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Semisimple & finite dimensional Hopf actions on Weyl algebras

joint w/ Juan Cuadra and Pavel Etingof   arxiv: 1409.1644, 1509.01165.

Take $k = \bar{k}$, $\operatorname{ch} k = 0$.

(understanding group actions is even tough... later in Roundtable (Wang))

Big Goal To understand (finite-dim'l) Hopf alg actions / (finite) quantum symmetries of classes of (non) commutative algs.

Given k-alg A:

Say: A admits **No Quantum Symmetry** if all Hopf actions on A factor through cocom. Hopf actions

—OR—

A admits **genuine q.sym** if $\exists$ an inner-faithful noncocom Hopf action on A;

($\nexists$ $I \neq 0$ Hopf ideal of H so that there is an induced action of H/I on A)

[As usual, H acts on A $\iff$ A is a left H-module alg]

Concentrating on finite quantum symmetries, recall that there are 2 tractable classes of fin-dim'l Hopf algs.

→ **Semisimple** → have a No Quantum Symmetry result when A is a commutative domain:

→ pointed.

Theorem 0 [Etingof–W, 2013] Any semisimple Hopf action on a com. domain factors through a group action.

→ holds in positive characteristic if H is also cosemisimple.

We extend this result to semisimple Hopf actions on a quantization of commutative domains:

the n-th Weyl algebra $A := A_n(k) = \dfrac{k\langle x_i, y_i\rangle_{i=1}^{n}}{\left([x_i, x_j] = [y_i, y_j] = 0,\ [y_i, x_j] = \delta_{ij}\right)}$

(general program will be discussed later in Roundtable (Etingof).)

Theorem 1 [Cuadra–Etingof–W]
Any semisimple Hopf action on $A_n(k)$ factors through a group action.

proved earlier by Etingof–W if we assume that
- * $A_n(k)$ has the standard filtration
- * Hopf action preserves this filtration

we don't assume this here

Pf of Thm 1 follows in three steps (made precise in a minute)

ⓐ Pass to characteristic p ("reduce modulo p") for $p \gg 0$.
to get $H_p \curvearrowright A_p$ over $\overline{\mathbb{F}}_p$

Reason: A_p is now a PI domain, so can localize!

ⓑ Study $H_p \curvearrowright$ division alg in positive characteristic & get H_p cocom.

ⓒ Obtain result in char 0: H is cocommutative.

(further results on Hopf actions on div. algs discussed later in Roundtable (Cuadra))

ⓑ **Proposition**
* H_F semisimple, cosemisimple Hopf alg of dim d, over alg. closed field F of arbitrary characteristic.
* D_F division alg over F of degree N over $Z(D_F) =: Z$ admitting action of H_F.

Then: $\gcd(d!, N) = 1 \implies H_F \curvearrowright D_F$ factors through action of cocom. Hopf alg.

(dropping subscripts)

Proof: degree argument using $\gcd(d!, N) = 1$. $\Rightarrow$ $D = ZD^H$

$\Rightarrow$ $\mathrm{Cent}_D(D^H) = Z$

$\Rightarrow$ Z is H-stable [can show $\forall z \in Z, h \in H, a \in D^H$: $(h \cdot z)a = a(h \cdot z)$]

Assume $H \curvearrowright D$ innerfaithfully. Take Hopf ideal I of H so that $I \cdot z = 0$

$D = ZD^H \Rightarrow I \cdot D = 0$

$\overset{\text{innerfaith.}}{\Longrightarrow}$ $I = 0$.

(co) semisimplicity used here. $\rightsquigarrow$ [Thm 0] $\Rightarrow$ $H \curvearrowright Z$ (field) innerfaithfully $\Rightarrow$ H is cocommutative. !

Step ⓐ Reduction modulo p. (for $p >> 0$). (sketch).

* Given H (ss) $\curvearrowright A_n(k)$ innerfaithful, get $\exists$ fin.gen. subring R of k so that $H_R \curvearrowright A_n(R)$

H_R: Hopf R-order in H (R-subalgebra of H)

$\rightsquigarrow$ $H \simeq k \otimes_R H_R$

* $\exists$ homomorphism $\varphi: R \longrightarrow \overline{\mathbb{F}_p}$ for $p >> 0$, using this define $H_{\varphi,p} = H_R \otimes_R \overline{\mathbb{F}_p} =: H_p$

* Get H_p (ss, coss) $\curvearrowright A_n(\overline{\mathbb{F}_p})$ for $p >> 0$ & this action is innerfaithful

∴ Now <u>proof of Theorem 1</u>. Assume that $\underset{ss.}{H} \curvearrowright An(k)$ is inner faithful.

(a)
- * <u>Reduce modulo p</u> to get:
  $\underset{ss, coss.}{H_p} \curvearrowright An(\overline{\mathbb{F}_p})$ inner faithful for $p \gg 0$
  ↖ a PI domain, can localize.
- * Take D_p = quot. div. ring of $An(\overline{\mathbb{F}_p})$.
- * [Skryabin-van Oystaeyen] ⟹ $H_p \curvearrowright D_p$ inner faithfully.

(b)
- * Since $p \gg \dim H$ and $\deg D_p$ = power of p, apply <u>Prop</u> to get:
  $H_p = H_{\psi, p}$ is cocommutative for any homom $\psi: R \to \overline{\mathbb{F}_p}$

(c)
- * ComAlg ⟹ dir. prod of all such ψ is an injection of R into a prod. of fields.
- * ⟹ H_R is cocommutative
  ⟹ H cocom. & finite dim'l
  ⟹ $H \cong kG$, as desired. ///

In the same fashion, we also get that:

(*)

Semisimple Hopf actions on * $An(k[z_1, \ldots, z_s])$ $n \geq 0$ * quotient div. ring of $An(k[z_1, \ldots, z_s])$ factor through group actions

Recently, we established a stronger No Quantum Symmetry result:

Theorem 2 [CEW] Any finite-dim'l Hopf action on $A_n(k)$ factors through a group action.

(But now the remark (*) doesn't apply:
take $n=0$: $\exists$ finite dim'l (non ss) inner faithful Hopf actions on $k[z_1, \dots, z_s]$.)

How does the proof of Thm 2 work? Say we have $H \curvearrowright A_n(k)$ inner faithfully, H finite dim'l.

⚠ Cannot use Thm 0 [EW] — no effects step ⓑ.

New step ⓐ

Reduce modulo prime p for $p \gg 0$.

Get $H_p \curvearrowright A_n(\mathbb{F}_p)$ inner faithful

⇓

Get $H_p \curvearrowright D_p$ inner faithful

⇓ $[\mathrm{Cent}_D(D^H) = Z]$

Get $H_p \curvearrowright Z(D_p) = Z$ inner faithful (as in proof of proposition)

Also need to Reduce modulo prime powers p^m

Get $H_{p^m} \curvearrowright A_n(W_{m,p})$ inner faithful, where $H_{p^m} = H_R \otimes_R W_{m,p}$ ($\mathbb{Z}/p^m\mathbb{Z}$-modules), $W_{m,p}$ = m^{th} truncated ring of Witt vectors of $\mathbb{F}_p$

⇓

Get $H_{p^m} \curvearrowright D_{p^m}$ (full localization of $A_n(W_{m,p})$) inner faithfully

⇓ $[\mathrm{Cent}_{D_{p^m}}(D_{p^m}^{H^{\{p^m\}}}) = Z_m]$

Get $Z_m = Z(D_{p^m})$ is H_{p^m}-stable

⇓ [computation: image of Z_m in D_p is $Z^{p^{m-1}}$]

Get Z^{p^m} is H_p-stable

New step ⓑ

We do have: Thm [CEW] * Let H_F be a fin dim'l Hopf alg over alg closed field F of char p

* Z a finitely generated field ext'n of F
* $H_F \curvearrowright Z$ inner faithfully
* (New) H_F preserves Z^{p^m} for all $m \geq 1$

Then: $p > \dim H_F \implies H_F$ cocommutative

(Take $F = \overline{\mathbb{F}}_p$ & build from here)

Now use Step ⓒ from before to get that $H \cong kG$, as desired ///.