otimes e_j$ for some scalars $\alpha_{i,j} \in \Bbbk$, we see that

$$\Delta(e_i) = \textstyle\sum_{i,j=1}^n (\alpha_{i,j} e_i \otimes e_j)(e_i \otimes e_i) = \alpha_{i,i} e_i \otimes e_i.$$

Since $\Delta(e_i) = \Delta(e_i^k) = \alpha_{i,i}^k e_i \otimes e_i$ for all $k \in \mathbb{Z}_+$, so $\alpha_{i,i} = 1$. Thus, $\Delta(e_i) = e_i \otimes e_i$.

(ii): Take $\{e_1, \dots, e_n\}$ as in part (i). Next define the $\Bbbk$-linear map $\varepsilon'_s : H \to H$ by $\varepsilon'_s(h) = 1_1 \varepsilon(1_2 h)$. It follows from Remark 1.28 that

$$(h \cdot 1_A)a = \varepsilon_t(h) \cdot a \quad \text{and} \quad a(h \cdot 1_A) = \varepsilon'_s(h) \cdot a, \tag{2.1}$$

for any $a \in A$ and $h \in H$. Note that for $h \in H$, we have

$$\varepsilon'_s(h) = 1_1 \varepsilon(1_2 h) = \textstyle\sum_{i=1}^n e_i \varepsilon(e_i h) = \sum_{i=1}^n \varepsilon(e_i h) e_i = \varepsilon(1_1 h) 1_2 = \varepsilon_t(h).$$

In particular, for any $1 \le j \le n$, we have

$$\varepsilon'_s(e_j) = e_j. \tag{2.2}$$

Therefore, by (2.1) we see that

$$(h \cdot 1_A)a = a(h \cdot 1_A) \tag{2.3}$$

for any $a \in A$ and $h \in H$.

By Definition 1.60, to show that A is an X-decomposable $\Bbbk$-algebra it suffices to check that $\{e_i \cdot 1_A \mid 1 \le i \le n\}$ is a complete set of orthogonal central idempotents of A. Indeed,

$$(e_i \cdot 1_A)(e_j \cdot 1_A) \overset{(2.1)}{=} \varepsilon_s'(e_j) \cdot (e_i \cdot 1_A) \overset{(2.2)}{=} e_j \cdot (e_i \cdot 1_A) = \delta_{i,j} e_i \cdot 1_A.$$

So $e_i \cdot 1_A$ and $e_i \cdot 1_A$ are mutually orthogonal idempotents. It follows from $\sum_{i=1}^n (e_i \cdot 1_A) = (\sum_{i=1}^n e_i) \cdot 1_A = 1_A$ that $\{e_i \cdot 1_A \mid e_i \in X\}$ is a complete set of idempotents. Letting $h = e_i$ in (2.3), we get $(e_i \cdot 1_A)a = a(e_i \cdot 1_A)$ for all $a \in A$. Hence each $e_i \cdot 1_A$ is central in A.

(iii): We give the proof in the case that $|X| = 2$. The proof easily generalizes to any finite set X. Let H_1 and H_2 be weak Hopf algebras and suppose that A_i is an H_i-module algebra for $i = 1, 2$. By part (ii), we know that A is Y-decomposable, where $Y = \{e_1, \dots, e_n\}$ is a complete set of primitive idempotents of H, and the $\Bbbk$-algebra decomposition of A is given by

$$A = \textstyle\bigoplus_{i=1}^n (e_i \cdot 1_A)A.$$

Suppose that $1_{H_1} = e_1 + \cdots + e_k$ and $1_{H_2} = e_{k+1} + \cdots + e_n$. Then

$$A = \textstyle\bigoplus_{i=1}^k (e_i \cdot 1_A)A \oplus \bigoplus_{i=k+1}^n (e_i \cdot 1_A)A = (1_{H_1} \cdot 1_A)A \oplus (1_{H_2} \cdot 1_A)A.$$

For $i = 1, 2$, let $A_i = (1_{H_i} \cdot 1_A)A$, so that $A = A_1 \oplus A_2$.

We claim that $H_i \cdot A_i \subseteq A_i$, but $H_i \cdot A_j = 0$ if $i \neq j$. Indeed, choosing $h_i \in H_i$ and $a_j = (1_{H_j} \cdot 1_A)a \in A_j$ for some $a \in A$, if $i \neq j \in \{1, 2\}$, then we compute:

$$h_i \cdot a_j = h_i \cdot ((1_{H_j} \cdot 1_A)a) = ((h_i)_1 \cdot (1_{H_j} \cdot 1_A))((h_i)_2 \cdot a) = (((h_i)_1 1_{H_j}) \cdot 1_A)((h_i)_2 \cdot a) = 0.$$

Without loss of generality, we assume that $h_1 \cdot a_1 = b_1 + b_2$ where $h_1 \in H_1$, $a_1 \in A_1$, and $b_i \in A_i$ for $i = 1, 2$. Then, $h_1 \cdot a_1 = 1_{H_1} \cdot b_1 + 1_{H_1} \cdot b_2 = b_1$. So, $H_1 \cdot A_1 \subseteq A_1$. Likewise, $H_2 \cdot A_2 \subseteq A_2$ as claimed. □

As a corollary to the previous proposition, we obtain that groupoid actions on domains factor through group actions (see similar results for inner faithful Hopf actions on various domains factoring through actions of groups [EW14, CEW15, CEW16, EW16a, EW16b]).

Corollary 2.25. *Suppose that $\mathcal{G}$ is an X-groupoid and A is a domain such that A is an inner faithful $\Bbbk\mathcal{G}$-module algebra. Then $\mathcal{G}$ is a disjoint union of groups, and at most one of the groups is nontrivial.*

Proof. By Proposition 2.24, if A is a $\Bbbk\mathcal{G}$-module algebra, then A is X-decomposable where $X = \{1, \dots, n\}$ is the set of objects of $\mathcal{G}$, that is, $A = \bigoplus_{i=1}^n A_i$. Since A is a domain, this implies that exactly one of the A_i is nonzero. Without loss of generality, suppose $A_1 \neq 0$. Now consider the ideal of $\Bbbk\mathcal{G}$ defined by

$$I = \langle g - e_i \mid g \in \mathcal{G} \text{ such that } s(g) = t(g) = i,\ 2 \le i \le n \rangle.$$

It is straightforward to check that I is in fact a weak Hopf ideal of $\Bbbk\mathcal{G}$. Further observe that if $a \in A_1$ and $g - e_i$ is a generator of I (so $i \neq 1$), then we have $(g - e_i) \cdot a = 0$. Hence, $I \cdot A_1 = 0$, whence $I \cdot A = 0$. Since A is inner faithful, this implies that for $2 \le i \le n$, the only element $g \in \mathcal{G}$ satisfying $s(g) = t(g) = i$ is the trivial path e_i. This shows that $\mathcal{G}$ is a disjoint union of trivial groups, together with a possibly nontrivial group at vertex 1. □

2.2.3 Key categories

Next, we define some categories that we first mentioned in Example 1.13 and Notation 2.1.

Definition 2.26 (X-Alg, X-WHA, X-GrpdAlg). We define the following categories.

(i) Let X-Alg be the category defined as follows:

- The objects are unital $\Bbbk$-algebras A with a given set $\{e_x^A\}_{x\in X}$ of nonzero orthogonal idempotents such that $1_A = \sum_{x\in X} e_x^A$; we call these *$X$-algebras*.
- The morphisms are unital $\Bbbk$-algebra maps $f : A \to B$ such that $f(e_x^A) = e_x^B$ for every $x \in X$; we call these maps *X-algebra morphisms*.

(ii) Let X-WHA be the category defined as follows:

- The objects are weak Hopf algebras H with a given set $\{e_x^H\}_{x\in X}$ of nonzero orthogonal idempotents such that $1_H = \sum_{x\in X} e_x^H$, $\Delta(e_x^H) = e_x^H \otimes e_x^H$, and $\varepsilon(e_x^H) = 1_\Bbbk$ for all $x \in X$; we call these *X-weak Hopf algebras*.
- The morphisms are unital weak Hopf algebra morphisms $f : H \to H'$ such that $f(e_x^H) = e_x^{H'}$ for all $x \in X$; we call these maps *X-weak Hopf algebra morphisms*.

(iii) Let X-GrpdAlg be the category of X-groupoid algebras, that is, the full subcategory of X-WHA consisting of groupoid algebras $\Bbbk\mathcal{G}$ over X-groupoids $\mathcal{G}$.

Remark 2.27. (i) Every X-decomposable $\Bbbk$-algebra is an X-algebra, but not conversely (as the idempotents of an X-algebra are not necessarily central).

(ii) We have that X-WHA is a subcategory of X-Alg.

(iii) By Proposition 2.24(i), a weak Hopf algebra H satisfying $H_s = H_t$ is an X-weak Hopf algebra, where X is the complete set of primitive idempotents of H.

(iv) By [NV02, Proposition 2.3.3], weak Hopf algebra morphisms preserve counital subalgebras. Hence, by considering all possible finite sets X, our results pertain to all weak Hopf algebras with commutative counital subalgebras, and all morphisms between such weak Hopf algebras.

(v) We remark that the category wha considered in [BGTLC14] has a weaker notion of morphisms than those in X-WHA here. The morphisms in wha need not be weak Hopf algebra morphisms (in particular, they are not necessarily $\Bbbk$-algebra morphisms; see [BGTLC14, Theorem 4.12]).

Example 2.28. For an X-groupoid $\mathcal{G}$, the groupoid algebra $\Bbbk\mathcal{G}$ belongs to X-WHA.

Example 2.29. Let $\mathcal{G}$ be an X-groupoid. Then an X-decomposable vector space $V = \bigoplus_{x\in X} V_x$ is a representation of $\mathcal{G}$ if and only if $\mathcal{G} \to \mathrm{GL}_X(V)$ is a morphism in X-Grpd. Moreover, an X-decomposable vector space $V = \bigoplus_{x\in X} V_x$ is a representation of $\Bbbk\mathcal{G}$ if and only if $\Bbbk\mathcal{G} \to \mathrm{End}(V)$ is a morphism in X-Alg.

Lemma 2.30. *Let $A, B \in X$-Alg with respective sets of idempotents $\{e_x^A\}_{x\in X}$ and $\{e_x^B\}_{x\in X}$. If $f : A \to B$ is an X-algebra map and $a \in A$ is a local unit in A, then $f(a)$ is a local unit in B.*

Proof. Let $a \in e_y^A A e_x^A$ for some $x, y \in X$. Then, $f(a) = f(e_y^A a e_x^A) = e_y^B f(a) e_x^B \in e_y^B B e_x^B$. Moreover, $f(a)f(\widehat{a}) = f(a\widehat{a}) = f(e_y^A) = e_y^B$, and similarly, $f(\widehat{a})f(a) = e_x^B$. □

2.2.4 Module algebras over groupoid algebras

We generalize well-known adjunctions of groups to the groupoid case, recovering the classical case when $|X| = 1$.

Theorem 2.31. *Let X be a finite nonempty set.*

(i) *The following functors are well-defined:*

$$\Bbbk(-) : X\text{-}\mathsf{Grpd} \longrightarrow X\text{-}\mathsf{Alg} \qquad \textit{(groupoid algebra);}$$
$$(-)_X^\times : X\text{-}\mathsf{Alg} \longrightarrow X\text{-}\mathsf{Grpd} \qquad \textit{(groupoid of local units).}$$

Moreover, $\Bbbk(-) \dashv (-)_X^\times$, that is, for an X-groupoid $\mathcal{G}$ and an X-algebra B, we have a bijection that is natural in each slot:

$$\mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\mathcal{G}, B_X^\times) \cong \mathrm{Hom}_{X\text{-}\mathsf{Alg}}(\Bbbk\mathcal{G}, B).$$

(ii) *The following functors are well-defined:*

$$\Bbbk(-) : X\text{-}\mathsf{Grpd} \longrightarrow X\text{-}\mathsf{WHA} \qquad \textit{(groupoid algebra);}$$
$$\Gamma(-) : X\text{-}\mathsf{WHA} \longrightarrow X\text{-}\mathsf{Grpd} \qquad \textit{(groupoid of grouplike elements).}$$

Moreover, $\Bbbk(-) \dashv \Gamma(-)$, that is, for an X-groupoid $\mathcal{G}$ and an X-weak Hopf algebra H, we have a bijection that is natural in each slot:

$$\mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\mathcal{G}, \Gamma(H)) \cong \mathrm{Hom}_{X\text{-}\mathsf{WHA}}(\Bbbk\mathcal{G}, H).$$

In particular, $\Gamma(\Bbbk\mathcal{G}) = \mathcal{G}$.

Proof. (i): First, we prove that $\Bbbk(-) : X\text{-}\mathsf{Grpd} \to X\text{-}\mathsf{Alg}$ is indeed a functor. As mentioned in Remark 2.16 and Example 2.19(a), $\Bbbk\mathcal{G}$ is an X-algebra with idempotent set $\{e_x^{\mathcal{G}}\}_{x\in X}$ given by the identity morphisms of $\mathcal{G}$, so $\Bbbk(-)$ sends objects to objects. Also, if $\phi : \mathcal{G} \to \mathcal{G}'$ is an X-groupoid morphism, then we can consider ϕ as a function between $\mathcal{G}_1$ and $\mathcal{G}'_1$ so the linear extension $\Bbbk\phi : \Bbbk\mathcal{G} \to \Bbbk\mathcal{G}'$ makes sense. It is straightforward to check that ϕ being a functor translates into $\Bbbk\phi$ being an X-algebra map. Hence, $\Bbbk(-)$ also sends morphisms to morphisms. Since the linear extension of a map behaves as the map itself when restricted to basis elements, $\Bbbk(-)$ respects compositions and identities.

Secondly, we check that $(-)^{\times}_X : X\text{-}\mathsf{Alg} \to X\text{-}\mathsf{Grpd}$ is a functor. By construction, the groupoid of local units is an X-groupoid, so $(-)^{\times}_X$ sends objects to objects. Moreover, if $\psi : A \to B$ is an X-algebra map, we can define the X-groupoid map $\psi^{\times}_X : A^{\times}_X \to B^{\times}_X$ where $a : x \to y$ maps to $\psi(a) : x \to y$ (which is a local unit by Lemma 2.30). So $\psi^{\times}_X$ fixes X and it is a functor due to the properties of ψ. Also, $(-)^{\times}_X$ respects compositions and identities.

Now, for any X-groupoid $\mathcal{G}$ and any X-algebra B, we want to display a bijection

$$\mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\mathcal{G}, B^{\times}_X) \cong \mathrm{Hom}_{X\text{-}\mathsf{Alg}}(\Bbbk\mathcal{G}, B).$$

Given an X-groupoid morphism $\phi \in \mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\mathcal{G}, B^{\times}_X)$, we consider it as a function $\phi : \mathcal{G}_1 \to (B^{\times}_X)_1$, and construct the linear extension $\phi' : \Bbbk\mathcal{G} \to B$. Note that for $g \in \mathcal{G}_1$, $\phi'(g) = \phi(g)$, which implies that ϕ' is indeed an X-algebra map. Such verification uses that $1_B = \sum_{x\in X} e^B_x$. Hence, we have constructed the assignment

$$\begin{aligned}\Phi_{\mathcal{G},B} : \mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\mathcal{G}, B^{\times}_X) &\to \mathrm{Hom}_{X\text{-}\mathsf{Alg}}(\Bbbk\mathcal{G}, B)\\ \phi &\mapsto \phi'.\end{aligned}$$

On the other hand, given an X-algebra map $\psi \in \mathrm{Hom}_{X\text{-}\mathsf{Alg}}(\Bbbk\mathcal{G}, B)$, consider the restriction $\psi|_{\mathcal{G}} : \mathcal{G} \to B^{\times}_X$ given by $\psi|_{\mathcal{G}}(g : x \to y) := \psi(g) : x \to y$. As above, this assignment is well-defined due to Lemma 2.30. The fact that $\psi|_{\mathcal{G}}$ is a functor follows from ψ being an X-algebra map. Hence, we have constructed the assignment

$$\begin{aligned}\Psi_{\mathcal{G},B} : \mathrm{Hom}_{X\text{-}\mathsf{Alg}}(\Bbbk\mathcal{G}, B) &\to \mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\mathcal{G}, B^{\times}_X)\\ \psi &\mapsto \psi|_{\mathcal{G}}.\end{aligned}$$

The following calculations show that these assignments are mutually-inverse. If $g : x \to y$ in $\mathcal{G}_1$, then

$$[(\Psi_{\mathcal{G},B} \circ \Phi_{\mathcal{G},B})(\phi)](g) = \Psi_{\mathcal{G},B}(\phi')(g) = \phi'|_{\mathcal{G}}(g) = \phi(g),$$
$$[(\Phi_{\mathcal{G},B} \circ \Psi_{\mathcal{G},B})(\psi)](g) = \Phi_{\mathcal{G},B}(\psi|_{\mathcal{G}})(g) = (\psi|_{\mathcal{G}})'(g) = \psi(g).$$

Finally, the bijection is natural since for any X-groupoid morphism $\varphi : \mathcal{G}' \to \mathcal{G}$, any X-algebra map $f : B \to B'$, and all $\psi \in \mathrm{Hom}_{X\text{-}\mathsf{Alg}}(\Bbbk\mathcal{G}, B)$ one can check the following equality:

$$(\Phi_{\mathcal{G}',B'} \circ \mathrm{Hom}_{X\text{-}\mathsf{Alg}}(\Bbbk\varphi, f))(\psi) = (\mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\varphi, f^{\times}_X) \circ \Phi_{\mathcal{G},B})(\psi).$$

Here, $\mathrm{Hom}_{X\text{-}\mathsf{Alg}}(\Bbbk\varphi, f) : \mathrm{Hom}_{X\text{-}\mathsf{Alg}}(\Bbbk\mathcal{G}, B) \to \mathrm{Hom}_{X\text{-}\mathsf{Alg}}(\Bbbk\mathcal{G}', B')$ is given by composition, that is, $\psi \mapsto f \circ \psi \circ \Bbbk\varphi$; a similar notion holds for $\mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\varphi, f^{\times}_X)$.

(ii): First, we show that $\Bbbk(-) : X\text{-}\mathsf{Grpd} \to X\text{-}\mathsf{WHA}$ is a functor. If $\mathcal{G}$ is an X-groupoid, then $\Bbbk\mathcal{G}$ is a weak Hopf algebra. The set $\{e^{\Bbbk\mathcal{G}}_x\}_{x\in X}$ of identity morphisms of $\mathcal{G}$ are idempotent elements of $\Bbbk\mathcal{G}$ satisfying $1_{\Bbbk\mathcal{G}} = \sum_{x\in X} e^{\Bbbk\mathcal{G}}_x$ and $\Delta(e^{\Bbbk\mathcal{G}}_x) = e^{\Bbbk\mathcal{G}}_x \otimes e^{\Bbbk\mathcal{G}}_x$ for each $x \in X$, and so $\Bbbk\mathcal{G}$ is an object of $X\text{-}\mathsf{WHA}$. If $\phi : \mathcal{G} \to \mathcal{G}'$ is an X-groupoid morphism, then by part (i), $\Bbbk\phi : \Bbbk\mathcal{G} \to \Bbbk\mathcal{G}'$ is a map of X-algebras. Since the elements of $\mathcal{G}$ form a

$\Bbbk$-basis of grouplike elements for $\Bbbk\mathcal{G}$, and $\Bbbk\phi$ maps these elements to grouplike elements of $\Bbbk\mathcal{G}'$, we have that $\Bbbk\phi$ is also a $\Bbbk$-coalgebra map. Since the antipode of $\Bbbk\mathcal{G}$ is defined by $S_{\Bbbk\mathcal{G}}(g) = g^{-1}$ for all $g \in \mathcal{G}$, therefore $S_{\Bbbk\mathcal{G}'} \circ \Bbbk\phi = \Bbbk\phi \circ S_{\Bbbk\mathcal{G}}$. Finally, $\phi(e_x^{\Bbbk\mathcal{G}}) = e_x^{\Bbbk\mathcal{G}'}$, so $\Bbbk\phi$ is a morphism in X-WHA.

Now we show that $\Gamma(-) : X\text{-}\mathsf{WHA} \to X\text{-}\mathsf{Grpd}$ is a functor. Note that $\Gamma(-)$ is the restriction of the functor $\mathsf{g}(-) : \mathsf{wha} \to \mathsf{Grpd}$ in [BGTLC14, Theorem 8.4] to X-WHA, where the category wha has a weaker notion of morphism; see Remark 2.27(v). So, we need only check that if $H \in X\text{-}\mathsf{WHA}$ then $\Gamma(H) \in X\text{-}\mathsf{Grpd}$. Indeed, if $H \in X\text{-}\mathsf{WHA}$, then $\Gamma(H)$ is a groupoid with object set $\{e_x^H\}_{x\in X}$, and so $\Gamma(H) \in X\text{-}\mathsf{Grpd}$. We remark that if $f : H \to H'$ is a morphism in X-WHA, then $\Gamma(f)$ is the restriction of f to $\Gamma(H)$.

Finally, for any $\mathcal{G} \in X\text{-}\mathsf{Grpd}$ and $H \in X\text{-}\mathsf{WHA}$, we wish to exhibit a bijection

$$\mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\mathcal{G}, \Gamma(H)) \cong \mathrm{Hom}_{X\text{-}\mathsf{WHA}}(\Bbbk\mathcal{G}, H).$$

Given $\phi \in \mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\mathcal{G}, \Gamma(H))$, the $\Bbbk$-linearization $\Bbbk\phi : \Bbbk\mathcal{G} \to \Bbbk\Gamma(H)$ is a morphism of X-weak Hopf algebras. Since composition in $\Gamma(H)$ was defined as restriction of multiplication in H, the map $\nu_H : \Bbbk\Gamma(H) \to H$ induced by the inclusion $\Gamma(H) \subseteq H$ is a $\Bbbk$-algebra map. It is also unital. Since the elements of $\Gamma(H)$ are grouplike in H, it is also a $\Bbbk$-coalgebra map. Since the inverse in $\Gamma(H)$ coincides with the antipode of H, ν_H intertwines the antipodes of $\Bbbk\Gamma(H)$ and H. Therefore, $\nu_H \circ \Bbbk\phi \in \mathrm{Hom}_{X\text{-}\mathsf{WHA}}(\Bbbk\mathcal{G}, H)$, and we have the assignment

$$\begin{aligned}\Phi_{\mathcal{G},H} : \mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\mathcal{G}, \Gamma(H)) &\to \mathrm{Hom}_{X\text{-}\mathsf{WHA}}(\Bbbk\mathcal{G}, H)\\ \phi &\mapsto \nu_H \circ \Bbbk\phi.\end{aligned}$$

Conversely, if $\psi \in \mathrm{Hom}_{X\text{-}\mathsf{WHA}}(\Bbbk\mathcal{G}, H)$, then $\Gamma(\psi) : \Gamma(\Bbbk\mathcal{G}) \to \Gamma(H)$ is an X-groupoid morphism. Note that $\Gamma(\Bbbk\mathcal{G}) = \mathcal{G}$. It is clear that the elements of $\mathcal{G}$ are grouplike elements of $\Bbbk\mathcal{G}$. On the other hand, any element of $\Bbbk\mathcal{G}$ is of the form $a = \sum_{g\in\mathcal{G}} \alpha_g g$, where

$$a \otimes a = \left(\sum_{g\in\mathcal{G}} \alpha_g g\right) \otimes \left(\sum_{h\in\mathcal{G}} \alpha_h h\right) = \sum_{g,h\in\mathcal{G}} \alpha_g \alpha_h (g \otimes h)$$

and $\Delta(a) = \sum_{g\in\mathcal{G}} \alpha_g (g \otimes g)$. By the linear independence of grouplike elements, if $a \in \Gamma(\Bbbk\mathcal{G})$ we have that $\alpha_g \alpha_h = \delta_{g,h}\alpha_g$, whence exactly one $\alpha_g = 1$ and the remaining coefficients are 0. Hence, the only grouplike elements of $\Bbbk\mathcal{G}$ are the elements of $\mathcal{G}$. We have therefore constructed an assignment

$$\begin{aligned}\Psi_{\mathcal{G},H} : \mathrm{Hom}_{X\text{-}\mathsf{WHA}}(\Bbbk\mathcal{G}, H) &\to \mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\mathcal{G}, \Gamma(H))\\ \psi &\mapsto \Gamma(\psi) \circ \eta_{\mathcal{G}},\end{aligned}$$

where $\eta_{\mathcal{G}} : \mathcal{G} \to \Gamma(\Bbbk\mathcal{G})$ is the identity map.

It is straightforward to check that $\Phi_{\mathcal{G},H}$ and $\Psi_{\mathcal{G},H}$ are mutually-inverse. Naturality holds by [BGTLC14, Theorem 8.4]. □

Remark 2.32. Note that Theorem 2.31(i) shows that for an X-groupoid $\mathcal{G}$, we have an isomorphism of categories, $\mathsf{rep}(\Bbbk\mathcal{G}) \cong \mathsf{rep}(\mathcal{G})$. Namely, if $V = \bigoplus_{x\in X} V_x$, let $B = \mathrm{End}(V)$. Then, $B_X^\times = \mathrm{GL}_X(V)$ and

$$\mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\mathcal{G}, \mathrm{GL}_X(V)) \cong \mathrm{Hom}_{X\text{-}\mathsf{Alg}}(\Bbbk\mathcal{G}, \mathrm{End}(V)).$$

See also Example 2.29.

Finally, by using the adjunction in Theorem 2.31(ii), together with Proposition 2.11, for an X-decomposable $\Bbbk$-algebra A, we are able to identify the object $\mathrm{Sym}_{X\text{-}\mathsf{GrpdAlg}}(A)$.

Proposition 2.33. *Let $A = \bigoplus_{x\in X} A_x$ be an X-decomposable $\Bbbk$-algebra and $\mathcal{G}$ be an X-groupoid. Then:*

(i) *The action of $\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$ on A can be extended linearly to make A a $\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$-module algebra. For the action map $\ell_{\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)),A} : \Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)) \otimes A \to A$, we use the notation $h \triangleright a := \ell_{\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)),A}(h \otimes a)$ for $h \in \Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$ and $a \in A$.*

(ii) *Suppose that A is a $\Bbbk\mathcal{G}$-module algebra via $\ell_{\Bbbk\mathcal{G},A}$, where we write $h \cdot a$ for $\ell(h\otimes a)$ for $h \in \Bbbk\mathcal{G}$ and $a \in A$. Then there is a unique X-weak Hopf algebra morphism $\widetilde{\pi} : \Bbbk\mathcal{G} \to \Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$ such that $h \cdot a = \widetilde{\pi}(h) \triangleright a$ for all $h \in \Bbbk\mathcal{G}$ and $a \in A$.*

(iii) *Every X-weak Hopf algebra morphism $\widetilde{\pi} : \Bbbk\mathcal{G} \to \Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$ gives A the structure of a $\Bbbk\mathcal{G}$-module algebra via $\ell_{\Bbbk\mathcal{G},A}(h \otimes a) = \ell_{\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)),A}(\widetilde{\pi}(h) \otimes a)$ for all $h \in \Bbbk\mathcal{G}$ and $a \in A$ (that is, $h \cdot a = \widetilde{\pi}(h) \triangleright a$).*

Hence $\mathrm{Sym}_{X\text{-}\mathsf{GrpdAlg}}(A) = \Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$.

Proof. (i): As mentioned in Remark 1.66, over any groupoid $\mathcal{G}$, the notion of a $\mathcal{G}$-module and a $\Bbbk\mathcal{G}$-module are equivalent. By Proposition 2.11, A is an $\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$-module via $*$, and so by linearizing this action, A is a $\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$-module via $\triangleright$. To see that A is actually a $\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$-module *algebra*, let $h \in \Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$ and $a, b \in A$. Write $h = \sum_{f\in\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)} \alpha_f f$ for some $\alpha_f \in \Bbbk$. Then

$$\begin{aligned} h \triangleright (ab) &= \Big(\sum \alpha_f f\Big) \triangleright (ab) = \sum \alpha_f\,(f \triangleright (ab)) = \sum \alpha_f\,(f * (ab)) \\ &= \sum \alpha_f (f * a)(f * b) = \sum \alpha_f (f_1 \triangleright a)(f_2 \triangleright b) = (h_1 \triangleright a)(h_2 \triangleright b) \end{aligned}$$

and

$$h \triangleright 1_A = \Big(\sum \alpha_f f\Big) \triangleright 1_A = \sum \alpha_f (f \triangleright 1_A) = \sum \alpha_f (f * 1_A) = \sum \alpha_f 1_{t(f)} = \varepsilon_t(h) \cdot 1_A.$$

(ii)-(iii): As shown above, A is a $\Bbbk\mathcal{G}$-module algebra if and only if A is a $\mathcal{G}$-module algebra. By Proposition 2.11, this happens if and only if the action of $\mathcal{G}$ on A factors through a unique X-groupoid morphism $\pi : \mathcal{G} \to \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$. Now, letting $H = \Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$ in Theorem 2.31(ii), we obtain a bijection:

$$\mathrm{Hom}_{X\text{-}\mathsf{Grpd}}(\mathcal{G}, \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)) \cong \mathrm{Hom}_{X\text{-}\mathsf{Alg}}(\Bbbk\mathcal{G}, \Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))), \quad \pi \mapsto \widetilde{\pi}.$$

Hence, there exists an X-groupoid morphism $\pi : \mathcal{G} \to \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$ if and only if there exists an X-weak Hopf algebra morphism $\widetilde{\pi} : \Bbbk\mathcal{G} \to \Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$ (which is simply the $\Bbbk$-linearization of π).

Note that since X-$\mathsf{GrpdAlg}$ is a full subcategory of X-WHA, the morphisms in X-$\mathsf{GrpdAlg}$ are simply X-weak Hopf algebra morphisms. As a consequence, (i), (ii), and (iii) above imply that $\mathrm{Sym}_{X\text{-}\mathsf{GrpdAlg}}(A) = \Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$. □

Remark 2.34. Proposition 2.33 has an alternative presentation appearing in [PF13, Proposition 2.2] using the language of partial actions.

As a corollary, we recover the following well-known result for actions of group algebras, which is the special case of Proposition 2.33 when $|X| = 1$.

Corollary 2.35. *Let A be a $\Bbbk$-algebra. Then A is a $\Bbbk\,\mathrm{Aut}_{\mathrm{Alg}}(A)$-module algebra and for a group G, the following are equivalent.*

(i) *A is a $\Bbbk G$-module algebra.*

(ii) *There exists a Hopf algebra morphism $\widetilde{\pi} : \Bbbk G \to \Bbbk(\mathrm{Aut}_{\mathrm{Alg}}(A))$.*

Hence $\mathrm{Sym}_{\mathsf{GrpAlg}}(A) = \Bbbk(\mathrm{Aut}_{\mathrm{Alg}}(A))$.

2.3 Weak quantum symmetries captured by Lie algebroids

The definition of a Lie algebroid was introduced in [Pra67], using the language of vector bundles over manifolds (see also [Mac87, Chapter III]). An equivalent, purely algebraic structure known as a Lie–Rinehart algebra, was introduced in [Rin63]. Although some Hopf-like structures have been defined for general Lie–Rinehart algebras by means of Hopf algebroids (see e.g. [Sar20, Sar21]), the study of their actions on $\Bbbk$-algebras is still an open problem. The focus of this section is to study actions of a special subclass of Lie algebroids [Definition 2.36]. We also extend this result for universal enveloping algebras of these Lie algebroids [Definition 2.49]. As in previous sections, here X denotes a finite non-empty set [Notation 1.43].

Definition 2.36. An *X-Lie algebroid* $\mathfrak{G}$ is a direct sum of vector spaces

$$\mathfrak{G} := \bigoplus\nolimits_{x\in X} \mathfrak{g}_x$$

where $\mathfrak{g}_x$ has structure of Lie algebra for all $x \in X$. We regard $\mathfrak{G}$ as having a partially defined bracket $[-,-]$, which is only defined on pairs of elements from the same component $\mathfrak{g}_x$. Namely, $[a,b] = [a,b]_{\mathfrak{g}_x}$ if $a, b \in \mathfrak{g}_x$ for some $x \in X$, and is undefined otherwise. Let $\mathfrak{G} = \bigoplus_{x\in X} \mathfrak{g}_x$ and $\mathfrak{G}' = \bigoplus_{x\in X} \mathfrak{g}'_x$ be two X-Lie algebroids. A linear map $\tau : \mathfrak{G} \to \mathfrak{G}'$ is an *X-Lie algebroid morphism* if there exists a collection $\{\tau_x : \mathfrak{g}_x \to \mathfrak{g}'_x\}_{x\in X}$ of Lie algebra morphisms such that $\tau|_{\mathfrak{g}_x} = \tau_x$ for all $x \in X$. Here, we write $\tau = (\tau_x)_{x\in X}$.

As remarked in [Nik01, Section 3.2] an X-Lie algebroid is an special kind of Lie algebroid in the sense of [Mac87].

Definition 2.37 (X-Lie, $\mathfrak{G}$). The X-Lie algebroids together with X-Lie algebroid morphisms form a category, which we denote by X-Lie. Throughout this section, $\mathfrak{G} = \bigoplus_{x\in X} \mathfrak{g}_x$ denotes an X-Lie algebroid.

Next, we briefly discuss modules and representations over X-Lie algebroids.

Definition 2.38 ($\mathfrak{G}$-module). A $\mathfrak{G}$-*module* is an X-decomposable $\Bbbk$-vector space $V = \bigoplus_{x\in X} V_x$ such that V_x is a $\mathfrak{g}_x$-module for each $x \in X$. Given two $\mathfrak{G}$-modules $V = \bigoplus_{x\in X} V_x$ and $W = \bigoplus_{x\in X} W_x$, a linear map $f : V \to W$ is a $\mathfrak{G}$-*module morphism* if there exist a collection $\{f_x : V_x \to W_x\}_{x\in X}$ of maps such that $f|_{V_x} = f_x$ and f_x is a $\mathfrak{g}_x$-module morphism for each $x \in X$. Here, we write $f = (f_x)_{x\in X}$.

The composition of two $\mathfrak{G}$-module morphisms is defined component-wise (and is again a $\mathfrak{G}$-module morphism).

Notation 2.39 ($\mathfrak{G}$-mod). Let $\mathfrak{G}$-mod be the category of $\mathfrak{G}$-modules with $\mathfrak{G}$-module morphisms. This category inherits a monoidal structure from the monoidal structures of each $\mathfrak{g}_x$-mod. Namely, for V, W in $\mathfrak{G}$-mod, we have that $V \otimes_{\mathfrak{G}\text{-mod}} W := \bigoplus_{x\in X}(V_x \otimes_{\mathfrak{g}_x\text{-mod}} W_x) = \bigoplus_{x\in X}(V_x \otimes W_x)$, and $\mathbb{1}_{\mathfrak{G}\text{-mod}} = \bigoplus_{x\in X} \mathbb{1}_{\mathfrak{g}_x\text{-mod}}$.

Remark 2.40. For a X-Lie algebroid $\mathfrak{G} = \bigoplus_{x\in X} \mathfrak{g}_x$ and an X-decomposable vector space $V = \bigoplus_{x\in X} V_x$, in the language of Remark 2.2, we define

$$\mathfrak{G} \boxtimes_{X\text{-Lie}} V = \{(p, v) \in \mathfrak{G} \times V \mid p \in \mathfrak{g}_x, v \in V_x\}.$$

Then a left action of $\mathfrak{G}$ on V can be viewed as a map $\mathfrak{G} \boxtimes_{X\text{-Lie}} V \to V$.

Now we introduce a generalization of the Lie algebra $\mathfrak{gl}(V)$.

Definition 2.41 ($\mathfrak{GL}_X(V)$, $\mathfrak{GL}_{(d_1,\ldots,d_n)}(\Bbbk)$). Let $V = \bigoplus_{x\in X} V_x$ be an X-decomposable vector space. The X-*general linear Lie algebroid* of V, denoted $\mathfrak{GL}(V)$, is the X-Lie algebroid $\mathfrak{GL}_X(V) := \bigoplus_{x\in X} \mathfrak{gl}(V_x)$. If $X = \{1, \ldots, n\}$ and V_i has dimension d_i, then we also use the notation $\mathfrak{GL}_{(d_1,\ldots,d_n)}(\Bbbk)$ where $d_1 \leq d_2 \leq \cdots \leq d_n$ for $\mathfrak{GL}_X(V)$.

Definition 2.42 (Representation of $\mathfrak{G}$). A *representation* of $\mathfrak{G}$ is an X-decomposable vector space $V = \bigoplus_{x\in X} V_x$ equipped with an X-Lie algebroid morphism $\tau : \mathfrak{G} \to \mathfrak{GL}_X(V)$.

Notation 2.43 (rep($\mathfrak{G}$)). Let rep($\mathfrak{G}$) be the monoidal category of representations of $\mathfrak{G}$, whose structure is built from $\{\mathfrak{g}_x\text{-mod}\}_{x\in X}$ [Notation 2.39]. Observe that rep($\mathfrak{G}$) $\cong$ $\mathfrak{G}$-mod as monoidal categories where if V is in $\mathfrak{G}$-mod, then $V = \bigoplus_{x\in X} V_x$ with each $V_x \in \mathfrak{g}_x$-mod, and so we get a Lie algebroid morphism $\tau_x : \mathfrak{g}_x \to \mathfrak{gl}_{x\in X}(V_x)$ which gives an X-Lie algebroid morphism $\mathfrak{G} \to \mathfrak{GL}_X(V)$. Conversely, if V is in rep($\mathfrak{G}$), then the component τ_x of $\tau : \mathfrak{G} \to \mathfrak{GL}_X(V)$ gives V_x the structure of a $\mathfrak{g}_x$-module for all $x \in X$, which makes V a $\mathfrak{G}$-module.

Now, we change our focus to actions of X-Lie algebroids on $\Bbbk$-algebras. Let $\mathfrak{G}$ be an X-Lie algebroid. As we did for the category $\mathcal{G}$-mod in Remark 1.64, we would like to understand the connection between algebra objects in $\mathfrak{G}$-mod and $\Bbbk$-algebras that are $\mathfrak{G}$-modules.

Remark 2.44. If $A = \bigoplus_{x\in X} A_x$ is an X-decomposable $\Bbbk$-algebra, the multiplication map

$m_A : A \otimes A \to A$ and unit map $u_A : \Bbbk \to A$ immediately decompose into the respective multiplication map $m_x : A_x \otimes A_x \to A_x$ and unit map $u_x : \Bbbk \to A_x$ of each $\Bbbk$-algebra A_x, for all $x \in X$. If additionally A is a $\mathfrak{G}$-module, using the notation of Lemma 2.39, m_A and u_A induce X-decomposable linear maps $\underline{m}_A := (m_x)_{x \in X} : A \otimes_{\mathfrak{G}\text{-mod}} A \to A$ and $\underline{u}_A := (u_x)_{x\in X} : \mathbb{1}_{\mathfrak{G}\text{-mod}} \to A$, which we call the *monoidal multiplication* and *monoidal unit* of A, respectively. It is clear that these maps satisfy associativity and unital condition. However, in general, these maps are not necessarily $\mathfrak{G}$-module morphisms.

Conversely, if A is $\mathfrak{G}$-module algebra, then it comes equipped with maps in $\mathfrak{G}$-mod $\underline{m}_A = (m_x)_{x\in X} A \otimes_{\mathfrak{G}\text{-mod}} A \to A$ and $\underline{u}_A = (u_x)_{x\in X} : \mathbb{1}_{\mathfrak{G}\text{-mod}} \to A$. These maps extend naturally to maps $m_A : A \otimes A \to A$ (where if $a \in A_x$ and $b \in A_y$ for $x \neq y$, we define $m_A(a \otimes b) = 0$) and $u_A : \Bbbk \to A$ (defined by $\Bbbk \to \mathbb{1}_{\mathfrak{G}\text{-mod}} \to A$ where $\Bbbk \to \mathbb{1}_{\mathfrak{G}\text{-mod}}$ maps $1_\Bbbk$ to $(1, 1, \cdots, 1)$).

By definition, it follows immediately that the monoidal multiplication and monoidal unit maps are $\mathfrak{G}$-module morphisms precisely when they make A a $\mathfrak{G}$-module algebra. Hence, we have proved the following result.

Lemma 2.45. *Let $A = \bigoplus_{x\in X} A_x$ be an X-decomposable $\Bbbk$-algebra and let $\mathfrak{G}$ be an X-Lie algebroid. Then the following statements are equivalent.*

(i) *A is a $\mathfrak{G}$-module algebra, via the monoidal product $m : A \otimes_{\mathfrak{G}\text{-mod}} A \to A$ and monoidal unit $u : \mathbb{1}_{\mathfrak{G}\text{-mod}} \to A$ of Remark 2.44.*

(ii) *A is a $\mathfrak{G}$-module, such that*

$$p \cdot (ab) = a(p \cdot b) + (p \cdot a)b, \tag{2.4}$$

$$p \cdot 1_x = 0, \tag{2.5}$$

for all $p \in \mathfrak{g}_x$ and $a, b \in A_x$.

As a consequence of this result, an X-decomposable $\Bbbk$-algebra satisfying the conditions in Lemma 2.45(ii) can be referred to as a $\mathfrak{G}$-module algebra. This terminology emphasizes that the $\Bbbk$-algebra is equipped with a compatible action of the X-Lie algebroid $\mathfrak{G}$, making it a monoid object within the category of $\mathfrak{G}$-modules (see also Remark 2.6). To proceed, we introduce a generalization of the Lie algebra $\mathrm{Der}(A)$ consisting of derivations of A with commutator bracket.

Notation 2.46 **($\mathrm{Der}_X(A)$).** Let $A = \bigoplus_{x\in X} A_x$ be an X-decomposable $\Bbbk$-algebra. We denote by $\mathrm{Der}_X(A)$ the X-Lie algebroid $\mathrm{Der}_X(A) = \bigoplus_{x\in X} \mathrm{Der}(A_x)$.

Since each $\mathrm{Der}(A_x)$ is a Lie subalgebra of $\mathfrak{gl}(A_x)$, we have that $\mathrm{Der}_X(A)$ is an X-Lie subalgebroid of $\mathfrak{GL}(A)$. Similar to the groupoid case, when $|X| = 1$ we recover the classical Lie structures.

Proposition 2.47. *Let $A = \bigoplus_{x\in X} A_x$ be an X-decomposable $\Bbbk$-algebra and let $\mathfrak{G} = \bigoplus_{x\in X} \mathfrak{g}_x$ be an X-Lie algebroid. Then:*

(i) *The natural action of* $\mathrm{Der}(A_x)$ *on* A_x *for all* $x \in X$ *makes* A *a* $\mathrm{Der}_X(A)$*-module algebra. We denote this action* $\ell_{\mathrm{Der}_X(A),A}$ *and write* $p \triangleright a := \ell_{\mathrm{Der}_X(A),A}(p \boxtimes a)$ *for* $p \boxtimes a \in \mathrm{Der}_X(A) \boxtimes_{X\text{-Lie}} A$.

(ii) *Suppose that* A *is a* $\mathfrak{G}$*-module algebra via* $\ell_{\mathfrak{G},A}$ *and denote* $p \cdot a := \ell_{\mathfrak{G},A}(p \boxtimes a)$ *for* $p \boxtimes a \in \mathfrak{G} \boxtimes_{X\text{-Lie}} A$*. Then there is a unique* X*-Lie algebroid morphism* $\tau : \mathfrak{G} \longrightarrow \mathrm{Der}_X(A)$ *such that* $p \cdot a = \tau(p) \triangleright a$ *for all* $p \boxtimes a \in \mathfrak{G} \boxtimes_{X\text{-Lie}} A$.

(iii) *Every* X*-Lie algebroid morphism* $\tau : \mathfrak{G} \longrightarrow \mathrm{Der}_X(A)$ *gives* A *the structure of a* $\mathfrak{G}$*-module algebra via* $\ell_{\mathfrak{G},A}(p \boxtimes a) = \ell_{\mathrm{Der}_X(A),A}(\tau(p) \boxtimes a)$ *for all* $p \boxtimes a \in \mathfrak{G} \boxtimes_{X\text{-Lie}} A$.

Hence, $\mathrm{Sym}_{X\text{-Lie}}(A) = \mathrm{Der}_X(A)$.

Proof. (i): The natural action of $\mathrm{Der}(A_x)$ on A_x is given by letting $p \triangleright a$ be the image of a under the derivation p, for $a \in A_x$ and $p \in \mathrm{Der}(A_x)$. It is clear that A is an $\mathrm{Der}_X(A)$-module and by Lemma 2.45, it is clear that in fact A is a $\mathrm{Der}_X(A)$-module algebra.

(ii): If A is a $\mathfrak{G}$-module via $\cdot$, then there is an associated sub-X-Lie algebroid of $\mathfrak{GL}_X(A)$ (see Notation 2.43). This induces a unique X-Lie algebroid morphism $\tau : \mathfrak{G} \longrightarrow \mathfrak{GL}_X(A)$. Moreover, if A is a $\mathfrak{G}$-module algebra, then it satisfies (2.4) and (2.5), which are equivalent to the map $\tau(p)$ being a derivation of A_x for each $p \in \mathfrak{g}_x$. Hence, the image of τ is contained in the sub-X-Lie algebroid $\mathrm{Der}_X(A)$ of $\mathfrak{GL}_X(A)$. Corestricting τ to a map $\mathfrak{G} \longrightarrow \mathfrak{GL}_X(A)$ gives the result.

(iii): If $\tau : \mathfrak{G} \longrightarrow \mathrm{Der}_X(A)$ is any X-Lie algebroid morphism, then we can define a map $\ell_{\mathfrak{G},A} : \mathfrak{G} \boxtimes_{X\text{-Lie}} A \longrightarrow A$ by $\ell_{\mathfrak{G},A}(p \boxtimes a) = \ell_{\mathrm{Der}_X(A),A}(\tau(p), a)$ for all $p \boxtimes a \in \mathfrak{G} \boxtimes_{X\text{-Lie}} A$. Since τ is an X-Lie algebroid morphism, we see that if $p, q \in \mathrm{Der}(A_x)$ and $a \in A_x$, then

$$\begin{aligned}[p,q] \cdot a = \tau([p,q]) \triangleright a = [\tau(p), \tau(q)] \triangleright a &= \tau(p) \triangleright (\tau(q) \triangleright a) - \tau(q) \triangleright (\tau(p) \triangleright a)\\ &= p \cdot (q \cdot a) - q \cdot (p \cdot a),\end{aligned}$$

so A is a $\mathfrak{G}$-module. Further, for all $p \in \mathfrak{g}_x$ and $a, b \in A_x$, we see

$$p \cdot (ab) = \tau(p) \triangleright (ab) = a(\tau(p) \triangleright b) + (\tau(p) \triangleright a)b = a(p \cdot b) + (p \cdot a)b$$

and

$$p \cdot 1_x = \tau(p) \triangleright 1_x = 0.$$

Hence, A is a $\mathfrak{G}$-module algebra. □

As a corollary, we recover the following well-known result for actions of Lie algebras, which is the special case of Proposition 2.47 when $|X| = 1$.

Corollary 2.48. *For a* $\Bbbk$*-algebra* A*,* A *is a* $\mathrm{Der}(A)$*-module algebra and for a Lie algebra* $\mathfrak{g}$*, the following are equivalent.*

(i) *A is a* $\mathfrak{g}$*-module algebra.*

(ii) *There exists a Lie algebra morphism* $\tau : \mathfrak{g} \longrightarrow \mathrm{Der}(A)$.

Hence, $\mathrm{Sym}_{\mathsf{Lie}}(A) = \mathrm{Der}(A)$.

Recall that by Proposition 1.31(i), a direct sum of Hopf algebras has a canonical weak Hopf algebra structure. Following [Nik01, Section 3.2] we recall a construction that generalizes the notion of the universal enveloping algebra of a Lie algebra to X-Lie algebroids.

Definition 2.49 **($U_X(\mathfrak{G})$, X-EnvLie).** The X*-universal enveloping algebra* of $\mathfrak{G}$ is the weak Hopf algebra $U_X(\mathfrak{G}) := \bigoplus_{x\in X} U(\mathfrak{g}_x)$, where each $U(\mathfrak{g}_x)$ is the classical universal enveloping algebra of the Lie algebra $\mathfrak{g}_x$.

Consider the following representation category for $U_X(\mathfrak{G})$.

Definition 2.50 **($\mathsf{rep}(U_X(\mathfrak{G}))$).** Let $\mathsf{rep}(U_X(\mathfrak{G}))$ be the category of *representations of* $U_X(\mathfrak{G})$, that is, objects are X-decomposable vector spaces $V = \bigoplus_{x\in X} V_x$ such that each V_x is a representation of $U(\mathfrak{g}_x)$.

Remark 2.51. Observe that $\mathsf{rep}(\mathfrak{G}) \cong \bigoplus_{x\in X} \mathsf{rep}(\mathfrak{g}_x) \cong \bigoplus_{x\in X} \mathsf{rep}(U(\mathfrak{g}_x)) \cong \mathsf{rep}(U_X(\mathfrak{G}))$; see Notation 2.43. With this, the monoidal structure of $\mathsf{rep}(\mathfrak{G})$ passes to $\mathsf{rep}(U_X(\mathfrak{G}))$.

There is a well-known adjunction between the categories Lie of Lie algebras and Hopf of Hopf algebras given by the universal enveloping algebra functor $U(-)$ and the Lie algebra of primitve elements functor $P(-)$. Namely, $U(-) \dashv P(-)$, that is, for a Lie algebra $\mathfrak{g}$ and a Hopf algebra H, we have a bijection that is natural in each slot:

$$\mathrm{Hom}_{\mathsf{Lie}}(\mathfrak{g}, P(H)) \;\cong\; \mathrm{Hom}_{\mathsf{Hopf}}(U(\mathfrak{g}), H).$$

In particular, taking $H = U(\mathrm{Der}(A))$ with its Hopf algebra structure yields a bijection

$$\mathrm{Hom}_{\mathsf{Lie}}(\mathfrak{g}, \mathrm{Der}(A)) \;\cong\; \mathrm{Hom}_{\mathsf{Hopf}}(U(\mathfrak{g}), U(\mathrm{Der}(A))).$$

Since, by Definition 2.36, X-Lie algebroid morphisms preserve the base set X, and by Definition 2.26, X-weak Hopf algebra morphisms also preserve X, it follows that for an X-Lie algebroid $\mathfrak{G}$ and an X-decomposable $\Bbbk$-algebra A, we have a bijection

$$\mathrm{Hom}_{X\text{-}\mathsf{Lie}}(\mathfrak{G}, \mathrm{Der}_X(A)) \cong \mathrm{Hom}_{X\text{-}\mathsf{WHA}}(U_X(\mathfrak{G}), U_X(\mathrm{Der}_X(A))). \tag{2.6}$$

Finally, we will construct the object $\mathrm{Sym}_{X\text{-}\mathsf{EnvLie}}(A)$ to actions on a $\Bbbk$-algebra A by X-universal enveloping algebras can be thought of equivalently in a category-theoretic and representation-theoretic way.

Proposition 2.52. *Let* $A = \bigoplus_{x\in X} A_x$ *be an* X*-decomposable* $\Bbbk$*-algebra and* $\mathfrak{G}$ *be an* X*-Lie algebroid.*

(i) *The action of* $U_X(\mathrm{Der}_X(A))$ *on* A *induced by the action of* $\mathrm{Der}_X(A)$ *on* A *makes* A *a* $U_X(\mathrm{Der}_X(A))$*-module algebra. For the action map* $\ell_{U_X(\mathrm{Der}_X(A)),A}$*, we use the notation* $h \triangleright a$ *for* $\ell_{U_X(\mathrm{Der}_X(A)),A}(h \otimes a)$*, where* $h \in U_X(\mathrm{Der}_X(A))$ *and* $a \in A$*.*

(ii) *Suppose that* A *is a* $U_X(\mathfrak{G})$*-module algebra via* $\ell_{U_X(\mathfrak{G}),A}$*, where we write* $h \cdot a$ *for* $\ell(h \otimes a)$ *for* $h \in U_X(\mathfrak{G})$ *and* $a \in A$*. Then there is a unique* X*-weak Hopf algebra morphism*

$\widetilde{\tau} : U_X(\mathfrak{G}) \to U_X(\mathrm{Der}_X(A))$ such that $h \cdot a = \widetilde{\tau}(h) \triangleright a$ for all $h \in U_X(\mathfrak{G})$ and $a \in A$.

(iii) *Every X-weak Hopf algebra morphism $\widetilde{\tau} : U_X(\mathfrak{G}) \to U_X(\mathrm{Der}_X(A))$ gives A the structure of a $U_X(\mathfrak{G})$-module algebra via $\ell_{U_X(\mathfrak{G}),A}(h \otimes a) = \ell_{U_X(\mathrm{Der}_X(A)),A}(\widetilde{\tau}(h) \otimes a)$ for all $h \in U_X(\mathfrak{G})$ and $a \in A$ (that is, $h \cdot a = \widetilde{\tau}(h) \triangleright a$).*

Hence, $\mathrm{Sym}_{X\text{-}\mathsf{EnvLie}}(A) = U_X(\mathrm{Der}_X(A))$.

Proof. (i): It is clear that each A_x is a $\mathrm{Der}(A_x)$-module algebra. By the classical theory, therefore each A_x is a $U(\mathrm{Der}(A_x))$-module algebra. Hence, by Proposition 2.24(iii), A is a $U_X(\mathrm{Der}_X(A))$-module algebra.

(ii)-(iii): By the same argument as above, A is a $U_X(\mathfrak{G})$-module algebra if and only if each A_x is a $U(\mathfrak{g}_x)$-module algebra if and only if each A_x is a $\mathfrak{g}_x$-module algebra. By Lemma 2.45, this happens if and only if A is a $\mathfrak{G}$-module algebra.

By Proposition 2.47, this happens if and only if the action of $\mathfrak{G}$ factors through a unique X-Lie algebroid morphism $\tau : \mathfrak{G} \to \mathrm{Der}_X(A)$. By (2.6), this if and only if there exists an X-weak Hopf algebra morphism $\widetilde{\tau} : U_X(\mathfrak{G}) \to U_X(\mathrm{Der}_X(A))$.

Note that since X-EnvLie is a full subcategory of X-WHA, the morphisms in X-EnvLie are simply X-weak Hopf algebra morphisms. As a consequence, (i), (ii), and (iii) above imply that $\mathrm{Sym}_{X\text{-}\mathsf{EnvLie}}(A) = U_X(\mathrm{Der}_X(A))$, as desired. □

As a corollary, we recover the following well-known result for actions of enveloping algebras of Lie algebras, which is the special case of Proposition 2.52 when $|X| = 1$.

Corollary 2.53. *Let A be a $\Bbbk$-algebra. Then A is a $U(\mathrm{Der}(A))$-module algebra and for a Lie algebra $\mathfrak{g}$, the following are equivalent.*

(i) *A is a $U(\mathfrak{g})$-module algebra.*

(ii) *There exists a Hopf algebra morphism $\widetilde{\tau} : U(\mathfrak{g}) \to U(\mathrm{Der}(A))$.*

Hence, $\mathrm{Sym}_{\mathsf{EnvLie}}(A) = U(\mathrm{Der}(A))$.

2.4 Weak quantum symmetries captured by cocommutative weak Hopf algebras

In this section our goal is to study the category $\mathcal{C} = X$-$\mathsf{CocomWHA}$ of cocommutative weak Hopf algebras. Our main result identifies $\mathrm{Sym}_{\mathcal{C}}(A)$. First,we recall the definition of the smash product over a weak Hopf algebra, which generalizes the construction of the classical smash product.

Definition 2.54 ([NV02, Section 4.2]). Let H be a weak Hopf algebra and let A be an H-module algebra. The *smash product* algebra $A\#H$ is defined as the $\Bbbk$-vector space $A \otimes_{H_t} H$, where A is a right H_t-module via $a \cdot z = S^{-1}(z) \cdot a = a(z \cdot 1_A)$, for $a \in A$ and $z \in H_t$.

Multiplication in $A\#H$ is defined by $(a\#h)(b\#g) = a(h_1 \cdot b)\#h_2 g$. Here $a\#h$ denotes a coset representative of $a \otimes_{H_t} h$ for $a \in A$ and $h \in H$.

To state Nikshych's a generalization of the Cartier–Gabriel–Kostant–Milnor–Moore theorem [Nik01], we describe the weak Hopf algebra structure on this smash product.

Definition 2.55. Let H be a weak Hopf algebra. We say that H is a *X-decomposable weak Hopf algebra* if there exists a family $\{H_x\}_{x\in X}$ of weak Hopf algebras (some of which may be 0) such that $H = \bigoplus_{x\in X} H_x$ as weak Hopf algebras.

Remark 2.56.

(i) Every X-decomposable weak Hopf algebra is an X-decomposable $\Bbbk$-algebra [Definition 1.60], whose X-decomposition also respects the weak Hopf structure.

(ii) Not every X-decomposable weak Hopf algebra is as an X-weak Hopf algebra. For example, for an X-groupoid $\mathcal{G}$, the groupoid algebra $\Bbbk\mathcal{G}$ is in X-WHA [Example 2.28]. However, $\Bbbk\mathcal{G}$ does not meet the requirements of an X-decomposable algebra. This is due to the fact that the idempotents $e_{x\,x\in X}$ in $\Bbbk\mathcal{G}$ may not be central (see Remark 1.61) Consequently, $\Bbbk\mathcal{G}$ cannot be classified as an X-decomposable weak Hopf algebra.

(iii) Conversely, not every X-weak Hopf algebra is an X-decomposable weak Hopf algebra. For instance, if we have an X-decomposable weak Hopf algebra $H = \bigoplus_{x\in X} H_x$, where some of the H_x are weak Hopf algebras, the idempotents 1_x in H may not necessarily be grouplike elements as required by Definition 2.26. Consequently, H would not satisfy the conditions to be classified as an X-weak Hopf algebra.

(iv) For every X-Lie algebroid $\mathfrak{G} = \bigoplus_{x\in X} \mathfrak{g}_x$, the X-universal enveloping algebra $U_X(\mathfrak{G})$ is an X-decomposable weak Hopf algebra.

Definition 2.57 ($\mathrm{Aut}_{X\text{-}\mathsf{WBA}}(H)$)**.** Let $H = \bigoplus_{x\in X} H_x$ be an X-decomposable weak Hopf algebra. We define $\mathrm{Aut}_{X\text{-}\mathsf{WBA}}(H)$, the *$X$-weak bialgebra automorphism groupoid* of H, as follows:

- the object set is X,
- for any $x, y \in X$, $\mathrm{Hom}_{\mathrm{Aut}_{X\text{-}\mathsf{WBA}}(H)}(H_x, H_y)$ is the space of weak bialgebra isomorphisms between the weak bialgebras H_x and H_y.

Clearly, $\mathrm{Aut}_{X\text{-}\mathsf{WBA}}(H)$ is a subgroupoid of $\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(H)$, as defined in Definition 2.9.

Proposition 2.58. *Let $\mathcal{G}$ be an X-groupoid and $H = \bigoplus_{x\in X} H_x$ be an X-weak Hopf algebra. If H is a $\Bbbk\mathcal{G}$-module algebra via the action $\cdot$, then the following statements are equivalent.*

(i) *For every $g \in \mathcal{G}_1$ and $a \in H_{s(g)}$ the following relations hold:*

$$\Delta_H(g \cdot a) = g \cdot a_1 \otimes g \cdot a_2, \tag{2.7}$$
$$\varepsilon_H(g \cdot a) = \varepsilon_{H_{s(g)}}(a). \tag{2.8}$$

(ii) *There exists an X-groupoid morphism $\pi : \mathcal{G} \longrightarrow \mathrm{Aut}_{X\text{-}\mathsf{WBA}}(H)$.*

Moreover, in this case the smash product algebra $H\#\Bbbk\mathcal{G}$ becomes a weak Hopf algebra with operations given by:

$$\Delta(a\#g) = a_1\#g \otimes a_2\#g, \quad \varepsilon(a\#g) = \varepsilon_H(a), \quad S(a\#g) = (1_H\#g^{-1})(S_H(a)\#1_{\Bbbk\mathcal{G}}), \tag{2.9}$$

for all $g \in \mathcal{G}_1$ and $a \in H_{t(g)}$.

Proof. Since H is a $\Bbbk\mathcal{G}$-module algebra, Proposition 2.33 guarantees the existence of a X-groupoid morphism $\pi : \mathcal{G} \to \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(H)$ that corresponds to the $\mathcal{G}$-action, that is, $\pi(g)(a) = g \cdot a$ for all $g \in \mathcal{G}_1$ and $a \in H_{s(g)}$; note that if $a \in H_x$ with $x \neq s(g)$ then $g \cdot a = 0$ (see Remark 1.47). Equations (2.7) and (2.8) are equivalent to the statement that for all $g \in \mathcal{G}_0$ the $\Bbbk$-algebra map $\pi(g) : H_{s(g)} \to H_{t(g)}$ is a $\Bbbk$-coalgebra map. But this is the same as requiring the image of π to be contained in the subgroupoid $\mathrm{Aut}_{X\text{-}\mathsf{WBA}}(H)$. This proves (i)-(ii).

Now, it is clear that $H\#\Bbbk\mathcal{G}$ is both a $\Bbbk$-algebra and a $\Bbbk$-coalgebra, and the maps of (2.9) are well-defined. Moreover, consider any $g, h \in \mathcal{G}_1$. Suppose that $t(h) = s(g)$, $a \in H_{t(g)}$ and $b \in H_{s(g)}$, so we have

$$\begin{aligned}\Delta((a\#g)(b\#h)) = a_1(g\cdot b)_1\#gh \otimes a_2(g\cdot b)_2\#gh &\overset{(2.7)}{=} a_1(g\cdot b_1)\#gh \otimes a_2(g\cdot b_2)\#gh \\ &= (a_1\#g \otimes a_2\#g)(b_1\#h \otimes b_2\#h) = \Delta(a\#g)\Delta(b\#h).\end{aligned}$$

Otherwise, both sides of the above equation are zero. Hence this proves that Δ is multiplicative. If $g, h, l \in \mathcal{G}_1$ satisfy $t(h) = s(g)$ and $t(l) = s(h)$, and $a \in H_{t(g)}$, $b \in H_{s(g)}$, $c \in H_{s(h)}$, then

$$\begin{aligned}\varepsilon((a\#g)(b\#h)(c\#l)) &= \varepsilon(a(g\cdot b)(gh\cdot c)\#ghl) = \varepsilon_H(a(g\cdot b)(gh\cdot c)) \\ &\overset{(2.7)}{=} \varepsilon_H(a(g\cdot b_1))\varepsilon_H((g\cdot b_2)(gh\cdot c)) = \varepsilon_H(a(g\cdot b_1))\varepsilon_H(g\cdot(b_2(h\cdot c))) \\ &\overset{(2.8)}{=} \varepsilon_H(a(g\cdot b_1))\varepsilon_H(b_2(h\cdot c)) = \varepsilon(a(g\cdot b_1)\#gh)\varepsilon(b_2(h\cdot c)\#hl) \\ &= \varepsilon((a\#g)(b\#h)_1)\varepsilon((b\#h)_2(c\#l)),\end{aligned}$$

and similarly, $\varepsilon((a\#g)(b\#h)(c\#l)) = \varepsilon((a\#g)(b\#h)_2)\varepsilon((b\#h)_1(c\#l))$, so ε is weak multiplicative. Further, letting $1 := 1_H$, $1_x = 1_{H_x}$ for all $x \in X$, and $1_{\Bbbk\mathcal{G}} = \sum_{x\in X} e_x$, then

$$\begin{aligned}&(\Delta(1\#1_{\Bbbk\mathcal{G}}) \otimes 1\#1_{\Bbbk\mathcal{G}})(1\#1_{\Bbbk\mathcal{G}} \otimes \Delta(1\#1_{\Bbbk\mathcal{G}})) \\ &= \left(\sum_{x\in X}\Delta(1\#e_x) \otimes \sum_{y\in X}1\#e_y\right)\left(\sum_{z\in X}1\#e_z \otimes \sum_{w\in X}\Delta(1\#e_w)\right) \\ &= \left(\sum_{x,y\in X}1_1\#e_x \otimes 1_2\#e_x \otimes 1\#e_y\right)\left(\sum_{z,w\in X}1\#e_z \otimes 1'_1\#e_w \otimes 1'_2\#e_w\right) \\ &= \sum_{x,y,z,w\in X}[1_1\#e_x][1\#e_z] \otimes [1_2\#e_x][1'_1\#e_w] \otimes [1\#e_y][1'_2\#e_w]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{x,y,z,w\in X} 1_1(e_x\cdot 1)\#e_xe_z \otimes 1_2(e_x\cdot 1'_1)\#e_xe_w \otimes e_y\cdot 1'_2\#e_ye_w\\
&\overset{(*)}{=} \sum_{x\in X}(1_x)_1\#e_x \otimes (1_x)_2(1'_x)_1\#e_x \otimes (1'_x)_2\#e_x\\
&\overset{(**)}{=} \sum_{x\in X}(1_x)_1\#e_x \otimes (1_x)_2\#e_x \otimes (1_x)_3\#e_x \overset{(***)}{=} \sum_{x\in X} 1_1\#e_x \otimes 1_2\#e_x \otimes 1_3\#e_x\\
&= \sum_{x\in X}\Delta(1_1\#e_x) \otimes 1_2\#e_x = \Delta^2(1\#1_{\Bbbk\mathcal{G}}).
\end{aligned}
$$

In $(*)$ we used that $e_x\cdot 1 = 1_x$ and $1_x \in H_t$, in $(**)$ that each H_x is a weak Hopf algebra, while in $(***)$ that $(1_y)_1\#e_x = 0$ if $x \neq y$; recall $(\Bbbk\mathcal{G})_t = \bigoplus_{x\in X}\Bbbk e_x$. This proves that Δ is weak comultiplicative and thus the smash product $H\#\Bbbk\mathcal{G}$ is a weak bialgebra. Finally, we will use the fact that H is an X-weak Hopf algebra, so $\Delta(1_H) = \sum_{x\in X} 1_x \otimes 1_x$ to show the following equation. If $g \in \mathcal{G}_1$ and $a \in H_{t(g)}$, then

$$
\begin{aligned}
S(a_1\#g)(a_2\#g) &= (1\#g^{-1})(S_H(a_1)\#1_{\Bbbk\mathcal{G}})(a_2\#g) = g^{-1}\cdot(S_H(a_1)a_2)\#e_{s(g)}\\
&= g^{-1}\cdot(\varepsilon_H)_s(a)\#e_{s(g)} = \sum_{x\in X}(g^{-1}\cdot 1_x\#e_{s(g)})\varepsilon_H(a1_x)\\
&\overset{(*)}{=} (1_{s(g)}\#e_{s(g)})\varepsilon_H(a1_{t(g)}) \overset{(**)}{=} (1_{s(g)}\#e_{s(g)})\varepsilon_H(a(g\cdot 1_{s(g)}))\\
&\overset{(***)}{=} (1_{s(g)}\#e_{s(g)})\varepsilon_H(a(g\cdot 1_{s(g)}))\varepsilon_{\Bbbk\mathcal{G}}(ge_{s(g)})\\
&= (1_{s(g)}\#e_{s(g)})\varepsilon_{H\#\Bbbk\mathcal{G}}(a(g\cdot 1_{s(g)})\#ge_{s(g)})\\
&\overset{(****)}{=} \sum_{x\in X}(1_x\#e_x)\varepsilon_{H\#\Bbbk\mathcal{G}}((a\#g)(1_x\#e_x))\\
&= \varepsilon_s(a\#g).
\end{aligned}
$$

Here $(*)$ holds due to that fact that $g^{-1}\cdot 1_x = \delta_{x,t(g)}1_{s(g)}$ and $a1_x = \delta_{x,t(g)}a1_{t(g)}$. In $(**)$, we use $g\cdot 1_{s(g)} = 1_{t(g)}$ and in $(***)$ we use $\varepsilon_{\Bbbk\mathcal{G}}(ge_{s(g)}) = 1$. Lastly, we use $(a\#g)(1_x\#e_x) = \delta_{x,s(g)}a(g\cdot 1_{s(g)})\#ge_{s(g)}$ in $(****)$ and $(1_y)\#e_x = \delta_{x,y}1_x\#e_x$ in the last equality.

Similarly, one can prove $(a_1\#g)S(a_2\#g) = \varepsilon_t(a\#g)$.

$$
\begin{aligned}
S((a\#g)_1)(a\#g)_2S((a\#g)_3) &= (1\#g^{-1})(S_H(a_1)\#1_{\Bbbk\mathcal{G}})(a_2\#g)(1\#g^{-1})(S_H(a_3)\#1_{\Bbbk\mathcal{G}})\\
&= (g^{-1}\cdot(S_H(a_1)a_2)\#1_{\Bbbk\mathcal{G}})(g^{-1}\cdot S_H(a_3)\#g^{-1})\\
&= g^{-1}\cdot(S_H(a_1)a_2S_H(a_3))\#g^{-1} \overset{(*)}{=} g^{-1}\cdot S_H(a)\#g^{-1}\\
&= (1\#g^{-1})(S_H(a)\#1_{\Bbbk\mathcal{G}}) = S(a\#g).
\end{aligned}
$$

Here, we used in $(*)$ that S_H is an antipode for H. Therefore, $H\#\Bbbk\mathcal{G}$ is a weak Hopf algebra. □

Lemma 2.59. *Let H be a cocommutative weak Hopf algebra of the form $U_X(\mathfrak{G})\#\Bbbk\mathcal{G}$ for an X-groupoid $\mathcal{G}$ acting on an X-Lie algebroid $\mathfrak{G}$. Then $\{f_x := 1_{U(\mathfrak{g}_x)}\otimes_{\Bbbk\mathcal{G}_0} e_x\}_{x\in X}$ is a complete set of orthogonal primitive idempotents of H satisfying $\Delta(f_x) = f_x\otimes f_x$. Hence, H is an X-weak Hopf* *algebra.*

Proof. Note that $U_X(\mathfrak{G})$ is a $\Bbbk\mathcal{G}$-module algebra, and also is a right $\Bbbk\mathcal{G}_0$-module via $p \cdot e_x = S^{-1}(e_x) \cdot p = p(e_x \cdot 1_{U_X(\mathfrak{G})})$ for all $x \in \mathcal{G}_0 = X$ and $p \in U_X(\mathfrak{G})$. Moreover, $U_X(\mathfrak{G})\#\Bbbk\mathcal{G} = U_X(\mathfrak{G}) \otimes_{\Bbbk\mathcal{G}_0} \Bbbk\mathcal{G}$ as coalgebras. Now we compute:

$$\Delta(1_H) = \Delta(1_{U_X(\mathfrak{G})} \otimes_{\Bbbk\mathcal{G}_0} 1_{\Bbbk\mathcal{G}}) = \textstyle\sum_{x,y\in X}(1_{U(\mathfrak{g}_x)} \otimes_{\Bbbk\mathcal{G}_0} e_y) \otimes (1_{U(\mathfrak{g}_x)} \otimes_{\Bbbk\mathcal{G}_0} e_y). \tag{2.10}$$

Therefore, $H_t = H_s = \mathrm{Span}_{\Bbbk}\{1_{U(\mathfrak{g}_x)} \otimes_{\Bbbk\mathcal{G}_0} e_y \mid x, y \in X\}$. Also, for $x, y \in X$, we have

$$\begin{aligned} 1_{U(\mathfrak{g}_x)}\#e_y &= 1_{U(\mathfrak{g}_x)} \otimes_{\Bbbk\mathcal{G}_0} e_y \overset{(e_y^2=e_y)}{=} 1_{U(\mathfrak{g}_x)} \cdot e_y \otimes_{\Bbbk\mathcal{G}_0} e_y \\ &= 1_{U(\mathfrak{g}_x)}(e_y \cdot 1_{U_X(\mathfrak{G})}) \otimes_{\Bbbk\mathcal{G}_0} e_y = \delta_{x,y}1_{U(\mathfrak{g}_x)} \otimes_{\Bbbk\mathcal{G}_0} e_x \\ &= \delta_{x,y}1_{U(\mathfrak{g}_x)}\#e_x. \end{aligned} \tag{2.11}$$

Hence, $\{f_x = 1_{U(\mathfrak{g}_x)} \otimes_{\Bbbk\mathcal{G}_0} e_x \mid x \in X\}$ is a basis of H_s. Moreover, we have that as $\Bbbk$-algebras, $H_t = H_s \cong U_X(\mathfrak{G})_s \cong (\Bbbk\mathcal{G})_s \cong \Bbbk\mathcal{G}_0$. Now the result follows from Proposition 2.24(i). □

Remark 2.60. (i) Let $\mathcal{G}$ be an X-groupoid acting by conjugation on an X-Lie algebroid $\mathfrak{G}$. This induces an action of the groupoid algebra $\Bbbk\mathcal{G}$ on the X-universal enveloping algebra $U_X(\mathfrak{G})$ satisfying the conditions of Proposition 2.58(i). Thus $U_X(\mathfrak{G})\#\Bbbk\mathcal{G}$ is a weak Hopf algebra.

(ii) Let G be a group and H be a Hopf algebra. If H is a $\Bbbk G$-module algebra, then Proposition 2.58 translates into the following classical well-known result: if there is a group morphism from G to the group of bialgebra automorphisms of H, $\pi : G \to \mathrm{Aut}_{\mathrm{Bialg}}(H)$, or equivalently, if the maps $\pi_g : H \to H$ given by $a \mapsto g \cdot a$ for $g \in G$ are all coalgebra morphisms, then the smash product algebra $H\#\Bbbk G$ has structure of Hopf algebra with operations as in (2.9).

Next, we recall Nikshych's generalization of the Cartier–Gabriel–Kostant–Milnor–Moore theorem to the weak Hopf setting.

Theorem 2.61 ([Nik01, Theorem 3.2.4]). *Any cocommutative weak Hopf algebra H is isomorphic to $U_X(\mathfrak{G})\#\Bbbk\mathcal{G}$ as weak Hopf algebras, for some X-Lie algebroid $\mathfrak{G}$ and X-groupoid $\mathcal{G}$.*

Lemma 2.62. *Let $A = \bigoplus_{x\in X} A_x$ be an X-decomposable $\Bbbk$-algebra. Then A is a $U_X(\mathrm{Der}_X(A))\#\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$-module algebra.*

Proof. It follows from Proposition 2.58 that $U_X(\mathrm{Der}_X(A))\#\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$ is a cocommutative weak Hopf algebra. By Propositions 2.52 and 2.33, A is a $U_X(\mathrm{Der}_X(A))$ and $\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$ module algebra. Define a $U_X(\mathrm{Der}_X(A))\#\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$ action on A via $(f\#g) \cdot a = f \cdot (g \cdot a)$ for all $a \in A$ and all $f\#g \in U_X(\mathrm{Der}_X(A))\#\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$. One can check that indeed, A is a $U_X(\mathrm{Der}_X(A))\#\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$-module algebra. □

Definition 2.63. Let R and S be two $\Bbbk$-algebras and let $\beta : R \to S$ be a $\Bbbk$-algebra morphism. Let $M \in R$-mod and $N \in S$-mod. We call a group morphism $\alpha : M \to N$ a *β-linear morphism* if $\alpha(r \cdot m) = \beta(r) \cdot \alpha(m)$ for all $m \in M$ and $r \in R$.

Using all the previous result, we are finally able to state our main and final results in this chapter.

Lemma 2.64. *Let H and H' be cocommutative X-weak Hopf algebras. Write $H = U_X(\mathfrak{G})\#\Bbbk\mathcal{G}$ and $H' = U_X(\mathfrak{G}')\#\Bbbk\mathcal{G}'$ where $\mathcal{G}$ and $\mathcal{G}'$ are X-groupoids, $\mathfrak{G}$ and $\mathfrak{G}'$ are X-Lie algebroids and $U_X(\mathfrak{G})$ is a $\mathcal{G}$-module algebra and $U_X(\mathfrak{G}')$ is a $\mathcal{G}'$-module algebra.*

(i) *Suppose that $\pi : \mathcal{G} \to \mathcal{G}'$ is an X-groupoid morphism and $\tau : \mathfrak{G} \to \mathfrak{G}'$ is an X-Lie algebroid morphism such that for the corresponding X-weak Hopf algebra morphisms $\widetilde{\pi} : \Bbbk\mathcal{G} \to \Bbbk\mathcal{G}'$ and $\widetilde{\tau} : U_X(\mathfrak{G}) \to U_X(\mathfrak{G}')$, $\widetilde{\tau}$ is a $\widetilde{\pi}$-linear morphism. Then there is an induced X-weak Hopf algebra morphism $H \to H'$.*

(ii) *Every X-weak Hopf algebra morphism $H \to H'$ arises in this way.*

Theorem 2.65. *Let $A = \bigoplus_{x\in X} A_x$ be an X-decomposable $\Bbbk$-algebra, let*

$$K = U_X(\mathrm{Der}_X(A))\#\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)),$$

and let H be a cocommutative weak Hopf algebra. Write $H = U_X(\mathfrak{G})\#\Bbbk\mathcal{G}$ for an X-Lie algebroid $\mathfrak{G}$ and an X-groupoid $\mathcal{G}$. Then:

(i) *The natural actions of $U_X(\mathrm{Der}_X(A))$ and $\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$ on A give A the structure of a K-module algebra. We denote this action $\ell_{K,A}$ and write $k \triangleright a := \ell_{K,A}(k \otimes a)$ where $k \otimes a \in K \otimes A$.*

(ii) *Suppose that A is an H-module algebra via $\ell_{H,A}$ and denote $h \cdot a := \ell_{H,A}(h \otimes a)$ for $h \otimes a \in H \otimes A$. Then there is a unique X-weak Hopf algebra morphism $\phi : H \to K$ such that $h \cdot a = \phi(h) \triangleright a$ for all $h \otimes a \in H \otimes A$.*

(iii) *Every X-weak Hopf algebra morphism $\phi : H \to K$ gives A the structure of an H-module algebra via $\ell_{H,A}(h \otimes a) = \ell_{K,A}(\phi(h) \otimes a)$ for all $h \otimes a \in H \otimes A$.*

Hence, $\mathrm{Sym}_{X\text{-}\mathsf{CocomWHA}}(A) = K$.

Proof. (i): We can define a $\Bbbk\mathcal{G}$-module algebra structure on A via $g \cdot a := (1_{U_X(\mathfrak{G})}\#g) \cdot a$ for $g \in \mathcal{G}$ and $a \in A$. It is clear that A is a $\Bbbk\mathcal{G}$-module. By (2.11), we have that

$$1_{U(\mathfrak{g}_x)}\#g = 1_{U(\mathfrak{g}_x)}\#e_{t(g)}g = \delta_{x,\,t(g)}1_{U(\mathfrak{g}_x)}\#g, \tag{2.12}$$

and therefore

$$1_{U_X(\mathfrak{G})}\#g = \sum_x 1_{U(\mathfrak{g}_x)}\#g = \sum_x \delta_{x,\,t(g)}1_{U(\mathfrak{g}_x)}\#g = 1_{U(\mathfrak{g}_{t(g)})}\#g. \tag{2.13}$$

Now we compute:

$$\begin{aligned}
g\cdot(ab) &= (1_{U_X(\mathfrak{G})}\#g)\cdot(ab)\\
&= \textstyle\sum_{x\in X}\left((1_{U(\mathfrak{g}_x)}\#g)\cdot a\right)\left((1_{U(\mathfrak{g}_x)}\#g)\cdot b\right)\\
&\overset{(2.12)}{=} ((1_{U(\mathfrak{g}_{t(g)})}\#g)\cdot a)((1_{U(\mathfrak{g}_{t(g)})}\#g)\cdot b)\\
&\overset{(2.13)}{=} ((1_{U_X(\mathfrak{G})}\#g)\cdot a)((1_{U_X(\mathfrak{G})}\#g)\cdot b)\\
&= (g\cdot a)(g\cdot b).
\end{aligned}$$

Note that $\varepsilon_t(g)=\sum_{x\in\mathcal{G}_0}\varepsilon(e_xg)e_x=e_{t(g)}$. Next we check that

$$\begin{aligned}
g\cdot 1_A = (1_{U_X(\mathfrak{G})}\#g)\cdot 1_A &\overset{1.28}{=} \varepsilon_t(1_{U_X(\mathfrak{G})}\#g)\cdot 1_A\\
&\overset{2.59}{=} \left(\textstyle\sum_{x\in X}\varepsilon((1_{U(\mathfrak{g}_x)}\#e_x)(1_{U_X(\mathfrak{G})}\#g))(1_{U(\mathfrak{g}_x)}\#e_x)\right)\cdot 1_A\\
&= \left(\textstyle\sum_{x\in X}\varepsilon(1_{U(\mathfrak{g}_x)}(e_x\cdot 1_{U_X(\mathfrak{G})})\#e_xg))(1_{U(\mathfrak{g}_x)}\#e_x)\right)\cdot 1_A\\
&\overset{(*)}{=} \left(\textstyle\sum_{x\in X}\varepsilon(1_{U(\mathfrak{g}_x)}\#e_xg)(1_{U(\mathfrak{g}_x)}\#e_x)\right)\cdot 1_A\\
&\overset{(**)}{=} \varepsilon(1_{U(\mathfrak{g}_{t(g)})}\#g)(1_{U(\mathfrak{g}_{t(g)})}\#e_{t(g)})\cdot 1_A\\
&\overset{(***)}{=} \left(1_{U(\mathfrak{g}_{t(g)})}\#e_{t(g)}\right)\cdot 1_A\\
&\overset{(2.13)}{=} (1_{U_X(\mathfrak{G})}\#e_{t(g)})\cdot 1_A\\
&= e_{t(g)}\cdot 1_A\\
&= \varepsilon_t(g)\cdot 1_A.
\end{aligned}$$

Here, $(*)$ holds since $e_x\cdot 1_{U_X(\mathfrak{G})}=1_{U(\mathfrak{g}_x)}$; $(**)$ holds since $e_xg=\delta_{x,t(g)}e_{t(g)}g=\delta_{x,t(g)}g$; and $(***)$ holds since $1_{U(\mathfrak{g}_{t(g)})}$ and g are grouplike, and thus, $\varepsilon(1_{U(\mathfrak{g}_{t(g)})}\#g)=1_{\Bbbk}$. So A is a $\Bbbk\mathcal{G}$-module algebra.

Next, one can define a $U_X(\mathfrak{G})$-module structure on A via $p_x\cdot b:=(p_x\#1_{\Bbbk G})\cdot b$ for all $p_x\in\mathfrak{g}_x$ with $b\in A$. By a similar argument of (2.11), we show $p_x\#e_y=\delta_{x,y}p_x\#e_x$ and so $(p_x\#1_{\Bbbk\mathcal{G}})=(p_x\#e_x)$ for $p_x\in\mathfrak{g}_x$. It is straightforward to check that

$$\begin{aligned}
p_y\cdot(ab) &= (p_y\cdot a)(1_{U(\mathfrak{g}_y)}\cdot b)+(1_{U(\mathfrak{g}_y)}\cdot a)(p_y\cdot b),\\
p_y\cdot 1_A &= 0=\varepsilon_t(p_y)\cdot 1_A,
\end{aligned}$$

since $\varepsilon_t(p_y)=(p_y)_1S((p_y)_2)=1_{U(\mathfrak{g}_y)}S(p_y)+p_yS(1_{U(\mathfrak{g}_y)}=-p_y+p_y=0$, for $p_y\in\mathfrak{g}_y$ and $a,b\in A$. By Propositions 2.47 and 2.52, A is then a $U_X(\mathfrak{G})$-module algebra.

Now there exists an X-groupoid map $\pi:\mathcal{G}\to\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$ by Proposition 2.11, and an X-Lie algebroid map $\tau=(\tau_x)_{x\in X}:\mathfrak{G}\to\mathrm{Der}_X(A)$ by Proposition 2.52. So there exists an X-weak Hopf algebra map $\widetilde{\pi}:=\nu_{\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))}\Bbbk(\pi):\Bbbk\mathcal{G}\to\Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$ by Theorem 2.31 and $\widetilde{\tau}:U_X(\mathfrak{G})\to U_X(\mathrm{Der}_X(A))$ by Proposition 2.52.

Further, by Lemma 1.58 and Notation 2.43,

$$\pi(g)(a)=g\cdot a=(1_{U_X(\mathfrak{G})}\#g)\cdot a \quad\text{and}\quad \tau(p)(b)=p\cdot b=(p\#1_{\Bbbk\mathcal{G}})\cdot b$$

for all $g \in \mathcal{G}$ with $a \in A_{s(g)}$, and all $p \in \mathfrak{g}_x$ with $b \in A_x$. Since for all $g \in \mathcal{G}$ and $p \in \mathfrak{g}_{s(g)}$, both $\widetilde{\tau}(g \cdot p)$ and $\widetilde{\pi}(g) \cdot \widetilde{\tau}(p)$ are in $U(\mathrm{Der}(A_{t(g)}))$, to obtain the remaining part of (ii), we shall show $\widetilde{\tau}(g \cdot p)(a) = (\widetilde{\pi}(g) \cdot \widetilde{\tau}(p))(a)$ for all $a \in A_{t(g)}$. Indeed:

$$\begin{aligned}(\widetilde{\pi}(g) \cdot \widetilde{\tau}(p))(a) &= \widetilde{\pi}(g) \circ \widetilde{\tau}(p) \circ \widetilde{\pi}(g^{-1})(a) = (1_{U_X(\mathfrak{G})} \# g)(p \# 1_{\Bbbk\mathcal{G}})(1_{U_X(\mathfrak{G})} \# g^{-1}) \cdot a \\ &= ((g \cdot p) \# 1_{\Bbbk\mathcal{G}}) \cdot a = \widetilde{\tau}(g \cdot p)(a).\end{aligned}$$

(ii): Given the X-weak Hopf algebra morphisms $\widetilde{\pi}, \widetilde{\tau}$ constructed in (i), define the map

$$\phi : U_X(\mathfrak{G}) \# \Bbbk\mathcal{G} \longrightarrow U_X(\mathrm{Der}_X(A)) \# \Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)), \quad p_x \# g \mapsto \widetilde{\tau}_x(p_x) \# \widetilde{\pi}(g),$$

where $p_x \in \mathfrak{g}_x$, $g \in \mathcal{G}$, for $x \in X$. First, we will show that ϕ is well-defined. Suppose that $p \otimes_{\Bbbk\mathcal{G}_0} g = q \otimes_{\Bbbk\mathcal{G}_0} h$ in $U_X(\mathfrak{G}) \# \Bbbk\mathcal{G}$. Then there exists $\beta \in \Bbbk\mathcal{G}_0$ such that $p = q \cdot \beta^{-1} = S^{-1}(\beta^{-1}) \cdot q$ and $g = \beta h$. Since $\widetilde{\tau}$ is a $\widetilde{\pi}$-linear morphism, we have $\widetilde{\tau}(g \cdot p) = \widetilde{\pi}(g) \cdot \widetilde{\tau}(p)$ for $p \in \mathfrak{G}$ and $g \in \mathcal{G}$. Then

$$\begin{aligned}\widetilde{\tau}(p) \# \widetilde{\pi}(g) &= \widetilde{\tau}(S^{-1}(\beta^{-1}) \cdot q) \# \widetilde{\pi}(\beta h) = (\widetilde{\pi}(S^{-1}(\beta^{-1})) \cdot \widetilde{\tau}(q)) \# \widetilde{\pi}(\beta h) \\ &= (\widetilde{\tau}(q) \cdot S(\widetilde{\pi}(S^{-1}(\beta^{-1})))) \# \widetilde{\pi}(\beta h) = \widetilde{\tau}(q) \# \widetilde{\pi}(\beta^{-1}) \widetilde{\pi}(\beta h) \\ &= \widetilde{\tau}(q) \# \widetilde{\pi}(h).\end{aligned}$$

The second-to-last equality holds since $\widetilde{\pi}$ is a weak Hopf algebra morphism.

Now ϕ is a $\Bbbk$-coalgebra map since $\widetilde{\pi}$ and $\widetilde{\tau}$ are coalgebra maps. Moreover, ϕ is unital as $\widetilde{\pi}$ and $\widetilde{\tau}$ are unital. Finally, ϕ is multiplicative:

$$\begin{aligned}\phi((p \# g)(q \# h)) &= \phi(p(g \cdot q) \# gh)) = \widetilde{\tau}(p(g \cdot q)) \# \widetilde{\pi}(gh) \\ &= \widetilde{\tau}(p) \widetilde{\tau}(g \cdot q) \# \widetilde{\pi}(g) \widetilde{\pi}(h) \overset{(*)}{=} \widetilde{\tau}(p)(\widetilde{\pi}(g) \cdot \widetilde{\tau}(q)) \# \widetilde{\pi}(g) \widetilde{\pi}(h) \\ &\overset{(**)}{=} (\widetilde{\tau}(p) \# \widetilde{\pi}(g))(\widetilde{\tau}(q) \# \widetilde{\pi}(h)) = \phi(p \# g) \phi(q \# h),\end{aligned}$$

for $g, h \in \mathcal{G}$ and $p, q \in \mathfrak{G}$. Here, $(*)$ holds because $\widetilde{\tau}$ is a $\widetilde{\pi}$-linear morphism, and $(**)$ holds because the definition of multiplication in the smash product algebra. Lastly, $\phi(1_{U(\mathfrak{g}_x)} \# e_x) = \widetilde{\tau}(1_{U(\mathfrak{g}_x)}) \# \widetilde{\pi}(e_x) = 1_{U(\mathrm{Der}(A_x))} \# e_x$. So ϕ preserves the base X.

(iii): By Lemma 2.62, $A = \bigoplus_{x \in X} A_x$ is a module algebra over the smash product weak Hopf algebra $K := U_X(\mathrm{Der}_X(A)) \# \Bbbk(\mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A))$ via $(\delta \# \alpha) \cdot a = \delta \cdot (\alpha \cdot a)$ for $\alpha \in \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A)$ with $a \in A_{s(\alpha)}$ and $\delta \in U(\mathrm{Der}(A_{t(\alpha)}))$, otherwise 0. Now given the weak Hopf algebra morphism ϕ, we have that A is an H-module algebra via $h \cdot a := \phi(h) \cdot a$, for all $h \in H$ and $a \in A$. Namely, $(hh') \cdot a = h \cdot (h' \cdot a)$ since ϕ is multiplicative; $1_H \cdot a = a$ since ϕ is unital; $h \cdot (aa') = (h_1 \cdot a)(h_2 \cdot a')$ since ϕ is comultiplicative, for $h, h' \in H$ and $a, a' \in A$. Since ϕ is a weak Hopf algebra morphism, it follows that

$$\phi((1_H)_1) \otimes \phi((1_H)_2) = \Delta(\phi(1_H)) = \Delta(1_K) = (1_K)_1 \otimes (1_K)_2. \tag{2.14}$$

Thus,

$$\begin{aligned}
h \cdot 1_A &= \phi(h) \cdot 1_A = (\varepsilon_K)_t(\phi(h)) \cdot 1_A \\
&\overset{(2.14)}{=} \left(\sum \varepsilon_K(\phi((1_H)_1)\phi(h))\phi((1_H)_2)\right) \cdot 1_A = (\varepsilon_K\phi \otimes \phi)((1_H)_1 h \otimes (1_H)_2) \cdot 1_A \\
&\overset{(*)}{=} (\varepsilon_H \otimes \phi)((1_H)_1 h \otimes (1_H)_2) \cdot 1_A = \phi\left((\varepsilon_H)_t(h)\right) \cdot 1_A \\
&\overset{(**)}{=} (\varepsilon_H)_t(h) \cdot 1_A
\end{aligned}$$

as desired. Here, $(*)$ follows from ϕ being a counital map, and $(**)$ is by the action of H on A.

Finally, $\mathrm{Sym}_{X\text{-CocomWHA}}(A) = K$ follows from this proof and Lemma 2.62. □

As an immediate consequence, we have the following corollary.

Corollary 2.66. *Let A be a $\Bbbk$-algebra. Let $K = U(\mathrm{Der}(A))\#\Bbbk(\mathrm{Aut}_{\mathrm{Alg}}(A))$. Then A is a K-module algebra and for a cocommutative Hopf algebra H, the following are equivalent.*

(i) *A is an H-module algebra.*

(ii) *There exists a Hopf algebra morphism $\phi : H \to K$.*

Hence, $\mathrm{Sym}_{\mathrm{CocomHopf}}(A) = K$.

Remark 2.67. As mentioned earlier, one potential research direction is to extend the results to the non-cocommutative case. For instance, there has been much recent activity on partial actions of Hopf-like structures in this case (see, e.g., [FMS21, FMF20, FMS22, MPDS22]).

2.5 Examples: Actions on polynomial algebras

In this section we illustrate the results above for actions on polynomial algebras by general linear Hopf-like structures. As a warm up, we begin by studying the (classical) indecomposable case before considering the X-decomposable case where $|X| \geq 2$.

2.5.1 Indecomposable module algebra case

Let $A := \Bbbk[x_1, \ldots, x_n]$ be a polynomial algebra, which is isomorphic to the symmetric algebra $S(V)$ on an n-dimensional vector space V. We consider the following group and Lie algebra, respectively,

$$T_n := \mathrm{Aut}_{\mathrm{Alg}}(A) \qquad \text{and} \qquad W_n := \mathrm{Der}(A).$$

When $n \geq 3$, the group T_n contains an automorphism of wild type [SU04] and so is not fully understood. Hence, we restrict our attention to the subgroup of *graded* automorphisms of

A, which we identify with the general linear group:

$$\mathrm{GL}_n(\Bbbk) = \mathrm{GL}(V) =: \mathrm{Aut}_{\mathrm{GrAlg}}(A).$$

On the other hand, W_n is well-known to be the infinite-dimensional Lie algebra consisting of derivations of the form $f_1(\underline{x})\frac{\partial}{\partial x_1} + \cdots + f_n(\underline{x})\frac{\partial}{\partial x_n}$, for $f_i(\underline{x}) \in A$ (see, e.g., [Bah21, Section 1.2]). The general linear Lie algebra is a Lie subalgebra of W_n given as follows:

$$\mathfrak{gl}_n(\Bbbk) = \mathrm{span}_{\Bbbk}\left\{x_i \frac{\partial}{\partial x_j}\right\}_{i,j=1,\dots,n} =: \mathrm{Der}_{\mathrm{Lin}}(A).$$

Moreover, $\mathrm{GL}_n(\Bbbk)$ acts on $\mathfrak{gl}_n(\Bbbk)$ by conjugation after identifying $x_i \frac{\partial}{\partial x_j}$ with the elementary matrix $E_{i,j}$. With the maps,

$$\pi : \mathrm{GL}_n(\Bbbk) \longrightarrow T_n \quad \text{(inclusion of groups)}, \qquad \tau : \mathfrak{gl}_n(\Bbbk) \longrightarrow W_n \quad \text{(inclusion of Lie algebras)},$$

we obtain the Hopf algebra maps,

$$\widetilde{\pi} : \Bbbk\,\mathrm{GL}_n(\Bbbk) \longrightarrow \Bbbk T_n \qquad \text{and} \qquad \widetilde{\tau} : U(\mathfrak{gl}_n(\Bbbk)) \longrightarrow U(W_n)$$

and $\widetilde{\tau}$ is $\widetilde{\pi}$-linear. Hence by Corollaries 2.13, 2.35, 2.48, 2.53, and 2.66, we obtain the following result.

Proposition 2.68. *The polynomial algebra $\Bbbk[x_1, \dots, x_n]$ is a module algebra over the following general linear Hopf-like structures:*

$$\mathrm{GL}_n(\Bbbk), \qquad \mathfrak{gl}_n(\Bbbk), \qquad \Bbbk\,\mathrm{GL}_n(\Bbbk), \qquad U(\mathfrak{gl}_n(\Bbbk)), \qquad U(\mathfrak{gl}_n(\Bbbk))\#\Bbbk\,\mathrm{GL}_n(\Bbbk).$$

2.5.2 Decomposable module algebra case

Now suppose $|X| > 2$. For $x \in X$, let $A_x := S(V_x)$ be the symmetric algebra on a finite-dimensional vector space V_x. Let n_x denote $\dim(V_x)$. Take $V := \bigoplus_{x \in X} V_x$ to be the corresponding X-decomposable vector space, and $A := \bigoplus_{x\in X} A_x$ to be the corresponding X-decomposable $\Bbbk$-algebra. Take

$$T_X := \mathrm{Aut}_{X\text{-}\mathsf{Alg}}(A),$$

to be the groupoid from Definition 2.9, that is, the objects of T_X are the elements of X and for $x, y \in X$, the morphisms $\mathrm{Hom}_{T_X}(x, y)$ are the unital $\Bbbk$-algebra isomorphisms $A_x \to A_y$.

Since $\mathrm{GL}(V_x)$ is a subgroupoid of the groupoid $\mathrm{GL}_X(V)$ from Definition 1.52, with objects X and morphisms being vector space isomorphisms between V_x and V_y, for $x, y \in X$. With this, the result below is straightforward, where the second statement follows from Theorem 2.31(ii) (or Proposition 2.33).

Lemma 2.69. *There exists an X-groupoid morphism*

$$\pi : \mathrm{GL}_X(V) \longrightarrow T_X.$$

This also yields an X-weak Hopf algebra morphism:

$$\widetilde{\pi} : \Bbbk\,\mathrm{GL}_X(V) \longrightarrow \Bbbk T_X.$$

On the other hand, take

$$W_X := \mathrm{Der}_X(A) := \bigoplus_{x\in X} \mathrm{Der}(A_x),$$

to be the X-Lie algebroid from Definition 2.36 and Notation 2.46. The next result is straightforward, with the second statement following from Proposition 2.52.

Lemma 2.70. *There is an X-Lie algebroid morphism*

$$\tau : \mathfrak{GL}_X(V) \to \mathrm{Der}_X(A),$$

given by inclusion. This also yields an X-weak Hopf algebra morphism:

$$\widetilde{\tau} : \ U_X(\mathfrak{GL}_X(V)) \longrightarrow U_X(\mathrm{Der}_X(A)).$$

Working component-wise, we also have the following result.

Lemma 2.71. *Given $\widetilde{\pi}$ from Lemma 2.69, we have that $\widetilde{\tau}$ from Lemma 2.70 is $\widetilde{\pi}$-linear.*

Finally, Lemmas 2.69, 2.70, and 2.71 yield the following consequences of Propositions 2.11, 2.33, and 2.47 and Theorem 2.65.

Proposition 2.72. *For an X-decomposable vector space $V := \bigoplus_{x\in X} V_x$, the X-decomposable $\Bbbk$-algebra $\bigoplus_{x\in X} S(V_x)$ is a module algebra over the following general linear X-Hopf-like structures:*

$$\mathrm{GL}_X(V), \quad \mathfrak{GL}_X(V), \quad \Bbbk\,\mathrm{GL}_X(V), \quad U_X(\mathfrak{GL}_X(V)), \quad \bigoplus_{x\in X} U_X(\mathfrak{GL}_X(V))\#\Bbbk\,\mathrm{GL}_X(V).$$

CHAPTER 3

Algebraic properties of face algebras

Recently, Huang, Walton, Wicks, and Won [HWWW23, Theorem 4.17] established a framework for studying universal coactions on non-connected graded algebras via coactions of weak bialgebras. Their main result is that the weak bialgebras that coact universally on the path algebra $\Bbbk Q$ (either from the left, from the right, or from both directions compatibly) are each isomorphic to Hayashi's face algebra $\mathfrak{H}(Q)$ attached to Q [Example 1.10]. It was inquired in [HWWW23, Question 6.5] whether $\mathfrak{H}(Q)$ and $\Bbbk Q$ share nice algebraic properties. Consequently, the primary objective of this chapter is to investigate this connection. To achieve this, we begin by delving into the study of Kronecker squares of a quiver, which serves as a fundamental technique throughout our results. Subsequently, we present our main findings and results, shedding light on the relationship between $\mathfrak{H}(Q)$ and $\Bbbk Q$. Finally, we provide several examples to illustrate the concepts discussed, including quivers of Dynkin type ADE.

3.1 On the Kronecker square of a quiver and its path algebra

The construction of the quiver below plays a key role in this work.

Definition 3.1 ($\widehat{Q}$). Let $Q = (Q_0, Q_1, s, t)$ be a quiver. We define the *Kronecker square* $\widehat{Q}$ of Q as the quiver $\widehat{Q} = (\widehat{Q}_0, \widehat{Q}_1, \widehat{s}, \widehat{t})$ given by

$$\widehat{Q}_0 = \{[i,j]\}_{i,j\in Q_0}, \qquad \widehat{Q}_1 = \{[p,q]\}_{p,q\in Q_1},$$
$$\widehat{s}([p,q]) = [s(p), s(q)], \quad \widehat{t}([p,q]) = [t(p), t(q)], \quad \text{for all } p, q \in Q_1.$$

Any path in $\widehat{Q}$ is of the form $[p_1\cdots p_k, q_1\cdots q_k] := [p_1,q_1][p_2,q_2]\cdots[p_k,q_k]$ where $p_1\cdots p_k$, $q_1\cdots q_k \in Q_k$. By construction, $|\widehat{Q}_0| = |Q_0|^2$ and $|\widehat{Q}_1| = |Q_1|^2$. Moreover, Q can be identified with the subquiver of $\widehat{Q}$ formed with the vertices $\{[i,i]\}_{i\in Q_0}$ and arrows $\{[p,p]\}_{p\in Q_1}$. Therefore, we have an embedding of path algebras $\Bbbk Q \hookrightarrow \Bbbk\widehat{Q}$. The reader may wish to refer to Table 3.1 below for examples of quivers Q, their Kronecker square $\widehat{Q}$, and their corresponding path algebras.

Next, we compare graph-theoretic properties of a quiver Q with that of its Kronecker square $\widehat{Q}$, which will be used in the proof of Theorem 3.5 below. Recall Notation 1.72.

Proposition 3.2. *Let $Q = (Q_0, Q_1, s, t)$ be a quiver. Then, the following statements hold.*

(i) (a) *Q is finite (resp., acyclic) if and only if $\widehat{Q}$ is finite (resp., acyclic).*

(b) $|\widehat{Q}_k| = |Q_k|^2$, $k \geq 0$.

(c) *The adjacency matrix $\widehat{C}^k$ of $\widehat{Q}_k$ is given by $\widehat{C}^k = C^k \otimes C^k$, for $k \geq 0$, where $\otimes$ is the tensor product of matrices.*

(ii) *Q satisfies the exclusive condition if and only if $\widehat{Q}$ satisfies the exclusive condition. In this case, if the maximal length of chains of cycles in Q is n, then the maximal length of chains of cycles in $\widehat{Q}$ is $2n - 1$.*

(iii) *Q has a source (resp., sink) cycle if and only if $\widehat{Q}$ has a source (resp., sink) cycle.*

(iv) (a) *Q is pairwise strongly connected if and only if $\widehat{Q}$ is strongly connected.*

(b) *If Q is strongly connected and has at least one cycle, then $\widehat{Q}$ is strongly connected. Conversely, if $\widehat{Q}$ is strongly connected, then Q is strongly connected.*

(v) *Q is path reversible if and only if $\widehat{Q}$ is path reversible.*

(vi) *Q is arrowless if and only if $\widehat{Q}$ is arrowless.*

Proof. (i)(a): The statement about the finite condition follows from the definition of $\widehat{Q}$.

If $\widehat{Q}$ is acyclic, then by identifying Q as a subquiver of $\widehat{Q}$, we see that Q is also acyclic. On the other hand, if there is a cycle $[p_1 p_2 \cdots p_k, q_1 q_2 \cdots q_k] \in \widehat{Q}_k$, then

$$[s(p_1), s(q_1)] = \widehat{s}([p_1, q_1]) = \widehat{t}([p_k, q_k]) = [t(p_k), t(q_k)].$$

This implies that Q contains the cycles $p_1 \cdots p_k$ and $q_1 \cdots q_k$.

(i)(b): Any path $[p_1 \cdots p_k, q_1 \cdots q_k]$ of length k in $\widehat{Q}$ uniquely corresponds to the pair of paths $p_1 \cdots p_k$ and $q_1 \cdots q_k$ in Q. This means that the sets $\widehat{Q}_k$ and $Q_k \times Q_k$ are in one-to-one correspondence. Thus, $|\widehat{Q}_k| = |Q_k|^2$.

(i)(c): Consider the adjacency matrix of paths of length k in $\widehat{Q}$:

$$\widehat{C}^k = \left(d^{(k)}_{[i,i'],[j,j']}\right)_{i,i',j,j' \in Q_0}.$$

On the other hand, each entry of $C^k \otimes C^k$ is of the form $c^{(k)}_{i,j} c^{(k)}_{i',j'}$, with $i, i', j, j' \in Q_0$. Suppose that $c^{(k)}_{i,j} = n \geq 0$ and $c^{(k)}_{i',j'} = m \geq 0$, that is, there are n paths of length k from i to j, and m paths of length k from i' to j'. This determines nm paths of length k from $[i, i']$ to $[j, j']$ in $\widehat{Q}$, and no other of such paths can be formed. So, $d^{(k)}_{[i,i'],[j,j']} = nm$. Hence, $\widehat{C}^k = C^k \otimes C^k$.

(ii): Suppose that there exists a non-exclusive cycle $\widehat{c} = [p_1 \cdots p_k, q_1 \cdots q_k]$ in $\widehat{Q}$. Then, there is another cycle $\widehat{c'} = [p'_1 \cdots p'_l, q'_1 \cdots q'_l]$ in $\widehat{Q}$ such that $[s(p_i), s(q_i)] = [s(p'_j), s(q'_j)]$ for some $i = 1, \ldots, k$ and $j = 1, \ldots, l$. This yields cycles $c_1 = p_1 \cdots p_k$ and $c_2 = q_1 \cdots q_k$ in Q that are non-exclusive: c_1 is not disjoint with $c'_1 = p'_1 \cdots p'_l$ for $s(p_i) = s(p'_j)$, and c_2 is not disjoint with $c'_2 = q'_1 \cdots q'_l$, for $s(q_i) = s(q'_j)$. Therefore, Q also fails the exclusive condition.

Conversely, suppose that Q fails the exclusive condition. Then, using the fact that Q can be identified with a sub-quiver of $\widehat{Q}$, we see that if two cycles in Q were not disjoint, they would be also non-disjoint in $\widehat{Q}$. Thus, $\widehat{Q}$ fails the exclusive condition.

For the second part, denote the maximum length of a chain of cycles in Q by $\operatorname{cyc.len}(Q)$. We aim to show that $\operatorname{cyc.len}(\widehat{Q}) = 2 \operatorname{cyc.len}(Q) - 1$. This is done with the notes below.

1. Given two distinct cycles $c = p_1 p_2 \cdots c_k$, $d = q_1 q_2 \cdots q_l$ in Q, construct the cycle $\widehat{c, d}$ in $\widehat{Q}$ given by

$$\widehat{s}([c^u, d^v]) = [s(p_1), s(q_1)] = [t(p_k), t(q_l)] = \widehat{t}([c^u, d^v]),$$

   for $u := \operatorname{lcm}(k, l)/k$ and $v := \operatorname{lcm}(k, l)/l$. Indeed,

$$\operatorname{length}(c^u) = u \operatorname{length}(c) = uk = vl = v \operatorname{length}(d) = \operatorname{length}(d^v).$$

2. Note that all simple cycles in $\widehat{Q}$ are of the form $\widehat{c, d}$ for some (not necessarily distinct) cycles c, d in Q.

3. Note that if c_1, c_2, d_1, d_2 are cycles in Q such that $c_1 \Rightarrow d_1$ and $c_2 \Rightarrow d_2$, we have that $\widehat{c_1, d_1} \Rightarrow \widehat{c_2, d_2}$ in $\widehat{Q}$.

4. If $\widehat{c_1, d_1}, \widehat{c_2, d_2}$ are simple cycles in $\widehat{Q}$ such that $\widehat{c_1, d_1} \Rightarrow \widehat{c_2, d_2}$, then $c_1 \Rightarrow c_2$ and $d_1 \Rightarrow d_2$ in Q.

5. Any chain of cycles of length n in Q induces a chain of cycles of length $2n - 1$ in $\widehat{Q}$. Indeed, let $c_1 \Rightarrow c_2 \Rightarrow \cdots \Rightarrow c_n$ be a chain of cycles of length n in Q. Then the chain of cycles

$$\widehat{c_1, c_1} \Rightarrow \widehat{c_1, c_2} \Rightarrow \cdots \Rightarrow \widehat{c_1, c_n} \Rightarrow \widehat{c_2, c_n} \Rightarrow \widehat{c_3, c_n} \Rightarrow \cdots \Rightarrow \widehat{c_n, c_n}$$

   in $\widehat{Q}$ has length $n + (n - 1) = 2n - 1$. The arrows were constructed as in Note 3.

To prove the claim, observe that by the result already proven in this part, $\widehat{Q}$ satisfies the exclusive condition, and hence all cycles $\widehat{Q}$ are either simple (of the form $\widehat{c, d}$ by Notes 1 and 2), or powers of simple cycles. Therefore, in any chain of cycles in $\widehat{Q}$, we can assume that all cycles are simple. Let $n := \operatorname{cyc.len}(Q)$, and suppose that there is a chain of cycles $\widehat{c_1, d_1} \Rightarrow \widehat{c_2, d_2} \Rightarrow \cdots \Rightarrow \widehat{c_m, d_m}$ in $\widehat{Q}$. Then by Note 4, we have $c_1 \Rightarrow c_2,\ c_2 \Rightarrow c_3,\ \ldots, c_{m-1} \Rightarrow c_m$ in Q. Observe that we only have at most n distinct choices for the c_i. A similar argument can be done for the d_i to deduce that there are at most n distinct choices. So when constructing the cycles, $\widehat{c_i, d_i}$, at most $n + (n-1) = 2n-1$ distinct pairs are possible, that is, $m \leq 2n-1$. Hence, $\operatorname{cyc.len}(\widehat{Q}) \leq 2 \operatorname{cyc.len}(Q) - 1$. But in Note 5 we used a chain of cycles of length n in Q to construct a chain of cycles of length exactly $2n-1$ in $\widehat{Q}$. So, $\operatorname{cyc.len}(\widehat{Q}) = 2 \operatorname{cyc.len}(Q) - 1$, as desired.

(iii): Suppose that Q has a source cycle $c := p_1 p_2 \cdots p_k$ with arrow $p \in Q_1$, not in c, such that $s(p) = s(p_i)$ for some $i = 1, \dots, k$. Then $\widehat{c} = [p_1 \cdots p_k, p_1 \cdots p_k]$ will be a source cycle in $\widehat{Q}$, where $[p, p]$ is an arrow leaving $\widehat{c}$ at $[p_i, p_i]$.

Conversely, suppose that $\widehat{Q}$ has a source cycle, that is, a cycle $\widehat{c} = [p_1 \cdots p_k, q_1 \cdots q_k]$ with arrow $[p, q] \in \widehat{Q}_1$, not in $\widehat{c}$, such that $[s(p), s(q)] = [s(p_i), s(q_i)]$ for some $i = 1, \dots, k$. This intermediately induces two source cycles in Q: $c_1 = p_1 \cdots p_k$ with arrow p leaving at p_i and $c_2 = q_1 \cdots q_k$ with arrow q leaving at q_i.

The argument for sink cycles is similar.

(iv)(a): Suppose that Q is pairwise strongly connected. Then for every pair of vertices $[i, i'], [j, j'] \in \widehat{Q}_0$ with $[i, i'] \neq [j, j']$, there exist paths of the same length $p_1 \cdots p_k$ and $q_1 \cdots q_k$ such that $s(p_1) = i$, $t(p_k) = j$, $s(q_1) = i'$ and $t(q_k) = j'$. Thus, we have the path $[p_1 \cdots p_k, q_1 \cdots q_k] \in \widehat{Q}_k$ connecting $[i, i']$ and $[j, j']$. Hence $\widehat{Q}$ is strongly connected.

Conversely, if $i, i', j, j' \in Q_0$ are such that $[i, i'] \neq [j, j']$ in $\widehat{Q}_0$, then $\widehat{Q}$ being strongly connected implies that there exists a path $[p_1 \cdots p_k, q_1 \cdots q_k] \in \widehat{Q}_k$ with $\widehat{s}[p_1, q_1] = [i, i']$ and $\widehat{t}[p_k, q_k] = [j, j']$. Therefore, the paths $p_1 \cdots p_k, q_1 \cdots q_k \in Q_k$ are such that $s(p_1) = i$, $t(p_k) = j$, $s(q_1) = i'$, and $t(q_k) = j'$. Hence, Q is pairwise strongly connected.

(iv)(b): Suppose that Q is strongly connected and has a cycle c. Take $i, i', j, j' \in Q_0$ such that $i \neq j$ and $i' \neq j'$. If $x \in Q_0$ is a vertex of the cycle c in Q, then take the following paths in Q: $p_1 \cdots p_n$ connecting i with x, and $\overline{p}_1 \cdots \overline{p}_m$ connecting x with j, $q_1 \cdots q_u$ connecting i' with x, and $\overline{q}_1 \cdots \overline{q}_v$ connecting x with j'. Then

$$p_1 \cdots p_n c^{u+v} \overline{p}_1 \cdots \overline{p}_m, \qquad q_1 \cdots q_u c^{n+m} \overline{q}_1 \cdots \overline{q}_v \in Q_{n+m+u+v}$$

are paths (of the same length) that connect i with j, and i' with j', respectively. Then, Q is pairwise strongly connected, and thus $\widehat{Q}$ is strongly connected by (iv.a).

Conversely, suppose that $\widehat{Q}$ is strongly connected. If $i, j \in Q_0$ are such that $i \neq j$, then $[i, i] \neq [j, j]$ in $\widehat{Q}_0$. By the hypothesis, there exists a path $[p_1 \cdots p_k, q_1 \cdots q_k]$ connecting $[i, i]$ with $[j, j]$ in $\widehat{Q}$. Hence, $p_1 \cdots p_k$ and $q_1 \cdots q_k$ are both paths connecting i with j in Q. That is, Q is strongly connected.

(v): Suppose that $\widehat{Q}$ is path reversible, and take a path $p_1 \cdots p_k$ in Q. Consider the corresponding path $[p_1 \cdots p_k, p_1 \cdots p_k]$ in $\widehat{Q}$. Then there exists a path $[q_1 \cdots q_l, q_1' \cdots q_l']$ in $\widehat{Q}$ such that $[s(p_1), s(p_1)] = \widehat{s}[p_1, p_1] = \widehat{t}[q_l, q_l'] = [t(q_l), t(q_l')]$ and $[s(q_1), s(q_1)] = \widehat{s}[q_1, q_1] = \widehat{t}[p_k, p_k] = [t(p_k), t(p_k)]$. That is, the paths $q_1 \cdots q_l, q_1' \cdots q_l' \in Q_l$ are both reverse paths for $p_1 \cdots p_k$, and Q is path reversible.

Towards the converse, we prove the following statement: if Q is path reversible, then given any two paths of the same length, their respective reverse paths can be taken also of the same length. Indeed, if $a, b \in Q_k$ are two paths of Q, there are reverse paths $a' \in Q_u$ and $b' \in Q_v$ resp., with u not necessarily is equal to v. If we take

$$a'' := a'(aa')^{k+v-1} \quad \text{and} \quad b'' := b'(bb')^{k+u-1},$$

notice that $s(a'') = s(a') = t(a)$, $t(a'') = t(a') = s(a)$, $s(b'') = s(b') = t(b)$ and $t(b'') = t(b') = s(b)$. Moreover,

$$\text{length}(a'') = u + (k + v - 1)(k + u) = v + (k + u - 1)(k + v) = \text{length}(b'').$$

Hence, a'', b'' are reverse paths of a, b, resp., of the same length.

Now take $[p_1 \cdots p_k, q_1 \cdots q_k] \in \widehat{Q}_k$ and suppose that Q is path reversible. Since $p_1 \cdots p_k$ and $q_1 \cdots q_k \in Q_k$ are paths of Q of the same length, by the above, there exist reverse path of the same length $p'_1 \cdots p'_l, q'_1 \cdots q'_l \in Q_l$. That is, $[p'_1 \cdots p'_l, q'_1 \cdots q'_l]$ is the reverse path of $[p_1 \cdots p_k, q_1 \cdots q_k]$ in $\widehat{Q}$.

(vi): This follows from the definition of $\widehat{Q}$. □

Remark 3.3. Regarding Proposition 3.2(iv)(b), being strongly connected does not necessarily imply that $\widehat{Q}$ is strongly connected, as shown in Figure 3.1 below.

Figure 3.1: On the strongly connected condition

3.2 On Hayashi's face algebra $\mathfrak{H}(Q)$

We provide the main result on algebraic properties of Hayashi's face algebras $\mathfrak{H}(Q)$ [Definition 1.10] in this section. Before this, we need the following preliminary result, which was also mentioned in [Pfe11, Remark 3.3].

Proposition 3.4. *Let Q be a quiver, and consider its Kronecker square $\widehat{Q}$* [Definition 3.1]*. Then, $\mathfrak{H}(Q) \cong \Bbbk\widehat{Q}$ as unital $\mathbb{N}$-graded $\Bbbk$-algebras.*

Proof. First, notice that the product of paths in $\Bbbk\widehat{Q}$ is defined in terms of the concatenation of paths in Q, that is, $[a, b][c, d] = \delta_{t(a),s(c)}\delta_{t(b),s(d)}[ac, bd]$ for any paths a, b, and paths c, d, of the same length. The unit of $\Bbbk\widehat{Q}$ is given by $1_{\Bbbk\widehat{Q}} = \sum_{i,j \in Q_0} e_{[i,j]}$, where $e_{[i,j]}$ denotes the trivial path at vertex $[i, j] \in \widehat{Q}_0$. Now consider the map $\varphi : \mathfrak{H}(Q) \to \Bbbk\widehat{Q}$ given by

$$\varphi(x_{a,b}) = [a, b], \quad a, b \in Q_k, \ k \geq 1 \qquad \text{and} \qquad \varphi(x_{i,j}) = e_{[i,j]}, \quad i, j \in Q_0.$$

Since $\mathfrak{H}(Q) = \bigoplus_{k\geq 0; a,b\in Q_k} \Bbbk x_{a,b}$ as $\Bbbk$-vector spaces, φ is a $\Bbbk$-linear map. Moreover,

$$\varphi(x_{a,b}x_{c,d}) = \delta_{t(a),s(c)}\delta_{t(b),s(d)}\varphi(x_{ac,bd}) = \delta_{t(a),s(c)}\delta_{t(b),s(d)}[ac,bd] = \varphi(x_{a,b})\varphi(x_{c,d}),$$
$$\varphi(1_{\mathfrak{H}(Q)}) = \varphi\left(\textstyle\sum_{i,j\in Q_0} x_{i,j}\right) = \textstyle\sum_{i,j\in Q_0}\varphi(x_{i,j}) = \sum_{i,j\in Q_0} e_{[i,j]} = 1_{\Bbbk\widehat{Q}}.$$

Therefore, φ is a unital $\Bbbk$-algebra map, which is clearly bijective and graded. □

This brings us to the main result on algebraic properties of the face algebra $\mathfrak{H}(Q)$.

Theorem 3.5. *Let Q be a quiver, and recall Notation 1.72. Then, the following hold.*

(i) *$\Bbbk Q$ is finite dimensional if and only if $\mathfrak{H}(Q)$ is finite dimensional. In this case,*

$$\dim_{\Bbbk}(\mathfrak{H}(Q)) = \textstyle\sum_{i,j\in Q_0,\ k\geq 0} (c_{i,j}^{(k)})^2.$$

In general, the Hilbert series of $\mathfrak{H}(Q)$ is given by

$$H_{\mathfrak{H}(Q)}(t) = (I\otimes I) + (C\otimes C)t + (C^2\otimes C^2)t^2 + \cdots,$$

where I is the $|Q_0|\times|Q_0|$ identity matrix, and $\otimes$ is the tensor product of matrices.

(ii) *$\Bbbk Q$ has finite GK-dimension if and only if $\mathfrak{H}(Q)$ has finite GK-dimension. In this case,* $\operatorname{GKdim}(\mathfrak{H}(Q)) = 2\operatorname{GKdim}(\Bbbk Q) - 1$.

(iii) *$\Bbbk Q$ is (left, right) Noetherian if and only if $\mathfrak{H}(Q)$ is (left, right) Noetherian.*

(iv) *If $\Bbbk Q$ is prime and Q has at least one cycle, then $\mathfrak{H}(Q)$ is prime. Conversely, if $\mathfrak{H}(Q)$ is prime, then $\Bbbk Q$ is prime.*

(v) *$\Bbbk Q$ is semiprime if and only if $\mathfrak{H}(Q)$ is semiprime.*

(vi) $\operatorname{gldim}(\Bbbk Q) = \operatorname{gldim}(\mathfrak{H}(Q))$*; in particular, $\mathfrak{H}(Q)$ is hereditary.*

(vii) *$\Bbbk Q$ and $\mathfrak{H}(Q)$ are Koszul.*

Proof. By Proposition 3.4, it suffices to establish the statements above with $\Bbbk\widehat{Q}$ in place of $\mathfrak{H}(Q)$. Now the statements (i)-(vi) follow from Propositions 1.73 and 3.2, respectively. Part (vii) holds as the path algebra $\Bbbk Q$ (resp., $\Bbbk\widehat{Q} \cong \mathfrak{H}(Q)$) is realized as a tensor algebra $T_{\Bbbk Q_0}(\Bbbk Q_1)$ (resp., $T_{\Bbbk\widehat{Q}_0}(\Bbbk\widehat{Q}_1)$), and tensor algebras are Koszul. □

Remark 3.6. For Q acyclic, $\Bbbk Q$ being prime may not imply $\mathfrak{H}(Q)$ is prime [Remark 3.3].

3.3 Examples

In Table 3.1 below, we present some quivers Q and $\widehat{Q}$ and their corresponding path algebras illustrating the results obtained in the previous sections. Moreover, in Table 3.2

below, we present a summary of the results for dimension of path algebras of quivers having as underlying graph a Dynkin diagram; note that orientation of the quiver is relevant.

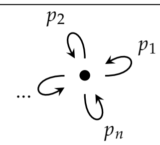
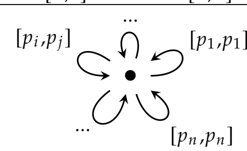
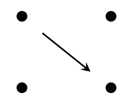
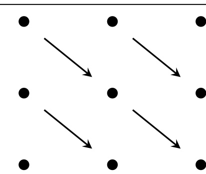
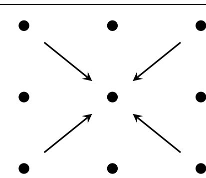
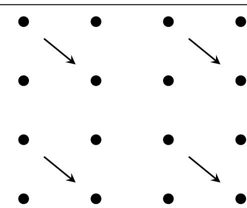
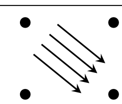
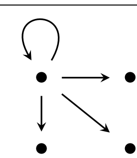
Q	$\Bbbk Q$	$\widehat{Q}$	$\Bbbk\widehat{Q} \cong \mathfrak{H}(Q)$
$\bullet_1 \quad \bullet_2 \quad \cdots \quad \bullet_n$	$\Bbbk^n$	$\begin{matrix}\bullet_{[1,1]} & \cdots & \bullet_{[1,n]}\\ \vdots & \ddots & \vdots \\ \bullet_{[n,1]} & \cdots & \bullet_{[n,n]}\end{matrix}$	$\Bbbk^{n^2}$
	$\Bbbk\langle t_1, \dots, t_n\rangle$		$\Bbbk\langle t_{i,j}\rangle_{i,j=1}^n$
$\bullet \longrightarrow \bullet$	$T_2(\Bbbk) = \begin{bmatrix}\Bbbk & \Bbbk\\ 0 & \Bbbk\end{bmatrix}$		$T_2(\Bbbk) \times \Bbbk^2$
$\bullet \longrightarrow \bullet \longrightarrow \bullet$	$T_3(\Bbbk) = \begin{bmatrix}\Bbbk & \Bbbk & \Bbbk\\ 0 & \Bbbk & \Bbbk\\ 0 & 0 & \Bbbk\end{bmatrix}$		$T_3(\Bbbk) \times T_2(\Bbbk)^2 \times \Bbbk^2$
$\bullet \longrightarrow \bullet \longleftarrow \bullet$	$\begin{bmatrix}\Bbbk & 0 & 0\\ \Bbbk & \Bbbk & 0\\ \Bbbk & 0 & \Bbbk\end{bmatrix}$		$\begin{bmatrix}\Bbbk & 0 & 0 & 0 & 0\\ \Bbbk & \Bbbk & 0 & 0 & 0\\ \Bbbk & 0 & \Bbbk & 0 & 0\\ \Bbbk & 0 & 0 & \Bbbk & 0\\ \Bbbk & 0 & 0 & 0 & \Bbbk\end{bmatrix} \times \Bbbk^4$
$\bullet \to \bullet \quad \bullet \to \bullet$	$T_2(\Bbbk)^2$		$T_2(\Bbbk)^4 \times \Bbbk^8$
$\bullet \rightrightarrows \bullet$	$\begin{bmatrix}\Bbbk & \Bbbk^2\\ 0 & \Bbbk\end{bmatrix}$		$\begin{bmatrix}\Bbbk & \Bbbk^4\\ 0 & \Bbbk\end{bmatrix} \times \Bbbk^2$
$\begin{tikzcd} \bullet \arrow[loop above] \arrow[r] & \bullet \end{tikzcd}$	$\begin{bmatrix}\Bbbk[t] & \Bbbk[t]\\ 0 & \Bbbk\end{bmatrix}$		$\begin{bmatrix}\Bbbk[t] & \Bbbk[t] & \Bbbk[t] & \Bbbk[t]\\ 0 & \Bbbk & 0 & 0\\ 0 & 0 & \Bbbk & 0\\ 0 & 0 & 0 & \Bbbk\end{bmatrix}$

Table 3.1: Examples of Q, $\widehat{Q}$, and their path algebras

Q	$\dim_{\Bbbk} \Bbbk Q$	$\dim_{\Bbbk} \mathfrak{H}(Q)$
$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \cdots \longrightarrow \bullet_n \quad n \geq 1$	$\frac{n(n+1)}{2}$	$\frac{n(n+1)(2n+1)}{6}$
$\bullet_1 \longrightarrow \bullet_2 \longleftarrow \cdots \longleftarrow \bullet \longrightarrow \bullet_{2n}$ or $\bullet_1 \longrightarrow \bullet_2 \longleftarrow \cdots \longrightarrow \bullet \longleftarrow \bullet_{2n-1}$ $\quad n \geq 2$	$2n-1$	$2n^2-2n+1$
$\begin{array}{lll} \bullet_1 & & \\ & \searrow & \\ & & \bullet_3 \longrightarrow \cdots \longrightarrow \bullet_n \quad n \geq 4 \\ & \nearrow & \\ \bullet_2 & & \end{array}$	$\frac{n(n+1)}{2} - 1$	$\frac{n(n+1)(2n+1)}{6} - 1$
$\begin{array}{lll} \bullet_1 & & \\ & \nwarrow & \\ & & \bullet_3 \longrightarrow \cdots \longrightarrow \bullet_n \quad n \geq 4 \\ & \swarrow & \\ \bullet_2 & & \end{array}$	$\frac{n^2-3n+10}{2}$	$\frac{2n^3-9n^2+61n-78}{6}$
$\begin{array}{ccccccccc} & & & & \bullet & & & & \\ & & & & \uparrow & & & & \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$	19	87
$\begin{array}{ccccccccccc} & & & & \bullet & & & & & & \\ & & & & \uparrow & & & & & & \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$	25	131
$\begin{array}{ccccccccccccc} & & & & \bullet & & & & & & & & \\ & & & & \uparrow & & & & & & & & \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$	32	188

Table 3.2: $\dim_{\Bbbk} \Bbbk Q$ and $\dim_{\Bbbk} \mathfrak{H}(Q)$ for some Q of Dynkin type ADE

Bibliography

[ABV15] M. Alves, E. Batista, and J. Vercruysse. Partial representations of hopf algebras. *J. Algebra*, 426:137–187, 2015. IV

[AFSS00] N. Andruskiewitsch, W. R. Ferrer-Santos, and H. J. Schneider, editors. *New trends in Hopf algebra theory. Proceedings of the Colloquium on Quantum Groups and Hopf Algebras held in La Falda, August 9–13, 1999*, volume 267 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 2000. III

[AS10] N. Andruskiewitsch and H. J. Schneider. On the classification of finite-dimensional pointed Hopf algebras. *Ann. of Math. (2)*, 171(1):375–417, 2010. III

[ASS06] I. Assem, A. Skowronski, and D. Simson. *Elements of the representation theory of associative algebras. Vol. 1: Techniques of representation theory*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. 30

[AST91] M. Artin, W. Schelter, and J. Tate. Quantum deformations of GL_n. *Comm. Pure Appl. Math.*, 44(8-9):879–895, 1991. IV

[BÏ8] G. Böhm. *Hopf Algebras and Their Generalizations from a Category Theoretical Point of View*, volume 2226 of *Lecture Notes in Mathematics*. Springer International Publishing, 2018. 11

[Bah21] Y. Bahturin. *Identical Relations in Lie Algebras*, volume 68 of *De Gruyter Expositions in Mathematics*. De Gruyter, sec edition, 2021. Translated from the Russian by Bahturin. 64

[BCJ11] G. Böhm, S. Caenepeel, and K. Janssen. Weak bialgebras and monoidal categories. *Comm. Algebra*, 39(12):4584–4607, 2011. III, 5, 6, 10, 13, 14, 18, 19

[BCM02] T. Brzeziński, S. Caenepeel, and G. Militaru. Doi-Koppinen modules for quantum groupoids. *J. Pure Appl. Algebra*, 175(1-3):45–62, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly. 20

[BFP10] D. Bagio, D. Florez, and A. Paques. Partial actions of ordered groupoids on rings. *J. Algebra Appl.*, 09(3):501–517, 2010. IV

[BG02] K. A. Brown and K. R. Goodearl. *Lectures on Algebraic Quantum Groups*. Advanced Courses in Mathematics - CRM Barcelona. Birkhäuser Basel, 2002. IV

[BGTLC14] G. Böhm, J. Gómez-Torrecillas, and E. López-Centella. On the category of weak bialgebras. *J. Algebra*, 399:801–844, 2014. 41, 45, 48

[BHS11] R. Brown, P. J. Higgins, and R. Sivera. *Nonabelian Algebraic Topology: Filtered Spaces, Crossed Complexes, Cubical Homotopy Groupoids*. Number 15 in Tracts in Mathematics. EMS Press, 2011. 22, 26

[BNS99] G. Böhm, F. Nill, and K. Szlachányi. Weak Hopf algebras. I. Integral theory and C^*-structure. *J. Algebra*, 221(2):385–438, 1999. III, 3, 4, 5, 6

[BP12] D. Bagio and A. Paques. Partial groupoid actions: Globalization, Morita theory, and Galois theory. *Comm. Algebra*, 40(10):3658–3678, oct 2012. 23, 26

[Bri12] M. Brion. Representations of quivers. In *Geometric methods in representation theory (I)*, volume 24 of *Séminaires & Congrès: Collection SMF*, pages 103–144. Société Mathématique de France, 2012. 31

[BS97] G. Böhm and K. Szlachányi. Weak C^*-Hopf algebras: the coassociative symmetry of non-integral dimensions. In R. Budzyński, W. Pusz, and S. Zakrzewski, editors, *Quantum groups and quantum spaces (Warsaw, 1995)*, volume 40 of *Banach Center Publ.*, pages 9–19. Polish Acad. Sci. Inst. Math., Warsaw, 1997. III, 3

[CEW15] J. Cuadra, P. Etingof, and C. Walton. Semisimple Hopf actions on Weyl algebras. *Adv. Math.*, 282:47–55, 2015. 44

[CEW16] J. Cuadra, P. Etingof, and C. Walton. Finite dimensional Hopf actions on Weyl algebras. *Adv. Math.*, 302:25–39, 2016. 44

[CG00] S. Caenepeel and E. De Groot. Modules over weak entwining structures. In N. Andruskiewitsch, W. R. Ferrer-Santos, and H. J. Schneider, editors, *New trends in Hopf algebra theory. Proceedings of the Colloquium on Quantum Groups and Hopf Algebras held in La Falda, August 9–13, 1999*, volume 267 of *Contemp. Math.*, pages 31–54. Amer. Math. Soc., Providence, RI, 2000. 15, 20

[CHWW23] F. Calderón, H. Huang, E. Wicks, and R. Won. Symmetries captured by actions of weak Hopf algebras. *arXiv preprint arXiv:2209.11903*, 2023. V

[CK98] A. Connes and D. Kreimer. Hopf algebras, renormalization and noncommutative geometry. *Comm. Math. Phys.*, 199(1):203–242, 1998. III

[CL00] F. U. Coelho and S. X. Liu. Generalized path algebras. In F. van Oystaeyen and M. Saorin, editors, *Interactions between ring theory and representations of algebras*, volume 210 of *Lecture Notes in Pure and Applied Mathematics*, pages 53–66. CRC, 2000. 31

[CL09] D. Cheng and F. Li. The structure of weak Hopf algebras corresponding to $U_q(sl_2)$. *Comm. Algebra*, 37(3):729–742, 2009. 27

[CM84] M. Cohen and S. Montgomery. Group-graded rings, smash products, and group actions. *Trans. Amer. Math. Soc.*, 282(1):237–258, 1984. IV, 37

[CPQS15] F. Castro, A. Paques, G. Quadros, and A. Sant'Ana. Partial actions of weak Hopf algebras: Smash product, globalization and Morita theory. *J. Pure Appl. Algebra*, 219(12):5511–5538, 2015. IV

[CR23] F. Calderón and A. Reyes. On the (partial) representation category of weak Hopf algebras. In preparation, 2023. V

[CW23] F. Calderón and C. Walton. Algebraic properties of face algebras. *J. Algebra Appl.*, 22(03):2350076, 2023. V

[DNR01] S. Dăscălescu, C. Năstăsescu, and Ş. Raianu. *Hopf Algebras: An Introduction*, volume 235 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., first edition, 2001. 1

[Dok18] M. Dokuchaev. Recent developments around partial actions. *São Paulo J. Math. Sci.*, 13(1):195–247, 2018. IV

[DP08] B. Day and C. Pastro. Note on Frobenius monoidal functors. *New York J. Math.*, 14:733–742, 2008. 10, 11, 12

[EE07] P. Etingof and C. H. Eu. Koszulity and the Hilbert series of preprojective algebras. *Math. Res. Lett.*, 14(4):589–596, 2007. 31

[EGNO15] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015. 1, 9, 10, 11

[EN55] S. Eilenberg and T. Nakayama. On the dimension of modules and algebras, II: Frobenius algebras and quasi-Frobenius rings. *Nagoya Math. J.*, 9:1–16, 1955. 3

[EW14] P. Etingof and C. Walton. Semisimple Hopf actions on commutative domains. *Adv. Math.*, 251:47–61, 2014. 44

[EW16a] P. Etingof and C. Walton. Finite dimensional Hopf actions on algebraic quantizations. *Algebra Number Theory*, 10(10):2287–2310, 2016. 44

[EW16b] P. Etingof and C. Walton. Finite dimensional Hopf actions on deformation quantizations. *Proc. Amer. Math. Soc.*, 145(5):1917–1925, 2016. 44

[FMF20] E. Fontes, G. Martini, and G. Fonseca. Partial actions of weak Hopf algebras on coalgebras. *J. Algebra Appl.*, 21(01):Paper No. 2250012, 35, oct 2020. IV, 63

[FMS21] G. Fonseca, G. Martini, and L. Silva. Partial (co)actions of Taft and Nichols Hopf algebras on their base fields. *Int. J. Algebra Comput.*, 31(07):1471–1496, 2021. 63

[FMS22] G. Fonseca, G. Martini, and L. Silva. Partial (co)actions of Taft and Nichols Hopf algebras on algebras. *ArXiv preprint arXiv:math/2208.05141*, 2022. 63

[Frö99] R. Fröberg. Koszul algebras. In D. Dobbs, M. Fontana, and S. E. Kabbaj, editors, *Advances in commutative ring theory*, volume 205 of *Lecture Notes in Pure and Applied Mathematics*, pages 337–350. Marcel Dekker, 1999. 31

[GW04] K. R. Goodearl and R. B. Jr Warfield. *An Introduction to Noncommutative Noetherian Rings*, volume 61 of *London Mathematical Society Student Texts*. Cambridge University Press, second edition, 2004. 31

[Hay93] T. Hayashi. Quantum group symmetry of partition functions of IRF models and its application to Jones' index theory. *Comm. Math. Phys.*, 157(2):331–345, 1993. 7

[Hay96] T. Hayashi. Compact quantum groups of face type. *Publ. Res. Inst. Math. Sci.*, 32(2):351–369, 1996. 7

[Hay99] T. Hayashi. Face algebras and unitarity of $SU(N)_L$-TQFT. *Comm. Math. Phys.*, 203(1):211–247, 1999. 21, 42

[HWWW23] H. Huang, C. Walton, E. Wicks, and R. Won. Universal quantum semigroupoids. *J. Pure Appl. Algebra*, 227:107193, 2023. III, 7, 66

[IR19a] A. Ibort and M. Rodríguez. *An introduction to groups, groupoids and their representations*. CRC Press, 2019. 23, 24, 26

[IR19b] A. Ibort and M. Rodríguez. On the structure of finite groupoids and their representations. *Symmetry*, 11(3):414, 2019. 24, 25, 26

[KR09] L. Kong and I. Runkel. Cardy algebras and sewing constraints. I. *Comm. Math. Phys.*, 292(3):871–912, 2009. 12

[Mac87] K. Mackenzie. *Lie Groupoids and Lie Algebroids in Differential Geometry*, volume 124 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1987. 50

[MFSM18] J. M. Moreno-Fernández and M. Siles-Molina. Graph algebras and the Gelfand-Kirillov dimension. *J. Algebra Appl.*, 17(05):1850095, 15, 2018. 31

[Mon93] S. Montgomery. *Hopf Algebras and Their Actions on Rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, 1993. III, 1, 4

[MPDS22] G. Martini, A. Paques, and L. Duarte-Silva. Partial actions of a Hopf algebra on its base field and the corresponding partial smash product algebra. *J. Algebra Appl.*, 22(06):2350140, 2022. 63

[MR01] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings*, volume 30 of *Graduate Studies in Mathematics*. American Mathematical Society, 2001. 31

[Nik01] D. Nikshych. *Quantum groupoids, their representation categories, symmetries of von Neumann factors, and dynamical quantum groups*. PhD thesis, University of California, 2001. 8, 50, 54, 56, 59

[Nik02] D. Nikshych. On the structure of weak Hopf algebras. *Adv. Math.*, 170(2):257–286, 2002. III, 5, 42

[Nik04] D. Nikshych. Semisimple weak Hopf algebras. *J. Algebra*, 275(2):639–667, 2004. 42

[Nil98] F. Nill. Axioms for weak bialgebras. *arXiv preprint arXiv:math/9805104*, 1998. 3, 13

[NSW98] F. Nill, K. Szlachanyi, and H. W. Wiesbrock. Weak Hopf Algebras and Reducible Jones Inclusions of Depth 2. I: From Crossed products to Jones towers. *arXiv preprint arXiv:math/9806130*, 1998. III

[NTV03] D. Nikshych, V. Turaev, and L. Vainerman. Invariants of knots and 3-manifolds from quantum groupoids. *Topology Appl.*, 127(1-2):91–123, 2003. III, 13

[NV02] D. Nikshych and L. Vainerman. Finite quantum groupoids and their applications. In S. Montgomery and H. J. Schneider, editors, *New directions in Hopf algebras*, volume 43 of *Mathematical Sciences Research Institute Publications*, pages 211–262. Cambridge Univ. Press, Cambridge, 2002. III, 5, 6, 13, 18, 45, 55

[PF13] A. Paques and D. Flôres. Duality for groupoid (co)actions. *Comm. Algebra*, 42(2):637–663, 2013. 24, 26, 50

[Pfe11] H. Pfeiffer. Fusion categories in terms of graphs and relations. *Quantum Topol.*, 2(4):339–379, 2011. 70

[Pra67] J. Pradines. Théorie de lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux. *C. R. Math. Acad. Sci. Paris*, 264:A245–A248, 1967. 50

[PT14] A. Paques and T. Tamusiunas. A Galois-Grothendieck-type correspondence for groupoid actions. *Algebra Discrete Math.*, 17(1):80–97, 2014. 22, 26

[Rin63] G. S. Rinehart. Differential forms on general commutative algebras. *Trans. Amer. Math. Soc.*, 108(2):195–222, 1963. 50

[RWZ21] D. Rogalski, R. Won, and J. J. Zhang. A proof of the Brown-Goodearl conjecture for module-finite weak Hopf algebras. *Algebra Number Theory*, 15(4):971–997, 2021. 6, 8

[Sar20] Paolo Saracco. On anchored Lie algebras and the Connes-Moscovici's bialgebroid construction. *ArXiv preprint arXiv:math/2009.14656*, 2020. 50

[Sar21] Paolo Saracco. Universal enveloping algebras of Lie-Rinehart algebras as a left adjoint functor. *ArXiv preprint arXiv:math/2102.01553*, 2021. 50

[SM08] M. Siles-Molina. Algebras of quotients of path algebras. *J. Algebra*, 319(12):5265–5278, 2008. 31

[Str07] R. Street. *Quantum Groups: a path to current algebra*, volume 19 of *Australian Mathematical Society Lecture Series*. Cambridge University Press, 2007. 10, 12

[SU04] I. Shestakov and U. Umirbaev. The tame and the wild automorphisms of polynomial rings in three variables. *J. Amer. Math. Soc.*, 17(1):197–227, 2004. 63

[Szl01] K. Szlachányi. Finite quantum groupoids and inclusions of finite type. In *Mathematical physics in mathematics and physics (Siena, 2000)*, volume 30 of *Fields Inst. Commun.*, pages 393–407. Amer. Math. Soc., Providence, RI, 2001. 6

[Szl05] K. Szlachányi. Adjointable monoidal functors and quantum groupoids. In *Hopf algebras in noncommutative geometry and physics*, volume 239 of *Lecture Notes in Pure and Appl. Math.*, pages 291–307. Dekker, New York, 2005. 10, 12, 14, 19

[Ufn82] V. Ufnarovskiĭ. A growth criterion for graphs and algebras defined by words. *Mat. Zametki*, 31(3):465–472, 476, 1982. 31

[Vár03] J. C. Várilly. Hopf algebras in noncommutative geometry. In A. Cardona, S. Paycha, and H. Ocampo, editors, *Geometric and Topological Methods for Quantum Field Theory*, pages 1–85. World Scientific, 2003. III

[WW16] C. Walton and X. Wang. On quantum groups associated to non-Noetherian regular algebras of dimension 2. *Math. Z.*, 284(1-2):543–574, 2016. IV

[WWW22] C. Walton, E. Wicks, and R. Won. Algebraic structures in comodule categories over weak bialgebras. *Comm. Algebra*, 50(7):2877–2910, 2022. III, 5, 6, 10, 18, 20, 21