# FILTERED FROBENIUS ALGEBRAS IN MONOIDAL CATEGORIES

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ARXIV: 2106.01999
(TO APPEAR IN IMRN)

ON THE POSTDOC MARKET

# A STORY ABOUT FROBENIUS ALGEBRAS : 1903

## QUESTION OF F.G. FROBENIUS (1849–1937)

- START WITH A FINITE-DIM'L $\mathbb{C}$-ALGEBRA $A$
- PICK BASIS $\{v_1, \ldots, v_n\}$ OF $A$
- GET SCALARS $\beta_{ij}^{(a)}, \gamma_{ij}^{(a)} \in \mathbb{C}$ SUCH THAT

$$v_i\, a = \sum_{j=1}^{n} \beta_{ij}^{(a)} v_j \quad \& \quad a v_i = \sum_{j=1}^{n} \gamma_{ji}^{(a)} v_j \qquad \forall a \in A$$

- GET $\mathbb{C}$-LINEAR MAPS

$$\beta : A \longrightarrow \mathrm{Mat}_n(\mathbb{C}), \quad a \longmapsto (\beta_{ij}^{(a)}) \qquad \& \qquad \gamma : A \longrightarrow \mathrm{Mat}_n(\mathbb{C}), \quad a \longmapsto (\gamma_{ij}^{(a)})$$

<u>Q:</u> WHEN DOES $\exists\, P \in GL_n(\mathbb{C})$ s.t. $P\,\beta(a) = \gamma(a)\,P \quad \forall a \in A$?

# A STORY ABOUT FROBENIUS ALGEBRAS : 1903

FINITE-DIM'L $\mathbb{k}$-ALGEBRA $A$

$\rightsquigarrow$ BASIS $\{v_1, \dots, v_n\}$ OF $A$

$\rightsquigarrow$ $\beta_{ij}^{(a)}, \gamma_{ij}^{(a)} \in \mathbb{C}$ SUCH THAT

$$v_i\, a = \sum_{j=1}^{n} \beta_{ij}^{(a)} v_j$$

$$\& \quad a v_j = \sum_{i=1}^{n} v_i\, \gamma_{ij}^{(a)} \quad \forall a \in A$$

$\rightsquigarrow$ GET $\mathbb{C}$-LINEAR MAPS

$$\beta : A \longrightarrow \mathrm{Mat}_n(\mathbb{C}), \quad a \longmapsto (\beta_{ij}^{(a)})$$

$$\gamma : A \longrightarrow \mathrm{Mat}_n(\mathbb{C}), \quad a \longmapsto (\gamma_{ij}^{(a)})$$

Q: WHEN DOES $\exists P \in GL_n(\mathbb{C})$ s.t.

$$P\beta(a) = \gamma(a)P \quad \forall a \in A\,?$$

IN THE MODERN LANGUAGE OF REP. THEORY:

Q $\Leftrightarrow$ WHEN ARE THE REG. REPS $\underset{\text{RIGHT}}{\beta}, \underset{\text{LEFT}}{\gamma}$ OF $A$ EQUIVALENT?

BACK IN 1903:

JUST HAD LINEAR ALGEBRA

ANSWER VIA MATRICES...

# A STORY ABOUT FROBENIUS ALGEBRAS : 1903

FINITE-DIM'L $\mathbb{k}$-ALGEBRA $A$

$\rightsquigarrow$ BASIS $\{v_1, \dots, v_n\}$ OF $A$

$\rightsquigarrow$ $\beta_{ij}^{(a)}, \gamma_{ij}^{(a)} \in \mathbb{C}$ SUCH THAT

$$v_i \, a = \sum_{j=1}^{n} \beta_{ij}^{(a)} v_j$$

$$\& \quad a v_j = \sum_{i=1}^{n} v_i \gamma_{ij}^{(a)} \quad \forall a \in A$$

$\rightsquigarrow$ GET $\mathbb{C}$-LINEAR MAPS

$$\beta : A \longrightarrow \mathrm{Mat}_n(\mathbb{C}), \quad a \longmapsto (\beta_{ij}^{(a)})$$

$$\gamma : A \longrightarrow \mathrm{Mat}_n(\mathbb{C}), \quad a \longmapsto (\gamma_{ij}^{(a)})$$

$\underline{Q}$: WHEN DOES $\exists P \in GL_n(\mathbb{C})$ s.t.

$$P \beta(a) = \gamma(a) P \quad \forall a \in A ?$$

GET SCALARS $p_{ij}^{k} \in \mathbb{C}$ SUCH THAT

$$v_i v_j = \sum_{k=1}^{n} p_{ij}^{k} v_k$$

FOR $\underline{c} = (c_1, \dots, c_n) \in \mathbb{C}^n$, DEFINE

PARATROPHIC MATRIX OF $A$ AT $\underline{c}$

$$P_{\underline{c}} = \left( \sum_{k=1}^{n} c_k p_{ij}^{k} \right)_{i,j} \in \mathrm{Mat}_n(\mathbb{C})$$

$\underline{A}$: [FROBENIUS, 1903]:

WHEN $\exists \underline{c} \in \mathbb{C}^n$ s.t. $P_{\underline{c}}$ IS INVERTIBLE

(IN THIS CASE, $P_{\underline{c}} = P$ ABOVE)

# A STORY ABOUT FROBENIUS ALGEBRAS : 1903

## DEFINITION 1

A FINITE DIM'L $\mathbb{C}$-ALG. $A$ IS FROBENIUS

$$\Updownarrow$$

$\exists\, \underline{c} \in \mathbb{C}^n$ s.t. $\det(P_{\underline{c}}) \neq 0$

## (NON)EXAMPLES

TRY!

- $\mathbb{C}[x,y]/(x^2,y^2)$ IS FROB.
- $\mathbb{C}[x,y]/(x^2,xy,y^2)$ IS NOT FROB

FINITE-DIM'L $\mathbb{C}$-ALGEBRA $A$

BASIS $\{v_1, \dots, v_n\}$ OF $A$

$\underline{Q}$: WHEN DOES $\exists\, P \in GL_n(k)$ s.t.
$P\,\beta(a) = \gamma(a)\,P \quad \forall a \in A$ ?

GET SCALARS $p_{ij}^k \in \mathbb{C}$ SUCH THAT

$$v_i v_j = \sum_{k=1}^n p_{ij}^k v_k$$

FOR $\underline{c} = (c_1, \dots, c_n) \in \mathbb{C}^n$, DEFINE PARATROPHIC MATRIX OF $A$ AT $\underline{c}$

$$P_{\underline{c}} = \Big(\sum_{k=1}^n c_k p_{ij}^k\Big)_{i,j} \in \mathrm{Mat}_n(\mathbb{C})$$

$\underline{A}$: [FROBENIUS, 1903]:
WHEN $\exists\, \underline{c} \in \mathbb{C}^n$ s.t. $P_{\underline{c}}$ IS INVERTIBLE
(IN THIS CASE, $P_{\underline{c}} = P$ ABOVE)

# FROBENIUS ALGEBRAS : 1937–1942

## DEFINITION 1

A FINITE DIM'L

$\mathbb{C}$-ALG. A

IS **FROBENIUS**

$$\Updownarrow$$

$$\exists\, \underline{e} \in \mathbb{C}^n \ni.$$

$$\det(P_{\underline{e}}) \neq 0$$

PARATROPHIC MATRIX

MORE TOOLS OF ABSTRACT ALG. NOW
RINGS, ALGEBRAS, MODULES ...

≡ ABLE TO WORK BASIS-FREE ≡

BRAUER, NAKAYAMA,
& NESBITT REVIVED
FROBENIUS ALGEBRAS

# FROBENIUS ALGEBRAS : 1937–1942

$/ \mathbb{k} = \overline{\mathbb{k}},\ \mathrm{ch}\,\mathbb{k} = 0$

## DEFINITION 1

FINITE DIM. $\mathbb{k}$-ALG. $A$ IS FROBENIUS

$$\Updownarrow$$

$$\exists\, \underline{e} \in \mathbb{k}^n \ni.\ \det(P_{\underline{e}}) \neq 0$$

PARATROPHIC MATRIX

## EXAMPLES

- MATRIX ALGEBRAS $\mathrm{Mat}_n(\mathbb{k})$
- EXTERIOR ALGEBRAS $\Lambda(V)$
- GROUP ALGEBRAS $\mathbb{k}G$
- COHOMOLOGY ALGEBRAS OF ORIENTED MANIFOLDS $H^*(X)$

## DEFINITIONS

**THEOREM** [BRAUER–NESBITT, NAKAYAMA, 1937–1942]

LET $A$ BE A FINITE DIM'L $\mathbb{k}$-ALGEBRA

THEN THE FOLLOWING ARE EQUIVALENT:

1. $A$ IS FROBENIUS

2. $\exists$ NONDEG. ASSOC. $\mathbb{k}$-BILINEAR FORM
$(-,-) : A \times A \longrightarrow \mathbb{k}$

3. $\exists$ ISOM. OF LEFT $A$-MODULES : $A \simeq A^*$

3'. $\exists$ ISOM. OF RIGHT $A$-MODULES : $A \simeq A^*$

4. $\exists$ $\mathbb{k}$-LINEAR FORM $\nu : A \longrightarrow \mathbb{k} \ni.\ \ker(\nu)$
DOES NOT CONTAIN $\neq 0$ LEFT IDEAL OF $A$

4'. $\exists$ $\mathbb{k}$-LINEAR FORM $\nu : A \longrightarrow \mathbb{k} \ni.\ \ker(\nu)$
DOES NOT CONTAIN $\neq 0$ RIGHT IDEAL OF $A$

# FROBENIUS ALGEBRAS : 1990s

DUE TO G. SEGAL, E. WITTEN, M. ATIYAH ORIGINALLY...

BUILD A CATEGORICAL MACHINE

TO PRODUCE INVARIANTS OF LOW-DIM'L MANIFOLDS

CATEGORY $\mathscr{C}$

$\equiv$ COLLECTION OF OBJECTS

& MAPS BETWEEN THESE OBJECTS.

(SUBJECT TO ADD'L NICE CONDITIONS)

MONOIDAL CATEGORY $(\mathscr{C}, \otimes, \mathbb{1})$

$\equiv$ CATEGORY $\mathscr{C}$ EQUIPPED WITH

BIFUNCTOR $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$

DISTINGUISHED OBJECT $\mathbb{1}$

THAT MIMIC THE STRUCTURE OF A MONOID

(SUBJECT TO COMPATIBILITY CONDITIONS)

SYMMETRIC MON'L CAT. $(\mathscr{C}, \otimes, \mathbb{1}, c)$

$\equiv$ MONOIDAL CATEGORY EQUIPPED WITH

NATURAL ISOMORPHISM : $\forall X, Y \in \mathscr{C}$

$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ $\ni.$ $c^2 = id$

FLIP MAP

EX. $(Vec_{\mathbb{k}}, \otimes_{\mathbb{k}}, \mathbb{k}, c)$

IS A SYM. MON'L CATEGORY

# FROBENIUS ALGEBRAS : 1990s

2-DIM'L TQFT $\equiv$ SYMMETRIC MONOIDAL FUNCTOR :

(T = TOPOLOGICAL)

$$Z : \left(\mathrm{Bord}^{or}_{1,2},\ \otimes = \text{DISJ. UNION},\ \mathbb{1} = \emptyset,\ c = \text{SWAP}\right) \longrightarrow (\mathcal{C}, \otimes, \mathbb{1}, c)$$

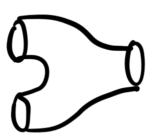
OBJECTS = CLOSED 1-MFLDS $\longmapsto$ OBJECT IN $\mathcal{C}$

MORPHISMS = ORIENTED 2-MFLDS AS COBORDISMS $\longmapsto$ MORPHISM IN $\mathcal{C}$

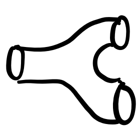
$$Z : \mathrm{Bord}^{or}_{1,2} \longrightarrow \mathrm{Vec}_{\mathbb{k}}$$

$\emptyset \longmapsto \mathbb{k}$

$S^1$  $\longmapsto$ $\mathbb{k}$-VS $A$

 $\longmapsto \left[A \otimes A \xrightarrow{\mu} A\right]$

 $\longmapsto \left[\mathbb{k} \xrightarrow{u} A\right]$

 $\longmapsto \left[A \xrightarrow{\Delta} A \otimes A\right]$

 $\longmapsto \left[A \xrightarrow{\varepsilon} \mathbb{k}\right]$

FORMS A COMMUTATIVE FROBENIUS ALGEBRA $/\mathbb{k}$ $\equiv$

ANOTHER CHARACTERIZATION

$(A, \mu, u)$ $\mathbb{k}$-ALG. & $(A, \Delta, \varepsilon)$ $\mathbb{k}$-COALG. s.t.

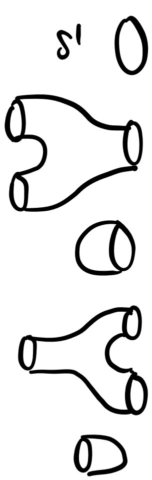
$$(\mu \otimes id)(id \otimes \Delta) = \Delta\mu = (id \otimes \mu)(\Delta \otimes id)$$

# FROBENIUS ALGEBRAS : 1990s

**THEOREM** [ABRAMS, QUINN, VORONOV]

> THERE IS AN EQUIVALENCE OF MONOIDAL CATEGORIES
>
> MON. CATEGORY OF 2-TQFTs WITH VALUE IN $\mathrm{VEC}_{\mathbb{k}}$ & MON. CATEGORY OF COMMUTATIVE FROBENIUS ALGEBRAS / $\mathbb{k}$

$$Z : \mathrm{Bord}^{or}_{1,2} \longrightarrow \mathrm{Vec}_{\mathbb{k}}$$



$$\varnothing \longmapsto \mathbb{k}$$

$$S^1 \quad \longmapsto \quad \mathbb{k}\text{-vs } A$$

$$\longmapsto [A \otimes A \xrightarrow{\mu} A]$$

$$\longmapsto [\mathbb{k} \xrightarrow{u} A]$$

$$\longmapsto [A \xrightarrow{\Delta} A \otimes A]$$

$$\longmapsto [A \xrightarrow{\varepsilon} \mathbb{k}]$$

FORMS A COMMUTATIVE FROBENIUS ALGEBRA / $\mathbb{k}$

ANOTHER CHARACTERIZATION

$\equiv$ $(A, \mu, u)$ $\mathbb{k}$-ALG.

&

$(A, \Delta, \varepsilon)$ $\mathbb{k}$-COALG

s.t.

$$(\mu \otimes id)(id \otimes \Delta) = \Delta \mu = (id \otimes \mu)(\Delta \otimes id)$$

# Frobenius Algebras: 1990s

Toy model for higher TQFTs...

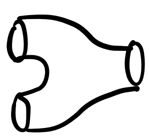
**Theorem** [Abrams, Quinn, Voronov]

> There is an equivalence of monoidal categories
>
> Mon. category of 2-TQFTs with value in $(\mathcal{C},\otimes,\mathbb{1},c)$ $\simeq$ Mon. category of commutative Frobenius algebras in $(\mathcal{C},\otimes,\mathbb{1},c)$

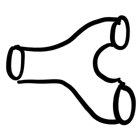
$Z: \mathrm{Bord}^{or}_{1,2} \longrightarrow \mathcal{C}$

$\emptyset \longmapsto \mathbb{1}$

$S^1$  $\longmapsto$ obs. $A$

 $\longmapsto [A\otimes A \xrightarrow{\mu} A]$

 $\longmapsto [\mathbb{1} \xrightarrow{u} A]$

 $\longmapsto [A \xrightarrow{\Delta} A\otimes A]$

 $\longmapsto [A \xrightarrow{\varepsilon} \mathbb{1}]$

} forms a commutative Frobenius algebra in $\mathcal{C}$

$\equiv$

$(A,\mu,u)$ alg. in $\mathcal{C}$ with $\mu = \mu\circ c$

&

$(A,\Delta,\varepsilon)$ coalg in $\mathcal{C}$

s.t.

$$(\mu\otimes id)(id\otimes\Delta) = \Delta\mu = (id\otimes\mu)(\Delta\otimes id)$$

# FROBENIUS ALGEBRAS : 2000s TO TODAY

ROLE IN 2D-CFT (CONFORMAL FIELD THEORY) DUE TO SCHWEIGERT, RUNKEL, FUCHS,...

GIVEN A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1})$

CAN TAKE REPS OF IT :

$(\mathcal{M}, \triangleright) = \mathcal{C}$-MODULE CATEG.

CATEGORY

BIFUNCTOR

$\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

SAT. ASSOC. + UNIT DIAGRAMS

CAN TAKE REPS IN IT :

FOR $(A, m, u)$ ALGEBRA IN $\mathcal{C}$,

FORM $(M, \triangleright) =$ A-MODULE IN $\mathcal{C}$

OBJECT IN $\mathcal{C}$

MORPHISM IN $\mathcal{C}$

$\triangleright : A \otimes M \to M$

SAT. ASSOC. + UNIT DIAGRAMS

UNDER NICE CONDITIONS :

$\mathcal{M} \simeq A\text{-mod}(\mathcal{C})$ AS $\mathcal{C}$-MOD. CATS.

FOR SOME $A \in \mathrm{Alg}(\mathcal{C})$

[OSTRIK, EO]

FOR APPLICATIONS TO 2D-CFTS

WORK IN $(\mathcal{C}, \otimes, \mathbb{1}, c)$ CERTAIN BRAIDED $\otimes$ CATS $\cancel{c^2 = id}$

& $A \in \mathrm{FrobAlg}(\mathcal{C})$

# FROBENIUS ALGEBRAS : 2000s TO TODAY

UNDER NICE CONDITIONS $(\star)$:

$\mathcal{M} \simeq A\text{-Mod}(\mathcal{C})$ AS $\mathcal{C}$-MOD. CATS.

FOR SOME $A \in \mathrm{Alg}(\mathcal{C})$

[OSTRIK, EO]

FOR APPLICATIONS TO 2D-CFTS

WORK IN $(\mathcal{C}, \otimes, \mathbb{1}, c)$ CERTAIN BRAIDED $\otimes$ CATS ~~$c^2 = \mathrm{id}$~~

& $A \in \mathrm{FrobAlg}(\mathcal{C})$

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1})$

+ ABELIAN
+ $\Bbbk$-LINEAR
+ LOCALLY-FINITE
+ RIGID

$\rightsquigarrow$ MULTITENSOR $(\mathcal{C}, \otimes, \mathbb{1})$

+ $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk$
+ SEMISIMPLE
+ FINITE

$\rightsquigarrow$ FUSION $(\mathcal{C}, \otimes, \mathbb{1})$ $\xleftarrow{\text{[OSTRIK]}} \star \rightarrow$ $(\mathcal{M}, \triangleright)$ SEMISIMPLE + ABELIAN

# FROBENIUS ALGEBRAS : 2000s TO TODAY

UNDER NICE CONDITIONS (★) :

$\mathcal{M} \simeq A\text{-Mod}(\mathcal{C})$ AS $\mathcal{C}$-MOD. CATS.

FOR SOME $A \in \mathrm{Alg}(\mathcal{C})$

[OSTRIK, EO]

FOR APPLICATIONS TO 2D-CFTS

WORK IN $(\mathcal{C}, \otimes, \mathbb{1}, c)$ CERTAIN BRAIDED $\otimes$ CATS ~~$c^2 = id$~~

& $A \in \mathrm{FrobAlg}(\mathcal{C})$

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1})$

+ ABELIAN
+ $\Bbbk$-LINEAR
+ LOCALLY-FINITE
+ RIGID

⇝ MULTITENSOR $(\mathcal{C}, \otimes, \mathbb{1})$ ← ★ → [ETINGOF-OSTRIK] ⇝ $(\mathcal{M}, \triangleright)$ EXACT

$\forall$ PROJECTIVE OBJECTS $X \in \mathcal{C}$, $\forall M \in \mathcal{M}$, GET $P \triangleright M$ PROJ IN $\mathcal{M}$

+ $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk$
+ SEMISIMPLE
+ FINITE

$\Uparrow$

ALL OBJECTS ARE PROJECTIVE

$\Uparrow$

⇝ FUSION $(\mathcal{C}, \otimes, \mathbb{1})$ ← ★ → [OSTRIK] $(\mathcal{M}, \triangleright)$ SEMISIMPLE + ABELIAN

# FROBENIUS ALGEBRAS:

## REPRESENTING MODULE CATEGORIES

**QUESTION:** GIVEN A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1})$

AND A $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright)$,

WHEN IS

$$\mathcal{M} \sim A\text{-Mod}(\mathcal{C})$$

AS $\mathcal{C}$-MODULE CATEGORIES

FOR SOME FROBENIUS ALGEBRA

$A = (A, m, u, \Delta, \varepsilon)$ IN $(\mathcal{C}, \otimes, \mathbb{1})$?

THAT IS, WHEN IS $\mathcal{M} \in \mathcal{C}\text{-Mod}$ "REPRESENTED" BY $A \in \mathrm{FrobAlg}(\mathcal{C})$?

# FROBENIUS ALGEBRAS :

## REPRESENTING MODULE CATEGORIES

QUESTION: WHEN IS $\mathcal{M} \in \mathcal{C}\text{-Mod}$ REP BY $A \in \mathrm{FrobAlg}(\mathcal{C})$ ?

ONE ANSWER: $\mathcal{C} = \mathrm{F.d.Vec}_{\Bbbk}$ (FUSION) $\mathcal{M}$ SEMISIMPLE

$\rightsquigarrow$ OSTRIK'S RESULT APPLIES

$\rightsquigarrow$ $\mathcal{M} \sim A\text{-Mod}(\mathcal{C})$ FOR A SEPARABLE ($\Rightarrow$ SS) ALG IN $\mathcal{C}$.

ARTIN-WEDDERBURN THEOREM $\rightsquigarrow$ $A = \prod_i^{\text{FINITE}} \mathrm{Mat}_{n_i}(\Bbbk)$

PRESERVES FROBENIUS

FROBENIUS

$\rightsquigarrow$ TRUE FOR $\mathcal{C} = \mathrm{F.d.Vec}_{\Bbbk}$ , $\mathcal{M}$ SEMISIMPLE

# FROBENIUS ALGEBRAS :

## REPRESENTING MODULE CATEGORIES

**QUESTION:** WHEN IS $\mathcal{M} \in \mathcal{C}$-Mod REP BY $A \in \mathrm{FrobAlg}(\mathcal{C})$ ?

**THEOREM** [W-YADAV]

THE ANSWER IS YES

WHEN:

$\mathcal{C}$ IS A SYMMETRIC

FINITE TENSOR CATEG.

AND

$\mathcal{M}$ IS AN EXACT

$\mathcal{C}$-MODULE CATEGORY

LIKE H-Mod

F.D. HOPF ALGEBRA

- MONOIDAL
- FINITELY MANY SIMPLES UP TO $\cong$
- $\exists$ DUAL OBJECTS

$\forall$ PROJECTIVE OBJECTS $P \in \mathcal{C}$

GET $P \triangleright M$ PROJECTIVE $\forall M \in \mathcal{M}$

(SEMISIMPLE $\Rightarrow$ EXACT)

# FROBENIUS ALGEBRAS :

## REPRESENTING MODULE CATEGORIES

**QUESTION:** WHEN IS $\mathcal{M} \in \mathcal{C}\text{-Mod}$ REP BY $A \in \mathrm{FrobAlg}(\mathcal{C})$ ?

**THEOREM** [W-YADAV]

THE ANSWER IS YES

WHEN:

$\mathcal{C}$ IS A SYMMETRIC FINITE TENSOR CATEG.

AND

$\mathcal{M}$ IS AN EXACT $\mathcal{C}$-MODULE CATEGORY

**STRATEGY**

[ETINGOF-OSTRIK]

$\rightsquigarrow \mathcal{C} \simeq (\Lambda(V) \# \mathbb{k}G)\text{-Mod}$

"SUPER HOPF ALG"

FOR $G$ FINITE GROUP, $V \in G\text{-Mod}$

$\rightsquigarrow \mathcal{M} \simeq A\text{-Mod}\big((\Lambda(V) \# \mathbb{k}G)\text{-Mod}\big)$ FOR

$A = \mathrm{Ind}_*^*\big(\mathrm{Ind}_*^*\big(\mathrm{Ind}_*^*(\mathrm{End}(V)) \# \mathrm{Cl}(V, B)\big)\big)$

PRESERVES FROB.

FROB.

"CLIFFORD ALG." IN $\mathcal{C}$

NEED TO SHOW FROB.

# FROBENIUS ALGEBRAS:

## PRESERVED UNDER FILTERED DEFORMATION

WHEN $\mathcal{C}$ IS A SYM. FIN. TENSOR CATEG.

AND

$\mathcal{M}$ IS AN EXACT $\mathcal{C}$-MODULE CATEG.

$\Downarrow$

GET $\mathcal{M} \sim A\text{-Mod}(\mathcal{C})$

FOR

$A = A(Cl(V,B))$

$\uparrow$

CLIFFORD ALG IN $\mathcal{C}$

$\underset{\text{FROB}}{Cl(V,B)} \Rightarrow \underset{\text{FROB}}{A}$

$\mathcal{C} = \mathrm{Vec}_{\Bbbk}$

GET THAT $gr(Cl(V,B)) \cong \Lambda(V)$

$\rightsquigarrow$ $Cl(V,B)$ IS A FILTERED DEF. OF $\Lambda(V)$

& $\Lambda(V)$ IS FROBENIUS.

# FROBENIUS ALGEBRAS :

## PRESERVED UNDER FILTERED DEFORMATION

WHEN $\mathcal{C}$ IS A SYM. FIN. TENSOR CATEG.

AND

$\mathcal{M}$ IS AN EXACT $\mathcal{C}$-MODULE CATEG.

$\downarrow$

GET $\mathcal{M} \sim A\text{-Mod}(\mathcal{C})$

FOR

$A = A(Cl(V,B))$

$\uparrow$

CLIFFORD ALG IN $\mathcal{C}$

$Cl(V,B)$ FROB $\Rightarrow$ $A$ FROB

$\mathcal{C} = Vec_{\mathbb{k}}$

GET THAT $gr(Cl(V,B)) \cong \Lambda(V)$

$\rightsquigarrow$ $Cl(V,B)$ IS A FILTERED DEF. OF $\Lambda(V)$

& $\Lambda(V)$ IS FROBENIUS.

THEOREM [BONGALE, 1964]

LET $A$ BE A FINITE DIMENSIONAL, CONNECTED, FILTERED $\mathbb{k}$-ALGEBRA

THEN $gr(A)$ FROB. $\Rightarrow$ $A$ FROB.

$\therefore$ $Cl(V,B) \in FrobAlg(Vec_{\mathbb{k}})$

# FROBENIUS ALGEBRAS :

## PRESERVED UNDER FILTERED DEFORMATION

WHEN $\mathcal{C}$ IS A SYM. FIN. TENSOR CATEG.
AND
$\mathcal{M}$ IS AN EXACT $\mathcal{C}$-MODULE CATEG.

$\downarrow$

GET $\mathcal{M} \sim A\text{-Mod}(\mathcal{C})$
FOR
$A = A(Cl(V,B))$

$\uparrow$ CLIFFORD ALG IN $\mathcal{C}$

$\underset{\text{FROB}}{Cl(V,B)} \Rightarrow \underset{\text{FROB}}{A}$

$\Uparrow$

$\Lambda(V)$ FROB IN $\mathcal{C}$ ✓

WANT TO GENERALIZE FOR MORE ARBITRARY MONOIDAL CATEGORIES $\mathcal{C}$

[BONGALE]
IN $\mathcal{C} = \mathrm{Vec}_{\Bbbk}$
TAKE $A$ FIN. DIM. CONNECTED FILTERED $\Bbbk$-ALG.

$\downarrow$

$\underset{\text{FROB}}{gr(A)} \Rightarrow \underset{\text{FROB}}{A}$

# FROBENIUS ALGEBRAS:

## PRESERVED UNDER FILTERED DEFORMATION

1ST TOOL DEVELOPED (W-YADAV) BUILDS ON SCHAUENBURG, ARDIZZONI-MENINI, GALATIUS ET. AL, HAUGSENG-MILLER, GWILLIAM-PAVLOV

FILTERED–GRADED TOOL—

FOR $\mathcal{C}$ ABELIAN, MONOIDAL, WITH $\otimes$ BIEXACT:

CONSTRUCTED A MONOIDAL ASSOC. GRADED FUNCTOR

$$gr: \mathrm{Fil}(\mathcal{C}) \longrightarrow \mathrm{Gr}(\mathcal{C})$$

$$(A, F_A) \longmapsto \coprod_{i \in \mathbb{N}_0} \mathrm{coker}(F_A(i-1) \to F_A(i))$$

[BONGALE]

IN $\mathcal{C} = \mathrm{Vec}_{\Bbbk}$

TAKE $A$ FIN. DIM. CONNECTED FILTERED $\Bbbk$-ALG.

$$\underset{\text{FROB}}{gr(A)} \Rightarrow \underset{\text{FROB}}{A}$$

WANT TO GENERALIZE

# FROBENIUS ALGEBRAS:

## PRESERVED UNDER FILTERED DEFORMATION

$1^{st}$ TOOL DEVELOPED (W-YADAV) BUILDS ON SCHAUENBURG, ARDIZZONI-MENINI, GALATIUS ET. AL, HAUGSENG-MILLER, GWILLIAM-PAVLOV

FILTERED-GRADED TOOL—

FOR $\mathcal{C}$ ABELIAN, MONOIDAL, WITH $\otimes$ BIEXACT:

CONSTRUCTED A MONOIDAL ASSOC. GRADED FUNCTOR

$$gr : \mathrm{Fil}(\mathcal{C}) \longrightarrow \mathrm{Gr}(\mathcal{C})$$

$$(A, F_A) \longmapsto \coprod_{i \in \mathbb{N}_0} \mathrm{coker}(F_A(i-1) \to F_A(i))$$

OBJ: $(X \in \mathcal{C},\ F_X : \underrightarrow{\mathbb{N}} \to \mathcal{C}$ FILT FUNC. s.t. $X \cong \mathrm{colim}_i F_X(i)$ in $\mathcal{C})$

HOMS: $X \to Y \in \mathcal{C}$ PRES. FILTRATION

OBJ: $X = \coprod_{i \in \mathbb{N}} X_i$, $X_i \in \mathcal{C}$

HOMS: $X \to Y$ COMPATIBLE w/ $\coprod$

IF $A$ IS AN ALGEBRA IN $\mathrm{Fil}(\mathcal{C})$, THEN $gr(A)$ IS AN ALGEBRA IN $\mathrm{Gr}(\mathcal{C})$.

[BONGALE]

IN $\mathcal{C} = \mathrm{Vec}_{\Bbbk}$

TAKE $A$ FIN. DIM. CONNECTED FILTERED $\Bbbk$-ALG.

$$\underset{\text{FROB}}{gr(A)} \Rightarrow \underset{\text{FROB}}{A}$$

WANT TO GENERALIZE

# FROBENIUS ALGEBRAS :

## PRESERVED UNDER FILTERED DEFORMATION

2nd TOOL DEVELOPED (W-YADAV) BUILDS ON FUCHS-STIGNER

CATEGORICAL CHARACTERIZATION OF FROBENIUS ALGEBRAS —

LET $\mathcal{C}$ BE A TENSOR CATEGORY. THEN:

$(A, m, u)$ ALGEBRA IN $\mathcal{C}$ IS FROBENIUS

$$\Updownarrow$$

$\exists\, \nu : A \to \mathbb{1}$ in $\mathcal{C}$ s.t.

ANY LEFT/RIGHT IDEAL* OF A

THAT FACTORS THROUGH $\ker(\nu)$ IS ZERO.

THAT IS, A HAS A "FROBENIUS FORM"

* MORPHISM: FROM IDEAL TO A NEED NOT BE A MONO

(... A PRIORI, $gr$ NEED NOT PRESERVE MONOS)

[BONGALE]

IN $\mathcal{C} = Vec_{\Bbbk}$

TAKE A FIN. DIM.

CONNECTED

FILTERED $\Bbbk$-ALG.

$$\underset{\text{FROB}}{gr(A)} \Rightarrow \underset{\text{FROB}}{A}$$

WANT TO GENERALIZE

# FROBENIUS ALGEBRAS :
## PRESERVED UNDER FILTERED DEFORMATION

MAIN THEOREM (W-YADAV)

LET $\mathcal{C}$ BE A TENSOR CATEGORY.
LET A BE A CONNECTED, FILTERED ALG. IN $\mathcal{C}$ W/ FINITE FILTRATION
IF $\mathrm{gr}(A)$ IS A FROBENIUS ALGEBRA IN $\mathcal{C}$, THEN SO IS A.

# FROBENIUS ALGEBRAS:

## PRESERVED UNDER FILTERED DEFORMATION

**MAIN THEOREM** (W-YADAV)

> LET $\mathcal{C}$ BE A TENSOR CATEGORY.
>
> LET $A$ BE A CONNECTED, FILTERED ALG. IN $\mathcal{C}$ W/ FINITE FILTRATION
>
> IF $gr(A)$ IS A FROBENIUS ALGEBRA IN $\mathcal{C}$, THEN SO IS $A$.

PROOF:

- $(A, F_A)$ FILTERED ALG. IN $\mathcal{C}$ W/ FIN. FILT. $\Rightarrow$ $A \simeq F_A(n)$ FOR SOME $n \in \mathbb{N}$
- TAKE MORPHISM $\nu : A \xrightarrow{\sim} F_A(n) \twoheadrightarrow F_A(n)/F_A(n-1) \overset{!}{\cong} \mathbb{1}$ A IS CONNECTED
- TAKE IDEAL $I$ OF $A \ni.$

$$\begin{tikzcd} \ker(\nu) \arrow[r] & A \arrow[r, "\nu"] & \mathbb{1}. \\ I \arrow[u, "2"] \arrow[ur, "\not{0}"'] & & \end{tikzcd}$$

BY TOOL 2,
SUFFICES TO SHOW $I = 0$

# FROBENIUS ALGEBRAS:

## PRESERVED UNDER FILTERED DEFORMATION

**MAIN THEOREM** (W-YADAV)

> LET $\mathcal{C}$ BE A TENSOR CATEGORY.
>
> LET $A$ BE A CONNECTED, FILTERED ALG. IN $\mathcal{C}$ W/ FINITE FILTRATION
>
> IF $gr(A)$ IS A FROBENIUS ALGEBRA IN $\mathcal{C}$, THEN SO IS $A$.

PROOF:

- $(A, F_A)$ FILTERED ALG. IN $\mathcal{C}$ W/ FIN. FILT. $\Rightarrow$ $A \simeq F_A(n)$ FOR SOME $n \in \mathbb{N}$
- TAKE MORPHISM $\nu : A \xrightarrow{\sim} F_A(n) \twoheadrightarrow F_A(n)/F_A(n-1) \cong \mathbb{1}$ ← $A$ IS CONNECTED
- TAKE IDEAL $I$ OF $A$ $\ni$. 

$$\begin{tikzcd} \ker(\nu) \arrow[r] & A \arrow[r, "\nu"] & \mathbb{1}. \\ I \arrow[u] \arrow[ur, "\varphi"'] \arrow[u, phantom, shift left=4, "2"] & & \end{tikzcd}$$

  By TOOL 2, SUFFICES TO SHOW $I = 0$

- TAKE $gr(\varphi) : gr(I) \to \underset{\text{FROB}}{gr(A)}$ VIA TOOL 1. $\rightsquigarrow$ $gr(\varphi)$ FACTORS THRU KERNEL OF FROB. FORM ON $gr(A)$

$\therefore$ TOOL 2 $\Rightarrow$ $gr(I) = 0$. $\therefore$ $I = 0$ ✓

# FROBENIUS ALGEBRAS :

## PRESERVED UNDER FILTERED DEFORMATION

WHEN $\mathscr{C}$ IS A SYM. FIN. TENSOR CATEG. AND $\mathcal{M}$ IS AN EXACT $\mathscr{C}$-MODULE CATEG.

$\downarrow$

GET $\mathcal{M} \sim A\text{-Mod}(\mathscr{C})$ FOR

$A = A(Cl(V,B))$

$\uparrow$

CLIFFORD ALG IN $\mathscr{C}$

$\underset{\text{FROB}}{Cl(V,B)} \Rightarrow \underset{\text{FROB}}{A}$

$\Uparrow$

$\Lambda(V)$ FROB IN $\mathscr{C}$ ✓

WANT TO GENERALIZE FOR MORE ARBITRARY MONOIDAL CATEGORIES $\mathscr{C}$

[BONGALE]

IN $\mathscr{C} = \mathrm{Vec}_{\Bbbk}$

TAKE $A$ FIN. DIM. CONNECTED FILTERED $\Bbbk$-ALG.

$\downarrow$

$\underset{\text{FROB}}{gr(A)} \Rightarrow \underset{\text{FROB}}{A}$

DONE!

# FROBENIUS ALGEBRAS : SUMMARY

* FROBENIUS ALGEBRAS HAVE A VERY RICH HISTORY
* APPEARED OVER 100 YEARS AGO & ~ DOZEN CHARACTER'NS
* MODERNS USES IN QFT — TOPOLOGICAL / CONFORMAL
* APPEAR IN CFT BY REPRESENTING MODULE CATEGORIES $\mathcal{M}$ OVER NICE MONOIDAL CATEGORIES $\mathcal{C}$
* CONVERSELY, WHEN $\mathcal{C}$, $\mathcal{M}$ ARE NICE ENOUGH ($\overset{\mathcal{C}}{\text{SYM. FIN.}\otimes}$, $\overset{\mathcal{M}}{\text{EXACT}}$) $\rightsquigarrow$ $\mathcal{M}$ IS ALWAYS REP. BY A FROB. ALG $A$ IN $\mathcal{C}$
* GET A FROB. $\Leftarrow$ CERTAIN CLIFFORD ALG IN $\mathcal{C}$ IS FROB.
* DEVELOPED FILTERED-GRADED TOOLS IN $\otimes$ CATS TO SHOW THIS

QUESTIONS ??

MORE FROM BONGALE

MORE ON gr

COMBINATORICS

REPRESENTING MODULE CATS.

MORE ON QFTs

# FILTERED FROBENIUS ALGEBRAS IN MONOIDAL CATEGORIES

CHELSEA WALTON
RICE UNIVERSITY

JOINT WORK WITH HARSHIT YADAV

ARXIV: 2106.01999

ON THE POSTDOC MARKET

≡ THANKS FOR LISTENING ≡

# FUTURE DIRECTIONS

MORE FROM BONGALE

MORE ON gr

COMBINATORICS

REPRESENTING MODULE CATS.

MORE ON QFTS

THEOREM [BONGALE, ~~1964~~ 1968]

LET A BE A FINITE DIMENSIONAL, ~~CONNECTED~~, FILTERED $\mathbb{k}$-ALGEBRA

THEN $gr(A)$ FROB. $\Rightarrow$ A FROB.

QUESTION

~~MAIN THEOREM~~ (W- YADAV)

LET $\mathcal{C}$ BE A TENSOR CATEGORY.

LET A BE A ~~CONNECTED~~, FILT. ALG. IN $\mathcal{C}$

W/ FINITE FILTRATION

THEN $gr(A)$ FROB. $\Rightarrow$ A FROB.

# FUTURE DIRECTIONS

MORE FROM BONGALE

**MORE ON $gr$**

COMBINATORICS

REPRESENTING MODULE CATS.

MORE ON QFTs

HAVE: MONOIDAL ASSOC. GRADED FUNCTOR

$$gr : \mathrm{Fil}(\mathcal{C}) \longrightarrow \mathrm{Gr}(\mathcal{C})$$

SHOWED:

$$\underset{\text{FROBENIUS}}{A} \Longleftarrow \underset{\text{FROBENIUS}}{gr(A)}$$

(UNDER CERTAIN CONDITIONS)

CAN THIS BE ACHIEVED VIA A "FROBENIUS MONOIDAL" ADJOINT TO $gr$?

ALSO IN $\mathrm{Vec}_{\Bbbk}$:

$$\underset{\left\{\begin{array}{l}\text{NOETHERIAN} \\ \text{DOMAIN} \\ \vdots\end{array}\right.}{A} \Longleftarrow \underset{\left\{\begin{array}{l}\text{NOETHERIAN} \\ \text{DOMAIN} \\ \vdots\end{array}\right.}{gr(A)}$$

WHAT ABOUT IN $(\mathcal{C}, \otimes, \mathbb{1})$?

(MAY NEED TO DEFINE PROPERTIES IN $\otimes$ CAT.)

# FUTURE DIRECTIONS

MORE FROM BONGALE

MORE ON $\mathfrak{gr}$

COMBINATORICS

REPRESENTING MODULE CATS.

MORE ON QFTs

(SEE V2 OF ARXIV PREPRINT)

THE EXTERIOR ALGEBRA

$$\Lambda(V) \in (\mathcal{C}, \otimes, \mathbb{1}, c)$$

USED IN THE MAIN THEOREM IS DEFINED EXPLICITLY & COMBINATORIALLY IN WORK OF BESPALOV ET. AL

(EVEN THOUGH WE DID NOT NEED THIS PRESENTATION)

CLIFFORD ALG.

DEFINE $Cl(V,B) \in (\mathcal{C}, \otimes, \mathbb{1}, c)$ EXPLICITLY & COMBINATORIALLY

# FUTURE DIRECTIONS

MORE FROM BONGALE

MORE ON $gr$

COMBINATORICS

**REPRESENTING MODULE CATS.**

MORE ON QFTs

## QUESTION:

WHEN IS $\mathcal{M} \in \mathcal{C}\text{-Mod}$ REPRESENTED BY SOME $A \in \mathrm{FrobAlg}(\mathcal{C})$?

(THAT IS, $\mathcal{M} \sim A\text{-mod}(\mathcal{C})$)

* $\mathcal{C} = \mathrm{Vec}_{\mathbb{R}}$, $\mathcal{M}$ SEMISIMPLE ✓
* $\mathcal{C}$ SYM. FIN $\otimes$, $\mathcal{M}$ EXACT ✓
* MORE? $\mathcal{C}$ FINITE $\otimes$, $\mathcal{M}$ EXACT ??

# FUTURE DIRECTIONS

MORE FROM BONGALE

MORE ON $\mathfrak{gr}$

COMBINATORICS

REPRESENTING MODULE CATS.

MORE ON QFTs

**THEOREM** [ABRAMS, QUINN, VORONOV]

> THERE IS AN EQUIVALENCE OF ⊗ CATS
> ⊗ CAT OF 2-TQFTS WITH VALUE IN $\mathcal{C}$ & ⊗ CAT OF COM FROB ALGS IN $\mathcal{C}$

**THEOREM** [TURAEV]

COBORDISMS ARE EQUIPPED W/ HOMOTOPY CLASSES OF MAPS TO A SPACE X

> THERE IS AN EQUIVALENCE OF ⊗ CATS
> ⊗ CAT OF 2-HQFTS WITH VALUE IN $\mathrm{Vec}_{\Bbbk}$ & ⊗ CAT OF CERTAIN G-GRADED COM FROB ALGS IN $\mathrm{Vec}_{\Bbbk}$

$\pi_1(X) = G$

**THEOREM** [LAZAROIU, LAUDA-PFEIFFER]

> THERE IS A CONNECTION BTW ⊗ CATS
> ⊗ CAT OF OPEN-CLOSED 2-TQFTS WITH VALUE IN $\mathrm{Vec}_{\Bbbk}$ & GRADED / FILTERED FROB ALGS IN $\mathcal{C}$
> "KNOWLEGDABLE FROB. ALGS"

* MORE??