FILTERED FROBENIUS ALGEBRAS IN MONOIDAL CATEGORIES

CHELSEA WALTON
RICE UNIVERSITY

JOINT WORK WITH HARSHIT YADAV

ARXIV: 2106.01999
(TO APPEAR IN IMRN)

ON THE POSTDOC MARKET
A STORY ABOUT FROBENIUS ALGEBRAS: 1903

QUESTION OF F.G. FROBENIUS (1849-1917)

- START WITH A FINITE-DIM'L C-ALGEBRA $A$
- PICK BASIS $\{v_1, ..., v_n\}$ OF $A$
- GET SCALARS $\beta_{ij}^{(a)}, \delta_{ij}^{(a)} \in \mathbb{C}$ SUCH THAT

\[ v_i a = \sum_{j=1}^{n} \beta_{ij}^{(a)} v_j \quad \text{and} \quad av_i = \sum_{j=1}^{n} \delta_{ij}^{(a)} v_j \quad \forall a \in A \]

- GET C-LINEAR MAPS

\[ \beta: A \rightarrow \text{Mat}_n(\mathbb{C}) \quad \delta: A \rightarrow \text{Mat}_n(\mathbb{C}) \]

\[ a \mapsto (\beta_{ij}^{(a)}) \quad a \mapsto (\delta_{ij}^{(a)}) \]

Q: WHEN DOES $FP \in \text{GL}_n(\mathbb{C})$ ? \( P \beta(a) = \delta(a)P \quad \forall a \in A \)?
A STORY ABOUT FROBENIUS ALGEBRAS: 1903

FINITE-DIM' L R-ALGEBRA A

→ BASIS \{v_1, ..., v_n\} OF A

→ \( b^{(a)}_{ij}, \delta^{(a)}_{ij} \in \mathbb{C} \) SUCH THAT

\[ v_i a = \sum_{j=1}^{n} b^{(a)}_{ij} v_j \]

\[ a v_j = \sum_{i=1}^{n} v_i \delta^{(a)}_{ij} \quad \forall a \in A \]

→ GET \( \mathbb{C} \)-LINEAR MAPS

\[ \beta: A \rightarrow \text{Mat}_n(\mathbb{C}) \]

\[ a \mapsto (b^{(a)}_{ij}) \]

\[ \delta: A \rightarrow \text{Mat}_n(\mathbb{C}) \]

\[ a \mapsto (\delta^{(a)}_{ij}) \]

Q: WHEN DOES \( \exists P \in \text{GL}_n(\mathbb{C}) \) \( \Rightarrow \)

\[ P b(a) = \delta(a) P \quad \forall a \in A \]?

IN THE MODERN LANGUAGE OF REP. THEORY:

Q: WHEN ARE THE REG. REPS \( \beta, \delta \) OF A RIGHT \& LEFT EQUIVALENT?

BACK IN 1903:

JUST HAD LINEAR ALGEBRA

ANSWER VIA MATRICES...
A STORY ABOUT FROBENIUS ALGEBRAS: 1903

\[ Q: \text{When does } \exists P \in \text{GL}_n(\mathbb{C}) \text{ s.t. } P \varphi(a) = \chi(a) P \quad \forall a \in A? \]

\[ \text{Get scalars } p_{ij} \in \mathbb{C} \text{ such that } \]
\[ v_i v_j = \sum_{k=1}^{n} p_{ij} v_k \]

For \( \mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{C}^n \), define the paratrophic matrix of \( A \) at \( \mathbf{c} \)
\[ P_\mathbf{c} = (\sum_{k=1}^{n} c_k p_{ij}^k) iij \in \mathfrak{M}_{n,n}(\mathbb{C}) \]

A: [FROBENIUS, 1903]:
When \( \exists \mathbf{c} \in \mathbb{C}^n \) s.t. \( P_\mathbf{c} \) is invertible

(in this case, \( P_\mathbf{c} = P \) above)
A STORY ABOUT FROBENIUS ALGEBRAS: 1903

DEFINITION

A finite dim'l C-alg. A is Frobenius

\[ \exists \alpha \in \mathbb{C}^n : \det(P_\alpha) \neq 0 \]

(NON)EXAMPLES

TRY!

- \( \mathbb{C}[x, y]/(x^2, y^2) \) is Frob.
- \( \mathbb{C}[x, y, z]/(x^2, xy, y^2) \) is NOT Frob

FINITE-DIM'L C-ALGEBRA A

Basis \( \{ v_1, \ldots, v_n \} \) of A

Q: WHEN DOES \( \exists P \in \text{GL}_n(\mathbb{C}) \) s.t.

\[ P \rho(a) = \rho(a) P \quad \forall a \in A? \]

GET SCALARS \( p_{ij} \in \mathbb{C} \) such that

\[ \sum_{k=1}^{n} p_{ij}^k v_k \]

For \( \xi = (c_1, \ldots, c_n) \in \mathbb{C}^n \), define

PARATROPHIC MATRIX of A at \( \xi \)

\[ P_\xi = (\sum_{k=1}^{n} c_k p_{ij}^k)_{ij} \in \text{Mat}_n(\mathbb{C}) \]

A: [FROBENIUS, 1903]:

WHEN \( \exists \xi \in \mathbb{C}^n \) s.t. \( P_\xi \) is invertible

(IN THIS CASE, \( P_\xi = P \) above)
**Frobenius Algebras: 1937-1942**

**Definition**

A finite dim'l $C$-alg. $A$ is Frobenius if

$$\exists e \in C^n \quad \det(e \cdot e) \neq 0$$

Paratrophic matrix

More tools of abstract alg. now rings, algebras, modules ... 

⇒ able to work basis-free

Brauer, Nakayama, & Nesbitt revived Frobenius algebras
DEFINITION

**FINITE DIM. IR-ALG.** A IS FROBENIUS

\[ \exists \alpha \in \mathbb{R}^n \in \det(P_\alpha) \neq 0 \]

PARATROPHIC MATRIX

**EXAMPLES**

- **MATRIX ALGEBRAS** \( \text{Mat}_n(\mathbb{R}) \)
- **EXTERIOR ALGEBRAS** \( \Lambda(V) \)
- **GROUP ALGEBRAS** \( \mathbb{R}G \)
- **COHOMOLOGY ALGEBRAS OF ORIENTED MANIFOLDS** \( H^*(X) \)

DEFINITIONS

**THEOREM** [BRAUER-NESBITT, NAKAYAMA, 1937-1942]

LET A BE A FINITE DIM. IR-ALGEBRA

THEN THE FOLLOWING ARE EQUIVALENT:

1. A IS FROBENIUS
2. \( \exists \) NONDEG. ASSOC. IR-BILINEAR FORM \((\cdot, \cdot) : A \times A \rightarrow \mathbb{R}\)
3. \( \exists \) ISOM. OF LEFT A-MODULES: \( A \cong A^* \)
4. \( \exists \) ISOM. OF RIGHT A-MODULES: \( A \cong A^* \)
5. \( \forall \) IR-LINEAR FORM \( \varphi : A \rightarrow \mathbb{R} \exists \in \ker(\varphi) \)

DOES NOT CONTAIN \( 0 \) Left IDEAL OF A

6. \( \forall \) IR-LINEAR FORM \( \varphi : A \rightarrow \mathbb{R} \exists \in \ker(\varphi) \)

DOES NOT CONTAIN \( 0 \) Right IDEAL OF A
FROBENIUS ALGEBRAS: 1990s

DUE TO G. SEGAL, E. WITTEN, M. ATIYAH ORIGINALLY...

BUILD A CATEGORICAL MACHINE

TO PRODUCE INVARIANTS OF LOW-DIM'L MANIFOLDS

CATEGORY $\mathcal{C}$ ----> MONOIDAL CATEGORY $(\mathcal{C}, \otimes, 1)$

$\equiv$ COLLECTION OF OBJECTS
$\&$ MAPS BETWEEN THESE OBJECTS.
(SUBJECT TO ADD'L NICE CONDITIONS)

$\equiv$ CATEGORY $\mathcal{C}$ EQUIPPED WITH
BIFUNCTOR $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$
DISTINGUISHED OBJECT $1$, THAT MIMIC THE STRUCTURE OF A MONOID
(SUBJECT TO COMPATIBILITY CONDITIONS)

EX. $(\text{Vec}_{\mathbb{R}}, \otimes_{\mathbb{R}}, 1_{\mathbb{R}}, \mathbb{C})$ IS A SYM. MON'L CATEGORY

SPECIALIZATION

SYMOMETRIC MON'L CAT. $(\mathcal{C}, \otimes, 1, \mathbb{C})$

$\equiv$ MONOIDAL CATEGORY EQUIPPED WITH
NATURAL ISOMORPHISM: $\forall x, y \in \mathcal{C}$
$C_{x,y}: x \otimes y \cong y \otimes x \quad \varepsilon. \quad C^2 = \text{id}$
**Frobenius Algebras: 1990s**

\[ \mathbb{Z} : (\text{Bord}_{1,2}^\text{or}, \otimes = \text{disj. union}, \mathcal{A} = \emptyset, c=\text{swap}) \rightarrow (\mathcal{C}, \otimes, 1, c) \]

- **Objects**: Closed 1-MFILDS
- **Morphisms**: Oriented 2-MFILDS as cobordisms

\[ \mathbb{Z} : \text{Bord}_{1,2}^\text{or} \rightarrow \text{Vec}_{\mathbb{K}} \]

Another characterization:

\[ (A, \mu, \eta) \}_{\mathbb{K}-\text{alg.}} \leftrightarrow (A, \Delta, \varepsilon) \}_{\mathbb{K}-\text{coalg.}} \]

\[ (\mu \otimes \text{id})(\text{id} \otimes \Delta) = \Delta \mu = (\text{id} \otimes \mu)(\Delta \otimes \text{id}) \]

\[ \otimes = \text{disj. union} \]
THEOREM [ABRAMS, QUINN, VORONOV]

There is an equivalence of monoidal categories

Mon. Category of 2-TQFTs \( \cong \) Mon. Category of Commutative Frobenius Algebras \( / \mathbb{K} \)

\[ \mathbb{E} : \text{Bord}_{1,2} \rightarrow \text{Vec}_{\mathbb{K}} \]

\[ \begin{array}{c}
\emptyset \\
\cup \\
\cap \\
\cap \\
\emptyset
\end{array} \rightarrow \begin{array}{c}
\mathbb{K} \\
\mathbb{K} \text{-vs } A \\
[A \otimes A \rightarrow A] \\
[A \rightarrow A \otimes A] \\
[A \rightarrow \mathbb{K}]
\end{array} \]

\[ (A, \mu, \eta) \ \mathbb{K}\text{-Alg.} \]

Another characterization

\[ (A, \Delta, \varepsilon) \ \mathbb{K}\text{-CoAlg.} \]

\[ (\mu \circ \text{id}) (\text{id} \otimes \Delta) = \Delta \eta = (\text{id} \otimes \mu)(\Delta \otimes \text{id}) \]
**Frobenius Algebras: 1990s**

**Theorem [Abrams, Quinn, Voronov]**

There is an equivalence of monoidal categories

Mon. Category of 2-TQFTs \& Mon. Category of Commutative

With value in $(\mathbb{C}, \otimes, 1, c)$ \quad Frobenius Algebras in $(\mathbb{C}, \otimes, 1, c)$

\[ \mathbb{E}: \text{Bord}_{1,2} \longrightarrow \mathbb{C} \]  

\[ \emptyset \longrightarrow 1 \]

\[ s' \quad 0 \longrightarrow \text{obs.} \quad A \]

\[ \longrightarrow [A \otimes A \xrightarrow{\mu} A] \]

\[ [1 \xrightarrow{\eta} A] \]

\[ A \xrightarrow{\Delta} A \otimes A \]

\[ A \xrightarrow{\varepsilon} 1 \]

\[ (A, \mu, \eta) \text{ alg. in } \mathbb{C} \]

\[ \text{with } \mu = \mu \circ c \]

\[ \text{forms a commutative Frobenius algebra in } \mathbb{C} \]

\[ \equiv (A, \Delta, \varepsilon) \text{ coalg. in } \mathbb{C} \]

\[ (\mu \circ \text{id})(\text{id} \otimes \Delta) \]

\[ = \Delta \mu \]

\[ = (\text{id} \otimes \mu)(\Delta \otimes \text{id}) \]
FROBENIUS ALGEBRAS: 2000s TO TODAY

ROLE IN 2D-CFT DUE TO SCHWEIGERT, RUNKEL, FUCHS, ...

GIVEN A MONOIDAL CATEGORY \((\mathcal{C}, \otimes, I)\)

- CAN TAKE REPS OF IT:
  \([m, D] = \mathcal{C}\)-MODULE CATEG.
  BIFUNCTOR
  CATEGORY \(D : \mathcal{B} \times \mathcal{M} \rightarrow \mathcal{M}\)
  SAT. ASSOC. + UNIT DIAGRAMS

- CAN TAKE REPS IN IT:
  FOR \((A, m, u)\) ALGEBRA IN \(\mathcal{C}\),
  FORM \((M, D) = A\)-MODULE IN \(\mathcal{C}\)
  OBJECT IN \(\mathcal{C}\)
  MORPHISM IN \(\mathcal{C}\)
  \(D : A \otimes M \rightarrow M\)
  SAT. ASSOC. + UNIT DIAGRAMS

UNDER NICE CONDITIONS:
\(\mathcal{M} = A\text{-MOD}(\mathcal{C})\) AS \(\mathcal{C}\)-MOD. CATS.
FOR SOME \(A \in \text{Alg}(\mathcal{C})\)

[OSTRIK, EO]

FOR APPLICATIONS TO 2D-CFTS
WORK IN \((\mathcal{C}, \otimes, I, \mathcal{C})\)
\(\& A \in \text{FrobAlg}(\mathcal{C})\)
CERTAIN BRAIDED \(\otimes\) CATS
\(c^2 = \text{id}\)
FROBENIUS ALGEBRAS: 2000s to Today

Under nice conditions:

\[ \mathcal{M} \cong \text{A-mod}(\mathcal{C}) \text{ as } \mathcal{C}\text{-mod. cats.} \]

For some \( A \in \text{Alg}(\mathcal{C}) \)

\[ (\mathcal{C}, \otimes, \mathbb{1}, \mathcal{C}) \]

\( \mathcal{M} \) is \( \mathcal{C} \)-mod. cats.

[OSTRIK, EO]

For applications to 2D-CFTs

Work in \( (\mathcal{C}, \otimes, \mathbb{1}, \mathcal{C}) \) certain braided \( \otimes \) cats

\( c^2 = \text{id} \)

\[ A \in \text{FrobAlg}(\mathcal{C}) \]

MONOIDAL CATEGORY \( (\mathcal{C}, \otimes, \mathbb{1}) \)

+ ABELIAN
+ \( \text{F}-\)LINEAR
+ LOCALLY-FINITE
+ RIGID

\( \Rightarrow \) MULTITENSOR \( (\mathcal{C}, \otimes, \mathbb{1}) \)

+ \( \text{End}_\mathcal{C}(\mathbb{1}) \cong \text{Id} \)
+ SEMISIMPLE
+ FINITE

\( \Rightarrow \) FUSION \( (\mathcal{C}, \otimes, \mathbb{1}) \)

\[ (\mathcal{M}, \mathcal{D}) \text{ SEMISIMPLE + ABELIAN} \]

[OSTRIK]
FROBENIUS ALGEBRAS: 2000s TO TODAY

UNDER NICE CONDITIONS:
\[ \mathcal{M} = \text{A-mod}(\mathcal{C}) \] as \( \mathcal{C} \)-mod. categories.

FOR SOME \( \mathcal{A} \in \text{Alg}(\mathcal{C}) \)

\[ \mathcal{M} \cong \text{FrobAlg}(\mathcal{C}) \] for \( \text{A} \in \text{FrobAlg}(\mathcal{C}) \)

\[ \mathcal{C}^2 = \text{id} \]

FOR APPLICATIONS TO 2D-CFTS WORKING \( (\mathcal{C}, \otimes, \mathbb{1}, \mathcal{C}) \) CERTAIN BRAIDED \( \otimes \) CATEGORIES

MONOIDAL CATEGORY \( (\mathcal{C}, \otimes, \mathbb{1}) \)

+ ABELIAN
+ \( \mathbb{K} \)-LINEAR
+ LOCALLY-FINITE
+ RIGID

\[ \Rightarrow \text{MULTITENSOR} (\mathcal{C}, \otimes, \mathbb{1}) \leftarrow \bigstar \rightarrow (\mathcal{M}, \mathcal{D}) \] EXACT

+ \( \text{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{K} \)
+ SEMISIMPLE
+ FINITE

\[ \Rightarrow \text{FUSION} (\mathcal{C}, \otimes, \mathbb{1}) \leftarrow \bigstar \rightarrow (\mathcal{M}, \mathcal{D}) \] SEMISIMPLE + ABELIAN

\[ \forall \text{PROJECTIVE OBJECTS } \times \mathbb{C}, \forall \text{MEM} \mathcal{M}, \text{GET PDM PROJ IN } \mathcal{M} \]

\[ \Rightarrow \text{ALL OBJECTS ARE PROJECTIVE} \]
**Frobenius Algebras:**

**Representing Module Categories**

**Question:** Given a monoidal category \((\mathcal{C}, \otimes, 1)\) and a \(\mathcal{C}\)-module category \((\mathcal{M}, \mathcal{D})\), when is

\[\mathcal{M} \simeq \mathcal{A} \cdot \text{-mod}(\mathcal{C})\]

as \(\mathcal{C}\)-module categories for some Frobenius algebra

\[\mathcal{A} = (\mathcal{A}, \mu, \eta, \Delta, \varepsilon)\] in \((\mathcal{C}, \otimes, 1)\)?

That is, when is \(\mathcal{M} \in \mathcal{C} \cdot \text{-mod}\) “represented” by \(\mathcal{A} \in \text{FrobAlg}(\mathcal{C})\)?
FROBENIUS ALGEBRAS:
REPRESENTING MODULE CATEGORIES

QUESTION: WHEN IS $\mathcal{M} \in \mathcal{C}$-Mod REP BY $A \in \text{FrobAlg}(\mathcal{C})$?

ONE ANSWER: $\mathcal{C} = \text{F.d. Vec}_{\mathbb{K}}$ (fusion) $\mathcal{M}$ semisimple
$\leadsto$ OSTRIK'S RESULT APPLIES

$\leadsto$ $\mathcal{M} \simeq A$-Mod($\mathcal{C}$) for a separable ($\equiv$ SS) alg in $\mathcal{C}$.

ARTIN-WEDDERBURN THEOREM $\leadsto$ $A = \bigoplus_{i=1}^{\text{finite}} \text{Mat}_{n_i}(\mathbb{K})$

$\leadsto$ TRUE FOR $\mathcal{C} = \text{F.d. Vec}_{\mathbb{K}}$, $\mathcal{M}$ semisimple
**Frobenius Algebras:**

**Representing Module Categories**

**Question:** When is $\mathcal{M} \in \mathcal{C}$-Mod Rep by $A \in \text{FrobAlg}(\mathcal{C})$?

**Theorem [W-Yadav]**

The answer is yes when:

- $\mathcal{C}$ is a symmetric finite tensor category.
- $\mathcal{M}$ is an exact $\mathcal{C}$-module category.

Like $\mathcal{H}$-Mod:

- Monoidal
- Finitely many simples up to $\cong$
- 3 dual objects
- All projective objects $P \in \mathcal{C}$ get $P \otimes M$ projective $\forall M \in \mathcal{M}$ (semisimple $\Rightarrow$ exact)
FROBENIUS ALGEBRAS:

REPRESENTING MODULE CATEGORIES

**QUESTION:** WHEN IS $\mathcal{M} \in \mathbb{G}$-Mod REP BY $A \in \text{FrobAlg}(\mathbb{G})$?

**THEOREM [W-YADAV]**

**STRATEGY**

[ETINGOF-OSTRIK]

$\Rightarrow \mathcal{G} = (\Lambda(V) \# 1_{\mathbb{G}})$-Mod

"SUPER HOPF ALG"

FOR $\mathbb{G}$ FINITE GROUP, $V \in \mathbb{G}$-mod

$\Rightarrow \mathcal{M} = A$-Mod$(\Lambda(V) \# 1_{\mathbb{G}})$-Mod

FOR $A = \text{Ind}_X^* \left( \text{Ind}_X^* \left( \text{Ind}_X^* (\text{End}(V)) \# \text{Cl}(V, B) \right) \right)$

PRESERVES FROB. "CLIFFORD ALG" IN $\mathbb{G}$

NEED TO SHOW FROB.
**Frobenius Algebras:**

**Preserved under filtered deformation**

When $\mathcal{C}$ is a sym. fin. tensor catecg. and $\mathcal{M}$ is an exact $\mathcal{C}$-module catecg.

\[ \Rightarrow \]

Get $\mathcal{M} \sim \text{A-Mod}(\mathcal{C})$

For $A = A(\text{Cl}(V, B))$

$\text{Cl}(V, B) \Rightarrow A$

$\text{Frob} \quad \text{Frob}$

\[ \mathcal{C} = \text{Vec}_{\mathbb{K}} \]

Get that $\text{gr}(\text{Cl}(V, B)) \cong \Lambda(V)$

$\Rightarrow \text{Cl}(V, B)$ is a filtered def. of $\Lambda(V)$

$\& \quad \Lambda(V)$ is Frobenius.
Frobenius Algebras:

Preserved under filtered deformation

When \( \mathcal{E} \) is a sym. fin. tensor category and \( \mathcal{M} \) is an exact \( \mathcal{E} \)-module category.

\[ \mathcal{E} = \text{Vec}_{\mathbb{K}} \]

get that \( \text{gr}(\text{cl}(V,B)) \cong \Lambda(V) \)

\( \Rightarrow \text{cl}(V,B) \) is a filtered def. of \( \Lambda(V) \)

\( \text{\&} \Lambda(V) \) is Frobenius.

Theorem [Bongale, 1964]

Let \( A \) be a finite dimensional, connected, filtered \( \mathbb{K} \)-algebra

then \( \text{gr}(A) \text{ Frob.} \Rightarrow A \text{ Frob.} \)

\( \therefore \text{cl}(V,B) \in \text{FrobAlg}(\text{Vec}_{\mathbb{K}}) \)
Frobenius Algebras:

Preserved under filtered deformation

When $\mathcal{C}$ is a sym. fin. tensor categ. and $\mathcal{M}$ is an exact $\mathcal{C}$-module categ.

\[ \xrightarrow{\downarrow} \]

Get $\mathcal{M} \cong A$-$\text{Mod}(\mathcal{C})$

For $A = A(C_{\mathcal{C}}(V,B))$

Clifford alg. in $\mathcal{C}$

$C_{\mathcal{C}}(V,B) \Rightarrow A$

Frob $\uparrow$ Frob

$A(V)$ Frob in $\mathcal{C}$

[Want to generalize for more arbitrary monoidal categories $\mathcal{C}$]

[Bongale] in $\mathcal{C} = \text{Vec}_{\mathbb{C}}$

Take a fin. dim. connected filtered $k$-alg.

$\text{gr}(A) \Rightarrow A$

Frob $\uparrow$ Frob
FROBENIUS ALGEBRAS:

PRESERVED UNDER FILTERED DEFORMATION

1ST TOOL DEVELOPED (W-YADAV) - BUILDS ON SCHAUENBURG, ARDIEZONI-MENINI, GALATIUS ET. AL, HANGSENG-MILNER, G-WILLIAM-PAVLOV

FILTERED- GRADED TOOL -

FOR ℂ ABELIAN, MONOIDAL, WITH ⊗ BIEXACT:
CONSTRUCTED A MONOIDAL ASSOC. GRADED Functor

\[ \text{gr}: 
\begin{array}{c}
\text{Fil}(\mathbb{C}) \\
A_i
\end{array} 
\Rightarrow 
\begin{array}{c}
\text{Gr}(\mathbb{C}) \\
F_A(i)
\end{array}
\]

[\text{BONGALE}]

IN \( \mathbb{C} = \text{Vec}_k \)
TAKES A FIN. DIM.
CONNECTED
FILTERED K-ALG.

\[ \text{gr} (A) \Rightarrow A \]

\[ \text{Frob} \Rightarrow \text{Frob} \]

WANT TO
GENERALIZE


**Frobenius Algebras:**

**Preserved Under Filtered Deformation**

1st Tool Developed (W-Yadav)

Building on Schauenburg, Ardizzoni-Menini, Galatius et al.

Hwang-Seng-Miller, Gwilliam-Pavlov

For \( \mathcal{C} \) abelian, monoidal, with \( \otimes \) biequivalent:

"constructed a monoidal assoc. graded functor"

\[
\text{gr: } \text{Fil}(\mathcal{C}) \to \text{Gr}(\mathcal{C})
\]

\[
(A, F_A) \mapsto \lim_{i \in \mathbb{N}_0} \text{coker}(F_A(i-1) \to F_A(i))
\]

**Obj:** \((X, F_X) \overset{f}{\rightarrow} (Y, F_Y)\) filter func.

\[f: X = \lim_{i \in \mathbb{N}} X_i \rightarrow Y = \lim_{i \in \mathbb{N}} Y_i\]

**Homs:** \(X \rightarrow Y \in \mathcal{C} \) pres. filtration

\[\text{homs: } X \rightarrow Y \text{ compatible with } \Pi\]

If \( A \) is an algebra in \( \text{Fil}(\mathcal{C}) \),

**then** \( \text{gr}(A) \) is an algebra in \( \text{Gr}(\mathcal{C}) \).

[conjunctive]

- IN \( \mathcal{C} = \text{Vec}_{\mathbb{R}} \)
- Take a fin. dim.
- Connected
- Filtered \( R \)-alg.

\[
\text{gr}(A) \Rightarrow A \text{ Frob Frob}
\]

WANT TO

**Generalize**
Frobenius Algebras:
Preserved under filtered deformation

2nd Tool Developed (W-Yadav)
Builds on Fuchs-Stinier

Categorical Characterization of Frobenius Algebras

Let $\mathcal{C}$ be a tensor category. Then:

$(A, \mu, \eta)$ algebra in $\mathcal{C}$ is Frobenius

$\exists \theta: A \to 1$ in $\mathcal{C}$.

Any left/right ideal* of $A$ that factors through $\ker(\theta)$ is zero.

That is, $A$ has a "Frobenius form"

*B: Morphism from ideal to $A$ need not be a mono
(... a priori, $\text{gr}$ need not preserve monos)

[Bongale]
In $\mathcal{C} = \text{Vec}_{\mathbb{R}}$
Take a fin. dim. connected filtered $k$-alg.
$\downarrow$
$\text{gr}(A) \Rightarrow A$

Frob Frob

Want to generalize
FROBENIUS ALGEBRAS:

PRESERVED UNDER FILTERED DEFORMATION

**Main Theorem (W-Yadav)**

Let \( \mathcal{C} \) be a tensor category.

Let \( A \) be a connected, filtered alg. in \( \mathcal{C} \) w/ finite filtration.

If \( \gr(A) \) is a Frobenius algebra in \( \mathcal{C} \), then so is \( A \).
FROBENIUS ALGEBRAS:

PRESERVED UNDER FILTERED DEFORMATION

**Main Theorem (W-Yadav)**

Let \( \mathcal{C} \) be a tensor category.
Let \( A \) be a connected, filtered alg. in \( \mathcal{C} \) w/ finite filtration.
If \( g_f(A) \) is a Frobenius algebra in \( \mathcal{C} \), then so is \( A \).

**Proof:**

- \((A, F_A)\) filtered alg. in \( \mathcal{C} \) w/ fin. filt. \( \Rightarrow \) \( A \cong F_A(n) \) for some \( n \in \mathbb{N} \)
- Take morphism \( \phi: A \xrightarrow{\sim} F_A(n) \xrightarrow{\cong} F_A(n)/F_A(n-1) \cong 1 \) \( \Rightarrow \) \( A \) is connected
- Take ideal \( I \) of \( A \) s.t. \( \ker(\phi) \to A \xrightarrow{\phi} 1 \).

By Tool 2, suffices to show \( I = 0 \).
FROBENIUS ALGEBRAS:

PRESERVED UNDER FILTERED DEFORMATION

**Main Theorem (W-Yadav)**

Let $C$ be a tensor category.
Let $A$ be a connected, filtered alg. in $C$ w/finite filtration.
If $gr(A)$ is a Frobenius algebra in $C$, then so is $A$.

**Proof:**

- $(A, F_A)$ filtered alg. in $C$ w/fin. filt. $\Rightarrow A \cong F_A(n)$ for some $n \in \mathbb{N}$
- Take morphism $\nu : A \twoheadrightarrow F_A(n) \rightarrowtail F_A(n)/F_A(n-1) \cong A$ is connected
- Take ideal $I$ of $A \ni \ker(\nu) \twoheadrightarrow A \twoheadrightarrow A/I$.

By Tool 2, suffices to show $I = 0$

- Take $gr(\nu) : gr(I) \rightarrow gr(A)$ via Tool 1.
- $gr(\nu)$ factors thru kernel of Frobenius form on $gr(A)$

$\therefore$ Tool 2 $\Rightarrow$ $gr(I) = 0$. $\therefore$ $I = 0$ $\checkmark$
**Frobenius Algebras:**

**Preserved under filtered deformation**

When \( \mathcal{C} \) is a sym. fin. tensor cateq.

\[ \text{and} \]

\( \mathcal{M} \) is an exact \( \mathcal{C} \)-module cateq.

\[ \Downarrow \]

Get \( \mathcal{M} \sim A-\text{Mod}(\mathcal{C}) \)

For

\[ A = A(\mathcal{C}(V,B)) \]

Clifford alg in \( \mathcal{C} \)

\[ \mathcal{C}(V,B) \Rightarrow A \]

Frob

\[ A(V) \text{ Frob in } \mathcal{C} \]

Want to generalize for more arbitrary monoidal categories \( \mathcal{C} \)

[Bongale]

In \( \mathcal{C} = \text{Vec}_{\mathbb{R}} \)

Take a fin. dim. connected filtered \( \mathbb{R} \)-alg.

\[ \Downarrow \]

\[ \text{gr}(A) \Rightarrow A \]

Frob Frob

Done!
**Frobenius Algebras: Summary**

* Frobenius algebras have a very rich history
* Appeared over 100 years ago & ~ dozen characters
* Moderns uses in QFT
  - Topological
  - Conformal
* Appear in CFT by representing module categories $\mathcal{M}$ over nice monoidal categories $\mathcal{C}$
* Conversely, when $\mathcal{C}, \mathcal{M}$ are nice enough (sym. fin. $\otimes$, exact)
  $\mathcal{M}$ is always rep. by a Frob. alg. A in $\mathcal{C}$
* Get a Frob. $\iff$ certain Clifford alg. in $\mathcal{C}$ is Frob.
* Developed filtered-graded tools in $\otimes$ cats to show this
FILTERED FROBENIUS ALGEBRAS IN MONOIDAL CATEGORIES

CHELSEA WALTON
RICE UNIVERSITY

JOINT WORK WITH HARSHIT JADAV

ARXIV: 2106.01999

ON THE POSTDOC MARKET

MORE ON GR
MORE FROM BONGALE
COMBINATORICS
REPRESENTING MODULE CATS.
MORE ON QFTS

QUESTIONS ??

= THANKS FOR LISTENING =
**Future Directions**

**More from Bongale**

- More on $gr$
- Combinatorics
- Representing Module Cats.
- More on QFTs

---

**Theorem [Bongale, 1964]**

Let $A$ be a finite dimensional, connected, filtered $k$-algebra. Then $gr(A)$ frob. $\Rightarrow$ $A$ frob.

---

**Question**

**Main Theorem (W-Yadav)**

Let $C$ be a tensor category. Let $A$ be a connected, filt. alg. in $C$ with finite filtration. Then $gr(A)$ frob. $\Rightarrow$ $A$ frob.
Future Directions

**More From Bongale**

**More on gr**

**Combinatorics**

**Representing Module Cats.**

**More on QFTs**

**Have:** Monoidal Assoc. Graded Functor

\[ \text{gr}: \text{Fil}(\mathbb{C}) \rightarrow \text{Gr}(\mathbb{C}) \]

**Showed:** A Frobenius \( \Leftarrow \) Frobenius (under certain conditions)

**Can this be achieved via a "Frobenius Monoidal" adjoint to \( \text{gr} \)?**

**Also in \( \text{Vec}_{\mathbb{R}} \):**

\[ \begin{align*}
\text{A} & \quad \text{NOETHERIAN DOMAIN} \\
\text{gr}(\text{A}) & \quad \text{NOETHERIAN DOMAIN}
\end{align*} \]

**What about in \( (\mathbb{C}, \otimes, \mathbb{C}) \)?**

(May need to define properties in \( \otimes \) Cat.)
The exterior algebra \( \Lambda(V) \in (\mathbb{R}, \otimes, \mu, c) \) used in the main theorem is defined explicitly & combinatorially in work of Bespalov et. al. (even though we did not need this presentation)

Clifford algebras

Define \( \text{Cl}(V, B) \in (\mathbb{R}, \otimes, \mu, c) \) explicitly & combinatorially

Future Directions

More from Bongale

More on Gr

Combinatorics

Representing module cats.

More on QFTs
**QUESTION:**

WHEN IS $\mathcal{M} \in \mathcal{C}$-Mod

REPRESENTED BY

SOME $A \in \text{FrobAlg}(\mathcal{C})$?

(that is, $\mathcal{M} \sim A\text{-Mod}(\mathcal{C})$)

* $\mathcal{C} = \text{Vec}_{(K)}$, $\mathcal{M}$ semisimple ✓

* $\mathcal{C}$ sym. fin. ⊗, $\mathcal{M}$ exact ✓

* More? $\mathcal{C}$ finite ⊗, $\mathcal{M}$ exact ??
**Future Directions**

- More from Bongale
- More on gr
- Combinatorics
- Representing module cats.
- More on QFTs

**Theorem [Abrams, Quinn, Voronov]**
There is an equivalence of $\otimes$ cats $\otimes$ Cat of 2-TQFTs $\simeq$ $\otimes$ Cat of $\text{Com}$ with value in $\mathcal{C}$ Frob algs in $\mathcal{C}$

**Theorem [Turaev]**
There is an equivalence of $\otimes$ cats $\otimes$ Cat of 2-HQFTs $\simeq$ $\otimes$ Cat of certain $G$-graded $\text{Com}$ Frob algs in $\text{Vec}_H$ $\pi_1(X) = G$

**Theorem [Lazarev, Lauda-Dfeiffer]**
There is a connection btw $\otimes$ cats $\otimes$ Cat of open-closed 2-TQFTs $\simeq$ $\otimes$ Cat of $\text{Graded/Filtered}$ Frob algs in $\mathcal{C}$ "knowlegdable Frob. algs"

* More??*