April 17, 2016 — Seattle workshop

$\boxed{\text{Finite dim'l Hopf actions on algebraic quantizations}}$ joint w/ Etingof (in preparation)

/ $k=\bar{k}$, char $k=0$.

Goal To establish "No Quantum Symmetry" (NQS) results:

Given $\underset{\text{Hopf alg}}{H}$ - action on $\underset{\text{alg}}{A}$ [A = H-module algebra]

Get the action factors through the action of a cocom. Hopf alg. "classical" setting

Def'n: we have $\boxed{\text{NFQS}}$ if we have NQS for action of a finite dim'l Hopf alg.
(i.e. Hopf alg. action factors through action of a group alg)

· we have $\boxed{\text{NSFQS}}$ if we have NQS for action of a semisimple Hopf alg. ⇒ fin dim'l

Previous Results w/ Etingof on NQS (* w/ Cuadra, ** w/ Goswami, Mandal)

ArXiv #	Module Algebra A	NSFQS?	NFQS?
1301.4161	commutative domain	✓	
1507.08486**	com. fin. gen. alg. w/ no homog. degree 2 relations	✓	
1409.1644*	Weyl algs $A_n(k[z_1, \ldots, z_s])$, $D(X)$ ring of diff'l ops	✓	
1509.01165*	Weyl algs $A_n(k)$, $D(X)$ ring of diff'l ops		✓
1602.00532	quantum (formal) deformations of com. domains A_0	✓	✓ under suff condition (Pois. center of $Q(A_0)$) is trivial
	filtered (PBW) deformation of com. domains		✓ if filt'n pres. (same)

New Results (and last in series of NQS papers).

Module Algebra A	NSFQS	NFQS
I. large class of filtered deformations of commutative domains = "algebra with PI reductions"	✓	✓ under a nondegeneracy condition
II. Skew poly'l rings $k_q[x_1,\dots,x_n] = \frac{k\langle x_1,\dots,x_n\rangle}{(x_ix_j - q_{ij}x_jx_i)}$, $q_{ij} \in k^\times$	✓ under a condition on the Zariski closure of the subgroup $\langle q\rangle$ in $\binom{n}{2}$-torus $(k^\times)^{\binom{n}{2}}$	✓ * along with a nondeg. condition
III $B(X, \mathcal{L}, \sigma)$ twisted homog coordinate ring; X abelian variety	✓ under a condition on the Zariski closure of $\{\sigma^i(s)\}_{i\in\mathbb{Z}}$, where σ is determined by shift by $s \in X$.	
$S(E, \mathcal{L}, \sigma)$ 3-dim'l Sklyanin algebra; E elliptic curve $\subseteq \mathbb{P}^2$	✓	

Outlining rest of talk

Sketch of NSFQS for $An(k)$ | Sketch of NFQS for $An(k)$

— Assume $H \curvearrowright A$ is ~~inner~~ faithful ($\nexists$ Hopf ideal I of H s.t. $IA = 0$) —

Reduce modulo p $\quad p \gg 0$

- $\exists$ fin.-gen. subring $R \subseteq k$ such that $H_R \curvearrowright An(R)$ innerfaith'l (H_R: R-subalg of H)

  $[H = k \otimes_R H_R]$

  $\longrightarrow$ ✓ (same).

- $\exists$ homomorphism $\psi : R \to \overline{\mathbb{F}_p}$ to define $H_{\psi,p} = H_R \otimes_R \overline{\mathbb{F}_p} =: H_p$ & get $H_p \curvearrowright An(\overline{\mathbb{F}_p})$ innerfaith'l. (H_p ss, coss; $\uparrow$ PI domain)

  $\longrightarrow$ Replace with $H_{p^m} \curvearrowright An(W_{m,p})$, $H_{p^m} = H_R \otimes_R W_{m,p}$

  $W_{m,p}$ = m-th truncated ring of Witt vec/$\overline{\mathbb{F}_p}$ (not defined / fields, are $\mathbb{Z}/p^m\mathbb{Z}$-modules)

Reduce modulo p^m $\quad p \gg 0$

Localize: Hopf actions on Div. Algs

- Get $H_p \curvearrowright D_p$ (quot. div alg of $An(\overline{\mathbb{F}_p})$) innerfaithfully. ($H_p$ ss, coss.)
- Have $D_p = Z(D_p)\, D_p^{H_p}$ & $Z(D_p)$ is H_p-stab.

  PI degree argument … $\deg D_p = p^n$ for some n & $\gcd(p^n, \dim H) = 1$ for $p \gg 0$
- Take any Hopf ideal I of H_p s.t. $I\, Z(D_p) = 0$.

  $\Rightarrow I \cdot D_p = 0 \overset{\text{innerfaith.}}{\Rightarrow} I = 0$

  $\Rightarrow H_p \curvearrowright Z(D_p)$ a field, innerfaithfully

  $\Rightarrow H_p$ cocommutative by [EW2013]. $\longrightarrow$! fails for H non-ss

Localize: Hopf actions on Artinian rings

Get $H_{p^m} \curvearrowright D_{p^m}$ = full local'n of $An(W_{m,p})$

- Most technical part: get
  - $Z(D_{p^m})$ is H_{p^m}-stable
  - Z^{p^m} is H_p-stable

  $\Rightarrow H/\overline{\mathbb{F}_p}$ is cocom.

Pass back to characteristic 0

Com Alg $\Rightarrow$ dir product of all $\psi : R \hookrightarrow \prod$ fields (prod. of fields)

$\Rightarrow H_R$ cocom.

$\Rightarrow H$ cocom ✓

Pass back to ch 0

$\longrightarrow$ ✓ (same)

Now NSFQS

Replace $A_n(k)$ with

A "residually PI" $\stackrel{def}{=}$

- A $\mathbb{Z}_+$ filtered
- gr(A) com. fin. gen. domain
- A admits R-order A_R so that $A_p := A_R \otimes_R \overline{\mathbb{F}_p}$ is PI for $p \gg 0$

Main difference btw this case & result for $A = A_n(k)$:

Have to show that quotient division alg D_p has degree a power of p

(don't have explicit presentation of A)

→ Done in Etingof 1602.06480

have to control deg D_p

Examples of residually PI algebras

1. any filtered deformation of a com. domain generated in filtered degree one
   - $U(\mathfrak{g})$ $\mathfrak{g}$ fin. dim'l Lie alg
   - $D_\omega(X)$ twisted diff'l ops
2. finite W-algs → ✓
3. Spherical symplectic reflection alg → ✓
4. (Any of the above) ⊗ (com. fin. gen. domain) → ✓

Question Is the "PI reductions" vacuous?

1602.06480, Q1: Let char F = p > 0.

Take A a filt. deform'n of a com. f.g. domain/F.

Is A PI? for $p \gg 0$

NFQS

Replace $A_n(k)$ with a residually PI algebra A that is "nondegenerate"

$$\bigcap_{m \geq 1} \pi(Z(D_{p^m})) = \overline{\mathbb{F}_p} \quad \left[\pi : D_{p^m} \twoheadrightarrow D_p\right] \quad \text{for almost all } p.$$

When does nondegeneracy hold?

Recall gr(A) is a Poisson alg.

$\parallel$

$\mathcal{O}(X)$ X irreducible Poisson variety

A nondegenerate ⇐ X is "generically symplectic" (having a symplectic dense open subset)

Examples

II. Will only discuss NSFQS

Thm Take $A = k_q[x_1, \dots, x_n] = k\langle x_1, \dots, x_n\rangle / (x_i x_j - q_{ij} x_j x_i)$ $i<j$
for a multiplicatively antisym $(q_{ij}) \in \mathrm{Mat}_n(k^\times)$

Let $\begin{cases} G_q := \text{Zariski closure of } \langle q\rangle \subseteq (k^\times)^{\binom{n}{2}} \leftarrow \binom{n}{2}\text{-torus} \\ G_q^\circ := \text{connected component of the identity of } G_q \end{cases}$

Suppose that a semisimple Hopf algebra H acts on A.

no condition on H-action preserving grading of A

If the order of $\overbrace{G_q / G_q^\circ \text{ is coprime to } (\dim H)!}^{(*)}$,
then any H-action on A factors through group action.

Ex. $n=2$. $A = k_q[x,y] = k\langle x,y\rangle / (xy - q yx)$.

- If q = non-root of unity, then $G_q^\circ = G_q \Rightarrow (*)$ holds for any H
  $\Rightarrow$ NSFQS for any H.

Recall $\exists$ inner faithful actions of the Kac-Paljutkin Hopf algebra H_8 on $k_{-1}[x,y]$ & $k_i[x,y]$

- If $\mathrm{ord}(q) = r < \infty$, then $(*) \Leftrightarrow r$ is coprime to $(\dim H)!$
  get NSFQS when $(r, (\dim H)!) = 1$.

Ex. If each q_{ij} is a root of unity, say of order $r_{ij} < \infty$,
then $|G_q / G_q^\circ| = \mathrm{lcm}\{r_{ij}\}_{i \le j}$

Ex. $A = k\langle x,y,z\rangle / \begin{pmatrix} xy - \zeta_{11} yx \\ yz - \zeta_7 zy \\ zx - \zeta_{13} xz \end{pmatrix}$: for ζ_j a prim $\sqrt[j]{1}$

Get NSFQS for H so that $((\dim H)!, 7 \cdot 11 \cdot 13) = 1$
(e.g. s.s. nontriv. Hopf of dim $5 \cdot 3^2 = 45$ cannot act inner-faithfully)

Ex. If q_{ij} has ∞ order & the set $\{q_{ij}\}$ is multiplicatively independent,
then $G_q^\circ = G_q \Rightarrow (*)$ holds for any H

Sketch of proof

same ideas as NSFQS for $A_n(k)$ result .... with modifications

Reduce modulo p

$\exists$ fin. gen subring $R \subseteq k \ni$.

$H_R \curvearrowright R_q[x_1, \dots, x_n]$ inner-faithfully

$[H = k \otimes_R H_R]$

! not so straightforward that we can reduce (i.e. $\exists$ homomorphism $\psi: R \to \overline{\mathbb{F}_p}$ to make ... $\ni$.

.... need to control $\deg D_p$ ......

Get $H_p \curvearrowright A_p$

Hopf alg/$\overline{\mathbb{F}_p}$ reduction of $k_q[x_1, \dots, x_n]$ alg/$\overline{\mathbb{F}_p}$

Localize: Hopf actions on division algs

Get $H_p \curvearrowright D_p$ full localization of A_p inner-faithfully

$\gcd(\deg D_p, (\dim H_p)!) = 1$ for $p >> 0$

Get H_p cocommutative

Pass back to char=0

Get H is cocommutative

go through a number field $\mathbb{K}$.

Given homom. $\xi: R \to \mathbb{K}$, let $R' := \mathrm{im}\,\xi$

Get $H_{R'} \curvearrowright R'_{\xi(q)}[x_1, \dots, x_n]$

$H_{R'} = H_R \otimes_R R'$

Get $|G_q / G_q^{\circ}| = |G_{\xi(q)} / G^{\circ}_{\xi(q)}| =: l$

Number-theoretic result of Perucca:

Get ∞ many primes $p \ni$.

for any generic homomorphism $\psi': R' \to \overline{\mathbb{F}_p}$ annihilating a prime ideal $\mathfrak{p} \subseteq R'$ lying over p.

we have that

$N := \mathrm{ord}\, \psi'(\xi(q))$ is finite & coprime to $(\dim H)!$

$N = N_p$

since l is coprime to $(\dim H)!$ by hypothesis (see Corollary 7.2 of 1605.00560)

Now form $A_p = R'_{\xi(q)}[x_1, \dots, x_n] \otimes_{R'} \overline{\mathbb{F}_p}$ via the map ψ'

$\hookrightarrow$ get A_p is a PI domain & PI degree A_p divides N^n

Get similar NSFQS results for semisimple Hopf actions
on $B(X, \mathcal{L}, \sigma)$ thcr. & on $S(E, \mathcal{L}, \sigma)$ 3-dim'l Skly alg

X: abelian variety; $\sigma \in \mathrm{Aut}X$: translation by a point $s \in X$ according to group law

E: elliptic curve $\subseteq \mathbb{P}^2$; $\sigma \in \mathrm{Aut}E$: translation by a point

How so? Basically, use the same proof as for NSFQS on $k_q[x_1, \dots, x_n]$
just swap $(k^\times)^{\binom{n}{2}}$ $\longrightarrow$ $\begin{cases} X & \text{abelian variety} \\ E \end{cases}$

$\binom{n}{2}$-torus

$\langle q \rangle$ $\longrightarrow$ $\{\sigma^i(s)\}_{i \in \mathbb{Z}}$ σ-orbit of s

subgrp of parameters

Ex. Take H ss. of dim d. If $|\sigma|$ is coprime to $d!$ or infinite,
then any H-action on $S(E, \mathcal{L}, \sigma) = S(a,b,c) = \dfrac{k\langle x,y,z \rangle}{\begin{pmatrix} ayz + bzy + cx^2 \\ azx + bxz + cy^2 \\ axy + byx + cz^2 \end{pmatrix}}$

$a,b,c \in k^\times$

factors through a group action.

Ex. Not so useful in studying semisimple Hopf algs H that acts on $S(1,1,c)$
innerfaithfully. . . . . . .

For $S(1,1,c)$, σ is given by translation by a points of order 2.
So, $|\sigma| = 2$ is coprime to $(\dim H)!$ only in the case
when $\dim H = 1$....

Useful for S with $|\sigma|$ large

★ If $H^{\ast}A$ gives rise to a Hopf-Galois extension, then "coprime to $(\dim H)!$" can be replaced with "coprime to $\dim H$"

for all above