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Twisting Systems in Closed Monoidal Categories

by

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Abstract

Zhang twists are a tool for "deforming" the product of graded $\Bbbk$-algebras while preserving desirable ring-theoretic properties. As the study of algebras in monoidal categories beyond $\mathsf{Vec}_{\Bbbk}$ grows in popularity, it has become increasingly important to find ways of twisting algebras in monoidal categories. To this end, this thesis generalizes Zhang twists to the setting of closed monoidal categories equipped with a canonical enriched structure.

Along the way, we also prove several results concerning closed monoidal categories and algebraic structures within them. We use these results to provide necessary and sufficient conditions for when graded algebras connected by Zhang twists produce equivalent categories of graded modules.

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Agradezco a mis padres, por su amor y sacrificio. A mis hermanas, las únicas dos en el mundo capaces de entender las idiosincrasias de nuestra juventud. Invertimos tanto en nuestra educación, sin saber que, al fin del día, solamente aprenderíamos del uno al otro como ser una familia. Les amo muchísimo.

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Friends, old and new, but barely young: you are my life. I wouldn't still be in school if I didn't have more to learn. A large integer-valued number of thank you's, you're welcome's, I'm sorry's, and see you soon's. To my partner, it's hard to find the words to express the joy you bring me. As a coauthor in the next chapter of our lives, I'll rely on you to help me find those words. To Oreo and Maggie, it's also hard to find the words to express to you the joy you bring me. But that's because you're cats.

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Chapter 1

Introduction

In 1991, Artin, Tate, and van den Bergh [ATVdB91, Section 8] introduced the notion of "twisting" a graded $\Bbbk$-algebra A using successive powers of a graded automorphism τ to produce a new algebra A^τ. In 1996, J. Zhang [Zha96] abstracted this procedure and introduced twisting systems: *families* of graded automorphisms of A that also produce a new algebra when they satisfy a certain compatibility condition. Throughout the years, twisting systems have found use in:

- Non-commutative Projective Geometry (e.g. in the study of noncommutative surfaces in [VdB01]),
- Non-commutative Algebra (e.g. in the study of degenerate Sklyanin algebras in [Wal09]),
- Quantum Groups (e.g. the study of twists of Manin's universal quantum group in [HNU$^+$22]), and
- Graded Morita Theory (e.g. in [Sie11]'s incorporation of twisting systems into the graded Morita theory developed in [Rio91]).

This work extends [Zha96]'s results to the class of closed monoidal categories, which are themselves ubiquitous and of wide interest.

1.1 Historical background

1.1.1 Artin-Schelter Regular Algebras

The polynomial algebras $\Bbbk[x_1, \ldots, x_n]$ are among the first objects of study in algebraic geometry and commutative algebra. As interest in non-commutative algebras grew in the 1980s, a natural question to ask became: which algebras serve as the non-commutative analogues to the polynomial algebras?

Artin-Schelter regular algebras arose as one answer to the above question. These were developed through the work of Artin and Schelter [AS87], while finding analogues of the polynomial algebras that shared in certain homological properties. These ideas were further developed in the work of Artin, van den Bergh, and Zhang [AdB90, AZ94], setting much groundwork for modern non-commutative projective geometry. The key insight here was that, graded commutative algebras A could be studied geometrically via their associated projective scheme $\mathsf{Proj}A$. The geometry of $\mathsf{Proj}A$, in turn, could be studied via a quotient of the category of graded modules of A. This latter category could be defined for certain classes of non-commutative algebras, thus allowing for a definition of $\mathsf{Proj}A$ in cases where A was non-commutative and graded.

While trying to characterize Artin-Schelter regular algebras constructed from automorphisms of certain elliptic curves, Artin, Tate, and van den Bergh [ATVdB91] introduced the notion of "twisting" a graded algebra A by an automorphism. Namely, if τ is a graded automorphism of A, then A admits a new multiplication $*$ using the rule:

$$a * b := a \cdot \tau^d(b)$$

for elements $a, b \in A$ where $\deg(a) = d$.

1.1.2 Twisting Systems

One issue with the twisting construction introduced in [ATVdB91] was that it did not define an equivalence relation on the collection of $\mathbb{N}$-graded $\Bbbk$-algebras modulo isomorphism. This made it hard to use in classification problems. One could instead try looking at families of graded automorphisms $\{\tau_n : A \to A\}_{n\in\mathbb{N}}$ of A, but the corresponding twisted product:

$$a * b := a \cdot \tau_m(b) \qquad \text{if } \deg(a) = m,$$

in general failed to be associative without any conditions helping relate the different automorphisms. In [Zha96], Zhang formulated the necessary condition to ensure associativity of the twisted products, namely:

$$\tau_n(a \cdot \tau_m(z)) = \tau_n(a) \cdot \tau_{n+m}(b),$$

for all $n, m, \ell \in \mathbb{N}$ with $\deg(a) = m$ and $\deg(b) = \ell$. Furthermore, he proved that algebras related by such twists shared in several ring-theoretic properties, and, under some mild conditions, also possessed equivalent categories of graded modules*.

In [Sie11], Susan Sierra expanded on Zhang's results and unified them with "classical" graded Morita theory. In particular, to graded algebras A and B, Sierra associated new "matrix algebras" $\bar{A}$ and $\bar{B}$, and showed that:

1. A and B are graded Morita equivalent if and only if $\bar{A} \cong \bar{B}$; and
2. $\bar{A} \cong \bar{B}$ if and only if B is isomorphic to a twist of A.

*These categories of graded modules only considered morphisms in degree 0, i.e. degree preserving morphisms.

1.2 Problem description

The broad objective of this thesis is to generalize the results of Zhang's original paper [Zha96] to categories beyond $\mathsf{Vec}_{\Bbbk}$.

To do this, we need our categories to have a notion of multiplication, so that we may define algebras in these categories. This serves to first limit our scope to *monoidal categories*. Recall that a monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ serving as a multiplication, and a distinguished object $\mathbb{1} \in \mathcal{C}$ serving as the unit of multiplication, all subject to axioms guaranteeing that the associativity of $\otimes$ and the unitality of $\mathbb{1}$ hold up to unique isomorphism.

The next insight comes from noticing that [Zha96, Theorem 3.3] requires an isomorphism between a graded $\Bbbk$-algebra A and an algebra constructed using hom-spaces between vector spaces. When $\mathcal{C} = \mathsf{Vec}_{\Bbbk}$, this is not an issue since for any $V, W \in \mathsf{Vec}_{\Bbbk}$, we can naturally think of $\mathrm{Hom}_{\mathsf{Vec}_{\Bbbk}}(V, W)$ as a $\Bbbk$-vector space. However, in the general case, we need a way of ensuring that we may speak of hom-spaces as if they were objects in the category. This leads to the notion of an *enriched category*. Recall that $\mathcal{C}$-category (or category enriched over $\mathcal{C}$) is a generalization of the concept of an ordinary category, wherein we designate objects in $\mathcal{C}$ to serve as hom-spaces between objects. In particular, in analogy with the case when $\mathcal{C} = \mathsf{Vec}_{\Bbbk}$, we seek categories that are "enriched over themselves", so that we may view both the objects and the hom-spaces as objects of the same category.

The next issue comes from realizing that our algebras have to be *graded*. A simple solution is to switch from the category $\mathcal{C}$ to some product category $\mathcal{D} = (\mathcal{C} \times \mathcal{C} \times \cdots)$. This shift allows us to view the objects in $\mathcal{D}$ as tuples whose pieces make up the degrees of a graded object. The price for this change, however, is that we now have to specify a monoidal structure for $\mathcal{D} = (\mathcal{C} \times \mathcal{C} \times \cdots)$. These various hurdles lead us to *right-closed* *monoidal categories.* A right-closed monoidal category is a monoidal category $\mathcal{C}$ for which the "tensor-hom adjunction" holds (i.e. for each object $X \in \mathcal{C}$, the functor $(- \otimes X)$ has a right adjoint). By analogy, the adjoint to the functor $(- \otimes X)$ may be thought of as the covariant hom-functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$, which produces an object in $\mathcal{C}$ when applied to any object. Thus, right-closed monoidal categories admit a "canonical" structure of a category enriched over themselves. Finally, under some mild algebraic conditions, we can show that the product categories $\mathcal{D} = (\mathcal{C} \times \mathcal{C} \cdots)$ are also right-closed. This allows us to discuss graded algebras and modules in $\mathcal{D}$, and generalize several of [Zha96]'s results to this new setting.

1.3 Main results

Though the main goal of this thesis is to generalize Zhang's results in [Zha96] to the setting of closed categories, we prove several results about closed, enriched, and graded categories along the way, which are of independent interest. These are based on recent work of the author with Walton in [LW25].

The first couple results pertain to closed-categories;

Theorem 1.3.1 (Theorem 3.2.3)**.** *If $\mathcal{C}$ is a right-closed monoidal category with certain limits, then for any algebra $A \in \mathsf{Alg}(\mathcal{C})$, the category $\mathsf{Mod}(\mathcal{C})_A$ of A-modules in $\mathcal{C}$ admits an enrichment over $\mathcal{C}$.* □

The second result categorifies the notion that A-linear endomorphisms $A \to A$ are fully specified by where they map the unit, thus defining a correspondence between such endomorphisms and elements of A.

Theorem 1.3.2 (Theorem 3.3.3)**.** *If $\mathcal{C}$ is a right-closed monoidal category with certain limits, then every algebra $A \in \mathsf{Alg}(\mathcal{C})$ is isomorphic to the algebra of A-linear endomorphisms of A.* □

Next, we move on to graded categories. As stated before, our approach is to define graded algebraic structures in $\mathcal{C}$ using tuples of elements $(A_0, A_1, \cdots) \in (\mathcal{C} \times \mathcal{C} \cdots)$. This allows us to easily define graded algebras and modules by defining products and actions degree by degree. For example, the multiplication of a graded algebra A is defined as a family of maps $A_m \otimes A_n \to A_{mn}$ parametrized by pairs (m, n) of non-negative integers. These graded algebras and graded modules form categories which we denote by $\mathsf{GrAlg}(\mathcal{C})$ and $\mathsf{GrMod}(\mathcal{C})_A$, respectively.

Although our approach allows us to define graded structures in terms of relatively simple components, it comes at the expense of losing the "traditional" setting of studying algebras and modules in monoidal categories. The next result remedies this issue.

Theorem 1.3.3 (Theorem 4.2.6)**.** *Let $\mathcal{C}$ be a right-closed category satisfying Hypotheses 4.0.1. Then there is a monoidal structure on $\mathcal{D} = (\mathcal{C} \times \mathcal{C} \cdots)$ such that:*

(a) $\mathsf{GrAlg}(\mathcal{C}) \cong \mathsf{Alg}(\mathcal{D})$, *and*

(b) *for any graded algebra $A \in \mathsf{GrAlg}(\mathcal{C})$, we get that $\mathsf{GrMod}(\mathcal{C})_A \cong \mathsf{Mod}(\mathcal{D})_A$.* □

The next result ties together closed categories and graded categories.

Proposition 1.3.4 (Proposition 4.2.2)**.** *Let $\mathcal{C}$ be a right-closed category satisfying Hypotheses 4.0.1. The monoidal structure on $\mathcal{D} = \mathcal{C} \times \mathcal{C} \cdots$ from Theorem 1.3.3 makes $\mathcal{D}$ a right-closed monoidal category.* □

In particular, the proposition above allows us to prove statements about the product category $\mathcal{D}$ through its right-closed or self-enriched structures, without having to deal with graded objects.

Finally, we extend and prove several of Zhang's [Zha96] definitions and results to the setting of closed categories.

Definition-Proposition 1.3.5 (Definition 5.1.1, Proposition 5.1.4)**.** Given a graded algebra $A \in \mathsf{GrAlg}(\mathcal{C})$, a *twisting system* on A consists of a family of graded isomorphisms $\tau = \{\tau_n : A \xrightarrow{\cong} A\}$ satisfying a *twisting condition*. Every twisting system on A induces a new algebra structure on A, obtained by "twisting" the multiplication of A using τ. We denote this new algebra by A^τ, and call it a *twist* of A. □

Finally, we explore several graded-Morita type results about algebras related by twists. To do this, we first introduce two ways of comparing graded algebras;

Definition 1.3.6 (Definition 5.2.1)**.** Two graded algebras A, B are *twist equivalent* if there is a twisting system τ on A making $B \cong A^\tau$.

Definition 1.3.7. Two graded algebras $A, B \in \mathsf{GrAlg}(\mathcal{C})$ are *Zhang-Morita equivalent* if there is an equivalence of categories $\mathsf{GrMod}(\mathcal{C})_A \simeq \mathsf{GrMod}(\mathcal{C})_B$.

The two main results are as follows.

Theorem 1.3.8 (Theorem 5.3.1)**.** *Twist equivalent algebras are Zhang-Morita equivalent. In fact, the categories* $\mathsf{GrMod}(\mathcal{C})_A$ *and* $\mathsf{GrMod}(\mathcal{C})_B$ *are isomorphic.* □

Theorem 1.3.9 (Corollary 5.3.5)**.** *Zhang-Morita equivalent algebras* $A, B \in \mathsf{GrAlg}(\mathcal{C})$ *are twist equivalent, provided that the corresponding equivalence of categories maps "shifts" of* A *to "shifts" of* B. □

1.4 Organization of the thesis

In Chapter 2, we provide background material on several categorical notions, including: additive categories, adjoint functors, monoidal categories, algebras and modules in monoidal categories, and enriched categories.

Chapter 3 is focused on introducing closed categories and proving general results about algebras coming from such categories. In particular, we prove Theorem 1.3.1 and Theorem 1.3.2.

Chapter 4 focuses on graded versions of our categories $\mathcal{C}$, and explore when these categories inherit monoidal, self-enriched, and right-closed structures of $\mathcal{C}$. In particular, we prove Proposition 1.3.4 and Theorem 1.3.3

Chapter 5 focuses on extending the results in [Zha96] to the closed monoidal setting. In particular, we prove Theorem 1.3.8 and Theorem 1.3.9.

Finally, we provide an index of notation and references.

Chapter 2

Background

In this chapter, we recall background material on monoidal and enriched categories. In Section 2.1, we review categorical notions that will be used throughout. Next, in Section 2.2 we discuss monoidal categories and algebraic structures within them. Finally, in Section 2.3, we give a primer on enriched categories.

2.1 Preliminaries

We assume the reader is familiar with standard terminology from category theory (categories, functors, natural transformations, equivalences, etc.) and algebra (rings, modules, algebras, graded algebras, additive and abelian categories, etc.). We present most of the relevant definitions here, but refer the reader to [ML13, Rie17, Wal24] for further details.

We begin by establishing categorical notation we will use throughout.

Notation 2.1.1. Categories will be denoted using calligraphic script ($\mathcal{C}, \mathcal{D}, \dots$). Objects X, Y in a category $\mathcal{C}$ will be denoted by $X, Y \in \mathcal{C}$. The identity morphism of $X \in \mathcal{C}$ will be denoted $1_X : X \to X$. Finally, the hom-set between two objects $X, Y \in \mathcal{C}$ will be denoted by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Next, we establish definitions and notation regarding certain (co)limits in categories.

Notation 2.1.2. Let $\{X_i\}_{i \in I}$ be objects in a category $\mathcal{C}$.

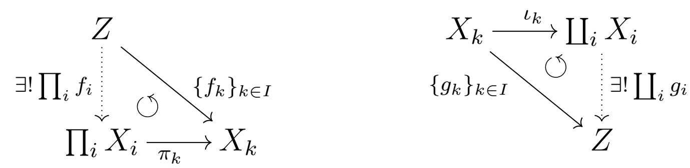


Figure 1 : Universal property of products and coproducts.

(a) We will denote the product of $\{X_i\}_{i\in I}$ by $\prod_i X_i$. It comes with *projection maps* $\{\pi_k : \prod_i X_i \to X_k\}_{k\in I}$. The universal property of $\prod_i X_i$ states that objects $Z \in \mathcal{C}$ equipped with morphisms $\{f_k : Z \to X_k\}_{k\in I}$ are in 1-1 correspondence with morphisms $\prod_i f_i : Z \to \prod_i X_i$ satisfying $f_k = (\pi_k \circ \prod_i f_i)$ for all $k \in I$, as shown in the left diagram of Figure 1.

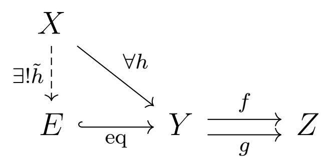
(b) We will denote the coproduct of $\{X_i\}_{i\in I}$ by $\coprod_i X_i$. It comes with *inclusion maps* $\{\iota_k : X_k \to \coprod_i X_i\}_{k\in I}$. The universal property of $\coprod_i X_i$ states that objects $Z \in \mathcal{C}$ equipped with morphisms $\{g_k : X_k \to Z\}_{k\in I}$ are in 1-1 correspondence with morphisms $\coprod_i g_i : \coprod_i X_i \to Z$ satisfying $g_k = (\coprod_i g_i \circ \iota_k)$ for all $k \in I$, as shown in the right diagram of Figure 1.

(c) If $\mathcal{C}$ has a zero object 0, there is a canonical map $\omega : \coprod_i X_i \to \prod_i X_i$ arising from combined universal properties of the product and coproduct. If this map is an isomorphism, we identify $\coprod_i X_i$ and $\prod_i X_i$, denote them as $\bigoplus_i X_i$, and call the result the *biproduct* of $\{X_i\}_{i\in I}$.

Finally, we introduce the notion of equalizers for pairs of morphisms.

Definition 2.1.3. Given morphisms $f, g : Y \to Z$ in a category $\mathcal{C}$, we say $h : X \to Y$ **equalizes** f and g if $fh = gh$. An **equalizer** for f and g consists of an object $E \in \mathcal{C}$ and a (mono)morphism $\mathrm{eq} : E \to Y$, such that every morphism equalizing f and g factors uniquely through eq. In other words, for any morphism $h : X \to Y$ satisfying

$fh = gh$, there exists a unique morphism $\tilde{h} : X \to E$ such that $h = \mathrm{eq} \circ \tilde{h}$.



Most of the categories we will be focusing on fall under the following type.

Definition 2.1.4. A category $\mathcal{C}$ is **pre-additive** if:

(a) each hom-set has the structure of an abelian group $(\mathrm{Hom}_{\mathcal{C}}(X, Y), +, 0)$,

(b) such that composition of morphisms distributes over addition, i.e.

$$g \circ (f + f') = (g \circ f) + (g \circ f') \quad \text{and} \quad (g' + g) \circ f = (g' \circ f) + (g \circ f),$$

for all $X, Y, Z \in \mathcal{C}$, $f, f' \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, and $g, g' \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$.

We call $\mathcal{C}$ **additive** if it is pre-additive and admits finite products and a zero object.

Remark 2.1.5. The following are standard results for which we omit proofs.

- In pre-additive categories, finite products and coproducts are biproducts. In these cases, the map ω from Notation 2.1.2(c) is an isomorphism.
- If a category has equalizers and arbitrary (resp. finite) products, then it has arbitrary (resp. finite) limits.

Example 2.1.6. The categories Ab and R-Mod of abelian groups and R-modules over a ring R are additive. In particular, so is the category $\mathsf{Vec}_{\Bbbk}$ of vector spaces over a field $\Bbbk$. In all these cases, the hom-sets form abelian groups under function addition.

Finally, we recall and establish notation for the notion of adjoint functors.

Definition 2.1.7. Two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are **adjoint** if there exist natural isomorphisms $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow 1_{\mathcal{D}}$ satisfying the *triangle identities*:

$$\text{Triangle Identity 1:} \qquad 1_F = \epsilon F \circ F\eta, \tag{2.1.1}$$

$$\text{Triangle identity 2:} \qquad 1_G = G\epsilon \circ \eta G. \tag{2.1.2}$$

We call η the *unit* of the adjunction and ϵ the *counit* of the adjunction. We say F is *left adjoint* to G and that G is *right adjoint* to F, and write $F \dashv G$.

The next proposition gives an alternative characterization of adjoint functors.

Proposition 2.1.8. *Functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are adjoint if and only if for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, we have natural bijections between hom-sets:*

$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, GY).$$

In terms of the unit and counit, the bijections above are given by the formulas:

$$(\psi : FX \to Y) \ \mapsto \ (\psi^\flat : X \xrightarrow{\eta_X} GFX \xrightarrow{G\psi} GY) \tag{2.1.3}$$

$$(\varphi^\# : FX \xrightarrow{F\varphi} FGY \xrightarrow{\epsilon_Y} Y) \ \leftarrow\!\shortmid \ (\varphi : X \to GY). \tag{2.1.4}$$

*We call $\varphi^\flat$ the **right transpose** of φ and call $\psi^\sharp$ the **left transpose** of ψ.*

Finally, the following minor result connects the notion of equalizers and adjoint functors. It will be used in Section 3.2.

Lemma 2.1.9. *If $F \dashv G$ are adjoint, then $h : X \to Y$ equalizes $f, g : Y \to GZ$ if and only if $F(h) : FX \to FY$ equalizes $f^\sharp, g^\sharp : FY \to Z$.*

Proof. Notice $fh = gh$ if and only if $(fh)^\sharp = (gh)^\sharp$. Using formula (2.1.4), the latter can be expressed as $\epsilon_Z \, Ff \, Fh = \epsilon_Z \, Fg \, Fh$. Using (2.1.4) again gives $f^\sharp \, Fh = g^\sharp \, Fh$. □

2.2 Monoidal categories

In this section, we review monoidal categories, along with algebras and modules in them. We refer the reader to [EGNO15, Chapter 2,7][Wal24, Chapter 3,4] for more details.

Definition 2.2.1. A **monoidal category** consists of a category $\mathcal{C}$, a choice of object $\mathbb{1} \in \mathcal{C}$ called the *unit object*, and a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product*, equipped with the following structure morphisms for all $X, Y, Z \in \mathcal{C}$:

- a natural isomorphism $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ called the *associator*,
- a natural isomorphism $\ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X$ called the *left unitor*, and
- a natural isomorphism $r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X$ called the *right unitor*,

satisfying coherence conditions called the pentagon and triangle axioms. We call $\mathcal{C}$ **strict** if the components of its associator and unitors are identity maps.

Notation 2.2.2. For the remainder of this section, $\mathcal{C}, \mathcal{D}$ denote monoidal categories

Definition 2.2.3. The bifunctoriality of the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ implies that for morphisms $f : X \to X'$ and $g : Y \to Y'$ in $\mathcal{C}$, we have the following equality:

$$(f \otimes g) := (f \otimes 1_{Y'})(1_X \otimes g) = (1_{X'} \otimes g)(f \otimes 1_Y). \tag{2.2.1}$$

We call this property **level exchange**.

Definition 2.2.4. A **monoidal functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$, a natural transformation $F^2_{X,Y} : F(X) \otimes_{\mathcal{D}} F(Y) \to F(X \otimes_{\mathcal{C}} Y)$, and a morphism $F^0 : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$ in $\mathcal{D}$, that satisfy associativity and unitality constraints. If $\mathcal{C}$ and $\mathcal{D}$ are strict, these amount to:

$$\text{Associativity:} \qquad F^2_{X,Y\otimes Z}(1_{FX} \otimes F^2_{Y,Z}) = F^2_{X\otimes Y,Z}(F^2_{X,Y} \otimes 1_{FZ}), \tag{2.2.2}$$

$$\text{Unitality:} \qquad F^2_{\mathbb{1},X}(F^0 \otimes 1_{FX}) = 1_{FX} = F^2_{X,\mathbb{1}}(1_{FX} \otimes F^0), \tag{2.2.3}$$

for all $X, Y, Z \in \mathcal{C}$. We refer to F^2 and F^0, respectively, as the *binary* and *nullary* components of F. We call F **strict** if $F^2_{X,Y}$ and F^0 are identity maps for all $X, Y \in \mathcal{C}$. Two monoidal categories are **monoidally equivalent** if there exists a monoidal functor between them, whose underlying functor is an equivalence of categories.

Remark 2.2.5. MacLane's strictness theorem states that every monoidal category is monoidally equivalent to a strict monoidal category. Therefore, we may assume without loss of generality that all our monoidal categories are strict.

Next we define algebras and modules in monoidal categories.

Definition 2.2.6. An **algebra** (A, m, u) in $\mathcal{C}$ consists of an object A in $\mathcal{C}$, a morphism $m : A \otimes A \to A$ in $\mathcal{C}$ called the *multiplication* of A, and a morphism $u : \mathbb{1} \to A$ in $\mathcal{C}$ called the *unit* of A. The structure morphisms of A are required to satisfy:

$$\text{Associativity:} \qquad m(m \otimes 1_A) = m(1_A \otimes m), \tag{2.2.4}$$

$$\text{Unitality:} \qquad m(u \otimes 1_A) = 1_A = m(1_A \otimes u). \tag{2.2.5}$$

Given algebras (A, m_A, u_A) and (B, m_B, u_B) in $\mathcal{C}$, an **algebra (iso)morphism** from A to B consists of an (iso)morphism $\varphi : A \to B$ in $\mathcal{C}$ that is multiplicative and unital, that is, $\varphi m_A = m_B(\varphi \otimes \varphi)$ and $\varphi u_A = u_B$. The algebras in $\mathcal{C}$ and their morphisms form a category $\mathsf{Alg}(\mathcal{C})$.

Definition 2.2.7. Given an algebra $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$, a **right A-module** (M, ρ) in $\mathcal{C}$ consists of an object M in $\mathcal{C}$, along with a morphism $\rho : M \otimes A \to M$ in $\mathcal{C}$ called the *action* of A on M. The action is required to satisfy:

$$\text{Associativity:} \qquad \rho(\rho \otimes 1_A) = \rho(1_M \otimes m), \tag{2.2.6}$$

$$\text{Unitality:} \qquad \rho(1_A \otimes u) = 1_M. \tag{2.2.7}$$

Given right A-modules (M, ρ_M) and (N, ρ_N) in $\mathcal{C}$, an **A-module (iso)morphism** from M to N consists of an (iso)morphism $\varphi : M \to N$ in $\mathcal{C}$ such that $\rho_N(\varphi \otimes 1_A) = \varphi\rho_M$. The right A-modules in $\mathcal{C}$ and their morphisms form a category $\mathsf{Mod}(\mathcal{C})_A$.

Example 2.2.8 (Regular right A-module)**.** For every algebra $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$, the multiplication map defines an action of A on itself. We call $(A, m) \in \mathsf{Mod}(\mathcal{C})_A$ the **regular right A-module.**

Monoidal functors transport algebras and modules, and induce functors between corresponding categories of algebras or modules.

Proposition 2.2.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor and $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$.*

(a) *Then $F(A) \in \mathsf{Alg}(\mathcal{D})$ when equipped with the structure morphisms:*

$$\text{Multiplication:} \qquad F(A) \otimes F(A) \xrightarrow{F^2_{A,A}} F(A \otimes A) \xrightarrow{F(m)} F(A), \tag{2.2.8}$$

$$\text{Unit:} \qquad \mathbb{1}_{\mathcal{D}} \xrightarrow{F^0} F(\mathbb{1}_{\mathcal{C}}) \xrightarrow{F(u)} F(A). \tag{2.2.9}$$

The assignment $A \mapsto F(A)$ defines a functor $F : \mathsf{Alg}(\mathcal{C}) \to \mathsf{Alg}(\mathcal{D})$.

(b) *If $(M, \rho) \in \mathsf{Mod}(\mathcal{C})_A$ then $F(M) \in \mathsf{Mod}(\mathcal{D})_{F(A)}$ via the action:*

$$F(M) \otimes F(A) \xrightarrow{F^2_{M,M}} F(M \otimes A) \xrightarrow{F(\rho)} F(A). \tag{2.2.10}$$

The assignment $M \mapsto F(M)$ defines a functor $F : \mathsf{Mod}(\mathcal{C})_A \to \mathsf{Mod}(\mathcal{D})_{F(A)}$. □

Proof. (a) Since the assignment $F : \mathcal{C} \to \mathcal{D}$ is a functorial, so is $F : \mathsf{Alg}(\mathcal{C}) \to \mathsf{Alg}(\mathcal{D})$. We verify associativity of $F(A)$ below:

$$\begin{aligned} m^{F(A)}(m^{F(A)} \otimes 1_{F(A)}) &= Fm \circ F^2_{A,A} \circ (Fm\, F^2_{A,A} \otimes 1_{F(A)}) \\ &= F\left(m(m \otimes 1_A)\right) \circ F^2_{A,A} \circ (F^2_{A,A} \otimes 1_{F(A)}) \\ &= F\left(m(1_A \otimes m)\right) \circ F^2_{A,A} \circ (1_{F(A)} \otimes F^2_{A,A}) \\ &= Fm \circ F^2_{A,A} \circ (1_{F(A)} \otimes Fm\, F^2_{A,A}) \\ &= m^{F(A)}(1_{F(A)} \otimes m^{F(A)}). \end{aligned}$$

Here the first and last equalities follow from definitions; the second and fourth from naturality of F^2; and the third from associativity (2.2.2) and (2.2.4) of F^2 and m. Next, we verify left unitality of $F(A)$:

$$\begin{aligned} m^{F(A)}(u^{F(A)} \otimes 1_{F(A)}) &= Fm \circ F^2_{A,A}(Fu\, F^0 \otimes 1_{F(A)}) \\ &= F(m(u \otimes 1_A))F^2_{A,A}(F^0 \otimes 1_{F(A)}) \\ &= 1_{F(A)}. \end{aligned}$$

Here the first equality follows from definitions; the second from naturality of F^2; and the third from unitality (2.2.3) and (2.2.5) of F^2 and m. The proof of right unitality is analogous, and the proof of (b) is also analogous to the proof of (a). □

2.3 Enriched categories

In this section, we review the basic definitions and results involving enriched categories. We refer the reader to [Kel82][Wal24, Section 3.11] for further reading.

Definition 2.3.1. Fix a monoidal category $\mathcal{C}$. A **$\mathcal{C}$-category** consists of a class of *objects* $\mathcal{E}$ equipped with the following structure:

- for pairs $X, Y \in \mathcal{E}$, a *Hom-object* $\mathcal{C}(X,Y)$ in $\mathcal{C}$,
- for each $X \in \mathcal{E}$, an *identity morphism* $\mathrm{id}_X : \mathbb{1}_{\mathcal{C}} \to \mathcal{C}(X,X)$ in $\mathcal{C}$, and
- for all triples $X, Y, Z \in \mathcal{E}$, a *composition morphism* in $\mathcal{C}$:

$$c_{X,Y,Z} : \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \to \mathcal{C}(X,Z).$$

The structure morphisms are subject to the following constraints for all $W, X, Y, Z \in \mathcal{E}$:

$$\text{Associativity:} \quad c_{W,X,Z}(c_{X,Y,Z} \otimes 1_{\mathcal{C}(W,X)}) \;=\; c_{W,Y,Z}(1_{\mathcal{C}(Y,Z)} \otimes c_{W,X,Y}), \tag{2.3.1}$$

$$\text{Left Identity:} \quad c_{X,Y,Y}(\mathrm{id}_Y \otimes 1_{\mathcal{C}(X,Y)}) \;=\; 1_{\mathcal{C}(X,Y)}, \tag{2.3.2}$$

$$\text{Right Identity:} \quad c_{X,X,Y}(1_{\mathcal{C}(X,Y)} \otimes \mathrm{id}_X) \;=\; 1_{\mathcal{C}(X,Y)}. \tag{2.3.3}$$

We also refer to $\mathcal{C}$-categories as **$\mathcal{C}$-enriched categories** or **categories enriched over $\mathcal{C}$**.

Notation 2.3.2. Since we will deal with multiple categories and enriched categories, we establish the following pieces of clarifying notation:

(a) When emphasizing the $\mathcal{C}$-enriched structure on $\mathcal{E}$, we write $\mathcal{E}^{[\mathcal{C}]}$ instead of $\mathcal{E}$.

(b) Identity morphisms in ordinary categories will be denoted by $1_X : X \to X$, while identity morphisms in the enriched setting will be denoted by $\mathrm{id}_X : \mathbb{1} \to \mathcal{C}(X,X)$.

Example 2.3.3. Recalling Definition 2.1.4, notice that every pre-additive category $\mathcal{E}$ has the structure of a category enriched over Ab, the category of abelian groups. Indeed, the hom-object between $X, Y \in \mathcal{E}$ can be chosen to be the hom-set $\mathrm{Hom}_{\mathcal{E}}(X,Y)$ equipped with its pre-additive abelian group structure. Continuing on Example 2.1.6, this means Ab and R-Mod are both enriched over Ab.

Next, we show how enriched categories give rise to ordinary categories.

Proposition 2.3.4. *Let $\mathcal{E}$ be a $\mathcal{C}$-category. There is a category $\mathcal{E}_0$ with:*

- *objects being the objects of $\mathcal{E}$,*
- *one morphism $f : X \to Y$ in $\mathcal{E}_0$ for each morphism $f : \mathbb{1}_{\mathcal{C}} \to \mathcal{C}(X, Y)$ in $\mathcal{C}$,*
- *the composition of $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ in $\mathcal{E}_0$ being defined as the morphism corresponding to $\mathbb{1}_{\mathcal{C}} \cong \mathbb{1}_{\mathcal{C}} \otimes \mathbb{1}_{\mathcal{C}} \xrightarrow{g \otimes f} \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \xrightarrow{c} \mathcal{C}(X, Z)$ in $\mathcal{E}$, and*
- *the identity morphism $1_X : X \to X$ of X in $\mathcal{E}_0$ being defined as the morphism corresponding to $\mathrm{id}_X : \mathbb{1}_{\mathcal{C}} \to \mathcal{C}(X, X)$ in $\mathcal{E}$.*

*We call $\mathcal{E}_0$ the **underlying category** of $\mathcal{E}$.*

Proof. Composition of morphisms in $\mathcal{C}_0$ is associative by (2.3.1). The fact that id_X is the identity morphism of X in $\mathcal{C}_0$ follows from (2.3.2) and (2.3.3). □

Monoidal functors transport enrichments, as shown in the following lemma.

Lemma 2.3.5. *Let $\mathcal{E}$ be a $\mathcal{C}$-category and let $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor. Then, F induces on $\mathcal{E}$ the structure of a $\mathcal{D}$-category as follows:*

- *the Hom-object from X to Y is $\mathcal{D}(FX, FY) := F(\mathcal{C}(X, Y))$,*
- *the identity morphism of $X \in \mathcal{E}$ is $\mathrm{id}_X^{\mathcal{D}} : \mathbb{1}_{\mathcal{D}} \xrightarrow{F^0} F(\mathbb{1}_{\mathcal{C}}) \xrightarrow{F(\mathrm{id}_X^{\mathcal{C}})} F(\mathcal{C}(X, X))$,*
- *the composition morphism $c_{X,Y,Z}^{\mathcal{D}}$ of $X, Y, Z \in \mathcal{E}$ is:*

$$F(\mathcal{C}(Y, Z)) \otimes F(\mathcal{C}(X, Y)) \xrightarrow{F^2_{\bullet,\bullet}} F(\mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y)) \xrightarrow{F(c^{\mathcal{C}}_{X,Y,Z})} F(\mathcal{C}(X, Z)).$$

*We call the $\mathcal{D}$-category structure on $\mathcal{E}$ the **change of base** of $\mathcal{E}$ along F, and denote it by $\mathcal{E}^{[F:\mathcal{C}\to\mathcal{D}]}$.*

Proof. We verify associativity of the composition morphisms below:

$$\begin{aligned}
c^{\mathcal{D}}_{W,X,Z}(c^{\mathcal{D}}_{X,Y,Z} \otimes 1_{\mathcal{D}(W,X)}) &= F c^{\mathcal{C}}_{W,X,Z} \circ F^2 \circ (F c^{\mathcal{C}}_{X,Y,Z}\, F^2 \otimes 1_{F(\mathcal{C}(W,X)}) \\
&= F(c^{\mathcal{C}}_{W,X,Z}(c^{\mathcal{C}}_{X,Y,Z} \otimes 1_{\mathcal{C}(W,X)})) \circ F^2 \circ (F^2 \otimes 1_{F(\mathcal{C}(W,X))}) \\
&= F(c^{\mathcal{C}}_{W,Y,Z}(1_{\mathcal{C}(Y,Z)} \otimes c^{\mathcal{C}}_{W,X,Y})) \circ F^2 \circ (1_{F(\mathcal{C}(Y,Z))} \otimes F^2) \\
&= F c^{\mathcal{C}}_{W,Y,Z} \circ F^2 \circ (1_{\mathcal{D}(Y,Z)} \otimes F c^{\mathcal{C}}_{W,X,Y} F^2) \\
&= c^{\mathcal{D}}_{W,Y,Z}(1_{\mathcal{D}(Y,Z)} \otimes c^{\mathcal{D}}_{W,X,Y}).
\end{aligned}$$

Here the first and last equations follow from definitions; the second and fourth by naturality of F^2; and the third by associativity (2.2.2) and (2.3.1) of F^2 and of $\mathcal{E}$ as an enriched $\mathcal{C}$-category.

Similarly, we verify the left identity axiom:

$$\begin{aligned}
c^{\mathcal{D}}_{X,Y,Y}(\mathrm{id}^{\mathcal{D}}_Y \otimes 1_{\mathcal{D}(X,Y)}) &= F c^{\mathcal{C}}_{X,Y,Y}\, F^2 \left(F(\mathrm{id}^{\mathcal{C}}_X)\, F^0 \otimes 1_{F(\mathcal{C}(X,Y))}\right) \\
&= F\left(c^{\mathcal{C}}_{X,Y,Y}(\mathrm{id}_X \otimes 1_{\mathcal{C}(X,Y)})\right) F^2\, (F^0 \otimes 1_{F(\mathcal{C}(X,Y))}) \\
&= 1_{F(\mathcal{C}(X,Y))} = 1_{\mathcal{D}(X,Y)}.
\end{aligned}$$

Here, the first and last equations follow from definitions; the second from naturality of F^2; and the third from the identity laws (2.2.3) and (2.3.2) of F^2 and of $\mathcal{E}$ as an enriched $\mathcal{C}$-category. The right identity axiom is proven similarly. ☐

Finally, we introduce the notion of functors between enriched categories.

Definition 2.3.6. Given $\mathcal{C}$-categories $\mathcal{E}$ and $\mathcal{E}'$, a **$\mathcal{C}$-functor** (or **$\mathcal{C}$-enriched functor**) from $\mathcal{E}$ to $\mathcal{E}'$ consists of the following data:

- a choice of object $FX \in \mathcal{E}'$ for each $X \in \mathcal{E}$,
- a choice of morphism $F_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{C}(FX, FY)$ for each pair $X, Y \in \mathcal{E}$.

The morphisms $F_{X,Y}$ are subject to the following constraints for all $X, Y, Z \in \mathcal{E}$:

$$\text{Respect composition:} \qquad F_{X,Z} \circ c_{X,Y,Z} = c_{FX,FY,FZ}(F_{Y,Z} \otimes F_{X,Y}) \tag{2.3.4}$$

$$\text{Respect identities:} \qquad \mathrm{id}_{FX} = F_{X,X} \circ \mathrm{id}_X. \tag{2.3.5}$$

We denote $\mathcal{C}$-functors by $F : \mathcal{E} \xrightarrow{\mathcal{C}} \mathcal{E}'$, and call F **fully faithful** if the structure morphisms $F_{X,Y}$ are isomorphisms for all $X, Y \in \mathcal{E}$.

Chapter 3

Results on Closed Categories

In this chapter, we discuss right-closed monoidal categories and prove some new results about modules over algebras coming from these categories. In Section 3.1, we go over basic definitions and results about right-closed monoidal categories. In Section 3.2, we show that the categories of modules over algebras coming from right-closed monoidal categories admit canonical enrichments. In Section 3.3, we show that algebras in right-closed monoidal categories can be reconstructed within the enriched setting of Section 3.2 by studying endomorphisms.

3.1 Closed categories

Given a monoidal category $\mathcal{C}$, every object $Y \in \mathcal{C}$ gives rise to an endofunctor $(- \otimes Y) : \mathcal{C} \to \mathcal{C}$ mapping objects $X \mapsto X \otimes Y$ and morphisms $\psi \mapsto \psi \otimes 1_Y$. Recalling the definition of adjoint functors introduced in Definition 2.1.7 and Proposition 2.1.8, we may naturally ask when $(- \otimes Y)$ admit adjoint functors. This motivates our first definition.

Definition 3.1.1. A **right-closed monoidal category** is a monoidal category $\mathcal{C}$, for which each of the functors $(- \otimes Y)$ admits a right adjoint $[Y, -] : \mathcal{C} \to \mathcal{C}$.

Notation 3.1.2. Let $\mathcal{C}$ be right-closed monoidal. The unit and counit of the adjunction $(- \otimes Y) \dashv [Y, -]$ will be denoted by $\eta^Y : 1_{\mathcal{C}} \Rightarrow [Y, - \otimes Y]$ and $\epsilon^Y : [Y, -] \otimes Y \Rightarrow 1_{\mathcal{C}}$ respectively. The natural bijections $\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, [Y, Z])$ give rise to left and right transpose maps which can be expressed by the following formulas:

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, [Y, Z])$$

$$\psi : X \otimes Y \to Z \quad \mapsto \quad \psi^{\flat} := [Y, \psi]\eta_X^Y, \tag{3.1.1}$$

$$\varphi^{\sharp} := \epsilon_Z^Y(\varphi \otimes 1_Y) \quad \leftarrow\!\shortmid \quad \varphi : X \to [Y, Z], \tag{3.1.2}$$

for all $X, Y, Z \in \mathcal{C}$. If $\psi' : Z \to Z'$ and $\varphi' : X' \to X$, the naturality of these adjunctions can be expressed in the following equations:

$$(\psi'\psi)^{\flat} = [Y, \psi'\psi]\eta_X^Y = [Y, \psi']\psi^{\flat}, \tag{3.1.3}$$

$$(\varphi\varphi')^{\sharp} = \epsilon_Z^Y(\varphi\varphi' \otimes 1_Y) = \varphi^{\sharp}(\varphi' \otimes 1_Y). \tag{3.1.4}$$

Example 3.1.3. Cartesian closed categories are, by definition, right-closed monoidal categories whose tensor product is given by the categorical product and whose unit object is a terminal object. Examples of these include:

- Cat: the category of small categories;
- Set: the category of sets and functions between them;
- G-Set: the category of sets equipped with a group action;
- $\mathsf{Set}^{\mathcal{C}}$: the categories of functors $\mathcal{C} \to \mathsf{Set}$, for $\mathcal{C}$ a small category;
- $\mathsf{DirGraph}$: the category of directed graphs;
- $\mathsf{SimplicialSet}$: the category of simplicial sets;
- $\mathsf{CG\text{-}Haus}$: the category of compactly generated Hausdorff topological spaces.

Example 3.1.4. For R a commutative ring, the category R-Mod of modules over R is both additive and right-closed monoidal. Here, the tensor product is the standard tensor product of modules, and right-closure expresses the standard "tensor-hom adjunction". In particular, $\mathsf{Ab} = \mathbb{Z}$-Mod is additive and right-closed monoidal.

The next proposition shows that right-closed monoidal categories admit canonical enrichments over themselves. To this end, recall the notions of left transposes $(-)^\flat$ and right transposes $(-)^\sharp$ introduced in Proposition 2.1.8.

Proposition 3.1.5. *If $\mathcal{C}$ is right-closed monoidal, then $\mathcal{C}$ admits the structure of a $\mathcal{C}$-category. The data of the enrichment is given as follows:*

- *The Hom-object from X to Y is $[X, Y]$.*
- *The identity morphism id_X for an object $X \in \mathcal{C}$ is the right transpose $(\ell_X)^\flat$ of the left unitor $\ell_X : \mathbb{1} \otimes X \to X$ under the adjunction $(- \otimes X) \dashv [X, -]$.*
- *The composition morphism $c_{X,Y,Z}$ of $X, Y, Z \in \mathcal{C}$ is the right transpose of:*

$$[Y, Z] \otimes [X, Y] \otimes X \xrightarrow{1_{[Y,Z]} \otimes \epsilon^X_Y} [Y, Z] \otimes Y \xrightarrow{\epsilon^Y_Z} Z,$$

under the adjunction $(- \otimes X) \dashv [X, -]$.

*We call the resulting $\mathcal{C}$-category the **self-enriched structure** on $\mathcal{C}$.*

Proof. Using formula (3.1.1), the identity and composition morphisms are defined by:

$$\mathrm{id}_X := (\ell_X)^\flat = [X, \ell_X] \eta^X_{\mathbb{1}} \tag{3.1.5}$$

$$c_{X,Y,Z} := [X, \epsilon^Y_Z]\, [X, 1_{[Y,Z]} \otimes \epsilon^X_Y]\, \eta^X_{[Y,Z] \otimes [X,Y]} \tag{3.1.6}$$

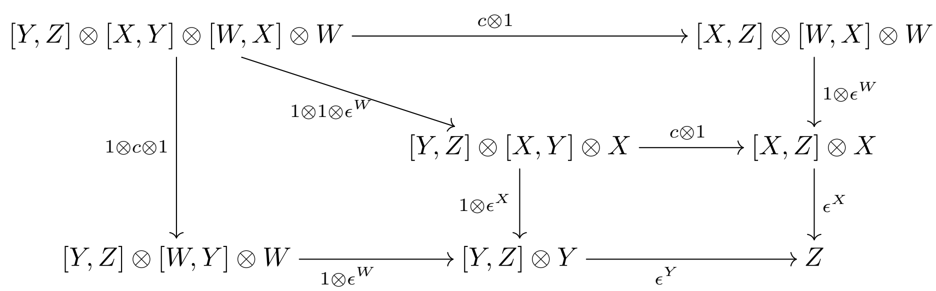
Using equation (3.1.2), we can express the left transposes $\mathrm{id}^\sharp_X$ and $c^\sharp_{X,Y,Z}$, respectively, as either of the composites in the following commutative diagrams:

$$\begin{array}{ccc} \mathbb{1} \otimes X & \xrightarrow{\mathrm{id}_X \otimes 1_X} & [X, X] \otimes X \\ & \searrow^{\ell_X} & \downarrow{\scriptstyle \epsilon^X_X} \\ & & X \end{array} \qquad \begin{array}{ccc} [Y, Z] \otimes [X, Y] \otimes X & \xrightarrow{c_{X,Y,Z} \otimes 1_Y} & [X, Z] \otimes X \\ {\scriptstyle 1_{[Y,Z]} \otimes \epsilon^X_Y} \downarrow & & \downarrow {\scriptstyle \epsilon^X_Z} \\ [Y, Z] \otimes Y & \xrightarrow[\epsilon^Y_Z]{} & Z \end{array} \tag{3.1.7}$$

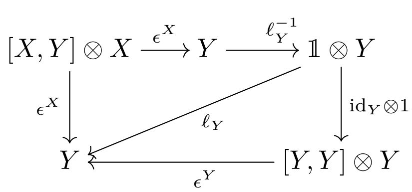
Instead of proving the associativity constraint (2.3.1) and identity constraints (2.3.2) and (2.3.3) of the enrichment directly, we take the left transpose of all the equations. After unpacking definitions, these constraints amount to:

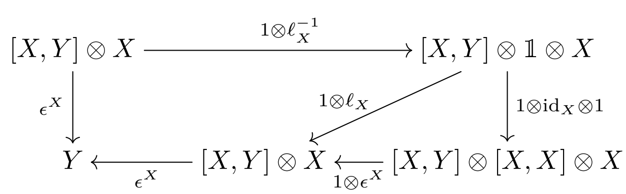
$$
\begin{aligned}
\text{Associativity:} \quad & \epsilon_Z^X(c_{X,Y,Z} \otimes \epsilon_X^W) = \epsilon_Z^Y(1_{[Y,Z]} \otimes \epsilon_Y^W)(1_{[Y,Z]} \otimes c_{W,X,Y} \otimes 1_W),\\
\text{Left Identity:} \quad & \epsilon_Y^Y(\mathrm{id}_Y \otimes \epsilon_Y^X) = \epsilon_Y^X,\\
\text{Right Identity:} \quad & \epsilon_Y^X(1_{[X,Y]} \otimes \epsilon_X^X)(1_{[X,Y]} \otimes \mathrm{id}_X \otimes 1_X) = \epsilon_Y^X.
\end{aligned}
$$

Associativity is shown in the diagram below, where the top-right square commutes by level exchange (2.2.1) and the other two commute by the right diagram of (3.1.7).



The left and right identity constraints are proven below. Here, both bottom right triangles commute by the left diagram of (3.1.7).





□

At this point, notice that $\mathcal{C}$ is both an ordinary category and a $\mathcal{C}$-category. Following Notation 2.3.2, we use $\mathcal{C}^{[\mathcal{C}]}$ when referring to the self-enriched structure on $\mathcal{C}$.

The data of $\mathcal{C}^{[\mathcal{C}]}$ retains information on the objects of $\mathcal{C}$, but not its morphisms. However, one may reconstruct $\mathcal{C}$ from the self-enriched structure by passing to its underlying category (defined in Proposition 2.3.4).

Proposition 3.1.6. *Let $\mathcal{C}^{[\mathcal{C}]}$ be the self-enriched structure of a right-closed monoidal category $\mathcal{C}$. There is an isomorphism of categories $(\mathcal{C}^{[\mathcal{C}]})_0 \cong \mathcal{C}$.*

Proof. As categories, $\mathcal{C}$ and $(\mathcal{C}^{[\mathcal{C}]})_0$ have the same class of objects, so it suffices to define bijective correspondences between hom-sets $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{(\mathcal{C}^{\mathcal{C}})_0}(X, Y)$ for all $X, Y \in \mathcal{C}$. But by definition, morphisms in $\mathrm{Hom}_{(\mathcal{C}^{[\mathcal{C}]})_0}(X, Y)$ correspond to morphisms in $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, [X, Y])$. Notice we have bijections:

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1} \otimes X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, [X, Y]),$$

induced by the left unitors $\ell_X : \mathbb{1} \otimes X \xrightarrow{\cong} X$ and the adjunctions $(- \otimes X) \dashv [X, -]$, respectively. A quick check shows this assignment is also functorial, which concludes the proof. □

Next, we show that in the right-closed self-enriched setting, the Hom-objects behave analogous to Hom-sets in ordinary categories. In particular, we are allowed to move between Hom-objects by (pre/post)-composing with specific morphisms.

Definition 3.1.7. Let $\mathcal{C}$ be a right-closed monoidal category equipped with its self-enriched structure. For every $X \in \mathcal{C}$ and $Y \xrightarrow{g} Z$ in $\mathcal{C}$, we define **post-composition** by g as the following morphism in $\mathcal{C}$:

$$g_* : [X, Y] \xrightarrow{[X,g]} [X, Z].$$

Equivalently, we may express $g_* = (g \circ \epsilon_Y^X)^\flat$. For every $Y \in \mathcal{C}$ and $W \xrightarrow{f} X$ in $\mathcal{C}$, we define **pre-composition** by f as the following morphism in $\mathcal{C}$:

$$f^* : [X, Y] \xrightarrow{\eta^W_{[X,Y]}} [W, [X, Y] \otimes W] \xrightarrow{[W, 1_{[X,Y]} \otimes f]} [W, [X, Y] \otimes X] \xrightarrow{[W, \epsilon^X_Y]} [W, Y].$$

Equivalently, we may express $f^* = \left(\epsilon^X_Y(1_{[X,Y]} \otimes f)\right)^\flat$.

The following proposition shows that the (pre/post)-composition morphisms satisfy several properties one might expect.

Proposition 3.1.8. *Let $\mathcal{C}$ be a right-closed monoidal category equipped with its self-enriched structure.*

(a) *Post-composition and pre-composition commute. That is, given morphisms $f : W \to X$ and $g : Y \to Z$ in $\mathcal{C}$, the following diagram commutes:*

$$\begin{array}{ccc} [X,Y] & \xrightarrow{f^*} & [W,Y] \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ [X,Z] & \xrightarrow[f^*]{} & [W,Z] \end{array}$$

(b) *If $h : X \xrightarrow{\cong} Y$ is an isomorphism, then $\mathrm{id}_X = (h^{-1})_* \circ h^* \circ \mathrm{id}_Y = h^* \circ (h^{-1})_* \circ \mathrm{id}_Y$.*

(c) *If $f : X' \to X$ and $g : Z \to Z'$ are morphisms in $\mathcal{C}$ then:*

$$c_{X',Y,Z'}(g_* \otimes f^*) = (f^* \circ g_*) \circ c_{X,Y,Z} = (g_* \circ f^*) \circ c_{X,Y,Z}.$$

(d) *If $h : W \xrightarrow{\cong} Y$ is an isomorphism, then for all $X, Z \in \mathcal{C}$:*

$$c_{X,W,Z}(h^* \otimes h_*^{-1}) = c_{X,Y,Z}.$$

Proof. (a) We use definitions to expand the diagram to the one shown below.

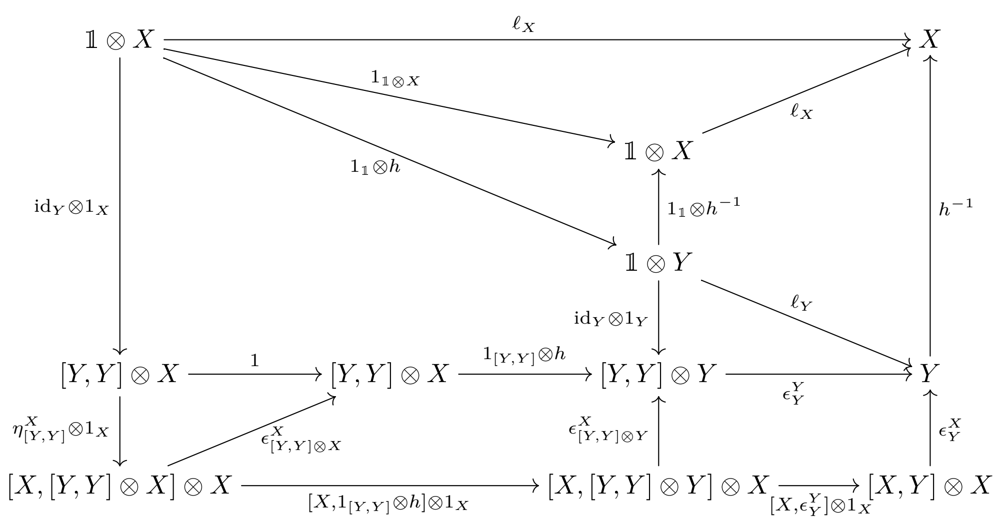
$$\begin{array}{ccccccc} [X,Y] & \xrightarrow{\eta^W_{[X,Y]}} & [W,[X,Y]\otimes W] & \xrightarrow{[W,1\otimes f]} & [W,[X,Y]\otimes X] & \xrightarrow{[W,\epsilon^X_Y]} & [W,Y] \\ {\scriptstyle [X,g]}\downarrow & & \downarrow{\scriptstyle [W,[X,g]\otimes 1_W]} & & \downarrow{\scriptstyle [W,[X,g]\otimes 1_X]} & & \downarrow{\scriptstyle [W,g]} \\ [X,Z] & \xrightarrow[\eta^W_{[X,Z]}]{} & [W,[X,Z]\otimes W] & \xrightarrow[{[W,1\otimes f]}]{} & [W,[X,Z]\otimes X] & \xrightarrow[{[W,\epsilon^X_Z]}]{} & [W,Z] \end{array}$$

Notice that the left square commutes by naturality of η^W, the right square commutes by naturality of ϵ^X, and the center square commutes by level exchange (2.2.1).

(b) By part (a) of this proposition, it suffices to show the first equality. Using Definition 3.1.7 and (3.1.5), we rewrite the equation as:

$$(\ell_X)^\flat = (h^{-1} \circ \epsilon^X_Y)^\flat \, (\epsilon^Y_Y(1_{[Y,Y]} \otimes h))^\flat \, \mathrm{id}_Y.$$

Taking left transposes and using naturality (3.1.4) of left transposes results in an equivalent equation which we place as the outer paths of the diagram below.

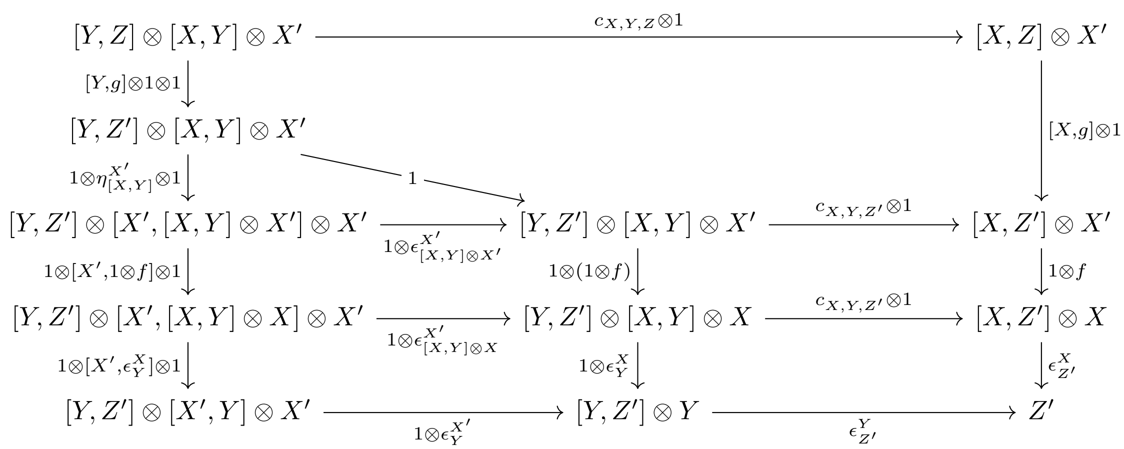


Here, the bottom two quadrangles commute by naturality of ϵ^X; the bottom left triangle commutes by a triangle identity (2.1.1); the upper left quadrangle commutes by level exchange (2.2.1); the center-right triangle commutes by (3.1.7); the top right quadrangle commutes by naturality of the left unitor ℓ; and the remaining top triangles commute trivially.

(c) By part (a), it suffices to show the first equality. Left transposing the equation and making use of naturality (3.1.4), we obtain the equivalent equation:

$$c^{\sharp}_{X',Y,Z'}(g_* \otimes f^* \otimes 1_{X'}) = (f^*)^{\sharp}\,(g_* \otimes 1_{X'})\,(c_{X,Y,Z} \otimes 1_{X'}).$$

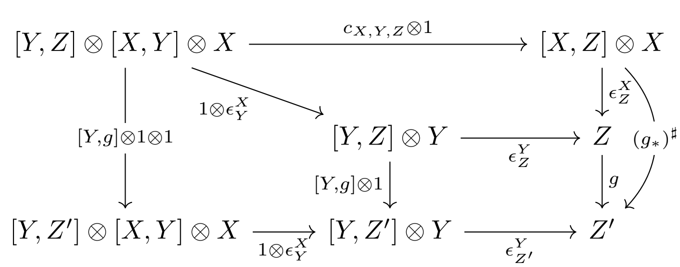
Expanding this equation using Definition 3.1.7 and (3.1.7), we obtain the outer paths of the following diagram:



Notice that the bottom left square and the square above it commute by naturality of $\epsilon^{X'}$; the bottom right square commutes by (3.1.7); the square above it commutes by level exchange (2.2.1); and the triangle commutes by the first triangle identity (2.1.1). To show that the top region commutes, it suffices to prove equation $(*)$ below:

$$[X,g]\,c_{X,Y,Z} =: g_*\,c_{X,Y,Z} \overset{(*)}{=} c_{X,Y,Z'}\,(g_* \otimes 1_{[X,Y]}) := c_{X,Y,Z'}\,([Y,g]\otimes 1_{[X,Y]}).$$

Left transposing and using (3.1.4) and (3.1.7) gives an equivalent equation which we place as the outer paths of the diagram below:



Here the outer curved regions commute by definition; the left quadrangle commutes by level exchange (2.2.1); the top right quadrangle commutes by (3.1.7); and the bottom right square commutes by naturality of ϵ^Y.

(d) By left transposing the equation and using naturality (3.1.4), we obtain:

$$c^\sharp_{X,W,Z}(h^* \otimes h_*^{-1} \otimes 1_X) = c^\sharp_{X,Y,Z}.$$

Using Definition 3.1.7 and (3.1.7), we expand the left-hand side to:

$$\begin{aligned}
&\epsilon^W_Z(1_{[W,Z]} \otimes \epsilon^X_W)\left([W, \epsilon^Y_Z(1_{[Y,Z]} \otimes h)]\eta^W_{[Y,Z]} \otimes 1_{[X,W]\otimes X}\right)(1_{[Y,Z]} \otimes [X, h^{-1}] \otimes 1_X) \\
&\overset{(1)}{=} \epsilon^W_Z([W, \epsilon^Y_Z] \otimes 1_W)(1_{[W,[Y,Z]\otimes Y]} \otimes \epsilon^X_W)\left([W, 1_{[Y,Z]} \otimes h]\eta^W_{[Y,Z]} \otimes [X, h^{-1}] \otimes 1_X\right) \\
&\overset{(2)}{=} \epsilon^Y_Z\epsilon^W_{[Y,Z]\otimes Y}(1_{[W,[Y,Z]\otimes Y]} \otimes \epsilon^X_W)\left([W, 1_{[Y,Z]} \otimes h]\eta^W_{[Y,Z]} \otimes [X, h^{-1}] \otimes 1_X\right) \\
&\overset{(3)}{=} \epsilon^Y_Z\epsilon^W_{[Y,Z]\otimes Y}\left([W, 1_{[Y,Z]} \otimes h] \otimes 1_W\right)(1_{[W,[Y,Z]\otimes W]} \otimes \epsilon^X_W)\left(\eta^W_{[Y,Z]} \otimes [X, h^{-1}] \otimes 1_X\right) \\
&\overset{(4)}{=} \epsilon^Y_Z(1_{[Y,Z]} \otimes h)\epsilon^W_{[Y,Z]\otimes W}(1_{[W,[Y,Z]\otimes W]} \otimes \epsilon^X_W)\left(\eta^W_{[Y,Z]} \otimes [X, h^{-1}] \otimes 1_X\right) \\
&\overset{(5)}{=} \epsilon^Y_Z(1_{[Y,Z]} \otimes h)(1_{[Y,Z]} \otimes \epsilon^X_W)(1_{[Y,Z]} \otimes [X, h^{-1}] \otimes 1_X) \\
&\overset{(6)}{=} \epsilon^Y_Z(1_{[Y,Z]} \otimes h)(1_{[Y,Z]} \otimes h^{-1})(1_{[Y,Z]} \otimes \epsilon^X_Y) \\
&\overset{(7)}{=} \epsilon^Y_Z(1_{[Y,Z]} \otimes \epsilon^X_Y) \\
&\overset{(8)}{=} c^\sharp_{X,Y,Z}.
\end{aligned}$$

Here, the first and third equalities follows by level exchange (2.2.1); the second, fourth, and sixth equalities follows by naturality of various counits ϵ; the fifth equality follows from a combination of level exchange and the first triangle identity (2.1.1); the seventh equality follows by composing h and h^{-1}; and the eighth equality follows by definition, as shown in the right diagram of (3.1.7). □

3.2 Enrichment of the category of modules

In this section, we prove a canonical enrichment for the categories of modules over a right-closed monoidal category. We begin with a motivating example.

Example 3.2.1. Fix a ring R and right R-modules M and N. Recall that R-module homomorphisms $\varphi : M \to N$ are group homomorphisms satisfying:

$$\varphi(m \cdot r) = \varphi(m) \cdot r \qquad \forall r \in R,\ \forall m \in M.$$

In particular, $\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M, N)$ is the subgroup of $\mathrm{Hom}_{\mathsf{Ab}}(M, N)$ consisting of morphisms for which the following square commutes.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{Ab}}(M,N) \times M \times R & \xrightarrow{1\times\mathrm{act}} & \mathrm{Hom}_{\mathsf{Ab}}(M,N) \times M \\ {\scriptstyle \mathrm{ev}\times 1}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ N \times R & \xrightarrow{\mathrm{act}} & N \end{array}$$

where "act" and "ev" refer to the respective action and evaluation maps. Put differently, $\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(M, N) \hookrightarrow \mathrm{Hom}_{\mathsf{Ab}}(M, N)$ serves as the equalizer of a pair of maps:

$$\mathrm{Hom}_{\mathsf{Ab}}(M, N) \rightrightarrows \mathrm{Hom}_{\mathsf{Ab}}(M \times R, N)$$

mapping $\varphi \in \mathrm{Hom}_{\mathsf{Ab}}(M, N)$ to $(m, r) \mapsto \varphi(m \cdot r)$ and $(m, r) \mapsto \varphi(m) \cdot r$ respectively. Since Ab possesses such equalizers, we may use the self-enriched structure of Ab to construct the Ab-enriched structure of R-Mod described in Example 2.3.3.

We now generalize the above example to right-closed monoidal categories.

Definition 3.2.2. Let A be an algebra in a right-closed monoidal category $\mathcal{C}$. For each pair of right A-modules $(M, \rho^M), (N, \rho^N) \in \mathsf{Mod}(\mathcal{C})_A$, consider the maps:

$$\mu_{M,N} : [M, N] \otimes M \otimes A \xrightarrow{1\otimes\rho^M} [M, N] \otimes M \xrightarrow{\epsilon^M_N} N, \tag{3.2.1}$$

$$\nu_{M,N} : [M,N] \otimes M \otimes A \xrightarrow{\epsilon_N^M \otimes 1} N \otimes A \xrightarrow{\rho^N} N, \tag{3.2.2}$$

whose right transposes give maps $\mu^\flat_{M,N}, \nu^\flat_{M,N} : [M,N] \to [M \otimes A, N]$. When it exists, the equalizer of $\mu^\flat_{M,N}$ and $\nu^\flat_{M,N}$ will be denoted $[M,N]_A$ and be called the **hom-object** from M to N.

The next result justifies the naming convention above.

Theorem 3.2.3. *Let A be an algebra in a right-closed monoidal category $\mathcal{C}$. Assume that for each pair of right A-modules, the equalizers of Definition 3.2.2 exist in $\mathcal{C}$. Then, the self-enriched structure of $\mathcal{C}$ induces a $\mathcal{C}$-enrichment on $\mathsf{Mod}(\mathcal{C})_A$.*

Proof. To define the $\mathcal{C}$-enrichment, we need to specify hom-objects, composition morphisms, and identity morphisms, subject to associativity and unitality axioms.

For each pair of A-modules $(M, \rho^M), (N, \rho^N) \in \mathsf{Mod}(\mathcal{C})_A$, we set the hom-object in $\mathcal{C}$ from M to N to be:

$$\mathsf{Mod}(\mathcal{C})_A(M,N) := [M,N]_A. \tag{3.2.3}$$

As an equalizer, $[M,N]_A$ comes equipped with a monomorphism:

$$\mathrm{eq}_{M,N} : [M,N]_A \hookrightarrow [M,N],$$

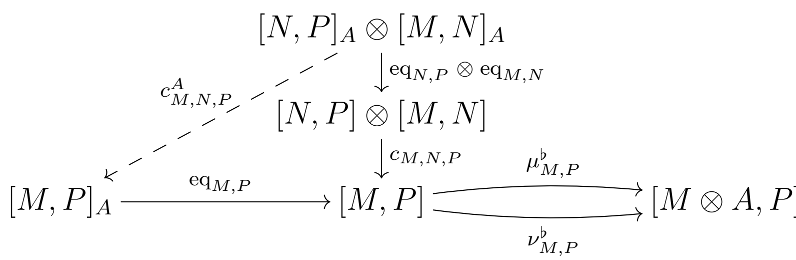
satisfying $\mu^\flat_{M,N} \circ \mathrm{eq}_{M,N} = \nu^\flat_{M,N} \circ \mathrm{eq}_{M,N}$. Using Lemma 2.1.9 on this equality, we get the auxiliary equation:

$$\mu_{M,N}(\mathrm{eq}_{M,N} \otimes 1_{M\otimes A}) = \nu_{M,N}(\mathrm{eq}_{M,N} \otimes 1_{M\otimes A}). \tag{3.2.4}$$

It remains to define the composition and identity morphisms for the enrichment, and verify the associativity and unitality axioms. The proofs of these are relegated to the next two claims. □

Claim 3.2.4. *Let $M, N, P, Q \in \mathsf{Mod}(\mathcal{C})_A$ be right A-modules in $\mathcal{C}$.*

(a) There is a composition morphism $c^A_{M,N,P} : [N,P]_A \otimes [M,N]_A \to [M,P]_A$ rendering the following diagram commutative.



(3.2.5)

(b) The composition morphisms are associative, i.e. they satisfy:

$$c^A_{M,N,Q} \circ (c^A_{N,P,Q} \otimes 1_{[M,N]_A}) \;=\; c^A_{M,P,Q} \circ (1_{[P,Q]_A} \otimes c^A_{M,N,P}). \tag{3.2.6}$$

Proof of Claim 3.2.4. (a) Looking at diagram (3.2.5), if we prove that the composition of the vertical morphisms $c_{M,N,P}(\mathrm{eq}_{N,P} \otimes \mathrm{eq}_{M,N})$ equalizes $\mu^\flat_{M,P}$ and $\nu^\flat_{M,P}$, then the universal property of $\mathrm{eq}_{M,P}$ guarantees the existence of the desired map $c^A_{M,N,P}$. By Lemma 2.1.9, it suffices to prove that $(c_{M,N,P} \otimes 1_{M\otimes A})(\mathrm{eq}_{N,P} \otimes \mathrm{eq}_{M,N} \otimes 1_{M\otimes A})$ equalizes $\mu_{M,P}$ and $\nu_{M,P}$. This is shown below:

$$\begin{aligned}
&\mu_{M,P}(c_{M,N,P} \otimes 1_{M\otimes A})(\mathrm{eq}_{N,P} \otimes \mathrm{eq}_{M,N} \otimes 1_{M\otimes A}) \\
&\overset{(1)}{=} \epsilon^M_P(1_{[M,P]} \otimes \rho^M)(c_{M,N,P} \otimes 1_{M\otimes A})(\mathrm{eq}_{N,P} \otimes \mathrm{eq}_{M,N} \otimes 1_{M\otimes A}) \\
&\overset{(2)}{=} \epsilon^M_P(c_{M,N,P} \otimes 1_M)(\mathrm{eq}_{N,P} \otimes \mathrm{eq}_{M,N} \otimes \rho^M) \\
&\overset{(3)}{=} (c_{M,N,P})^\sharp(\mathrm{eq}_{N,P} \otimes \mathrm{eq}_{M,N} \otimes \rho^M) \\
&\overset{(4)}{=} \epsilon^N_P(1_{[N,P]} \otimes \epsilon^M_N)(\mathrm{eq}_{N,P} \otimes \mathrm{eq}_{M,N} \otimes \rho^M)
\end{aligned}$$

$$\overset{(5)}{=} \epsilon_P^N(\mathrm{eq}_{N,P} \otimes 1_N)(1_{[N,P]_A} \otimes \epsilon_N^M)(1_{[N,P]_A \otimes [M,N]} \otimes \rho^M)(1_{[N,P]_A} \otimes \mathrm{eq}_{M,N} \otimes 1_{M \otimes A})$$
$$\overset{(6)}{=} \epsilon_P^N(\mathrm{eq}_{N,P} \otimes 1_N)(1_{[N,P]_A} \otimes \mu_{M,N})(1_{[N,P]_A} \otimes \mathrm{eq}_{M,N} \otimes 1_{M \otimes A})$$
$$\overset{(7)}{=} \epsilon_P^N(\mathrm{eq}_{N,P} \otimes 1_N)(1_{[N,P]_A} \otimes \nu_{M,N})(1_{[N,P]_A} \otimes \mathrm{eq}_{M,N} \otimes 1_{M \otimes A})$$
$$\overset{(8)}{=} \epsilon_P^N(\mathrm{eq}_{N,P} \otimes 1_N)(1_{[N,P]_A} \otimes \rho^N)(1_{[N,P]_A} \otimes \epsilon_N^M \otimes 1_A)(1_{[N,P]_A} \otimes \mathrm{eq}_{M,N} \otimes 1_{M \otimes A})$$
$$\overset{(9)}{=} \epsilon_P^N(1_{[N,P]} \otimes \rho^N)(\mathrm{eq}_{N,P} \otimes 1_{N \otimes A})(1_{[N,P]_A} \otimes \epsilon_N^M \otimes 1_A)(1_{[N,P]_A} \otimes \mathrm{eq}_{M,N} \otimes 1_{M \otimes A})$$
$$\overset{(10)}{=} \mu_{N,P}(\mathrm{eq}_{N,P} \otimes 1_{N \otimes A})(1_{[N,P]_A} \otimes \epsilon_N^M \otimes 1_A)(1_{[N,P]_A} \otimes \mathrm{eq}_{M,N} \otimes 1_{M \otimes A})$$
$$\overset{(11)}{=} \nu_{N,P}(\mathrm{eq}_{N,P} \otimes 1_{N \otimes A})(1_{[N,P]_A} \otimes \epsilon_N^M \otimes 1_A)(1_{[N,P]_A} \otimes \mathrm{eq}_{M,N} \otimes 1_{M \otimes A})$$
$$\overset{(12)}{=} \rho^N(\epsilon_P^N \otimes 1_A)(\mathrm{eq}_{N,P} \otimes 1_{N \otimes A})(1_{[N,P]_A} \otimes \epsilon_N^M \otimes 1_A)(1_{[N,P]_A} \otimes \mathrm{eq}_{M,N} \otimes 1_{M \otimes A})$$
$$\overset{(13)}{=} \rho^N(\epsilon_P^N \otimes 1_A)(1_{[N,P]} \otimes \epsilon_N^M \otimes 1_A)(\mathrm{eq}_{N,P} \otimes \mathrm{eq}_{M,N} \otimes 1_{M \otimes A})$$
$$\overset{(14)}{=} \rho^N((c_{M,N,P})^\sharp \otimes 1_A)(\mathrm{eq}_{N,P} \otimes \mathrm{eq}_{M,N} \otimes 1_{M \otimes A})$$
$$\overset{(15)}{=} \rho^P(\epsilon_P^M \otimes 1_A)(c_{M,N,P} \otimes 1_{M \otimes A})(\mathrm{eq}_{N,P} \otimes \mathrm{eq}_{M,N} \otimes 1_{M \otimes A})$$
$$\overset{(16)}{=} \nu_{M,P}(c_{M,N,P} \otimes 1_{M \otimes A})(\mathrm{eq}_{N,P} \otimes \mathrm{eq}_{M,N} \otimes 1_{M \otimes A}).$$

Here equations 1, 6, 8, 10, 12, and 16 follow from the definitions (3.2.1, 3.2.2) of μ and ν; equations 2,5,9, and 13 follow from level exchange (2.2.1); equations 3 and 15 follow from the definition (3.1.2) of left transposes; equations 4 and 14 follow from the definition of composition in the self-enriched setting (3.1.6); and equations 7 and 11 follow from the auxiliary equation (3.2.4).

(b) Associativity of the composition morphisms is proven below, keeping in mind that $\mathrm{eq}_{M,Q}$ is a monomorphism, hence left-cancellable.

$$\mathrm{eq}_{M,Q} \circ c^A_{M,N,Q} \circ (c^A_{N,P,Q} \otimes 1_{[M,N]_A})$$
$$\overset{(1)}{=} c_{M,N,Q} \circ (\mathrm{eq}_{N,Q} \otimes \mathrm{eq}_{M,N})(c^A_{N,P,Q} \otimes 1_{[M,N]_A})$$
$$\overset{(2)}{=} c_{M,N,Q} \circ (1_{[N,Q]} \otimes \mathrm{eq}_{[M,N]})(c_{N,P,Q} \otimes 1_{[M,N]_A})(\mathrm{eq}_{P,Q} \otimes \mathrm{eq}_{N,P} \otimes 1_{[M,N]_A})$$

$$\overset{(3)}{=} c_{M,N,Q}(c_{N,P,Q} \otimes 1_{[M,N]})(1_{[P,Q]\otimes[N,P]} \otimes \mathrm{eq}_{M,N})(\mathrm{eq}_{P,Q} \otimes \mathrm{eq}_{N,P} \otimes 1_{[M,N]_A})$$

$$\overset{(4)}{=} c_{M,P,Q}(1_{[P,Q]\otimes[N,P]} \otimes c_{M,N,P})(1_{[P,Q]\otimes[N,P]} \otimes \mathrm{eq}_{M,N})(\mathrm{eq}_{P,Q} \otimes \mathrm{eq}_{N,P} \otimes 1_{[M,N]_A})$$

$$\overset{(5)}{=} c_{M,P,Q}(1_{[P,Q]\otimes[N,P]} \otimes c_{M,N,P})(\mathrm{eq}_{P,Q} \otimes \mathrm{eq}_{N,P} \otimes \mathrm{eq}_{M,N})$$

$$\overset{(6)}{=} c_{M,P,Q}(1_{[P,Q]} \otimes \mathrm{eq}_{M,P})(1_{[P,Q]} \otimes c^A_{M,N,P})(\mathrm{eq}_{P,Q} \otimes 1_{[N,P]_A \otimes [M,N_A]})$$

$$\overset{(7)}{=} c_{M,P,Q}(\mathrm{eq}_{P,Q} \otimes \mathrm{eq}_{M,P})(1_{[P,Q]_A} \otimes c^A_{M,N,P})$$

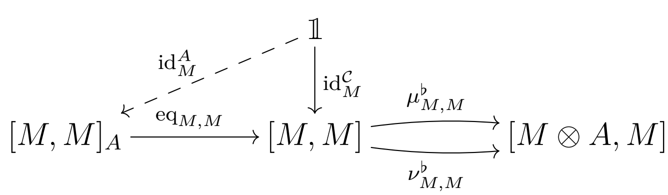
$$\overset{(8)}{=} \mathrm{eq}_{M,Q} \circ c^A_{M,P,Q} \circ (1_{[P,Q]_A} \otimes c^A_{M,N,P}).$$

Here, first, second, sixth, and eighth equalities follow from the definition of the enriched compositions c^A; the third, fifth, and seventh equalities follow from level exchange (2.2.1); and the fourth equality follows from associativity of composition (2.3.1). □

Next, we define the identity morphisms of the enrichment.

Claim 3.2.5. *Let $M, N \in \mathsf{Mod}(\mathcal{C})_A$ be right A-modules in $\mathcal{C}$.*

(a) There is an identity morphism $\mathrm{id}^A_M : \mathbb{1} \to [M,M]_A$ rendering the following diagram commutative.



(3.2.7)

(b) The identity morphisms satisfy the identity axioms, i.e. they satisfy:

$$c^A_{M,N,N} \circ (\mathrm{id}^A_N \otimes 1_{[M,N]_A}) = \ell_{[M,N]_A}, \tag{3.2.8}$$

$$c^A_{M,M,N} \circ (1_{[M,N]_A} \otimes \mathrm{id}^A_M) = r_{[M,N]_A}. \tag{3.2.9}$$

Proof of Claim 3.2.5. (a) Looking at diagram (3.2.7), if we prove $\mathrm{id}^{\mathcal{C}}_M$ equalizes $\mu^\flat_{M,M}$

and $\nu^\flat_{M,M}$, then the universal property of $\mathrm{eq}_{M,P}$ guarantees the existence of the desired map id^A_M. By Lemma 2.1.9, it suffices to prove that $\mathrm{id}^{\mathcal{C}}_M \otimes 1_{M\otimes A}$ equalizes $\mu_{M,M}$ and $\nu_{M,M}$. This is shown below.

$$\begin{aligned}
\mu_{M,M}(\mathrm{id}^{\mathcal{C}}_M \otimes 1_{M\otimes A}) &= \epsilon^M_M(\mathrm{id}^{\mathcal{C}}_M \otimes \rho^M) \\
&= \ell_M(1_{\mathbb{1}} \otimes \rho^M) \\
&= \rho^M \circ \ell_{M\otimes A} \\
&= \rho^M(\epsilon^M_M \otimes 1_A)(\mathrm{id}^{\mathcal{C}}_M \otimes 1_{M\otimes A}) \\
&= \nu_{M,M}(\mathrm{id}^{\mathcal{C}}_M \otimes 1_{M\otimes A}).
\end{aligned}$$

Here, the first and last equations follow from the definitions (3.2.1,3.2.2) of $\mu_{M,M}$ and $\nu_{M,M}$; the second using the definition of left transposes (3.1.2) and identity morphisms in the self-enriched setting (3.1.5); the third by the naturality of the left unitor; and the fourth using the same reasoning as equality two, plus the fact that $\ell_{M\otimes A} = \ell_M \otimes 1_A$.

(b) We focus the left identity axiom, since the right identity holds similarly. Using the fact that $\mathrm{eq}_{M,N}$ is a monomorphism (hence left cancellable), we have:

$$\begin{aligned}
\mathrm{eq}_{M,N}\, c^A_{M,N,N}\,(\mathrm{id}^A_N \otimes 1_{[M,N]_A}) &= c_{M,N,N}\,(\mathrm{eq}_{N,N} \otimes \mathrm{eq}_{M,N})\,(\mathrm{id}^A_N \otimes 1_{[M,N]_A}) \\
&= c_{M,N,N}\,(\mathrm{id}^{\mathcal{C}}_N \otimes \mathrm{eq}_{M,N}) \\
&= \ell_{[M,N]}\,(1_{\mathbb{1}} \otimes \mathrm{eq}_{M,N}) \\
&= \mathrm{eq}_{M,N}\,\ell_{[M,N]}.
\end{aligned}$$

Here the first equality follow from the definition (3.2.5) of $c^A_{M,N,N}$; the second by (3.2.7) of id^A_N; the third by the left identity axiom in the self-enriched setting; and the fourth by the naturality of the left unitor. $\square$

3.3 Endomorphism algebras

For this section, we fix a right-closed monoidal category $\mathcal{C}$, an algebra (A, m, u) in $\mathcal{C}$, and assume the category of modules $\mathsf{Mod}(\mathcal{C})_A$ comes equipped with the $\mathcal{C}$-enrichment of Theorem 3.2.3.

Definition-Proposition 3.3.1. The hom-object $[A, A]_A \in \mathsf{Mod}(\mathcal{C})_A$ of the right regular A-module to itself admits the structure of an algebra over $\mathcal{C}$, with multiplication and unit given by composition and identity morphisms in $\mathsf{Mod}(\mathcal{C})_A$. We call this the **$\mathcal{C}$-endomorphism algebra** of A.

Proof. We need to show that $c^A_{A,A,A}$ and id^A_A satisfy the associativity and unitality axioms defined in (2.2.4) and (2.2.5). Explicitly, these are:

$$\text{Associativity:} \qquad c^A_{A,A,A}(c^A_{A,A,A} \otimes 1_{[A,A]_A}) = c^A_{A,A,A}(1_{[A,A]_A} \otimes c^A_{A,A,A}),$$

$$\text{Unitality:} \qquad c^A_{A,A,A}(\mathrm{id}^A_A \otimes 1_{[A,A]_A}) = 1_{[A,A]_A} = c^A_{A,A,A}(1_{[A,A]_A} \otimes \mathrm{id}^A_A),$$

which amount to instances of equations (3.2.6), (3.2.8), and (3.2.9). This was proved in Claim 3.2.4(b) and Claim 3.2.5(b). □

Example 3.3.2. If $\mathcal{C} = \mathsf{Vec}_{\Bbbk}$, then $A \in \mathsf{Alg}(\mathcal{C})$ is a $\Bbbk$-algebra, and $[A, A]_A$ is the algebra of all A-linear endomorphisms of A. Notice that $[A, A]_A \cong A$ as $\Bbbk$-algebras, expressing the fact that the A-linear endomorphisms of A are uniquely determined by where they send the unit element.

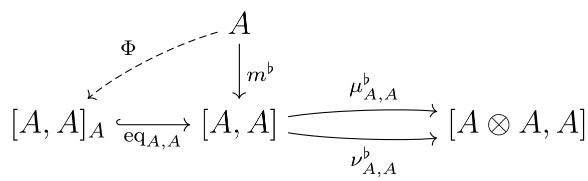
The main result of this section generalizes the example above to right-closed monoidal categories.

Theorem 3.3.3. *For all $A \in \mathsf{Alg}(\mathcal{C})$, we get that $[A, A]_A \cong A$ in $\mathsf{Alg}(\mathcal{C})$.*

Proof. We need to construct an isomorphism of algebras $\Phi : A \to [A, A]_A$. We start off by defining the inverse map $\Phi^{-1} : [A, A]_A \to A$ as the composite:

$$\Phi^{-1} : [A, A]_A = [A, A]_A \otimes \mathbb{1} \xrightarrow{\mathrm{eq}_{A,A} \otimes u} [A, A] \otimes A \xrightarrow{\epsilon^A_A} A.$$

To construct Φ, we first consider the right transpose $m^\flat : A \to [A, A]$ of the multiplication map. Using the universal property of $\mathrm{eq}_{A,A} : [A, A]_A \hookrightarrow [A, A]$, we are guaranteed a map $A \to [A, A]_A$ if we can check that $\mu^\flat_{A,A} m^\flat = \nu^\flat_{A,A} m^\flat$. This is visualized in the diagram below.

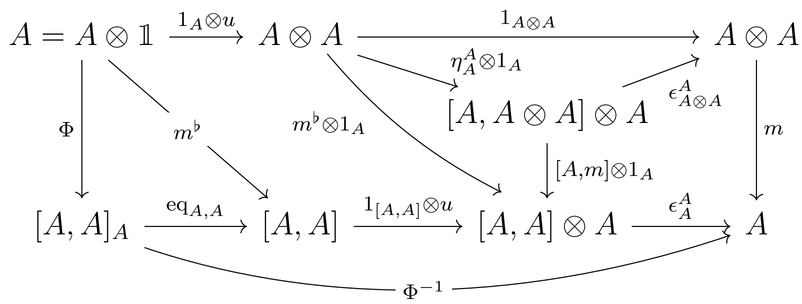


By Lemma 2.1.9, $\mu^\flat_{A,A} m^\flat = \nu^\flat_{A,A} m^\flat$ is equivalent to $\mu_{A,A}(m^\flat \otimes 1_{A\otimes A}) = \nu_{A,A}(m^\flat \otimes 1_{A\otimes A})$. We verify the latter equality below.

$$\begin{aligned} \mu_{A,A}(m^\flat \otimes 1_{A\otimes A}) &\overset{(1)}{=} \epsilon^A_A(1_{[A,A]} \otimes m)(m^\flat \otimes 1_{A\otimes A}) &&\overset{(2)}{=} \epsilon^A_A(m^\flat \otimes 1_A)(1_A \otimes m) \\ &\overset{(3)}{=} m(1_A \otimes m) &&\overset{(4)}{=} m(m \otimes 1_A) \\ &\overset{(5)}{=} m((m^\flat)^\sharp \otimes 1_A) &&\overset{(6)}{=} m(\epsilon^A_A \otimes 1_A)\big((m^\flat \otimes 1_A) \otimes 1_A\big) \\ &\overset{(7)}{=} \nu_{A,A}(m^\flat \otimes 1_{A\otimes A}). \end{aligned}$$

The first equality follows from the definition (3.2.1) of $\mu_{A,A}$; the second by level exchange (2.2.1); the third, fifth, and sixth using the definition (3.1.2) of left transposes or the fact that $(m^\flat)^\sharp = m$; the fourth by associativity (2.2.4) of the multiplication of A; and the seventh from the definition (3.2.2) of $\nu_{A,A}$.

Next, we verify that Φ^{-1} is an inverse to Φ below. We show $\Phi^{-1}\Phi = 1_A$ below.



Here, the leftmost triangle commutes by construction of Φ; the left quadrilateral commutes by level exchange (2.2.1); the center triangle commutes by definition of $m^\flat$; the top right triangle commutes by a triangle identity (2.1.1); and the rightmost quadrilateral commutes by naturality of ϵ^A. By unitality (2.2.5) of A, the upper path of the diagram is $m(1_A \otimes u) = 1_A$, thus showing that $\Phi^{-1}\Phi = 1_A$.

Next, we need show $\Phi\Phi^{-1} = 1_{[A,A]_A}$. Applying $\mathrm{eq}_{A,A}$ to both sides and taking left transposes $(-)^\sharp$ gives an equivalent equation, which we verify below:

$$
\begin{aligned}
(\mathrm{eq}_{A,A}\Phi \circ \Phi^{-1})^\sharp &\overset{(1)}{=} (m^\flat \circ \Phi^{-1})^\sharp \\
&\overset{(2)}{=} m \circ (\Phi^{-1} \otimes 1_A) \\
&\overset{(3)}{=} m(\epsilon^A_A \otimes 1_A)(\mathrm{eq}_{A,A} \otimes u \otimes 1_A) \\
&\overset{(4)}{=} \nu_{A,A}(\mathrm{eq}_{A,A} \otimes 1_{A \otimes A})(1_{[A,A]_A} \otimes u \otimes 1_A) \\
&\overset{(5)}{=} \mu_{A,A}(\mathrm{eq}_{A,A} \otimes 1_{A \otimes A})(1_{[A,A]_A} \otimes u \otimes 1_A) \\
&\overset{(6)}{=} \epsilon^A_A(1_{[A,A]} \otimes m)(\mathrm{eq}_{A,A} \otimes u \otimes 1_A) \\
&\overset{(7)}{=} \epsilon^A_A(\mathrm{eq}_{A,A} \otimes 1_A) \\
&\overset{(8)}{=} (\mathrm{eq}_{A,A})^\sharp.
\end{aligned}
$$

Here, the first equality follows from the construction of Φ; the second by naturality

(3.1.4) of left transposes; the third by definition of Φ^{-1}; the fourth by level exchange (2.2.1) and the definition (3.2.2) of $\nu_{A,A}$; the fifth by the auxiliary equation (3.2.4) characterizing $\mathrm{eq}_{A,A}$; the sixth by level exchange (2.2.1) and the definition (3.2.1) of $\mu_{A,A}$; the seventh by a unitality axiom (2.2.5) of A; and the eighth by definition (3.1.2) of left transposes.

Next, we need to show the map Φ is unital, i.e. that $\Phi \circ u = \mathrm{id}_A^A$. Instead of proving this directly, we again apply the monomorphism $\mathrm{eq}_{A,A}$ and take left transposes $(-)^\sharp$ of both sides of the equation. The resulting equality is verified below:

$$(\mathrm{eq}_{A,A} \circ \Phi \circ u)^\sharp = (m^\flat \circ u)^\sharp = m(u \otimes 1_A) = 1_A = (\mathrm{id}_A)^\sharp = (\mathrm{eq}_{A,A} \circ \mathrm{id}_A^A)^\sharp.$$

The first equality follows from the construction of Φ; the second using naturality of left transposes (3.1.4) and the fact that $(m^\flat)^\sharp = m$; the third by unitality (2.2.5) of A; and the last two by the definitions of identity morphisms in the $\mathcal{C}$-enrichments of $\mathcal{C}$ and $\mathsf{Mod}(\mathcal{C})_A$ respectively (see equation (3.1.5) and diagram (3.2.7).

Next, we check that Φ is multiplicative, i.e. $\Phi \circ m = c^A_{A,A,A}(\Phi \otimes \Phi)$. We deal with each side of the equation separately, but again proceed by first applying $\mathrm{eq}_{A,A}$ and take left transposes. The left-hand side of the equation becomes:

$$(\mathrm{eq}_{A,A} \circ \Phi \circ m)^\sharp = (m^\flat \circ m)^\sharp = m(m \otimes 1_A) = m(1_A \otimes m),$$

where the first equality follows by construction of Φ, the second using naturality (3.1.4) of left transposes, and the third by associativity (2.2.4) of the multiplication of A. Meanwhile, the right-hand side becomes:

$$\left(\mathrm{eq}_{A,A} \circ c^A_{A,A,A}(\Phi \otimes \Phi)\right)^\sharp \overset{(1)}{=} \left(c_{A,A,A}(\mathrm{eq}_{A,A} \otimes \mathrm{eq}_{A,A})(\Phi \otimes \Phi)\right)^\sharp$$

$$\begin{aligned}
&\overset{(2)}{=} \Big(c_{A,A,A}(m^\flat \otimes m^\flat)\Big)^\sharp \\
&\overset{(3)}{=} c^\sharp_{A,A,A}(m^\flat \otimes m^\flat \otimes 1_A) \\
&\overset{(4)}{=} \epsilon^A_A(1_{[A,A]} \otimes \epsilon^A_A)(m^\flat \otimes m^\flat \otimes 1_A) \\
&\overset{(5)}{=} \epsilon^A_A(m^\flat \otimes 1_A)(1_A \otimes \epsilon^A_A)(1_A \otimes m^\flat \otimes 1_A) \\
&\overset{(6)}{=} (m^\flat)^\sharp(1_A \otimes (m^\flat)^\sharp) \\
&\overset{(7)}{=} m(1_A \otimes m),
\end{aligned}$$

thus showing Φ is multiplicative. Here the first equality follows from the definition $c^A_{A,A,A}$ (see diagram (3.2.5)); the second by our construction of Φ; the third by naturality (3.1.4) of left transposes; the fourth by diagram (3.1.7); the fifth by level exchange (2.2.1); the sixth by definition (3.1.2) of left transposes; and the seventh since $(m^\flat)^\sharp = m$.

□

Chapter 4

Graded Categories

In this chapter, we focus on generalizing previous results to categories whose objects and morphisms are graded by some group G. These categories have been studied extensively in several contexts, including higher category theory [AHLF18], Hopf monoids [AM13, AM14], knot theory [Str12], Morita theory [Szl04], and perturbative quantum field theory [Nor20].

For the rest of this chapter, we will focus only on categories satisfying the hypotheses outlined below.

Hypothesis 4.0.1. Fix a group G with identity element $e \in G$. We will assume that our categories $\mathcal{C}$ satisfy the following three conditions:

(a) $\mathcal{C}$ is right-closed monoidal,

(b) for every tuple of objects $\{X_g\}_{g\in G}$, both their product and coproduct exist in $\mathcal{C}$,

(c) for each $X \in \mathcal{C}$, the endofunctors $(X \otimes -) : \mathcal{C} \to \mathcal{C}$ preserve coproducts.

When referring to (b) above, we sometimes say that $\mathcal{C}$ has G**-indexed (co)products**. If the group G is finite, we may equivalently assume that $\mathcal{C}$ has finite (co)products.

The organization of the chapter is as follows: in Section 4.1 we introduce notions of graded algebraic structures in categories; in Section 4.2, we recast these definitions using a monoidal structure on a single graded category; in Section 4.3, we give a sufficient condition for a graded category to inherit a right-closed monoidal structure; and in Section 4.4, we introduce shift endofunctors that serve to permute degrees of graded objects.

4.1 Graded algebraic structures

We begin by introducing a category of objects and morphisms in $\mathcal{C}$ graded by G.

Definition 4.1.1. The category $\mathsf{Gr}(\mathcal{C}) = \prod_{g\in G} \mathcal{C}$ of **graded objects in** $\mathcal{C}$ has:

as objects: tuples $X = (X_g)_{g\in G}$ of objects in $\mathcal{C}$ indexed by G, and

as morphisms: morphisms $\varphi : X \to Y$ in $\mathsf{Gr}(\mathcal{C})$ consist of tuples of morphisms $(\varphi|_g : X_g \to Y_g)_{g\in G}$ in $\mathcal{C}$.

We refer to X_g and $\varphi|_g$ as **degree** g of X and φ, respectively.

In $\mathsf{Gr}(\mathcal{C})$, we may now define graded algebras and modules as objects in $\mathsf{Gr}(\mathcal{C})$, equipped with families of multiplication or action maps that satisfy the expected associativity and unital constraints, but also respect the degrees of the grading.

Definition 4.1.2. A G**-graded algebra in** $\mathcal{C}$ consists of an object $A = (A_g)_{g\in G}$ in $\mathsf{Gr}(\mathcal{C})$, a *multiplication map* $m_{g,h} : A_g \otimes A_h \to A_{gh}$ in $\mathcal{C}$ for each pair $g, h \in G$, and a unit map $u : \mathbb{1} \to A_e$ in $\mathcal{C}$, satisfying the following constraints for all $g, h, k \in G$:

$$\text{Associativity:} \qquad m_{gh,k}(m_{g,h} \otimes \mathrm{id}_{A_k}) = m_{g,hk}(\mathrm{id}_{A_g} \otimes m_{h,k}), \tag{4.1.1}$$

$$\text{Unitality:} \qquad m_{e,g}(u_e \otimes \mathrm{id}_{A_g}) = \mathrm{id}_{A_g} = m_{g,e}(\mathrm{id}_{A_g} \otimes u_e). \tag{4.1.2}$$

Given two graded algebras $(A, m^A_{g,h}, u^A)$ and $(B, m^B_{g,h}, u^B)$, an **(iso)morphism of** G**-graded algebras** from A to B is a tuple of (iso)morphisms $\varphi = (\varphi|_g : A_g \to B_g)_{g\in G}$ in $\mathcal{C}$ that respect the multiplication and unit morphisms:

$$\text{Respect multiplication:} \qquad m^B_{g,h}(\varphi|_g \otimes \varphi|_h) = \varphi|_{gh}\, m^A_{g,h},$$

$$\text{Respect unit:} \qquad \varphi|_e\, u^A_e = u^B_e.$$

for all $g, h \in G$. Note that G-graded algebras and their morphisms form a category, which we denote $\mathsf{GrAlg}(\mathcal{C})$.

Definition 4.1.3. If A is a G-graded algebra in $\mathcal{C}$, a **G-graded right A-module in $\mathcal{C}$** consists of an object $M = (M_g)_{g \in G}$ in $\mathsf{Gr}(\mathcal{C})$, along with a collection of *action maps* $\rho_{g,h} : M_g \otimes A_h \to M_{gh}$ in $\mathcal{C}$ for all $g, h \in G$, satisfying:

$$\text{Associativity:} \qquad \rho_{gh,k}(\rho_{g,h} \otimes \mathrm{id}_{A_k}) = \rho_{g,hk}(\mathrm{id}_{M_g} \otimes m_{h,k}), \tag{4.1.3}$$

$$\text{Unitality:} \qquad \rho_{g,e}(\mathrm{id}_{M_g} \otimes u_e) = \mathrm{id}_{M_g}. \tag{4.1.4}$$

Given two graded A-modules $(M, \rho^M_{g,h})$ and $(N, \rho^N_{g,h})$ in $\mathcal{C}$, an **(iso)morphism of G-graded modules** from M to N is a tuple of (iso)morphisms $\varphi = (\varphi|_g : M_g \to N_g)_{g \in G}$ in $\mathcal{C}$ that respect the action maps:

$$\text{Respect actions:} \qquad \rho^N_{g,h}\,(\varphi|_g \otimes \mathrm{id}_{A_h}) = \varphi|_{gh}\,\rho^M_{g,h}$$

for all $g, h \in G$. Graded right A-modules and their morphisms form a category, which we denote $\mathsf{GrMod}(\mathcal{C})_A$.

4.2 Monoidal structures on graded categories

In this section, we endow $\mathsf{Gr}(\mathcal{C})$ with a monoidal structure called the *Cauchy monoidal structure*. This can be understood as a generalization of the Cauchy product of power series. Furthermore, we show that the graded algebras and modules defined in the previous section admit cleaner representations, as algebras and modules over the graded category $\mathsf{Gr}(\mathcal{C})$.

We begin with an auixiliary lemma for which we omit the proof.

Lemma 4.2.1. *The category $\mathsf{Gr}(\mathcal{C})$ inherits any limits and colimits that $\mathcal{C}$ has.* $\square$

We now introduce the monoidal structure on $\mathsf{Gr}(\mathcal{C})$.

Proposition 4.2.2. *The category $\mathsf{Gr}(\mathcal{C})$ admits the following monoidal structure:*

(a) *The tensor product $X \otimes Y \in \mathsf{Gr}(\mathcal{C})$ of $X = (X_g)_{g\in G}$ and $Y = (Y_g)_{g\in G}$ is the tuple whose degree g is:*

$$(X \otimes Y)_g := \coprod_{p\in G} X_p \otimes Y_{p^{-1}g}. \tag{4.2.1}$$

(b) *The monoidal unit $\mathbb{1}_{\mathsf{Gr}(\mathcal{C})}$ has $\mathbb{1}_{\mathcal{C}}$ in degree e and the zero object 0 elsewhere:*

$$\mathbb{1}_{\mathsf{Gr}(\mathcal{C})} := (\delta_{g,e}\mathbb{1}_{\mathcal{C}})_{g\in G}.$$

The monoidal structure on $\mathsf{Gr}(\mathcal{C})$ outlined above is commonly refer to as the *Cauchy monoidal structure on graded objects of $\mathcal{C}$.*

Proof. For simplicity, we assume $\mathcal{C}$ is strict monoidal. We first verify that $\delta_{g,e}\mathbb{1}_{\mathcal{C}}$ serves as a monoidal unit for $\mathsf{Gr}(\mathcal{C})$. Given $X \in \mathsf{Gr}(\mathcal{C})$, take the tensor product with $\mathbb{1}_{\mathsf{Gr}(\mathcal{C})}$ and calculate degree $g \in G$ to obtain:

$$(\mathbb{1}_{\mathsf{Gr}(\mathcal{C})} \otimes X)_g := \coprod_{p\in G} (\mathbb{1}_{\mathcal{C}})_p \otimes X_{p^{-1}g} \cong \mathbb{1}_{\mathcal{C}} \otimes X_g = X_g.$$

For the isomorphism above, we use the fact that if 0 is an initial object, then there is a canonical isomorphism $0 \sqcup Y \cong Y$ for any $Y \in \mathcal{C}$. A similar calculation shows $X \otimes \mathbb{1}_{\mathsf{Gr}(\mathcal{C})} \cong X$ in $\mathsf{Gr}(\mathcal{C})$. Next, we verify the tensor product is associative:

$$\begin{aligned}
\big((X \otimes Y) \otimes Z\big)_g &\overset{(1)}{=} \coprod_{p\in G} (X \otimes Y)_p \otimes Z_{p^{-1}g} \overset{(2)}{=} \coprod_{p\in G} \left(\coprod_{q\in G} X_q \otimes Y_{q^{-1}p}\right) \otimes Z_{p^{-1}g} \\
&\overset{(3)}{=} \coprod_{p\in G}\coprod_{q\in G} X_q \otimes Y_{q^{-1}p} \otimes Z_{p^{-1}g} \overset{(4)}{=} \coprod_{p\in G}\coprod_{q\in G} X_q \otimes Y_p \otimes Z_{p^{-1}q^{-1}g} \\
&\overset{(5)}{=} \coprod_{q\in G}\coprod_{p\in G} X_p \otimes Y_q \otimes Z_{q^{-1}p^{-1}g} \overset{(6)}{=} \coprod_{p\in G} X_p \otimes \left(\coprod_{q\in G} Y_q \otimes Z_{q^{-1}p^{-1}g}\right)
\end{aligned}$$

$$\overset{(7)}{=} \coprod_{p \in G} X_p \otimes (Y \otimes Z)_{p^{-1}g} \overset{(8)}{=} \Big(X \otimes (Y \otimes Z)\Big)_g.$$

Equations 1,2,7 and 8 follow from definition of the Cauchy monoidal structure; equation three follows because $(- \otimes Z_{p^{-1}g})$ are left adjoint functors, which thus preserve coproducts; equation four follows by re-indexing $p \in G$ with $qp \in G$; equation five follows by swapping the roles of $p, q \in G$; and equation six follows by Hypothesis 4.0.1(c). □

Remark 4.2.3. The Cauchy monoidal structure on $\mathsf{Gr}(\mathcal{C})$ may be defined for any monoid G. In this case, the tensor product of $X, Y \in \mathsf{Gr}(\mathcal{C})$ is defined by:

$$(X \otimes Y)_g = \coprod_{pq=g} X_p \otimes Y_q.$$

Proposition 4.2.4. *The functor $(-)|_e : \mathsf{Gr}(\mathcal{C}) \to \mathcal{C}$ projecting graded objects and morphisms onto the neutral degree $e \in G$ is monoidal. The binary and nullary components of $(-)|_e$ are defined as follows:*

Nullary: $(-)|_e^0 : \mathbb{1}_{\mathcal{C}} \to \mathbb{1}_{\mathcal{C}} =: (\mathbb{1}_{\mathsf{Gr}(\mathcal{C})})|_e$ is the identity map,

Binary: $(-)|_e^2 : X_e \otimes Y_e \xrightarrow{\iota_{X,Y}} \coprod_{p \in G} X_p \otimes Y_{p^{-1}} =: (X \otimes_{\mathsf{Gr}(\mathcal{C})} Y)|_e,$

where the map $\iota_{X,Y}$ above is the canonical coproduct inclusion. □

Proof. Functoriality, left unitality, and right unitality of $(-)|_e$ follow trivially from definitions. Associativity (2.2.2) boils down to the statement that the two ways of including into a doubly indexed coproduct are equal:

$$\begin{array}{ccc} X_e \otimes Y_e \otimes Z_e & \xrightarrow{1 \otimes \iota_{Y,Z}} & X_e \otimes (Y \otimes Z)|_e \\ {\scriptstyle \iota_{X,Y} \otimes 1}\downarrow & & \downarrow{\scriptstyle \iota} \\ (X \otimes Y)|_e \otimes Z_e & \xrightarrow[\iota]{} & (X \otimes Y \otimes Z)|_e \end{array}$$

By checking the appropriate universal property, both paths above can be shown to be equal to the canonical coproduct inclusion $X \otimes Y_e \otimes Z_e \xrightarrow{\iota} (X \otimes Y \otimes Z)|_e$. □

Notation 4.2.5. Equipping $\mathsf{Gr}(\mathcal{C})$ with its Cauchy monoidal structure allows us to talk about algebras and modules in $\mathsf{Gr}(\mathcal{C})$. Following the same notation as before, we denote algebras by $(A, m, u) \in \mathsf{Alg}(\mathsf{Gr}(\mathcal{C}))$, with the added understanding that the structure maps $m : A \otimes A \to A$ and $u : \mathbb{1}_{\mathsf{Gr}(\mathcal{C})} \to A$ decompose by degrees into tuples $\left(m|_g : \coprod_{p\in G} A_p \otimes A_{p^{-1}g} \to A_g\right)_{g\in G}$ and $(u|_g : \delta_{g,e}\mathbb{1}_{\mathcal{C}} \to A_g)_{g\in G}$. In particular, since $\delta_{g,e}\mathbb{1}_{\mathcal{C}} = 0$ for all $g \neq e$, the data of the unit map amounts to a single morphism $u|_e : \mathbb{1} \to A_e$ in degree $e \in G$.

Theorem 4.2.6. *Let $\mathcal{C}$ be a monoidal category $\mathcal{C}$ satisfying Hypotheses 4.0.1.*

(a) We have an isomorphism of categories:

$$\mathsf{Alg}(\mathsf{Gr}(\mathcal{C})) \cong \mathsf{GrAlg}(\mathcal{C}).$$

(b) For any algebra $(A, m, u) \in \mathsf{Alg}(\mathsf{Gr}(\mathcal{C}))$, the corresponding G-graded algebra structure on A induces an isomorphism of categories:

$$\mathsf{Mod}(\mathsf{Gr}(\mathcal{C}))_A \cong \mathsf{GrMod}(\mathcal{C})_A.$$

Proof. (a) Given an algebra $(A, m, u) \in \mathsf{Alg}(\mathsf{Gr}(\mathcal{C}))$, we can take degrees to decompose the data of the multiplication map into a tuple of morphisms in $\mathcal{C}$:

$$\left(m|_q : \coprod_{\ell\in G} A_\ell \otimes A_{\ell^{-1}q} \to A_q\right)_{q\in G}.$$

Now considering the coproduct inclusions: $\left(\iota_{p,q} : A_p \otimes A_{p^{-1}q} \hookrightarrow \coprod_{\ell\in G} A_\ell \otimes A_{\ell^{-1}q}\right)_{p,q\in G}$, the universal property of the coproducts tells us that the multiplication map m is uniquely determined by the following collection of maps:

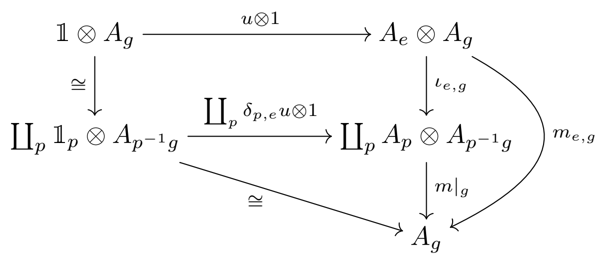
$$\left(m_{g,h} : A_g \otimes A_h \xrightarrow{\iota_{g,gh}} \coprod_{\ell \in G} A_\ell \otimes A_{\ell^{-1}gh} \xrightarrow{m_{gh}} A_{gh}\right)_{g,h \in G}.$$

Similarly, after taking degrees, the data of the unit morphism $u : \mathbb{1}_{\mathsf{Gr}(\mathcal{C})} \to A$ amounts to a single map $u|_e : \mathbb{1} \to A_e$, since for $g \neq e$ the maps $u|_g : 0 \to A_g$ are predetermined.

We define the isomorphism of categories by mapping:

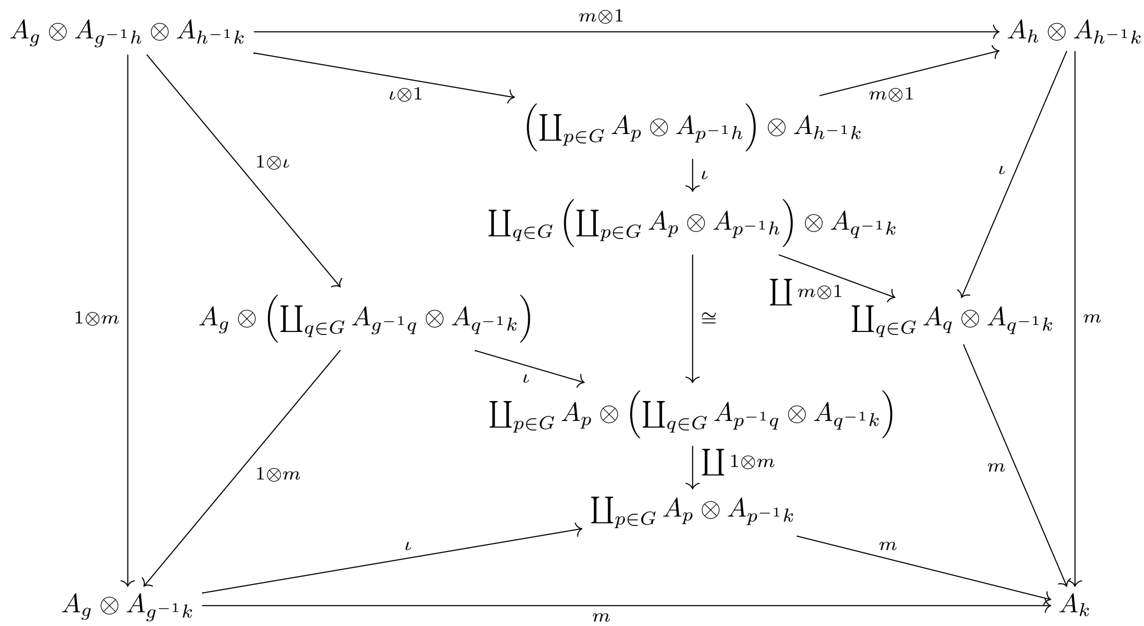
$$(A, m, u) \in \mathsf{Alg}(\mathsf{Gr}(\mathcal{C})) \mapsto (A, m_{g,h}, u|_e) \in \mathsf{GrAlg}(\mathcal{C}).$$

The left unital constraint (4.1.2) is shown below:



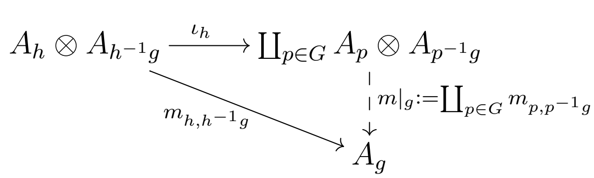
The square region commutes by universal property of coproducts; the right region commutes by definition; and the bottom triangle commutes by a unit axiom. Right unitality follows likewise.

The associativity constraint (4.1.1) is shown below:



All four outer triangles commute by definition; the top right and bottom left quadrangles commute by the universal property of coproducts; and the top left and bottom right pentagons commute by associativity of coproducts and multiplication, respectively.

Conversely, starting with a G-graded algebra $(\{A_g\}_{g\in G}, m_{g,h}, u) \in \mathsf{GrAlg}(\mathcal{C})$, we begin by setting $A = (A_g)_{g\in G}$. To define the product map $m : A \otimes A \to A$, we use the universal property of $\coprod_{p\in G} A_p \otimes A_{p^{-1}g}$ to define degree $g \in G$ of m as the coproduct of the morphisms $\{m_{p,p^{-1}g}\}_{p\in G}$.



We define the unit map $\mathbb{1} \to A$ in $\mathsf{Gr}(\mathcal{C})$ with $u : \mathbb{1} \to A_e$ in degree $e \in G$ and zero morphisms in other degrees. The associativity and unitality constraints for A follow from (4.1.1) and (4.1.2). Finally, it is a straightforward check that our assignments are mutually inverse. □

4.3 Closed graded categories

In this section, we give a sufficient condition for $\mathsf{Gr}(\mathcal{C})$ to be right-closed monoidal.

Theorem 4.3.1. *Suppose $\mathcal{C}$ is a right-closed monoidal category. When $\mathsf{Gr}(\mathcal{C})$ is equipped with the Cauchy tensor product, it is right-closed monoidal category. For $Y \in \mathsf{Gr}(\mathcal{C})$, the right adjoint of $(- \otimes Y)$ is defined as follows:*

$$\text{On objects:} \qquad \text{Degree } g \in G \text{ of } [Y, Z] \text{ is } \prod_{p \in G} [Y_{g^{-1}p}, Z_p].$$

$$\text{On morphisms:} \qquad \text{Degree } g \in G \text{ of } [Y, \varphi] \text{ is } \prod_{p \in G} [Y_{g^{-1}p}, \varphi|_p].$$

Remark 4.3.2. The above result is known in severals contexts, including higher category theory [AM14, AHLF18], perturbative quantum field theory [Nor20], and combinatorial species [Sch93, Joy06]. Several of the combinatorial contexts impose the restriction that G be finite, while higher-category contexts use the theory of Day convolutions, which require $\mathcal{C}$ be symmetric monoidal.

Here, we provide a constructive proof, under the simplifying assumption that our products and coproducts be biproducts. That is, for $X, Y \in \mathsf{Gr}(\mathcal{C})$, we let:

$$(X \otimes Y)_g = \bigoplus_{p \in G} X_p \otimes Y_{p^{-1}g} \qquad \text{and} \qquad [X, Y]_g = \bigoplus_{p \in G} [Y_{g^{-1}p}, Z_p],$$

in degree $g \in G$. If one wishes to maintain coproducts and products separate, the proof below should be altered to move between these structures whenever necessary using the canonical map presented in Notation 2.1.2(c).

Proof. We need to establish a natural bijection:

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Gr}(\mathcal{C})}(X \otimes Y, Z) &\cong \mathrm{Hom}_{\mathsf{Gr}(\mathcal{C})}(X, [Y, Z]) \\ \psi : X \otimes Y \to Z \quad &\mapsto \quad \psi^\flat = (\psi^\flat|_g)_{g \in G} \\ \varphi^\# = (\varphi^\#|_g)_{g \in G} \quad &\leftarrow\!\shortmid \quad \varphi : X \to [Y, Z] \end{aligned} \tag{4.3.1}$$

for all $X, Y, Z \in \mathsf{Gr}(\mathcal{C})$. Start with a morphism $\psi : X \otimes Y \to Z$. Taking degree $g \in G$ gives a map $\psi|_g : \bigoplus_{p \in G} X_p \otimes Y_{p^{-1}g} \to Z_g$. The universal property of the coproduct allows us to capture the data of $\psi|_g$ as a collection of maps $\{\psi_{g,h}\}_{h \in G}$ in $\mathcal{C}$:

$$\psi|_g : \bigoplus_{p \in G} X_p \otimes Y_{p^{-1}g} \to Z_g \quad \overset{1-1}{\leftrightsquigarrow} \quad \{\psi_{g,h} : X_h \otimes Y_{h^{-1}g} \to Z_g\}_{h \in G},$$

where $\psi_{g,h} := \psi|_g \circ \iota_{g,h}$ and $\iota_{g,h} : X_h \otimes Y_{h^{-1}g} \to \bigoplus_{p \in G} X_p \otimes Y_{p^{-1}g}$ is the canonical coproduct inclusion. In particular, using Notation 2.1.2(b) we express $\psi|_g = \bigoplus_{h \in G} \psi_{g,h}$.

Likewise, starting with a map $\varphi : X \to [Y, Z]$ and taking degree $g \in G$ gives a map $\varphi|_g : X_g \to \bigoplus_{p \in G} [Y_{g^{-1}p}, Z_p]$. By, the universal property of the product, the data of $\varphi|_g$ is equivalent to a collection of maps $\{\varphi_{g,h}\}$ in $\mathcal{C}$:

$$\varphi|_g : X_g \to \bigoplus_{p \in G} [Y_{g^{-1}p}, Z_p] \quad \overset{1-1}{\leftrightsquigarrow} \quad \{\varphi_{g,h} : X_g \to [Y_{g^{-1}h}, Z_h]\}_{h \in G},$$

where $\varphi_{g,h} := \pi_{g,h} \circ \varphi|_g$ and $\pi_{g,h} : \prod_{p \in G} [Y_{g^{-1}p}, Z_p] \to [Y_{g^{-1}h}, Z_h]$ is the canonical product projection. Using Notation 2.1.2(a), we write $\varphi|_g = \bigoplus_{h \in G} \varphi_{g,h}$.

Next, we establish the correspondence (4.3.1). For every pair $g, h \in G$, we define:

$$(\psi^\flat)_{g,h} := (\psi_{h,g})^\flat \qquad \text{and} \qquad (\varphi^\sharp)_{g,h} := (\varphi_{h,g})^\sharp,$$

where $(\psi_{h,g})^\flat$ and $(\varphi_{h,g})^\sharp$ are defined as the transposes of the maps $\psi_{h,g}$ and $\varphi_{h,g}$ above, using the right-closure of $\mathcal{C}$. We use these to define:

$$\psi^\flat|_g := \bigoplus_{h\in G}(\psi^\flat)_{g,h} = \bigoplus_{h\in G}(\psi_{h,g})^\flat \qquad \text{and} \qquad \varphi^\sharp|_g := \bigoplus_{h\in G}(\varphi^\sharp)_{g,h} = \bigoplus_{h\in G}(\varphi_{h,g})^\sharp.$$

From these definitions, checking the bijectivity of the correspondence is routine. For example:

$$(\psi^\flat)^\sharp|_g = \bigoplus_{h\in G}((\psi^\flat)_{h,g})^\sharp = \bigoplus_{h\in G}((\psi_{g,h})^\flat)^\sharp = \bigoplus_{h\in G}\psi_{g,h} =: \psi|_g.$$

Similarly $(\varphi^\sharp)^\flat|_g = \varphi|_g$.

Finally, we check the correspondence is natural. Given morphisms $\chi : X' \to X$ and $\zeta : Z \to Z'$ in $\mathsf{Gr}(\mathcal{C})$, this amounts to checking the following squares commute:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{Gr}(\mathcal{C})}(X\otimes Y, Z) & \xrightarrow{\text{1-1}} & \mathrm{Hom}_{\mathsf{Gr}(\mathcal{C})}(X, [Y,Z]) \\
{\scriptstyle (\chi\otimes 1_Y)^*}\big\downarrow & & \big\downarrow{\scriptstyle \chi^*} \\
\mathrm{Hom}_{\mathsf{Gr}(\mathcal{C})}(X'\otimes Y, Z) & \xrightarrow[\text{1-1}]{} & \mathrm{Hom}_{\mathsf{Gr}(\mathcal{C})}(X', [Y,Z])
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{Gr}(\mathcal{C})}(X, [Y,Z]) & \xrightarrow{\text{1-1}} & \mathrm{Hom}_{\mathsf{Gr}(\mathcal{C})}(X\otimes Y, Z) \\
{\scriptstyle [Y,\zeta]_*}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_*} \\
\mathrm{Hom}_{\mathsf{Gr}(\mathcal{C})}(X, [Y,Z']) & \xrightarrow[\text{1-1}]{} & \mathrm{Hom}_{\mathsf{Gr}(\mathcal{C})}(X\otimes Y, Z')
\end{array}$$

We check naturality in X here:

$$(\psi\circ(\chi\otimes 1_Y))^\flat|_g = \bigoplus_{h\in G}((\psi\circ(\chi\otimes 1_Y))_{h,g})^\flat = \bigoplus_{h\in G}(\psi_{h,g}\circ(\chi_g\otimes 1_{Y_{g^{-1}h}}))^\flat$$
$$\overset{(*)}{=} \bigoplus_{h\in G}(\psi_{h,g})^\flat\circ\chi_g = (\psi^\flat\circ\chi)|_g.$$

Here, equation $(*)$ follows from naturality in $\mathcal{C}$. Naturality in Z follows similarly. $\square$

If $\mathcal{C}$ has (finite) limits, then so does $\mathsf{Gr}(\mathcal{C})$ by Lemma 4.2.1. In particular, the results from Section 3.2 apply to $\mathsf{Gr}(\mathcal{C})$.

Corollary 4.3.3. *Let $\mathcal{C}$ satisfy Hypotheses 4.0.1. Then $\mathsf{Gr}(\mathcal{C})$ admits an enrichment over $\mathcal{C}$, obtained by equipping $\mathsf{Gr}(\mathcal{C})$ with its right-closed self-enriched structure, then changing base along the projection functor $(-)|_e : \mathsf{Gr}(\mathcal{C}) \to \mathcal{C}$.* □

Corollary 4.3.4. *Let $\mathcal{C}$ satisfy Hypotheses 4.0.1 and let $A \in \mathsf{GrAlg}(\mathcal{C})$. Then:*

(a) *The category $\mathsf{GrMod}(\mathcal{C})_A$ is enriched over $\mathsf{Gr}(\mathcal{C})$.*

(b) *The category $\mathsf{GrMod}(\mathcal{C})_A$ admits an enrichment over $\mathcal{C}$ by changing base along the projection functor $(-)|_e : \mathsf{Gr}(\mathcal{C}) \to \mathcal{C}$.*

Proof. Part (a) follows by applying Theorem 3.2.3 to $\mathsf{Gr}(\mathcal{C})$. For part (b), recall that $(-)|_e$ is a monoidal functor (Proposition 4.2.4). Thus, we may apply Lemma 2.3.5 to $(-)|_e$ and the enrichment in part (a) to get part (b). □

Remark 4.3.5. When viewing $\mathsf{GrMod}(\mathcal{C})_A$ as a $\mathsf{Gr}(\mathcal{C})$-category, the Hom-object between two graded modules $M, N \in \mathsf{GrMod}(\mathcal{C})_A$ is $[M, N]_A$, which we defined in (3.2.3) as the equalizer of the maps $\mu^{\flat}_{M,N}$ and $\nu^{\flat}_{M,N}$ defined in Definition 3.2.2.

In contrast, when viewing $\mathsf{GrMod}(\mathcal{C})_A$ as a $\mathcal{C}$-category, the Hom-object from M to N is the projection $[M, N]_A|_e$ of the aforementioned object onto degree $e \in G$.

Using the $\mathcal{C}$-enrichments of the categories of graded modules, we now define a way to compare when graded algebras have "the same" categories of graded modules.

Definition 4.3.6. Let $A, B \in \mathsf{GrAlg}(\mathcal{C})$ be graded algebras. Equip their categories of graded modules with the $\mathcal{C}$-enrichments of Corollary 4.3.4(b). We call A and B *Zhang-Morita equivalent* if there is an enriched equivalence of $\mathcal{C}$-categories:

$$\mathsf{GrMod}(\mathcal{C})_A \simeq \mathsf{GrMod}(\mathcal{C})_B.$$

4.4 Graded categories with shifts

We begin with a definition.

Definition 4.4.1. A category $\mathcal{E}$ enriched over $\mathcal{D}$ is G**-graded** if it comes equipped with a family of $\mathcal{D}$-enriched endofunctors $(\sigma_g : \mathcal{E} \xrightarrow{\mathcal{D}} \mathcal{E})_{g \in G}$ satisfying the following:

(a) σ_e is the identity $\mathcal{D}$-functor,

(b) σ_g is fully faithful for all $X \in \mathcal{E}$ and $g \in G$,

(c) $\sigma_g \sigma_h(X) = \sigma_{gh}(X)$ for all $X, Y \in \mathcal{E}$ and $g, h \in G$,

(d) $(\sigma_g)_{\sigma_h X, \sigma_h Y}(\sigma_h)_{X,Y} = (\sigma_{gh})_{X,Y}$ for all $X, Y \in \mathcal{E}$ and $g, h \in G$.

We call σ_g the **shift functor** of $\mathcal{E}$ in degree $g \in G$.

Our goal for this section is to show the categories $\mathsf{GrMod}(\mathcal{C})_A$ of graded modules are G-graded $\mathcal{C}$-categories when equipped with the enrichment of Corollary 4.3.4(b). As a first step, we begin by showing that $\mathsf{Gr}(\mathcal{C})$ is a G-graded $\mathcal{C}$-category when equipped with the enriched structure of Corollary 4.3.3.

Proposition 4.4.2. *The following assignments give endofunctors* $S_g : \mathsf{Gr}(\mathcal{C}) \to \mathsf{Gr}(\mathcal{C})$ *for each* $g \in G$*:*

$$\text{On objects:} \qquad X = (X_d)_{d \in G} \mapsto S_g(X) = (X_{g^{-1}d})_{d \in G}.$$

$$\text{On morphisms:} \qquad \varphi = (\varphi_d)_{d \in G} \mapsto S_g(\varphi) = (\varphi_{g^{-1}d})_{d \in G}.$$

Furthermore, for each graded algebra $A \in \mathsf{GrAlg}(\mathcal{C})$*, these endofunctors lift to endofunctors* $S_g : \mathsf{GrMod}(\mathcal{C})_A \to \mathsf{GrMod}(\mathcal{C})_A$ *by mapping:*

$$(M, \rho^M) \mapsto \left(S_g M, \rho^{S_g M} := S_g(\rho^M)\right),$$

for each graded module $(M, \rho^M) \in \mathsf{GrMod}(\mathcal{C})_A$.

Proof. Let $X \in \mathsf{Gr}(\mathcal{C})$ and $g \in G$. Then for all $d \in G$, we have:

$$S_g(1_X)|_d = 1_X|_{g^{-1}d} = 1_{X_{g^{-1}d}} = 1_{S_g(X)|_d},$$

which proves that $S_g(1_X) = 1_{S_g(X)}$. Next, for all $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ in $\mathsf{Gr}(\mathcal{C})$, we have:

$$S_g(\psi\varphi)|_d = (\psi\varphi)|_{g^{-1}d} = \psi|_{g^{-1}d}\varphi|_{g^{-1}d} = S_g(\psi)|_d S_g(\varphi)_d = (S_g(\psi) \circ S_g(\varphi))|_d.$$

This proves each S_g defines an endofunctor of $\mathsf{Gr}(\mathcal{C})$. The same computations show that S_g is functorial in $\mathsf{GrMod}(\mathcal{C})_A$. Finally, note that (S_gM, ρ^{S_gM}) is indeed an object in $\mathsf{GrMod}(\mathcal{C})_A$. This follows since the module associativity and identity axioms of (S_gM, ρ^{S_gM}) in degree $d \in G$ are precisely those of (M, ρ^M) in degree $g^{-1}d$. □

Example 4.4.3. When $G = \mathbb{Z}$, the functor $S_n : \mathsf{Gr}(\mathcal{C}) \to \mathsf{Gr}(\mathcal{C})$ shifts sequences "to the left" by n, in the sense that $S_n(X)|_m = (X_{m-n})$ for $m, n \in \mathbb{Z}$ and $X \in \mathsf{Gr}(\mathcal{C})$.

Proposition 4.4.4. *The functors $S_g : \mathsf{Gr}(\mathcal{C}) \to \mathsf{Gr}(\mathcal{C})$ satisfy:*

(a) $S_g(X \otimes Y) = S_g(X) \otimes Y$,

(b) $[X, S_gY] = S_g[X, Y]$,

(c) $[S_gX, Y]_d = [X, Y]_{dg}$ *for all* $d \in G$,

(d) $[S_gX, S_gY]_d = [X, Y]_{g^{-1}dg}$ *for all* $d \in G$.

for all $X, Y \in \mathsf{Gr}(\mathcal{C})$. Furthermore, for each graded algebra $A \in \mathsf{GrAlg}(\mathcal{C})$, the functors $S_g : \mathsf{GrMod}(\mathcal{C})_A \to \mathsf{GrMod}(\mathcal{C})_A$ satisfy:

(e) $[M, S_gN]_A = S_g[M, N]_A$;

(f) $[S_gM, N]_A|_d = [M, N]_A|_{dg}$, *for all* $d \in G$;

(g) $[S_gM, S_gN]_A|_d = [M, N]_A|_{g^{-1}dg}$, *for all* $d \in G$.

Proof. Parts (a)-(d) follow from the definition of the tensor product (4.2.1) in $\mathsf{Gr}(\mathcal{C})$ and those of Hom-objects in $\mathsf{Gr}(\mathcal{C})$ (see Theorem 4.3.1). For example, for part (a):

$$S_g(X \otimes Y)|_d := X \otimes Y|_{g^{-1}d} = \coprod_{p \in G} X_p \otimes Y_{p^{-1}g^{-1}d} \overset{(*)}{=} \coprod_{p \in G} X_{g^{-1}p} \otimes Y_{p^{-1}d}$$
$$= \coprod_{p \in G} S_g(X)|_p \otimes Y_{p^{-1}d} =: S_g(X) \otimes Y|_d,$$

where equality $(*)$ above follows by reindexing the coproduct using the substitution $p \mapsto g^{-1}p$. We can use the same substitution to prove part (b):

$$S_g([X,Y]|_d := [X,Y]_{g^{-1}d} = \prod_{p \in G} [X_{d^{-1}gp}, Y_p] \overset{(*)}{=} \prod_{p \in G} [X_{d^{-1}p}, Y_{g^{-1}p}]$$
$$= \prod_{p \in G} [X_{d^{-1}p}, S_g(Y)|_p] =: [X, S_g(Y)]|_d.$$

Parts (c) and (d) are proven similarly.

For parts (e)-(f), first recall that $[M,N]_A$ was defined as the equalizer of $\mu^\flat_{M,N}$ and $\nu^\flat_{M,N}$, where:

$$\mu_{M,N} = \epsilon^M_N(1_{[M,N]} \otimes \rho^M) \quad \text{and} \quad \nu_{M,N} = \rho^N(\epsilon^M_N \otimes 1_A).$$

It follows that $S_g[M,N]_A$ is an equalizer of $S_g(\mu^\flat_{M,N})$ and $S_g(\mu^\flat_{M,N})$. Similarly, we define $[M, S_gN]_A$ as the equalizer of $\mu^\flat_{M,S_gN}$ and $\nu^\flat_{M,S_gN}$. We show $S_g[M,N]_A = [M, S_gN]_A$ by showing that $S_g(\mu^\flat_{M,N}) = \mu^\flat_{M,S_gN}$ and $S_g(\nu^\flat_{M,N}) = \nu^\flat_{M,S_gN}$, so that both objects are equalizers for the same pairs of maps.

Taking degree $d \in G$ of these maps, we obtain:

$$S_g(\mu^\flat_{M,N})|_d = \mu^\flat_{M,N}|_{g^{-1}d} = [M \otimes A, \epsilon^M_N(1_{[M,N]} \otimes \rho^M)]|_{g^{-1}d} \circ \eta^{M \otimes A}_{[M,N]}|_{g^{-1}d},$$
$$\mu^\flat_{M,S_gN} = [M \otimes A, \epsilon^M_{S_gN}(1_{[M,S_gN]} \otimes \rho^M)]|_d \circ \eta^{M \otimes A}_{[M,S_gN]}|_d$$

$$S_g(\nu^\flat_{M,N}) = [M \otimes A, \rho^N(\epsilon^M_N \otimes 1_A)]|_{g^{-1}d} \circ \eta^{M\otimes A}_{[M,N]}$$

$$\nu^\flat_{M,S_gN} = [M \otimes A, \rho^{S_gN}(\epsilon^M_{S_gN} \otimes 1_A)]|_d \circ \eta^{M\otimes A}_{[M,S_gN]}.$$

Using Proposition 4.4.4(a)-(d), unpacking definitions, and reindexing (co)products when necessary like we did in the proofs of Proposition 4.4.4(a), we can show that:

$$\eta^{M\otimes A}_{[M,N]}|_{g^{-1}d} = \prod_{p\in G} [(M\otimes A)|_{d^{-1}gp}, \iota_{p,g^{-1}d}]\, \eta^{(M\otimes A)|_{d^{-1}gp}}_{[M,S_gN]|_d} = \eta^{M\otimes A}_{[M,S_gN]}|_{g^{-1}d},$$

$$(1_{[M,N]} \otimes \rho^M)|_{g^{-1}d} = \coprod_{p\in G} 1_{[M,N]}|_{g^{-1}p} \otimes \rho^M|_{p^{-1}d} = (1_{[M,S_gN]} \otimes \rho^M)|_d$$

$$\epsilon^M_N|_{g^{-1}d} = \coprod_{p\in G} \epsilon^{M_{p^{-1}d}}_{N_{g^{-1}d}}(\pi_{g^{-1}p,g^{-1}d} \otimes 1_{M_{p^{-1}d}}) = \epsilon^M_{S_gN}|_d$$

$$\rho^N|_{g^{-1}d} = \rho^{S_gN}|_d.$$

Finally, we can use these substitutions to conclude that $S_g(\mu^\flat_{M,N}) = \mu^\flat_{M,S_gN}$ and $S_g(\nu^\flat_{M,N}) = \nu^\flat_{M,S_gN}$. The proof of part (f) follows using similar strategies. Finally part (g) follows directly from parts (e) and (f). □

Example 4.4.5. Consider $\mathsf{Gr}(\mathcal{C})$ as the $\mathcal{C}$-category of Corollary 4.3.3. Then, $\mathsf{Gr}(\mathcal{C})$ is a G-graded $\mathcal{C}$-category when equipped with the shift functors $(S_g)_{g\in G}$ of Proposition 4.4.2. To verify this, we first need to recast $(S_g)_{g\in G}$ as $\mathcal{C}$-enriched functors, by defining their action on hom-objects. For each pair, $X, Y \in \mathsf{Gr}(\mathcal{C})$, we need to pick morphisms in $\mathcal{C}$:

$$(S_g)_{X,Y} : [X,Y]|_e \to [S_gX, S_Y]|_e$$

By Proposition 4.4.2(d), notice that $[S_gX, S_Y]|_e = [X,Y]|_e$, so we can set $(S_g)_{X,Y}$ to be the identity morphism $1_{[X,Y]|_e}$. From this, we see that parts (a), (b), and (d) of

Definition 4.4.1 are immediately satisfied. Part (c) is verified in the next computation:

$$S_g(S_h(X))|_d = S_h(X)|_{g^{-1}d} = X|_{h^{-1}g^{-1}d} = X_{(gh)^{-1}d} = S_{gh}X|_d.$$

Finally, the following example shows that shift functors (S_g) on $\mathsf{Gr}(\mathcal{C})$ also make the category $\mathsf{GrMod}(\mathcal{C})_A$ of graded modules a graded $\mathcal{C}$-category.

Example 4.4.6. Consider $\mathsf{GrMod}(\mathcal{C})_A$ as the $\mathcal{C}$-category of Corollary 4.3.4(b). Then, $\mathsf{GrMod}(\mathcal{C})_A$ is a G-graded $\mathcal{C}$-category when equipped with the shift functors $(S_g)_{g\in G}$ of Proposition 4.4.2.

Like above, we first recast $(S_g)_{g\in G}$ as $\mathcal{C}$-enriched functors. By Proposition 4.4.4(g), notice that $[S_gM, S_gN]_A|_e = [M,N]_A|_e$. Thus, we may set the action of S_g on hom-objects $(S_g)_{M,N} : [S_gM, S_gN]_A|_e \to [M,N]_A|_e$ as the identity morphism $1_{[M,N]_A|_e}$, for all modules $M, N \in \mathsf{GrMod}(\mathcal{C})_A$. Parts (a), (b), and (d) of Definition 4.4.1 follow immediately since $(S_g)_{M,N} := 1_{[M,N]_A}|_e$. Part (c) follows from a computation analogous to the one in Example 4.4.5. Thus, $\mathsf{GrMod}(\mathcal{C})_A$ is a G-graded $\mathcal{C}$-category.

Chapter 5

Twists

In this chapter, we find necessary and sufficient conditions for when algebras are Zhang-Morita equivalent. To this effect, we generalize [Zha96]'s notion of a twisting system to the setting of right-closed monoidal categories.

5.1 Twisted algebras and modules

For this section, we fix a right-closed monoidal category $\mathcal{C}$ satisfying Hypotheses 4.0.1, a group G with neutral element $e \in G$, and a G-graded algebra $(A, m_{g,h}, u) \in \mathsf{GrAlg}(\mathcal{C})$. Finally, recall that graded morphisms $\varphi : A \to A$ in $\mathsf{Gr}(\mathcal{C})$ consist of tuples of morphisms $\{\varphi|_g : A_g \to A_g\}_{g \in G}$, where we call $\varphi|_g$ the g-th degree of φ.

Definition 5.1.1. A **twisting system** on A consists of a collection of isomorphisms $\tau := \{\tau_g : A \to A\}_{g \in G}$ in $\mathsf{Gr}(\mathcal{C})$, satisfying the following **twisting condition** for all $g, h, k \in G$:

$$\tau_g|_{hk} \circ m_{h,k}(1_{A_h} \otimes \tau_h|_k) = m_{h,k}(\tau_g|_h \otimes \tau_{gh}|_k). \tag{5.1.1}$$

We can express the twisting condition in various other ways, as showcased in the following proposition, which generalizes [Zha96, (2.1.2), (2.1.3), (2.1.4)].

Proposition 5.1.2. *Let $\tau = \{\tau_g\}$ be a twisting system on A. Then (5.1.1) holding for all $g, h, k \in G$ is equivalent any of the following holding for all $g, h, k \in G$:*

$$\tau_g|_{hk} \circ m_{h,k} = m_{h,k}\,(\tau_g|_h \otimes \tau_{gh}|_k\,\tau_h^{-1}|_k) \tag{5.1.2}$$

$$m_{h,k}\,(\tau_g^{-1}|_h \otimes \tau_h|_k) = \tau_g^{-1}|_{hk} \circ m_{h,k}\,(1_{A_h} \otimes \tau_{gh}|_k) \tag{5.1.3}$$

$$m_{h,k}\,(\tau_g^{-1}|_h \otimes \tau_h|_k\,\tau_{gh}^{-1}|_k) = \tau_g^{-1}|_{hk} \circ m_{h,k} \tag{5.1.4}$$

We refer to the equations above as the ***second, third, and fourth twisting conditions****, respectively.*

Proof. Starting with (5.1.1), we may precompose by $(1_{A_h} \otimes \tau_h^{-1}|_k)$ to obtain (5.1.2). Starting with (5.1.3), precomposing by $(\tau_g|_h \otimes 1_{A_k})$ and postcomposing by $\tau_g|_{hk}$ returns (5.1.1). Finally, starting with (5.1.4) and precomposing by $(1_{A_h} \otimes \tau_{gh}^{-1}|_k)$ returns (5.1.3). □

The next proposition explores how twisting systems interact with the unit of the algebra $A \in \mathsf{GrAlg}(\mathcal{C})$. This proposition generalizes [Zha96, Proposition 2.2]

Proposition 5.1.3. *Let $\tau = \{\tau_g\}_{g\in G}$ be a twisting system on A.*

(a) *For all $g \in G$: $u = m_{e,e}\,(\tau_g^{-1}|_e \otimes \tau_e|_e)\,(u \otimes u)$.*
(b) *For all $g \in G$: $\tau_g^{-1}|_e \circ u = \tau_e^{-1}|_e \circ u$.*
(c) *For all $g \in G$: $\tau_e|_g = m_{e,g}\,(\tau_e|_e\,u \otimes 1_{A_g})$.*
(d) *For all $g \in G$, if $\tau_e|_e\,u = u$, then $\tau_e = 1_A$ and $\tau_g|_e\,u = u$.*

Proof. For (a) we have:

$$\begin{aligned} u &= \tau_g^{-1}|_e \circ 1_{A_e} \circ \tau_g|_e \circ u \\ &= \tau_g^{-1}|_e \circ m_{e,e}\,(u \otimes 1_{A_e})\,(1_{\mathbb{1}} \otimes \tau_g|_e u) \\ &= \tau_g^{-1}|_e \circ m_{e,e}\,(1_{A_e} \otimes \tau_g|_e)\,(u \otimes u) \\ &= m_{e,e}\,(\tau_g^{-1}|_e \otimes \tau_e|_e)\,(u \otimes u), \end{aligned}$$

where the second equality follows from the left unitality constraint (4.1.2); the third equality follows from level exchange (2.2.1); and the fourth equality follows from (5.1.3) with $h = k = e$. For part (b), begin by noticing that:

$$\begin{aligned} u &= \tau_g^{-1}|_e \circ \tau_g|_e \circ u \\ &= \tau_g^{-1}|_e \circ m_{e,e} \left(1_{A_e} \otimes u\right) \left(\tau_g|_e \, u \otimes 1_{\mathbb{1}}\right) \\ &= m_{e,e} \left(\tau_g^{-1}|_e \otimes \tau_e|_e \, \tau_g^{-1}|_e\right) \left(\tau_g|_e \, u \otimes u\right) \\ &= m_{e,e} \left(u \otimes \tau_e|_e \, \tau_g^{-1}|_e \, u\right) \\ &= \tau_e|_e \, \tau_g^{-1}|_e u. \end{aligned}$$

Here, the second and last equalities holds by unital constraints (4.1.2). The third equality holds by level exchange and the twisting condition (5.1.4) with $h = k = e$. Applying $\tau_e^{-1}|_e$ to both sides of the equation above proves part (b). For part (c):

$$\begin{aligned} \tau_e|_g &= \tau_e|_g \, m_{e,g} \left(u \otimes 1_{A_g}\right) \\ &= \tau_e|_g \, m_{e,g} \left(u \otimes \tau_e|_g \, \tau_e^{-1}|_g\right) \\ &= \tau_e|_g \, m_{e,g} \left(1_{A_e} \otimes \tau_e|_g\right) \left(u \otimes \tau_e^{-1}|_g\right) \\ &= m_{e,g} \left(\tau_e|_e \otimes \tau_e|_g\right) \left(u \otimes \tau_e^{-1}|_g\right) \\ &= m_{e,g} \left(\tau_e|_e \, u \otimes 1_{A_g)}\right). \end{aligned}$$

Here, the first equality holds by unital constraint (4.1.2); the third equality holds by level exchange; and the fourth equality holds by twisting condition (5.1.1).

For part (d), suppose that $\tau_e|_e \, u = u$. Applying $\tau_g|_e \, \tau_e^{-1}|_e$ to both sides results in:

$$\tau_g|_e \, u = \tau_g|_e \, \tau_e^{-1}|_e \, u = \tau_g|_e \, \tau_g^{-1}|_e \, u = u,$$

where the second equality follows from Proposition 5.1.3(b). Finally, if $\tau_e|_e \, u = u$, then:

$$\tau_e|_g\, u = m_{e,g}\left(\tau_e|_e\, u \otimes 1_{A_g}\right) = m_{e,g}\left(u \otimes 1_{A_g}\right) = 1_{A_g},$$

where the first equality follows from Proposition 5.1.3(c), and the third equality follows from the left unitality constraint (4.1.2) of A. □

Proposition 5.1.4. *Every twisting system τ on $(A, m, u) \in \mathsf{GrAlg}(\mathcal{C})$ gives rise to an algebra $(A^\tau, m^\tau, u^\tau) \in \mathsf{GrAlg}(\mathcal{C})$, whose underlying object is $A^\tau := (A_g)_{g\in G}$, and whose structure maps are given by:*

$$\begin{aligned} \text{Product:} \qquad & m^\tau_{g,h} : A_g \otimes A_h \xrightarrow{1\otimes \tau_g|_h} A_g \otimes A_h \xrightarrow{m_{g,h}} A_{gh}, \\ \text{Unit:} \qquad & u^\tau : \mathbb{1} \xrightarrow{u} A_e \xrightarrow{\tau_e^{-1}|_e} A_e. \end{aligned}$$

We call A^τ a ***twisted algebra*** *of A, or say A^τ is the* ***twist*** *of A by τ.*

Proof. Associativity of the twisted product m^τ is verified below:

$$\begin{array}{ccccc}
A_g \otimes A_h \otimes A_\ell & \xrightarrow{1\otimes\tau_g|_h\otimes 1} & A_g \otimes A_h \otimes A_\ell & \xrightarrow{m_{g,h}\otimes 1} & A_{gh} \otimes A_\ell \\
\downarrow{\scriptstyle 1\otimes 1\otimes \tau_h|_\ell} & & \downarrow{\scriptstyle 1\otimes 1\otimes \tau_{gh}|_\ell} & & \downarrow{\scriptstyle 1\otimes \tau_{gh}|_\ell} \\
A_g \otimes A_h \otimes A_\ell & & A_g \otimes A_h \otimes A_\ell & \xrightarrow[m_{g,h}\otimes 1]{} & A_{gh} \otimes A_\ell \\
\downarrow{\scriptstyle 1\otimes m_{h,\ell}} & & \downarrow{\scriptstyle 1\otimes m_{h,\ell}} & & \downarrow{\scriptstyle m_{gh,\ell}} \\
A_g \otimes A_{h\ell} & \xrightarrow[1\otimes\tau_g(h\ell)]{} & A_g \otimes A_{h\ell} & \xrightarrow[m_{g,h\ell}]{} & A_{gh\ell}
\end{array}$$

Here, the left square commutes by the twisting condition, the top right square commutes by level exchange, and the bottom right square commutes by associativity of $m_{g,h}$.

The left unital constraint on the twisted unit is shown below:

$$m^\tau_{e,g}\left(u^\tau \otimes 1_{A_g}\right) = m_{e,g}\left(1_{A_g} \otimes \tau_e|_g\right)\left(\tau_e^{-1}|_e \otimes 1_{A_g}\right)\left(u \otimes 1_{A_g}\right)$$

$$= m_{e,g}\,(\tau_e^{-1}|_e \otimes \tau_e|_g)\,(u \otimes 1_{A_g})$$
$$= \tau_e^{-1}|_g \circ m_{e,g}\,(1_{A_e} \otimes \tau_e|_g)\,(u \otimes 1_{A_g})$$
$$= \tau_e^{-1}|_g \circ m_{e,g}\,(u \otimes 1_{A_g})\,(1_{A_e} \otimes \tau_e|_g)$$
$$= \tau_e^{-1}|_g\,\tau_e|_g = 1_{A_g}.$$

The second and fourth equalities follow from level exchange (2.2.1); the third equality follows from the third twisting condition (5.1.3); and the fifth equality follows from the left unitality (4.1.2) of A. The right unital constraint is shown below:

$$m^\tau_{g,e}\,(1_{A_g} \otimes u^\tau) := m_{g,e}\,(1 \otimes \tau_g|_e)\,(1_{A_g} \otimes \tau_e^{-1}|_e)\,(1_{A_g} \otimes u)$$
$$= m_{g,e}\,(1_{A_g} \otimes \tau_g|_e)\,(1_{A_g} \otimes \tau_g^{-1}|_e)\,(1_{A_g} \otimes u)$$
$$= m_{e,g}\,(u \otimes 1_{A_g}) = 1_{A_g}.$$

Here, the second equality follows from Proposition 5.1.3(b), and the fourth equality follows from the right unitality (4.1.2) of A. □

The next result generalizes [Zha96, Proposition 2.4], and proves that we may always safely assume that a twisted algebra A^τ of A has the same unit as A.

Proposition 5.1.5. *For every twisting system τ on A, there is another twisting system π on A, that further satisfies $u^\pi = u$, and whose twisted algebra A^π is isomorphic A^τ.*

Proof. Fix $\ell \in G$. Define the twisting system $\pi = \{\pi_g : A \to A\}_{g\in G}$ by $\pi_g := \tau_{\ell g}\,\tau_\ell^{-1}$. We prove that π satisfies the twisting condition below. Let $g, h, k \in G$. Then:

$$\pi_g|_{hk}\,m_{h,k}\,(1_{A_h} \otimes \pi_h|_k) \overset{(1)}{=} (\tau_{\ell g}\tau_\ell^{-1})|_{hk}\,m_{h,k}\,(1_{A_h} \otimes \tau_{\ell h}\tau_\ell^{-1})|_k$$
$$\overset{(2)}{=} \tau_{\ell g}|_{hk}\,\tau_\ell^{-1}|_{hk}\,m_{h,k}\,(1_{A_h} \otimes \tau_{\ell h}|_k\tau_\ell^{-1}|_k)$$

$$
\begin{aligned}
&\overset{(3)}{=} \tau_{\ell g}|_{hk}\, m_{h,k}\, (\tau_\ell^{-1}|_h \otimes \tau_h|_k\, \tau_{\ell h}^{-1}|_k)\, (1_{A_h} \otimes \tau_{\ell h}|_k\, \tau_\ell^{-1}|_k) \\
&\overset{(4)}{=} \tau_{\ell g}|_{hk}\, m_{h,k}\, (\tau_\ell^{-1}|_h \otimes \tau_h|_k)\, (1_{A_h} \otimes \tau_\ell^{-1}|_k) \\
&\overset{(5)}{=} \tau_{\ell g}|_{hk}\, m_{h,k}\, (1_{A_h} \otimes \tau_h|_k)\, (\tau_\ell^{-1}|_h \otimes \tau_\ell^{-1}|_k) \\
&\overset{(6)}{=} m_{h,k}\, (\tau_{\ell g}|_h \otimes \tau_{\ell gh}|_k)\, (\tau_\ell^{-1}|_h \otimes \tau_\ell^{-1}|_k) \\
&\overset{(7)}{=} m_{h,k}\, (\tau_{\ell g}|_h\, \tau_\ell^{-1}|_h \otimes \tau_{\ell gh}|_k\, \tau_\ell^{-1}|_k) \\
&\overset{(8)}{=} m_{h,k}\, (\pi_g|_h \otimes \pi_{gh}|_k).
\end{aligned}
$$

The first, second, and last equalities follow by the definition of π_g; the third equality follows by the fourth twisting condition (5.1.4); the fourth and seventh equality follows by composition; the fifth equality follows by level exchange (2.2.1); the sixth equality follows by the twisting condition (5.1.1). Thus, π is a twisting system on A.

Next, note that $\pi_e = \tau_\ell \tau_\ell^{-1} = 1_A$, so $u^\pi := \pi_e^{-1}|_e u = u$, as required. Finally, to show $A^\tau \cong A^\pi$, we show that $\tau_\ell = (\tau_\ell|_g : A_g \xrightarrow{\cong} A_g)_{g\in G}$ is a morphism of graded algebras from A^τ to A^π. First, we show τ_ℓ respects units:

$$
\tau_\ell|_e\, u^\tau = \tau_\ell|_e\, \tau_e^{-1}|_e\, u = \tau_\ell|_e\, \tau_\ell^{-1}|_e\, u = u = u^\pi,
$$

Here, the second equality follows from Proposition 5.1.3(b). Finally, we show τ_ℓ respects multiplication:

$$
\begin{aligned}
\tau_\ell|_{gh}\, m^\tau_{g,h} &= \tau_\ell|_{gh}\, m_{g,h}\, (1_{A_g} \otimes \tau_g|_h) \\
&= m_{g,h}\, (\tau_\ell|_g \otimes \tau_{\ell g}|_h) \\
&= m_{g,h}\, (1_{A_g} \otimes \tau_{\ell g}|_h)\, (\tau_\ell|_g \otimes 1_{A_h})\, (1_{A_g} \otimes \tau_\ell^{-1}|_h\, \tau_\ell|_h) \\
&= m_{g,h}\, (1_{A_g} \otimes \tau_{\ell g}|_h\, \tau_\ell^{-1}|_h)\, (\tau_\ell|_g \otimes \tau_\ell|_h) \\
&= m_{g,h}\, (1_{A_g} \otimes \pi_g|_h)\, (\tau_\ell|_g \otimes \tau_\ell|_h)
\end{aligned}
$$

$$= m^{\pi}_{g,h}\,(\tau_\ell|_g \otimes \tau_\ell|_h).$$

Here, the first and last equalities follow from the definition of the twisted multiplications; the second equality follows from the twisting condition (5.1.1); the third equality follows from level exchange (2.2.1) and introducing $1_{A_h} = \tau_\ell^{-1}|_h\,\tau_\ell|_h$; the fourth equality follows from level exchange; and the fifth equality follows from the definition of π_g. □

Finally, we show that twisting systems τ on A can twist A-modules in a manner analogous to how they twist the algebra A.

Proposition 5.1.6. *Let τ be a twisting system on A. Then for every graded A-module $(M,\rho) \in \mathsf{GrMod}(\mathcal{C})_A$, there is a graded A^τ-module $(M^\tau, \rho^\tau) \in \mathsf{GrMod}(\mathcal{C})_{A^\tau}$, whose underlying object is $M^\tau = M$ and whose action maps $\rho^\tau_{g,h}$ are given by:*

$$\rho^\tau_{g,h} : M_g \otimes A_h \xrightarrow{1\otimes\tau_g|_h} M_g \otimes A_h \xrightarrow{\rho_{g,h}} M_{gh}.$$

We call M^τ a ***twisted module*** *of M or say that M^τ is the* ***twist*** *of M by τ.*

Proof. We need to verify the action on M^τ is associative and unital. For associativity:

$$\begin{aligned}
\rho^\tau_{gh,k}\,(\rho^\tau_{g,h} \otimes 1_{A_k}) &= \rho_{gh,k}\,(1_{M_{gh}} \otimes \tau_{gh}|_k)\,(\rho_{g,h} \otimes 1_{A_k})\,(1_{M_g} \otimes \tau_g|_h \otimes 1_{A_k}) \\
&= \rho_{gh,k}\,(\rho_{g,h} \otimes 1_{A_k})\,(1_{M_g} \otimes \tau_g|_h \otimes \tau_{gh}|_k) \\
&= \rho_{g,hk}\,(1_{M_g} \otimes m_{h,k})\,(1_{M_g} \otimes \tau_g|_h \otimes \tau_{gh}|_k) \\
&= \rho_{g,hk}\,(1_{M_g} \otimes \tau_g|_{hk} m_{h,k})\,(1_{M_g} \otimes 1_{A_h} \otimes \tau_h|_k) \\
&= \rho^\tau_{g,hk}\,(1_{M_g} \otimes m^\tau_{h,k}).
\end{aligned}$$

The first and last equalities follow from the definitions of $\rho^\tau_{g,h}$ and $m^\tau_{g,h}$; the second by level exchange (2.2.1); the third from the associativity (4.1.3) of $\rho_{g,h}$; and the fourth from the twisting condition (5.1.1). Similarly, for unitality:

$$
\begin{aligned}
\rho^{\tau}_{g,e}\left(1_{M_g} \otimes u^{\tau}\right) &= \rho_{g,e}\left(1_{M_g} \otimes \tau_g|_e\right)\left(1_{M_g} \otimes \tau_e^{-1}|_e u\right) \\
&= \rho_{g,e}\left(1_{M_g} \otimes \tau_g|_e\, \tau_g^{-1}|_e\, u\right) \\
&= \rho_{g,e}\left(1_{M_g} \otimes u\right) = 1_{M_g},
\end{aligned}
$$

where the first equality follows from definitions; the second from Proposition 5.1.3(b); and the fourth by the unitality of $\rho_{g,h}$. □

5.2 Twist equivalence

The next definition uses twisting systems to relate graded algebras.

Definition 5.2.1. We call two graded algebras $A, B \in \mathsf{GrAlg}(\mathcal{C})$ **twist equivalent** if there is a twisting system τ on A inducing an isomorphism of graded algebras $B \cong A^{\tau}$.

Example 5.2.2. Given any algebra $A \in \mathsf{GrAlg}(\mathcal{C})$, the identity maps:

$$\tau^{\mathrm{id}} := \{1_A : A \to A\}_{g \in G}$$

form a twisting system on A called the *identity twisting system*.

Example 5.2.3. If $\tau = \{\tau_g : A \to A\}_{g \in G}$ is a twisting system on A, then

$$\tau^{-1} := \{\tau_g^{-1} : A \to A\}_{g \in G}$$

is a twisting system on the twisted algebra A^{τ} called the *inverse twisting system* of τ.

The twisting condition is verified below:

$$\begin{aligned}
\tau_g^{-1}|_{hk}\, m^{\tau}_{h,k}\,(1_{A_h}\otimes\tau_h^{-1}|_k) &= \tau_g^{-1}|_{hk}\, m_{h,k}\,(1_{A_h}\otimes\tau_h|_k)\,(1_{A_h}\otimes\tau_h^{-1}|_k)\\
&= \tau_g^{-1}|_{hk}\, m_{h,k}\\
&= m_{h,k}\,(\tau_g^{-1}|_h\otimes\tau_h|_k\,\tau_{gh}^{-1}|_k)\\
&= m_{h,k}\,(1_{A_h}\otimes\tau_h|_k)\,(\tau_g^{-1}|_h\otimes\tau_{gh}^{-1}|_k)\\
&= m^{\tau}_{h,k}\,(\tau_g^{-1}|_h\otimes\tau_{gh}^{-1}|_k),
\end{aligned}$$

where the first and last equalities follow by definition of m^τ; the third equality follows from the fourth twisting condition (5.1.4); and the fourth equality follows from level exchange (2.2.1). Notice that the twisted algebra $(A^\tau)^{\tau^{-1}} = A$ by definition.

Example 5.2.4. If τ is a twisting system on A, and σ is a twisting system on A^τ, then $\tau\sigma := \{\tau_g\sigma_g : A \to A\}_{g\in G}$ is a twisting system on A called the *composite twisting system* of τ and σ. The twisting condition is verified below:

$$\begin{aligned}
\tau_g|_{hk}\,\sigma_g|_{hk}\, m_{h,k}(1_{A_h}\otimes\tau_h|_k\,\sigma_h|_k) &= \tau_g|_{hk}\,\sigma_g|_{hk}\, m^{\tau}_{h,k}\,(1_{A_h}\otimes\sigma_h|_k)\\
&= \tau_g|_{hk}\, m^{\tau}_{h,k}\,(\sigma_g|_h\otimes\sigma_{gh}|_k)\\
&= \tau_g|_{hk}\, m_{h,k}\,(1_{A_h}\otimes\tau_h|_k)\,(\sigma_g|_h\otimes\sigma_{gh}|_k)\\
&= m_{h,k}\,(\tau_g|_h\otimes\tau_{gh}|_k)\,(\sigma_g|_h\otimes\sigma_{gh}|_k)\\
&= m_{h,k}\,(\tau_g|_h\,\sigma_g|_h\otimes\tau_{gh}|_k\,\sigma_{gh}|_k).
\end{aligned}$$

Here, the first and third equalities use the definition of the twisted product m^τ; and the second and fourth equalities use the twisting conditions of σ and τ.

As a consequence of the above three examples, we have the following result:

Proposition 5.2.5. *Twist equivalence defines an equivalence relation on* $\mathsf{GrAlg}(\mathcal{C})$. □

Next, we introduce an alternative characterization of twist equivalence that is more useful for computations. This generalizes [Zha96, Proposition 2.8].

Proposition 5.2.6. *Algebras $A, B \in \mathsf{GrAlg}(\mathcal{C})$ are twist equivalent if and only if there exist a collection of isomorphisms $\{\phi_g : B \xrightarrow{\cong} A\}_{g\in G}$ in $\mathsf{Gr}(\mathcal{C})$, satisfying:*

$$m^A_{h,k}\,(\phi_g|_h \otimes \phi_{gh}|_k) = \phi_g|_{hk}\, m^B_{h,k} \quad \text{and} \quad u^A = \phi_e|_e\, u^B, \tag{5.2.1}$$

for all $g, h, k \in G$.

Proof. Suppose τ is a twisting system on A and that $B \cong A^\tau$ via some isomorphism of graded algebras $\varphi = (\varphi|_h : B_h \to (A^\tau)_h = A_h)_{h\in G}$. Define:

$$\phi_g|_h : B_h \to A_h \quad \text{as} \quad \phi_g|_h := \tau_g|_h\, \varphi|_h.$$

We verify the first condition of (5.2.1):

$$\begin{aligned} m^A_{h,k}\,(\phi_g|_h \otimes \phi_{gh}|_k) &= m^A_{h,k}\,(\tau_g|_h\, \varphi|_h \otimes \tau_{gh}|_k\, \varphi|_k) \\ &= m^A_{h,k}\,(\tau_g|_h \otimes \tau_{gh}|_k)\,(\varphi|_h \otimes \varphi|_k) \\ &= \tau_g|_{hk}\, m^A_{h,k}\,(1_{A_h} \otimes \tau_h|_k)\,(\varphi|_h \otimes \varphi|_k) \\ &= \tau_g|_{hk}\, m^{A^\tau}_{h,k}\,(\varphi|_h \otimes \varphi|_k) \\ &= \tau_g|_{hk}\, \varphi|_{hk}\, m^B_{h,k} \\ &= \phi_g|_{hk}\, m^B_{h,k}. \end{aligned}$$

Here, the first and last equalities follow from the definition of $\phi_g|_h$; the second follows from level exchange (2.2.1); the third by the twisting condition (5.1.1); the fourth by the definition of the twisted product $m^{A^\tau}_{g,h}$; and the fifth since $\varphi : B \to A^\tau$ is an algebra map. Next, we verify the second condition of (5.2.1):

$$u^A = \tau_e|_e \, \tau_e^{-1}|_e \, u^A = \tau_e|_e \, u^{A^\tau} = \tau_e|_e \, \varphi|_e \, u^B = \phi_e|_e \, u^B.$$

The first equality follows from the definition of the twisted unit u^{A^τ}; the second since φ is an algebra map; and the third by the definition of $\phi_e|_e$.

Conversely, suppose we have isomorphisms $\{\phi_g : B \to A\}_{g \in G}$ satisfying (5.2.1). Define a twisting system on A by $\tau = \{\tau_g := \phi_g \, \phi_e^{-1} : A \to A\}_{g \in G}$. We check the twisting condition below:

$$\begin{aligned}
\tau_g|_{hk} \circ m^A_{h,k} \, (1_{A_h} \otimes \tau_h|_k) &\overset{(1)}{=} \phi_g|_{hk} \, \phi_e^{-1}|_{hk} \, m^A_{h,k} \, (1_{A_h} \otimes \phi_h|_k \, \phi_e^{-1}|_k) \\
&\overset{(2)}{=} \phi_g|_{hk} \, m^B_{h,k} \, (\phi_e^{-1}|_h \otimes \phi_h^{-1}|_k) \, (1_{A_h} \otimes \phi_h|_k \, \phi_e^{-1}|_k) \\
&\overset{(3)}{=} \phi_g|_{hk} \, m^B_{h,k} \, (\phi_e^{-1}|_h \otimes \phi_h^{-1}|_k \, \phi_h|_k \, \phi_e^{-1}|_k) \\
&\overset{(4)}{=} \phi_g|_{hk} \, m^B_{h,k} \, (\phi_e^{-1}|_h \otimes \phi_e^{-1}|_k) \\
&\overset{(5)}{=} m^A_{h,k} \, (\phi_g|_h \otimes \phi_{gh}|_k) \, (\phi_e^{-1}|_h \otimes \phi_e^{-1}|_k) \\
&\overset{(6)}{=} m^A_{h,k} \, (\phi_g|_h \, \phi_e^{-1}|_h \otimes \phi_{gh}|_k \, \phi_e^{-1}|_k) \\
&\overset{(7)}{=} m^A_{h,k} \, (\tau_g|_h \otimes \tau_{gh}|_k).
\end{aligned}$$

Here, the first and last equalities follow from the definition of τ_g; the second and fifth from (5.2.1); the third and sixth from level exchange (2.2.1); and the fourth from composition. Thus, τ is a twisting system on A.

Finally, we claim that $\phi_e : B \to A$ is an algebra isomorphism from B to A^τ, which makes them twist equivalent. First, we check ϕ_e respects multiplication:

$$\begin{aligned}
m^{A^\tau}_{g,h} \, (\phi_e|_g \otimes \phi_e|_h) &= m^A_{g,h} \, (1_{A_g} \otimes \tau_g|_h) \, (\phi_e|_g \otimes \phi_e|_h) \\
&= m^A_{g,h} \, (1_{A_g} \otimes \phi_g|_h \, \phi_e^{-1}|_h) \, (\phi_e|_g \otimes \phi_e|_h)
\end{aligned}$$

$$= m^A_{g,h}\,(\phi_e|_g \otimes \phi_g|_h)$$
$$= \phi_e|_{gh}\, m^B_{g,h}.$$

Here, the first equality follows from the definition of the twisted product $m^\tau_{g,h}$; the second from the definition of the twisting system τ; the third from level exchange (2.2.1); and the fourth from (5.2.1).

Finally, we check that ϕ_e respects units:

$$u^{A^\tau} := \tau_e^{-1}|_e\, u^A = u^A = \phi_e|_e\, u^B.$$

Here, the second equality follows since $\tau_e^{-1} := \phi_e\phi_e^{-1} = 1_A$, and the third equality follows from (5.2.1). Thus, ϕ_e is an algebra isomorphism, showing that $B \cong A^\tau$ and completing the proof. □

5.3 Zhang-Morita equivalence

In this section, we compare the notions of twist equivalence and Zhang-Morita equivalence. We begin by showing that twist equivalence always yields Zhang-Morita equivalence.

Theorem 5.3.1. *Let $A, B \in \mathsf{GrAlg}(\mathcal{C})$. If A and B are twist equivalent via some twisting system τ on A, then A and B are Zhang-Morita equivalent. In fact, we have an isomorphism of categories*

$$\mathsf{GrMod}(\mathcal{C})_A \cong \mathsf{GrMod}(\mathcal{C})_B.$$

Proof. Suppose $B \cong A^\tau$ for some twisting system τ on A. We construct an isomorphism of categories $\Phi : \mathsf{GrMod}(\mathcal{C})_A \xrightarrow{\cong} \mathsf{GrMod}(\mathcal{C})_{A^\tau}$ as follows:

$$\begin{aligned} \text{On objects:} \quad & \Phi(M,\rho) := (M^\tau, \rho^\tau), \\ \text{On morphisms} \quad & \Phi(\varphi : M \to N) := (\varphi : M^\tau \to N^\tau). \end{aligned}$$

By Proposition 5.1.6, $M^\tau = M$ as objects of $\mathsf{Gr}(\mathcal{C})$, and $(M^\tau, \rho^\tau) \in \mathsf{GrMod}(\mathcal{C})_{A^\tau}$. Furthermore, for each, $g, h \in G$ we verify that $\varphi : M^\tau \to N^\tau$ is a map of graded A^τ-modules:

$$\begin{aligned} \rho^\tau_{g,h}\,(\varphi_g \otimes 1_{A_h}) &= \rho_{g,h}\,(1_{M_g} \otimes \tau_g|_h)\,(\varphi_g \otimes 1_{A_h}) \\ &= \rho_{g,h}\,(\varphi_g \otimes 1_{A_h})\,(1_{M_g} \otimes \tau_g|_h) \\ &= \varphi_{gh}\,\rho_{g,h}\,(1_{M_g} \otimes \tau_g|_h) \\ &= \varphi_{gh}\ \rho^\tau_{g,h}. \end{aligned}$$

Here, the first and last equalities follow from the definition of ρ^τ, the second from level exchange (2.2.1), and the third since $\varphi : M \to N$ was a map of graded A-modules. The above work shows that Φ indeed define assignments from $\mathsf{GrMod}(\mathcal{C})_A$ to $\mathsf{GrMod}(\mathcal{C})_{A^\tau}$. Furthermore, since Φ acts as the identity on morphisms, the assignment is functorial.

Using the inverse twisting system τ^{-1} of Example 5.2.3, we can similarly construct a functor $\Phi^{-1} : \mathsf{GrMod}(\mathcal{C})_{A^\tau} \to \mathsf{GrMod}(\mathcal{C})_A$ mapping $(M,\rho) \mapsto (M^{\tau^{-1}}, \rho^{\tau^{-1}})$ and $\varphi \mapsto \varphi$. Since $\tau\,\tau^{-1} = 1_A = \tau^{-1}\,\tau$, it follows that $(M^\tau)^{\tau^{-1}} = M = (M^{\tau^{-1}})^\tau$, proving that Φ^{-1} is an inverse for Φ. □

Our next goal is to find when Zhang-Morita equivalence is sufficient to guarantee twist equivalence. Before we do this, we prove a couple auxiliary results, the first of which relies on the notion of G-graded enriched category defined in Section 4.4.

Proposition 5.3.2. *Let $\mathcal{E}$ be a G-graded $\mathcal{C}$-category with shift functors $(\sigma_g : \mathcal{E} \xrightarrow{\mathcal{C}} \mathcal{E})$. Then, for every object $X \in \mathcal{E}$, there is graded algebra $\Gamma(X)$ in $\mathcal{C}$, defined in degree $g \in G$ as the Hom-object in $\mathcal{C}$ from $\sigma_g X$ to X:*

$$\Gamma(X)|_g := \mathcal{C}(\sigma_g X, X).$$

The unit map of $\Gamma(X)$ is defined as the identity morphism of X:

$$u^{\Gamma(X)} : \mathbb{1}_{\mathcal{C}} \xrightarrow{\mathrm{id}_X} \mathcal{C}(X, X) = \Gamma(X)|_e.$$

For each pair $g, h \in G$, the multiplication morphism $m^{\Gamma(X)}_{g,h}$ of $\Gamma(X)$ is defined as:

$$\Gamma(X)|_g \otimes \Gamma(X)|_h \xrightarrow{1 \otimes \sigma_g} \mathcal{C}(\sigma_g X, X) \otimes \mathcal{C}(\sigma_{gh} X, \sigma_g X) \xrightarrow{c} \mathcal{C}(\sigma_{gh} X, X) = \Gamma(X)|_{gh}.$$

Proof. The following computation shows the multiplication morphisms satisfy associativity (4.1.1):

$$\begin{aligned}
m^{\Gamma(X)}_{gh,k}\,(m^{\Gamma(X)}_{g,h} \otimes 1) &= c\,(1 \otimes \sigma_{gh})\,(c \otimes 1)\,(1 \otimes \sigma_g \otimes 1) \\
&= c\,(c \otimes 1)(1 \otimes 1 \otimes \sigma_{gh})(1 \otimes \sigma_g \otimes 1) \\
&= c\,(c \otimes 1)(1 \otimes \sigma_g \otimes \sigma_g)(1 \otimes 1 \otimes \sigma_h) \\
&= c\,(1 \otimes c)(1 \otimes \sigma_g \otimes \sigma_g)(1 \otimes 1 \otimes \sigma_h) \\
&= c\,(1 \otimes \sigma_g)(1 \otimes c)(1 \otimes 1 \otimes \sigma_h) \\
&= m^{\Gamma(X)}_{g,hk}\,(1 \otimes m^{\Gamma(X)}_{h,k}).
\end{aligned}$$

Here, the first and last equalities follow from the definition of $m^{\Gamma(X)}$; the second from level exchange (2.2.1); the third from level exchange and the property of shift functors in Definition 4.4.1(d); the fourth from associativity of composition (2.3.1) of the enrichment; and the fifth since σ_g is an enriched functor that respects composition (2.3.4).

The next computation verifies the left unital axiom (4.1.2):

$$\begin{aligned} m^{\Gamma(X)}_{e,g}\,(u^{\Gamma(X)} \otimes 1_{\Gamma(X)|_g}) &= c\,(1 \otimes \sigma_e)(\mathrm{id}_X \otimes 1_{\Gamma(X)|_g}) \\ &= c\,(\mathrm{id}_X \otimes 1_{\Gamma(X)|_g}) \\ &= 1_{\Gamma(X)|_g}. \end{aligned}$$

Here, the first equation follows from definitions of $m^{\Gamma(X)}$ and $u^{\Gamma(X)}$; the second since σ_e is the identity functor by Definition 4.4.1(a); and the third by left identity (2.3.2) of the enrichment.

Finally, we verify the right unital axiom (4.1.2):

$$\begin{aligned} m^{\Gamma(X)}_{g,e}\,(1_{\Gamma(X)|_g} \otimes u^{\Gamma(X)}) &= c\,(1_{\Gamma(X)|_g} \otimes \sigma_g)(1_{\Gamma(X)|_g} \otimes \mathrm{id}_X) \\ &= c\,(1_{\Gamma(X)|_g} \otimes \mathrm{id}_{\sigma_g(X)}) \\ &= 1_{\mathcal{C}(\sigma_g X, X)} \\ &= 1_{\Gamma(X)|_g}. \end{aligned}$$

Here, the first equality follows from definitions of $m^{\Gamma(X)}$ and $u^{\Gamma(X)}$; the second since σ_g is an enriched functor and thus respects identities (2.3.5); the third by right identity of the enrichment (2.3.3); and the fourth by definition of $\Gamma(X)|_g$. □

Next, show an example of $\Gamma(X)$ in the case when $\mathcal{E} = \mathsf{GrMod}(\mathcal{C})_A$.

Example 5.3.3. Let $A \in \mathsf{GrAlg}(\mathcal{C})$. By Corollary 4.3.4(b), the category $\mathsf{GrMod}(\mathcal{C})_A$ is enriched over $\mathcal{C}$. Furthermore, by Proposition 4.4.2 and Example 4.4.6, $\mathsf{GrMod}(\mathcal{C})_A$ is G-graded, with shift functors $(S_g)_{g\in G}$ given by $S_g X|_h = X|_{g^{-1}h}$ and $S_g\varphi|_h = \varphi|_{g^{-1}h}$.

Thus, we may apply Proposition 5.3.2 to $\mathsf{GrMod}(\mathcal{C})_A$.

Since A is a graded algebra, we may consider it as a graded A-module, where the action is given by the multiplication map of A. We may thus construct the graded algebra $\Gamma(A)$ in $\mathcal{C}$. At the level of objects, we have:

$$\Gamma(A)|_g = [S_gA, A]_A|_e = [A, A]_A|_g,$$

where the second equality follow from Proposition 4.4.4(f). Thus, $\Gamma(A) = [A, A]_A$ as objects in $\mathsf{Gr}(\mathcal{C})$. For each $g, h \in G$, the multiplication map of $\Gamma(A)$ is given by:

$$\begin{aligned} m^{\Gamma(A)}_{g,h} &= c^A_{S_{gh}A, S_gA, A}\left(1_{\Gamma(A)|_g} \otimes (S_g)_{S_hA,A}\right) \\ &= c^A_{S_{gh}A, S_gA, A}\left(1_{[A,A]_A|_g} \otimes 1_{[S_hA,A]_A|_e}\right) \\ &= c^A_{S_{gh}A, S_gA, A}\left(1_{[A,A]_A|_g} \otimes 1_{[A,A]_A|_h}\right) \\ &= (c^A_{A,A,A})_{g,h}. \end{aligned}$$

Here, the first equality follows from the definition of $m^{\Gamma(A)}_{g,h}$; the second from the definition of S_g on Hom-objects found in Example 4.4.6; the third from Proposition 4.4.4(f); and the fourth by tracking the composition morphism c^A through the isomorphism $\mathsf{GrAlg}(\mathcal{C}) \cong \mathsf{Alg}(\mathsf{Gr}(\mathcal{C}))$ of Theorem 4.2.6(a).

Similarly, the unit map of $\Gamma(A)$ is given by $\mathbb{1} \xrightarrow{\mathrm{id}^A_A|_e} [A, A]_A|_e = \Gamma(A)|_e$.

The computations above show that the algebra $\Gamma(A)$ is *equal to* the endomorphism algebra $[A, A]_A$ of Definition 3.3.1. Furthermore, in light of Theorem 3.3.3, we obtain that:

$$\Gamma(A) = [A, A]_A \cong A,$$

as graded algebras in $\mathcal{C}$.

We use the construction in the example above to work towards a partial converse to Theorem 5.3.1.

Theorem 5.3.4. *Let $\mathcal{E}$ and $\mathcal{E}'$ be G-graded $\mathcal{C}$-categories, with shift functors $(\sigma_g)_{g\in G}$ and $(\sigma'_g)_{g\in G}$ respectively. Given objects $X \in \mathcal{E}$ and $X' \in \mathcal{E}'$, if there is a fully faithful $\mathcal{C}$-functor $\Phi : \mathcal{E} \xrightarrow{\mathcal{C}} \mathcal{E}'$ such that $\Phi(\sigma_g X) \cong \sigma'_g X'$ in $\mathcal{E}'_0$ for all $g \in G$, then $\Gamma(X)$ is twist equivalent to $\Gamma(X')$ as algebras in $\mathsf{Gr}(\mathcal{C})$.*

Proof. Let $\Phi : \mathcal{E} \xrightarrow{\mathcal{C}} \mathcal{E}'$ be a fully faithful $\mathcal{C}$-functor, and assume that for every $g \in G$, we have an isomorphism $t_g : \Phi(\sigma_g X) \xrightarrow{\cong} \sigma'_g X'$ in the underlying category $\mathcal{E}'_0$.

By Proposition 5.2.6, we can show $\Gamma(X)$ is twist equivalent to $\Gamma(X')$ by producing isomorphisms: $(\phi_g : \Gamma(X') \to \Gamma(X))_{g\in G}$ satisfying equations (5.2.1), which we reproduce below:

$$m^{\Gamma(X)}_{h,k}\,(\phi_g|_h \otimes \phi_{gh}|_k) = \phi_g|_{hk}\, m^{\Gamma(X')}_{h,k}, \tag{5.3.1}$$

$$u^{\Gamma(X)} = \phi_e|_e\, u^{\Gamma(X')}. \tag{5.3.2}$$

We define $\phi_g : \Gamma(X')|_h \to \Gamma(X)|_h$ as follows:

$$\begin{aligned}\Gamma(X')|_h = \mathcal{C}(\sigma'_h X', X') &\xrightarrow{\sigma'_g} \mathcal{C}(\sigma'_{gh} X', \sigma'_g X') \xrightarrow{(t_{gh})^*} \mathcal{C}(\Phi(\sigma_{gh} X), \sigma'_g X') \\ &\xrightarrow{(t_g^{-1})_*} \mathcal{C}(\Phi(\sigma_{gh} X), \Phi(\sigma_g X)) \xrightarrow{\Phi^{-1}} \mathcal{C}(\sigma_{gh} X, \sigma_g X) \\ &\xrightarrow{\sigma_{g^{-1}}} \mathcal{C}(\sigma_h X, X) = \Gamma(X)|_h\end{aligned}$$

Here, the morphism $(t_{gh})^*$ and $(t_g^{-1})_*$ represent the pre-composition and post-composition morphisms introduced in Definition 3.1.7. We verify (5.3.2) below:

$$\begin{aligned}\phi_e|_e\, u^{\Gamma(X')} &= \sigma_e\, \Phi^{-1}\, (t_e^{-1})_* (t_e)^*\, \sigma'_e\, \mathrm{id}_{X'} = \Phi^{-1}\, (t_e^{-1})_* (t_e)^*\, \mathrm{id}_{X'} \\ &= \Phi^{-1}\, \mathrm{id}_{\Phi X} = \mathrm{id}_X = u^{\Gamma(X)}.\end{aligned}$$

Here, the first equality follows from the definitions of ϕ and $u^{\Gamma(X')}$; the second since σ_e and σ'_e are identity morphisms by Definition 4.4.1(a); the third using Proposition 3.1.8(b); the fourth since Φ is an enriched functor and respects identities (2.3.5) and the fifth by definition of $u^{\Gamma(X)}$.

Next, we verify (5.3.1), taking an alternate approach to [LW25]:

$$
\begin{aligned}
& m^{\Gamma(X)}_{h,k}\,(\phi_g|_h \otimes \phi_{gh}|_k) \\
& \overset{(1)}{=} c\,(1\otimes\sigma_h)(\phi_g|_h \otimes \phi_{gh}|_k) \\
& \overset{(2)}{=} c\,(1\otimes\sigma_h)\left(\sigma_{g^{-1}}\,\Phi^{-1}\,(t_g^{-1})_*(t_{gh})^*\,\sigma'_g \otimes \sigma_{h^{-1}g^{-1}}\Phi^{-1}\,(t_{gh}^{-1})_*(t_{ghk})^*\,\sigma'_{gh}\right) \\
& \overset{(3)}{=} c\,\left(\sigma_{g^{-1}}\Phi^{-1}(t_g^{-1})_*(t_{gh})^*\,\sigma'_g \otimes \sigma_{g^{-1}}\Phi^{-1}(t_{gh}^{-1})_*(t_{ghk})^*\,\sigma'_{gh}\right) \\
& \overset{(4)}{=} c\,(\sigma_{g^{-1}}\otimes\sigma_{g^{-1}})\left(\Phi^{-1}(t_g^{-1})_*(t_{gh})^*\,\sigma'_g \otimes \Phi^{-1}\,(t_{gh}^{-1})_*(t_{ghk})^*\,\sigma'_{gh}\right) \\
& \overset{(5)}{=} (\sigma_{g^{-1}}\,c)\circ\left(\Phi^{-1}(t_g^{-1})_*(t_{gh})^*\,\sigma'_g \otimes \Phi^{-1}(t_{gh}^{-1})_*(t_{ghk})^*\,\sigma'_{gh}\right) \\
& \overset{(6)}{=} \sigma_{g^{-1}}\,c\,\left(\Phi^{-1}\otimes\Phi^{-1}\right)\left((t_g^{-1})_*(t_{gh})^*\,\sigma'_g \otimes (t_{gh}^{-1})_*(t_{ghk})^*\,\sigma'_{gh}\right) \\
& \overset{(7)}{=} \sigma_{g^{-1}}\,\Phi^{-1}\,c\,\left((t_g^{-1})_*(t_{gh})^*\,\sigma'_g \otimes (t_{gh}^{-1})_*(t_{ghk})^*\,\sigma'_{gh}\right) \\
& \overset{(8)}{=} \sigma_{g^{-1}}\,\Phi^{-1}\,c\,\left((t_g^{-1})_*(t_{gh})^*\,\sigma'_g \otimes ((t_{ghk})^*(t_{gh}^{-1})_*\,\sigma'_{gh}\right) \\
& \overset{(9)}{=} \sigma_{g^{-1}}\,\Phi^{-1}\,c\,\left((t_g^{-1})_*\otimes(t_{ghk})^*\right)\left((t_{gh})^*\otimes(t_{gh}^{-1})_*\right)\left(\sigma'_g\otimes\sigma'_{gh}\right) \\
& \overset{(10)}{=} \sigma_{g^{-1}}\,\Phi^{-1}\,\left((t_g^{-1})_*(t_{ghk})^*\right)\,c\,\left((t_{gh})^*\otimes(t_{gh}^{-1})_*\right)\left(\sigma'_g\otimes\sigma'_{gh}\right) \\
& \overset{(11)}{=} \sigma_{g^{-1}}\,\Phi^{-1}\,\left((t_g^{-1})_*(t_{ghk})^*\right)\left((t_{gh})^*(t_{gh}^{-1})_*\right)\,c\,\left(\sigma'_g\otimes\sigma'_{gh}\right) \\
& \overset{(12)}{=} \sigma_{g^{-1}}\,\Phi^{-1}\,\left((t_g^{-1})_*(t_{ghk})^*\right)\,c\,(\sigma'_g\otimes\sigma'_{gh}) \\
& \overset{(13)}{=} \sigma_{g^{-1}}\,\Phi^{-1}\,\left((t_g^{-1})_*(t_{ghk})^*\right)\,c\,(\sigma'_g\otimes\sigma'_g)(1\otimes\sigma'_h) \\
& \overset{(14)}{=} \left(\sigma_{g^{-1}}\,\Phi^{-1}\,\left((t_g^{-1})_*(t_{ghk})^*\right)\,\sigma'_g\right)\circ\left(c\,(1\otimes\sigma'_h)\right) \\
& \overset{(15)}{=} \phi_g|_{hk}\circ m^{\Gamma(X')}_{h,k}.
\end{aligned}
$$

Here, equations 1,2, and 15 follow from the definitions of $\{\phi|_g\}_{g\in G}$, $m^{\Gamma(X)}$, and

$m^{\Gamma(X')}$; equation 3 follows by composition and the way shift functors compose (see Definition 4.4.1(c)); equations 4,6,9, and 13 follow by undoing a composition; and equation 5,7, and 14 follow since σ_g,σ'_g, and Φ^{-1} are enriched functors that respect composition (see equation (2.3.4)).

The rest of the equalities follow from the properties of pre-composition and post-composition proved in Proposition 3.1.8. In particular, equation 8 follows from Proposition 3.1.8(a), equation 10 from Proposition 3.1.8(c), equation 11 from Proposition 3.1.8(d), and equation 12 from Proposition 3.1.8(b).

Thus, by Proposition 5.2.6, $\Gamma(X)$ and $\Gamma(X')$ are twist equivalent, concluding the proof. $\square$

We use the theorem above to prove a sufficient condition for Zhang-Morita equivalence to guarantee twist equivalence.

Corollary 5.3.5. *Let $A, B \in \mathsf{GrAlg}(\mathcal{C})$, and consider their categories of graded modules as $\mathcal{C}$-categories via the enrichments defined in Corollary 4.3.4(b). If A and B are Zhang-Morita equivalent, via some $\mathcal{C}$-functor $\Phi : \mathsf{GrMod}(\mathcal{C})_A \xrightarrow{\cong} \mathsf{GrMod}(\mathcal{C})_B$ such that $\Phi(S_g A) \cong S_g B$ as objects in $(\mathsf{GrMod}(\mathcal{C})_B)_0$ for all $g \in G$, then A is twist equivalent to B.*

Proof. By Theorem 5.3.4, $\Gamma(A)$ and $\Gamma(B)$ are twist equivalent. But by Example 5.3.3, $\Gamma(A) \cong A$ and $\Gamma(B) \cong B$ as graded algebras, so A and B are twist equivalent. $\square$

Index of Notation

Category Theory

$\mathcal{C}, \mathcal{D}, \ldots$	abstract categories.
$X, Y \in \mathcal{C}$	objects X, Y in a category $\mathcal{C}$.
1_X	identity morphism $X \to X$ of an object X.
$\mathrm{Hom}_{\mathcal{C}}(X, Y)$	collection of all morphisms in $\mathcal{C}$ from X to Y.
$F, G : \mathcal{C} \to \mathcal{D}$	functors F, G between $\mathcal{C}$ and $\mathcal{D}$.
$F \dashv G$	adjoint functors with F left adjoint and G right adjoint.
$\prod_i X_i$	categorical product of a collection of objects $\{X_i\}_i$.
$\coprod_i X_i$	categorical coproduct of a collection of objects $\{X_i\}_i$.
$\bigoplus_i X_i$	categorical biproduct of a collection of objects $\{X_i\}_i$.
$(-)^\flat, (-)^\sharp$	right and left transposes of a morphism $(-)$, respectively.
eq	monomorphism associated to an equalizer.

Monoidal categories

$\otimes_{\mathcal{C}}$	monoidal product of a monoidal category $\mathcal{C}$.
$\mathbb{1}_{\mathcal{C}}$	monoidal unit of $\mathcal{C}$.
$a_{X,Y,Z}$	associator $(X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$ of $X, Y, Z \in \mathcal{C}$.
ℓ_X	left unitor $\mathbb{1} \otimes X \xrightarrow{\cong} X$ of $X \in \mathcal{C}$.
r_X	right unitor $X \otimes \mathbb{1} \xrightarrow{\cong} X$ of $X \in \mathcal{C}$.

(F, F^2, F^0)	monoidal functor $F : \mathcal{C} \to \mathcal{D}$ with binary component $F^2 : F(-)\otimes_{\mathcal{D}} F(-) \Rightarrow F(-\otimes_{\mathcal{C}}-)$ and nullary component $F^0 : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$.
(A, m, u)	algebra in a monoidal category with underlying object A, multiplication map $m : A \otimes A \to A$, and unit map $u : \mathbb{1} \to A$.
(M, ρ^M)	right A-module with underlying object M and action map $\rho : M \otimes A \to M$.
$\mathsf{Alg}(\mathcal{C})$	categories of algebras over a monoidal category $\mathcal{C}$.
$\mathsf{Mod}(\mathcal{C})_A$	category of right modules over an algebra A in $\mathcal{C}$.

Enriched categories

$\mathcal{E}^{[\mathcal{C}]}$	a $\mathcal{C}$-enriched category $\mathcal{E}$.
$\mathcal{C}(X, Y)$	Hom-object from X to Y in a $\mathcal{C}$-category.
id_X	identity morphism $\mathbb{1} \to \mathcal{C}(X, X)$ of $X \in \mathcal{E}$.
$c_{X,Y,Z}$	composition morphism $\mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ for objects X, Y, Z in some $\mathcal{C}$-category.
$\mathcal{E}_0$	underlying category of $\mathcal{E}$ (see Proposition 2.3.4).
$\mathcal{E}^{[F:\mathcal{C}\to\mathcal{D}]}$	change of base of a $\mathcal{C}$-category $\mathcal{E}$ along a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ (see Lemma 2.3.5)
$F : \mathcal{E} \xrightarrow{\mathcal{C}} \mathcal{E}'$	enriched $\mathcal{C}$-functor between $\mathcal{C}$-categories $\mathcal{E}$ and $\mathcal{E}'$.

Closed-monoidal categories

$\mathcal{C}^{[\mathcal{C}]}$	the category $\mathcal{C}$ equipped with the canonical self-enriched structure (see Proposition 3.1.5).

$[Y,-]$	right adjoint to functor $(-\otimes Y):\mathcal{C}\to\mathcal{C}$.
η^Y,ϵ^Y	unit and counit of an adjunction $(-\otimes Y)\dashv[Y,-]$.
$(-)^*,(-)_*$	pre and post-composition by a morphism $(-)$.
$[M,N]_A$	Hom-object from M to N in the $\mathcal{C}$-enrichment of $\mathsf{Mod}(\mathcal{C})_A$ of Theorem 3.2.3.
id^A_M	identity morphism of M in the $\mathcal{C}$-enrichment of $\mathsf{Mod}(\mathcal{C})_A$ of Theorem 3.2.3.
$c^A_{M,N,P}$	composition morphism of M,N,N in the $\mathcal{C}$-enrichment of $\mathsf{Mod}(\mathcal{C})_A$ of Theorem 3.2.3.

Graded categories

$\mathsf{Gr}(\mathcal{C})$	category of G-graded objects in $\mathcal{C}$, for G a group.
$X\|_g$	degree $g\in G$ of an object $X\in\mathsf{Gr}(\mathcal{C})$.
$(-)\|_e$	projection functor $\mathsf{Gr}(\mathcal{C})\to\mathcal{C}$ onto neutral degree $e\in G$.
σ_g,S_g	shift endofunctors in degree $g\in G$.
$\mathsf{GrAlg}(\mathcal{C})$	category of graded algebras in $\mathcal{C}$.
$\mathsf{GrMod}(\mathcal{C})_A$	category of graded modules in $\mathcal{C}$ over a graded algebra $A\in\mathsf{GrAlg}(\mathcal{C})$.

Twisting Systems

$\tau=\{\tau_g\}_{g\in G}$	twisting system on an algebra A.
(A^τ,m^τ,u^τ)	twisted algebra of A by the twisting system τ.
(M^τ,ρ^τ)	twisted A^τ-module for an A-module M.
$\Gamma(X)$	graded algebra associated to an object X, introduced in Proposition 5.3.2.

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