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Actions of finite dimensional Hopf algebras on commutative domains

joint with Pavel Etingof: 1301.4161, 1403.4673, work in preparation

Interested in Hopf algebra actions on algebras in general

* difficult because computations can be horrible
* depends on the classification of Hopf algebras

In any case, there are three goals one could shoot for:

Fix field k, $\mathcal{H}$ = class of Hopf algs / k, $\mathcal{A}$ = class of algs / k.

$\boxed{\text{NO QUANTUM SYMMETRY}}$

$H \curvearrowright A$ factors through a cocommutative Hopf algebra.
$\underset{\mathcal{H}}{\overset{\cap}{H}}$ $\underset{\mathcal{A}}{\overset{\cap}{A}}$ $[\tau \circ \Delta = \Delta]$

In other words, if $H \curvearrowright A$ inner faithfully
$[\nexists$ nonzero Hopf ideal I of H with $IA = 0]$
then H is cocommutative.

$\boxed{\text{SOME QUANTUM SYMMETRY}}$

Classify pairs (H, A) so that $H \curvearrowright A$ inner faithfully
$\underset{\mathcal{H}}{\overset{\cap}{H}}$ $\underset{\mathcal{A}}{\overset{\cap}{A}}$ (with at least one H is non-cocom)

$\boxed{\text{ALL QUANTUM SYMMETRY}}$

Fix $H \in \mathcal{H}$, $A \in \mathcal{A}$, classify all inner faithful $H \curvearrowright A$

Today, $k=\bar{k}$, $\mathrm{ch}\,k=0$

$\mathcal{H}$ = finite dim'l Hopf algs/k $\qquad$ $\mathcal{A}$ = commutative domains/k

Two tractable subclasses.

(Classification programs are active for these two subclasses)

$\boxed{\mathcal{H}' = \text{semisimple Hopf algebras}/k}$ (as a k-alg)

$\boxed{\mathcal{H}' = \text{pointed Hopf algebras}/k}$ (all simple H-comodules are 1-dim'l $\Leftrightarrow$ H^* is basic)

Ex. kG group algebra
$(kG)^*$ dual
$(kG)^J$ twists
extensions along s.e.s
H_8 noncom-noncocom.

'GROUP THEORETIC'

Ex. (variants of) small quantum groups
= finite dim'l Hopf quotients of $U_q(\mathfrak{g})$ at q a root of 1

Ex. $u_q(\mathfrak{sl}_2)$

'LIE THEORETIC'

In fact,

Thm [Cartier-Kostant-Milnor-Moore]

① H Hopf algebra/k cocommutative $\Rightarrow$ $H \cong \underset{\text{group part}}{kG} \ltimes \underset{\text{Lie part}}{U(\mathfrak{g})}$,
$G \curvearrowright \mathfrak{g}$, G finite

$\Rightarrow$ ② H finite dim'l, cocom, Hopf alg/k $\Rightarrow$ $H \cong kG$.

$\boxed{\text{We show}}$

$k=\bar{k}$, $\mathrm{ch}\,k=0$ & $\mathcal{A}$ = commutative domains/k

* $\mathcal{H}$ = semisimple Hopf algs/k ($\Rightarrow$ fin. dim) $\rightsquigarrow$ NO QUANTUM SYMMETRY
* $\mathcal{H}$ = finite dim'l, pointed Hopf algs/k $\rightsquigarrow$ SOME QUANTUM SYMMETRY.

finite dim'l.

Semisimple Hopf alg actions on commutative domains $(k=\bar{k}, \operatorname{ch} k=0)$

Thm [Etingof-W] $\underset{\text{ss}}{H} \curvearrowright \underset{\text{comdomain}}{A}$ inner faithfully $\Rightarrow H \simeq kG$ G finite group.

Sketch of Proof: Want to show H^* is commutative (by [CKum thm])

Recall that a right coideal subalgebra B of H^* is a subalgebra of H^* so that

$$\underset{H^* \otimes H^*}{\Delta(B)} \subseteq B \otimes H^*.$$

Ex. A = commutative domain, $H \curvearrowright A$ from the left

$\rightsquigarrow$ $\rho: A \to A \otimes H^*$, the coaction of H^* on A from the right

$\chi: A \to k$, character of A

$\rightsquigarrow$ $\rho_\chi: (\chi \otimes id) \circ \rho: A \to A \otimes H^* \to H^*$

$\rightsquigarrow$ $\boxed{\rho_\chi(A) =: A_\chi}$ is a right coideal subalg of H^*.

Thm [EW / Schneider (in unpublished lecture notes)] A semisimple Hopf alg has finitely many coideal subalgebras.

To proceed with proof: Reduce to case where A is finitely generated

- Let $X = \operatorname{Spec}_m(A)$ (affine irreducible alg variety $/k$)

  closed pts $\leftrightarrow$ characters $\chi: A \to k$

- Let $\gamma: X \to \coprod_d \underbrace{CS_d(H^*)}_{\text{the set / variety of dimension } d \text{ coideal subalgebras of } H^*}$

  $\chi \mapsto A_\chi$

$d_0 := \max\{\dim_k A_x\}$ and $X_0 = \{x \in X \mid \dim_k A_x = d_0\}$

Get that: $\cdot$ X_0 irreducible ($X_0 \neq \emptyset$, open & dense)
$\cdot$ $\gamma|_{X_0}$ regular
$\cdot$ $CS_{d_0}(H^*)$ is finite (since H^* semisimple)

$\Rightarrow$ $\gamma|_{X_0}$ is constant; $\forall x \in X_0 : \gamma|_{X_0}(x) = B$ for some coideal subalg B of H^* of dimension d_0

Argue that: $\underset{\text{coaction}}{\rho : A \to A \otimes H^*}$ — restricts to — $\rho' : A \to A \otimes B$ (since B generated by all $\{A_x\}_{x \in X}$).

inner faithfulness $\Rightarrow$ $H^* = B$
$\Leftarrow$ the image of A (commutative), ($= A_x$ for some $x \in X_0$)

$\Rightarrow$ H^* is commutative, as desired.

<u>Finite dim'l pointed Hopf alg actions on commutative domains</u>

* Study boils down to actions on <u>fields</u>

<u>Lem</u> [Skryabin] A com. domain. Q_A quotient field
H finite dim'l $\curvearrowright A$ inner faithful $\Rightarrow$ $H \curvearrowright Q_A$ inner faithful.

$\rightsquigarrow$ <u>Def'n</u> A Hopf alg is $\boxed{\text{Galois-Theoretical}}$ (GT) if it acts inner faithfully and k-linearly on a field containing k.

<u>Prop</u> ① Any finite group algebra is GT
② H semisimple & GT $\Rightarrow$ $H \cong kG$
③ GT preserved under
— Hopf subalg
— ⊗

, not preserved under
— Hopf dual
— twists — 2-cocycle (twists mult)
— Drinfeld (twists Δ).

What about the invariant fields?

<u>Thm</u> H finite dim'l pointed GT Hopf algebra w/ H-modulefield L.

Then ① $L^H = L^{G(H)}$, $G(H)$ = group of grouplike elts of H
$= \{g \in H \mid \Delta(g) = g \otimes g\}$.

② $L^H \subseteq L$ is Galois with Galois group $G(H)$.

Have similar results for H not nec. pointed & $L \rightsquigarrow A$ a comm domain / Azumaya

$\xrightarrow{\text{action}}$	on commutative domain A	on Azumaya algebra A ($Z = Z(A)$)
H finite dim'l	$A^H = A^{H_0}$	$\underset{Z^H}{\underset{\parallel}{Z \cap A^H}} = \underset{Z^{H_0}}{\underset{\parallel}{Z \cap A^{H_0}}}$
H finite dim'l pointed $\Rightarrow H_0 = kG$	$A^H = A^G$	$Z^H = Z^G$

<u>Example</u> $\boxed{H = u_q(sl_2) \text{ is GT}}$

$u_q(sl_2)$, q root of 1 with $\mathrm{ord}(q^2) = m$:
generated by E, F, K
relations $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$, $KE = q^2 EK$, $KF = q^{-2} FK$, $K^m = 1$, $E^m = F^m = 0$
$\Delta(K) = K \otimes K$ $\quad \Delta(E) = K \otimes E + E \otimes 1$ $\quad \Delta(F) = 1 \otimes F + F \otimes K^{-1}$

$\rightsquigarrow G(H) = \mathbb{Z}_m$

$H \overset{\text{inner faithfully}}{\curvearrowright} k(u)$ by $K \cdot u = q^{-2} u$ $\quad E \cdot u = 1$ $\quad F \cdot u = -q u^2$

$\rightsquigarrow k(u)^{u_q(sl_2)} = k(u)^{\mathbb{Z}_m} \hookrightarrow k(u)$
cyclic Galois ext of degree m.

We have examples / classification results of finite dim'l, pointed Hopf algs of finite Cartan type

Defn/Thm [Andruskiewitsch-Schneider, Angiono]
H finite dim'l pointed with $G(H)$ abelian
$\Rightarrow H$ generated by grouplike elts & skew primitive elts $\{x_1, \dots, x_n\}$
$[\Delta(x_i) = g_i \otimes x_i + x_i \otimes 1 \quad g_i \in G(H)]$

The most extensively studied class of finite dim'l pointed Hopf algs

Say H is of finite Cartan type if $g_i x_j = q_{ij} x_j g_i$, $q_{ii} \neq 1$
where $\boxed{q_{ij} q_{ji} = q_{ii}^{a_{ij}}}$ & (a_{ij}) = Cartan matrix of ass. Lie alg.

Example $\boxed{H = u_q(sl_2)}$ Take $g_1 = g_2 = K$ $x_1 = E$ $x_2 = KF$
$\rightsquigarrow \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} q^2 & q^{-2} \\ q^2 & q^{-2} \end{pmatrix} \rightsquigarrow (a_{ij}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
$\rightsquigarrow$ $\boxed{\text{type } A_1 \times A_1}$

Thm $\boxed{\text{Examples}}$ of finite dim'l pointed GT Hopf algs of finite Cartan type:
- Taft algebras ($\cong u_q^{\geq 0}(sl_2)$) (type A_1)
- $u_q(sl_2)$ (type $A_1 \times A_1$)
- some Drinfeld twists of $u_q(gl_n)$, $u_q(sl_n)$, $u_q^{\geq 0}(g)$ g ss Lie alg
  (type $A_{n-1} \times A_{n-1}$) (same type as g)
- Nichols Hopf algs $E(n)$ (type $A_1^{\times n}$)
  - generated by g (grplike), $\{x_1, \dots, x_n\}$ ($(g,1)$ skew-prim): $g^2 = 1$ $x_i^2 = 0$ $gx_i + x_i g = 0 \ \forall i$

$\boxed{\text{Nonexamples}}$ include $gr(u_q(sl_2))$ & generalized Taft algebras ($\neq$ Taft algs)

Classification results for type $A_1^{\times n}$, of "rank two" in preparation.
Ex. $u_q(g)$ is GT $\Leftrightarrow$ $g = sl_2$
g simple Lie algebra.