

April 17, 2016

Seattle Workshop

Finite dim'l Hopf actions on algebraic quantizations

joint w/ Etingof
(in preparation)

$k = \bar{k}$, $\text{char } k = 0$.

Goal To establish "No Quantum Symmetry" (NQS) results:

Given H -action on A [$A = H$ -module algebra]
Hopf alg alg

Get the action factors through the action of a cocomp. Hopf alg.
"classical" action

Def'n: we have **NFQS** if we have NQS for action of a finite dim'l Hopf alg.
(i.e. Hopf alg. action factors through action of a group alg.)

we have **NSFQS** if we have NQS for action of a semisimple Hopf alg.
 \rightarrow finite dim'l

Previous Results w/ Etingof on NQS (* w/ Cuadra, ** w/ Goswami, Mandal)

ArXiv #	Module Algebra A	NSFQS?	NFQS?
1301.4161	Commutative domain	✓	
1507.08486**	fin-gen. com. alg. w/ no homog. degree 2 relations	✓	
1409.1644*	Weyl algs $A_n(k[z_1, \dots, z_s])$, $D(X)$ ring of diff'l ops	✓	
1509.01165*	Weyl algs $A_n(k^s)$, $D(X)$ ring of diff'l ops		✓
1602.00532	quantum (formal) deformations of com. domains (filtered (PBW) deformation of com. domains)	✓	✓ under stiff condition (Pois. centr. of $S(A_0)$ is trivial)
			✓ if filt' in pos. (same)

New Results (and last in series of NSF papers).

Outlining rest of talk

Module Algebra A

NSFQS

NFQS

large class of filtered deformations of commutative domains
= "algebra with PI reductions"

✓

✓ under a nondegeneracy condition

I.

II.

Skew poly'l rings

$$k_q[x_1, \dots, x_n] = \frac{k[x_1, \dots, x_n]}{(x_i x_j - q_{ij} x_j x_i)} \quad q_{ij} \in k^*$$

✓ under a condition on the Zariski closure of the subgroup $\langle q \rangle$ in $(\mathbb{Z}/2\mathbb{Z})^{\binom{n}{2}}$

✓ along with a nondeg. condition

III.

$B(X, L, \sigma)$ twisted homog coordinatizing algebra variety

✓ under a condition on the Zariski closure of

$S(E, L, \sigma)$ 3dim Sklyanin algebra
↑
elliptic curve $\subseteq \mathbb{P}^2$

✓ $\{ \sigma^i(s) \}_{i \in \mathbb{Z}}$, where σ is determined by shift by $s \in X$.

Sketch of NSFOS for $A_n(k)$

Sketch of NFS for $A_n(k)$

— Assume $H \curvearrowright A$ is innerfaithful (\neq Hopf ideal I of $H \curvearrowright A$)
 $IA=0$

Reduce modulo p $p \gg 0$

- \exists fin-gen. subalg $R \subseteq k$ such that $H_R \curvearrowright A_n(R)$ innerfaithful R -subalg of H
- $[H = R \otimes_R H_R]$
- \exists homomorphism $\psi: R \rightarrow \overline{\mathbb{F}}_p$ to define $H_{\psi, p} = H_R \otimes_R \overline{\mathbb{F}}_p = H_p$
- get $H_p \curvearrowright A_n(\overline{\mathbb{F}}_p)$ innerfaithful.
ss, cocom \uparrow PI domain

Reduce modulo p^m $p \gg 0$

→ ✓ (same).

Replace with $H_{p^m} \curvearrowright A_n(W_{m,p})$

$H_R \otimes_R W_{m,p}$

$W_{m,p} = m$ -th truncated ring of Witt vec / $\overline{\mathbb{F}}_p$
 (not deg/fields, or $\mathbb{Z}/p^m\mathbb{Z}$ -modules)

Localize: Hopf actions on Div. Algs

Localize: Hopf actions on Artinian rings

- Get $H_p \curvearrowright D_p$ just div. alg of $A_n(\overline{\mathbb{F}}_p)$ innerfaithful.
ss, cocom.
- Have $D_p = Z(D_p) D_p^{H_p}$ & $Z(D_p)$ is H_p -stab.
- Take any Hopf ideal I of $H_p \curvearrowright A_n(\overline{\mathbb{F}}_p)$. $I \cap Z(D_p) = 0$.
innerfaithful.
 $\Rightarrow I \cdot D_p = 0 \Rightarrow I = 0$
 $\Rightarrow H_p \curvearrowright Z(D_p)$ a field, innerfaithfully
 $\Rightarrow H_p$ cocommutative by [EW2013].

- Get $H_{p^m} \curvearrowright D_{p^m} =$ full localh of $A_n(W_{m,p})$
- Most technical part: get
 - $Z(D_{p^m})$ is H_{p^m} -stable
 - \mathbb{Z}/p^m is H_p -stable

PI degree argument
 $\deg D_p = p^n$
 for noncomm.
 \neq
 $\text{rad}(p^m, \dim(H_p) = 1)$
 for $p \gg 0$

! fails to H -non-ss $H/\overline{\mathbb{F}}_p$ is cocomm.

Pass back to characteristic 0

Pass back to \mathbb{C}

Com Alg \Rightarrow dir product of all $\psi: R \rightarrow \overline{\mathbb{F}}_p$ fields
 $\Rightarrow H_R$ cocomm.
 $\Rightarrow H$ cocomm ✓

→ ✓ (same)

Now NSFQS

Replace $A_n(k)$ with

A "residually PI" $\stackrel{\text{def}}{=}$

- A \mathbb{Z}_+ filtered
- $\text{gr}(A)$ com. fin-gen. domain
- A admits R-order AR so that $A_p := AR \otimes_R \mathbb{F}_p$ is PI for $p \gg 0$

Main difference b/w this case &

result for $A = A_n(k)$:

Have to show that quotient division alg

has degree a power of p

(don't have explicit presentation of A)

→ Done in Etingof 1602.06480

have to control $\deg p$

Examples of residually PI algebras

① any filtered deformation of a com-domain generated in filtered degree one

* $U(\mathfrak{g})$ of fin-dim Lie alge

* $D_m(X)$ twisted diff'ls

② finite W-algebras

③ spherical symplectic reflection algebras

④ (Any of the above) \otimes (com. fin-gen. domain)

→ ✓
→ ✓
→ ✓

Question Is the "PI reductions" vacuous?

1602.06480, Q1: Let $\text{ch} F = p > 0$.

Take A a filt. deform'n of a com. f.g. domain \neq

Is A PI?

for $p \gg 0$

NFQS

Replace $A_n(k)$ with

a residually PI algebra A

that is "nondegenerate"

// def

$$\bigcap_{m \geq 1} \pi(Z(D_{p^m})) = \overline{\mathbb{F}_p}$$

for almost all p .

When does nondegeneracy hold?

Recall $\text{gr}(A)$ is a Poisson alg.

||

$\mathcal{O}(X)$ X irreducible Poisson variety

A nondegenerate $\iff X$ is "generically symplectic"

(having a symplectic dense open subset)

Examples

II. Will only discuss NSFQS

Thm Take $A = k_q[x_1, \dots, x_n] = k\langle x_1, \dots, x_n \rangle / (x_i x_j - q_{ij} x_j x_i) \quad i < j$

for a multiplicatively antisym $(q_{ij}) \in \text{Mat}_n(k^\times)$

Let $G_q := \text{Zariski closure of } \langle q \rangle \subseteq (k^\times)^{\binom{n}{2}} \leftarrow \binom{n}{2}\text{-tors}$

$G_q^0 := \text{connected component of the identity of } G_q$

Suppose that a semisimple Hopf algebra H acts on A .

no condition
H-action
on A
preserving
grading of A

If the order of G_q/G_q^0 is coprime to $(\dim H)!$,
then any H -action on A factors through k group action.

Ex. $n=2$. $A = k_q[x, y] = k\langle x, y \rangle / (xy - qyx)$

• If $q = \text{non-root of unity}$, then $G_q^0 = G_q \Rightarrow (*)$ holds for any H
 \Rightarrow NSFQS for any H .

Recall that inner faithful
actions of the Kac-Paljutkin
Hopf algebra H_8 on
 $k\langle x, y \rangle = k[x, y]$

• If $\text{ord}(q) = r < \infty$, then $(*) \Leftrightarrow r$ is coprime to $(\dim H)!$
get NSFQS when $(r, (\dim H)!) = 1$.

Ex. If each q_{ij} is a root of unity, say of order $r_{ij} < \infty$,
then $|G_q/G_q^0| = \text{lcm}\{r_{ij} \mid i < j\}$

ex. $A = k\langle x, y, z \rangle / \begin{pmatrix} xy - s_{11}yx \\ yz - s_{22}zy \\ zx - s_{33}xz \end{pmatrix}$; for s_j a primitive j th

get NSFQS for H so that $(\dim H)!, 7 \cdot 11 \cdot 13 = 1$
(e.g. do. Hopf of $\dim 5 \cdot 3^2 = 45$ cannot act innerfaithfully)

Ex. If $\{q_{ij}\}$ has no order d
the set $\{q_{ij}\}$ is multiplicatively independent,
then $G_q^0 = G_q \Rightarrow (*)$ holds for any H

Sketch of proof

same ideas as NSFOS for An(k) result ... with modifications

Reduce modulo p

\exists fin. gen. subalg $R \subseteq k \Rightarrow$
 $H_R \xrightarrow{\sim} R[x_1, \dots, x_n]$ inner faithfully
 $[H = k \otimes_R H_R]$

! not straightforward that we can reduce?
 (ie. \exists homomorphism $\psi: R \rightarrow \mathbb{F}_p$ to make...)

\rightarrow go through a number field K .
 Given homom. $\xi: R \rightarrow K$, let $R' := \text{im } \xi$

Get $H_{R'} \xrightarrow{\sim} R'[\xi(q), x_1, \dots, x_n]$
 \parallel
 $H_R \otimes_R R'$

Get $|G_q/G_q^0| = |G_{\xi(q)}/G_{\xi(q)}^0| =: l$

Number-theoretic heart of Perucca:

Get ∞ many primes $p \Rightarrow$
 for any generic homomorphism
 $\psi: R' \rightarrow \mathbb{F}_p$ annihilating a
 prime ideal $\mathfrak{p} \subseteq R'$ lying over p

... need to control $\deg D_p$

Get $H_p \xrightarrow{\sim} A_p$
 $H \otimes_{k_0} \mathbb{F}_p$ reduction
 $[k_0[x_1, \dots, x_n] \xrightarrow{\sim} \text{alg}/\mathbb{F}_p]$

Localize: Hopf actions on division algs

Get $H_p \xrightarrow{\sim} D_p$ full localization of A_p
 inner faithfully

$\#(\deg D_p, (\dim H_p)!) = 1$ for $p \gg 0$

Get H_p cocommutative

Pass back to char=0

Get H is cocommutative

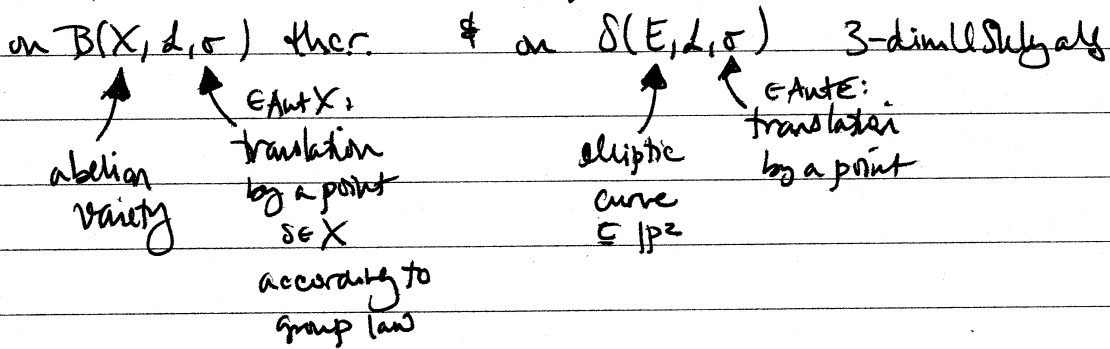
we have that
 $N := \text{ord } \psi(\xi(q))$ is finite &
 \parallel
 $\parallel_{\mathbb{F}_p}$ coprime to $(\dim H)!$

since l is coprime to $(\dim H)!$ by hypothesis
 (see Corollary 7.2 of 1605.00560)

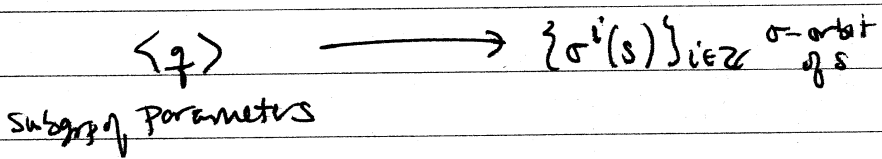
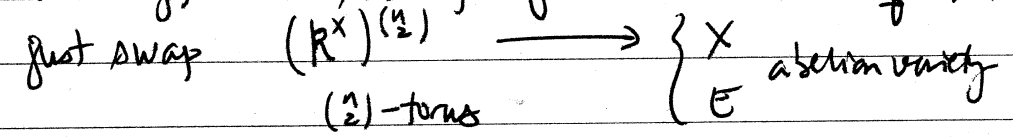
Now form $A_p = R'_{\xi(q)}[x_1, \dots, x_n] \otimes_{R'} \mathbb{F}_p$
 via the map ψ

\hookrightarrow get A_p is a PI domain
 & PI degree A_p divides N^m

Get similar NSFOS results for semisimple Hopf algebras



How so? Basically, use the same proof as for NSFOS on $k\langle x_1, \dots, x_n \rangle$



Ex. Take H as. of dim d . If $|o|$ is coprime to $d!$ or infinite,

then any H -action on $\delta(E, d, \sigma) = \delta(a, b, c) = k\langle x, y, z \rangle$

$a, b, c \in k^x$ $\begin{pmatrix} ayz + byz + cx^2 \\ axz + bxz + cy^2 \\ axy + byx + cz^2 \end{pmatrix}$

factors through a group action.

Ex. Not so useful in ^{studying} semisimple Hopf algs H_n that acts on $\delta(1, 1, c)$ innerfaithfully.

For $\delta(1, 1, c)$, σ is given by translation by a point of order 2.
 So, $|o|=2$ is coprime to $(\dim H)!$ only in the case when $\dim H = 1, \dots$

useful for S with $|o|$ large.

★ If H^2A gives rise to a Hopf-Galois extension, then "coprime to $(\dim H)!$ " can be replaced with "coprime to $\dim H$ ".

for all above