FILTERED FROBENIUS ALGEBRAS
IN MONOIDAL CATEGORIES

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MOTIVATION 1

(FROM THE PERSPECTIVE OF NONCOMMUTATIVE RING THEORY)

NICE ALGEBRAIC PROPERTIES PRESERVED UNDER
FILTERED DEFORMATION...

TAKE AN $\mathbb{N}_0$-FILTERED ALGEBRA/ MODULE

$A = \bigcup_{i \in \mathbb{N}_0} A_i$, $A_i \cdot A_j \subseteq A_{i+j}$

$1_A \in \mathbb{N}_0$

ITS ASSOCIATED GRADED ALGEBRA

$gr(A) = \bigoplus_{i \in \mathbb{N}_0} A_i/A_{i-1}$

... "IF $gr(A)$ IS (X),

THEN SO IS $A$.

EXAMPLES

$A$ | $gr(A)$
--- | ---
$U_q(g)$ | $\delta_q(g)$
$Weyl(v)$ | $\delta(v)$
$Cl(V, B)$ | $\Lambda(v)$

EXAMPLES OF (X)
AN INTEGRAL DOMAIN
PRIME
NOETHERIAN
**Motivation 2**

*(From the perspective of Quantum Algebra)*

**Building Frobenius Algebras in Monoidal Categories...**

**Monoidal Category** $(\mathcal{C}, \otimes, \mathbb{1})$
- Category $\mathcal{C}$ equipped with
- Bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$
- Distinguished object $\mathbb{1}$
- That mimic the structure of a monoid
  (subject to compatibility conditions)

**Frobenius Algebra** in $\mathcal{C}$
- Is a 5-tuple:
  - $\mathbb{1}$, object in $\mathcal{C}$
  - $\Delta : A \otimes A \rightarrow A$, morphism in $\mathcal{C}$
  - $\varepsilon : A \rightarrow \mathbb{1}$, morphism in $\mathcal{C}$

**Applications in**

- Morita Theory
- TQFTs & CFTs
- Computer Science

**Motivation 1**

*(From Noncommutative Ring Theory)*

Nice algebraic properties preserved under filtered deformation

**Goal:** Generalize

**Theorem** (Bongale, 1947)

Let $A$ be a finite-dim/l, connected, filtered algebra over $\mathbb{1}$.

If $q(A)$ is a Frobenius algebra over $\mathbb{1}$, then so is $A$.  

**Motivation 2**

*(From Quantum Algebra)*

Building Frobenius Algebras in Monoidal Categories

**Frobenius Algebras** over $\mathcal{C}$

$$\text{Frobenius Algebras over } \mathcal{C} \\ (\text{Vec}_{\mathbb{1}}, \otimes = \otimes_{\mathbb{1}}, 1 = \mathbb{1})$$

\[ \text{where: } \Delta \text{m} \quad (\alpha \otimes \beta) \otimes \gamma \quad \text{and} \quad (\alpha \otimes \beta)^{\otimes 3} \]

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\[ \beta \otimes (\alpha \otimes \gamma) \quad \text{and} \quad \beta \otimes (\alpha \otimes \gamma) \]

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**Main Theorem (W-Yadav)**

Let \( \mathcal{C} \) be an Abelian, rigid monoidal category.

\[ A = \mathbb{1} \]

Let \( A \) be a connected, filtered alg. in \( \mathcal{C} \) with finite filtration.

If \( \text{gr}(A) \) is a Frobenius algebra in \( \mathcal{C} \), then so is \( A \).

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**Need to Develop Categorical Tools**

\[ \equiv \text{Categorical Filtered-Graded Structures} \equiv \]

\[ \equiv \text{Categorical Characterization} \equiv \]

**Of Frobenius Algebras**

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**1st Tool Developed (W-Yadav)**

Filtered-Graded Tool -

For \( \mathcal{C} \) Abelian, monoidal, with \( \otimes \) Biexact:

- Constructed a monoidal associated graded functor \( \text{gr} : \text{Fil}(\mathcal{C}) \rightarrow \text{Gr}(\mathcal{C}) \)

\[ \begin{array}{c}
(A, F_A) \\
\text{objects: } (x \in \mathcal{C}, F_x : A \rightarrow \mathcal{C} \text{ function}) \\
\text{objects: } x = \bigsqcup_{i \in \mathbb{N}_0} x_i \in \text{Ind}(\mathcal{C}) \\
\text{morphisms: } (x \rightarrow y \in \mathcal{C} \text{ that prei. filin}) \\
\text{morphisms: } (x \rightarrow y \in \mathcal{C} \text{ compatible w/ II decomposition})
\end{array} \]

\[ \begin{array}{c}
(x, F_x) \otimes (y, F_y) = (x \otimes y, \text{the conv of } F_x + F_y) \\
\otimes \text{ given by } (x \otimes y)_k = \bigsqcup_{i+j=k} (x_i \otimes y_j)
\end{array} \]

(IF \( A \) is an algebra in \( \text{Fil}(\mathcal{C}) \),

THEN \( \text{gr}(A) \) is an algebra in \( \text{Gr}(\mathcal{C}) \).
**2nd Tool Developed (W-Yadav)**

**Categorical Characterization of Frobenius Algebras**

**Recall:**

Frobenius algebra in \( \mathcal{C} \)

**Is a 5-tuple:**

\[
\begin{align*}
(A, \text{object in } \mathcal{C}) & \quad \text{forms an alg in } \mathcal{C} \\
\Delta: A \otimes A & \rightarrow A, \text{ morphism in } \mathcal{C} \\
\varepsilon: A & \rightarrow \text{Id}_{\mathcal{C}}, \text{ morphism in } \mathcal{C} \\
\end{align*}
\]

**Where:**

\[
\begin{align*}
\Delta \otimes \text{Id}_{\mathcal{C}} & \rightarrow \Delta \\
\text{Id}_{\mathcal{C}} \otimes \Delta & \rightarrow \Delta \\
\end{align*}
\]

\[
\begin{align*}
\varepsilon \circ \Delta & = \text{Id}_{A} \\
\varepsilon \circ \text{Id}_{A} & = \text{Id}_{\mathcal{C}} \\
\end{align*}
\]

**Let \( \mathcal{C} \) be a rigid monoidal category.**

\((A, m, u)\) algebra in \( \mathcal{C} \) is Frobenius

\[\Leftrightarrow \exists \mu: A \rightarrow 1 \text{ in } \mathcal{C} \implies \text{any left/right ideal of } A \text{ that factors through } \ker(\mu) \text{ is zero.} \]

That is, \( A \) has a "Frobenius form."

**Main Theorem (W-Yadav)**

Let \( \mathcal{C} \) be an abelian, rigid monoidal category.

Let \( A \) be a connected, filtered alg. in \( \mathcal{C} \) w/ finite filtration

If \( g_f(A) \) is a Frobenius algebra in \( \mathcal{C} \), then so is \( A \).

**Rough Sketch of Proof:**

- \((A_f, F)\) filtered alg. in \( \mathcal{C} \) w/ finite filtration \( \Rightarrow \) \( A \simeq F_k(n) \) for some \( k \in \mathbb{N}_0 \).

- **Take morphism** \( \psi: A \rightarrow F_k(n) \) \( \Rightarrow F_k(n)/F_{k-1}(n) \rightarrow \text{Id} \) because \( A \) is connected.

- **Take ideal** \( I \) of \( A \) so that \( \ker(\psi) \rightarrow A \rightarrow 1 \).

  **By Tool 2,**

  It suffices to show \( I = 0 \)

- **Consider** \( g_f(\psi): g_f(I) \rightarrow g_f(A) \) **via Tool 2.**

  Show \( g_f(\psi) \) factors through Frobenius kernel of Frobenius form on \( g_f(A) \)

  \[\therefore \text{ Tool 2 } \Rightarrow g_f(I) = 0\]

  \[\therefore I = 0 \checkmark\]
**Main Theorem (W-Yadav)**

Let $\mathcal{C}$ be an abelian, rigid monoidal category.

Let $A$ be a connected, filtered alg. in $\mathcal{C}$ w/ finite filtration.

If $\gamma(A)$ is a Frobenius algebra in $\mathcal{C}$, then so is $A$.

**Source of Examples:** "Braided Clifford Algebras"

Recall for $\mathcal{C} = \text{Vec}_k$, and $V \in \text{Vec}_k$, $B : V \otimes V \to k$ sym. bilinear form.

$$\text{Cl}(V,B) = T(V)/(v\otimes w + w \otimes v - B(v \otimes w)),$$

$V$ is a filtered deformation of Clifford algebra.

$$\wedge(V) = T(V)/(v \otimes w + w \otimes v), \forall v,w \in V$$

$\wedge(V)$ Frobenius $\Rightarrow$ $\text{Cl}(V,B)$ Frobenius.

**Bangale**

Main Thm for $\mathcal{C} = \text{Vec}_k$.

Take $(\mathcal{C}, \otimes, 1, \circ)$ symmetric monoidal category.

Natural isom. $c_{X,Y} : X \otimes Y \cong Y \otimes X$. $c^* = \text{id}$ (ex. for $\mathcal{C} = \text{Vec}_k$).

Take $V \in \mathcal{C}$:

Bespalov et al. (1997, 2000) defined braided exterior algebra $\wedge_c(V)$.

Take morphism $B : V \otimes V \to A$ in $\mathcal{C}$.

$B = B_{c_{V,V}}$.

W-Yadav pose combinatorial problem to define braided Clifford algebra $\text{Cl}_c(V,B)$.

**Proposition (W-Yadav)**

Under certain finiteness conditions,

we have that $\wedge_c(V)$ is a Frobenius algebra in $\mathcal{C}$, and if problem resolved, so is $\text{Cl}_c(V,B)$.
**Motivation 1**
(from noncommutative ring theory)

Nice algebraic properties
preserved under
filtered deformation

**Motivation 2**
(from quantum algebra)

Building Frobenius algebras
in monoidal categories

**Main Theorem (W. Yadav)**

Let $\mathcal{C}$ be an abelian, rigid monoidal category.

Let $A$ be a connected, filtered alg. in $\mathcal{C}$ w/ finite filtration.

If $\mathcal{F}(A)$ is a Frobenius algebra in $\mathcal{C}$, then so is $A$.

**Further Direction**

Generalize & study other
ring-theoretic properties that
lift under filtered deformation
(ex. domain, prime, noetherian)

In the context of alg. in $\mathcal{C}$ cateQs.
MAIN THEOREM (W. YADAV)

Let \( \mathcal{C} \) be an abelian, rigid monoidal category.

Let \( A \) be a connected, filtered alg. in \( \mathcal{C} \) w/ finite filtration.

If \( \gamma_c(A) \) is a Frobenius algebra in \( \mathcal{C} \), then so is \( A \).

FURTHER DIRECTION

2-d TQFTs w/ values in \( \mathcal{C} \)

= Comm. Frob. Algebras in \( \mathcal{C} \).

Graded/filtered Frob. algs in \( \mathcal{C} \)

play a role in “open-closed” TQFTs.

Study connection to main thm.

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Thanks for listening!

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