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A Century of Frobenius Algebras:
I. Frobenius Algebras over a Field.

Let's start at the beginning! ... A 1903-Style Game
due to F.G. Frobenius (1849-1917)

* called "hypercomplex numbers"

Goal: To understand finite-dim'l $\mathbb{C}$-algebras* A via matrices

- Fix a $\mathbb{C}$-vs basis $\{v_1, \dots, v_n\}$ of A [for the $\mathbb{C}$-vs structure of A]

- For $a \in A$, collect scalars $\beta^a_{i,j}, \gamma^a_{i,j} \in \mathbb{C}$ such that
$$v_i a = \sum_{i=1}^{n} \beta^a_{i,j} v_j \quad \& \quad a v_j = \sum_{i=1}^{n} \gamma^a_{j,i} v_j$$
[for the multip of A]

- Form $\mathbb{C}$-linear maps $\beta: A \to \mathrm{Mat}_n(\mathbb{C})$ & $\gamma: A \to \mathrm{Mat}_n(\mathbb{C})$
$$a \mapsto (\beta^a_{i,j})_{i,j} \qquad a \mapsto (\gamma^a_{i,j})_{i,j}$$

$\boxed{\text{Question}}$ When does $\exists P \in GL_n(\mathbb{C})$ such that
$$P\beta(a) = \gamma(a)P \quad \forall a \in A \, ?$$

* Supplies A with an interesting symmetry when this happens.
* Frobenius noticed that this happens for finite group algs.
* In modern language, this is asking:
When are the left & right regular representations of A equivalent?

no Representation Theory in 1903

... just Linear Algebra ...

Towards an answer:

– Collect scalars $\mu_{i,j}^{k} \in \mathbb{C}$ such that $v_i v_j = \sum_{k=1}^{n} \mu_{i,j}^{k} v_k$

$\boxed{\text{Def'n}}$ For $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, the parastrophic matrix of A at $\underline{\lambda}$ is

$$\boxed{P_{\underline{\lambda}}^{A} := \left(\sum_{k=1}^{n} \lambda_k \mu_{i,j}^{k} \right)_{i,j}}$$

$\boxed{\text{Answer}}$ Yes $\Leftrightarrow$ $\exists \lambda \in \mathbb{C}^n$ such that $P_{\underline{\lambda}}^{A}$ is invertible.

$\boxed{\text{Def'n}}$ A is Frobenius~~ean~~* when

* Named by Nesbitt, Nakayama later, but shortened.

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Example : Take $\boxed{A = \mathbb{C}[x]/(x^2)}$ (2-dim'l $\mathbb{C}$-algebra) :

$\mathbb{C}$-vs : $\{1_A, x\}$ & multip. table

	1_A	x
1_A	1_A	x
x	x	0

Here, $\mu_{1,1}^{1} = \mu_{1,x}^{x} = \mu_{x,1}^{x} = 1$ & $\mu_{*,*}^{*} = 0$ otherwise.

For $\lambda_1, \lambda_x \in \mathbb{C}$, get $P_{\underline{\lambda}}^{A} = \begin{pmatrix} \lambda_1 & \lambda_x \\ \lambda_x & 0 \end{pmatrix}$ ← invertible for $\lambda_x \neq 0$.

$\therefore$ A is Frobenius

Example : If you're bored, show that the 3-dim'l $\mathbb{C}$-alg :

$A = \mathbb{C}[x,y]/(x^2, xy, y^2)$ is not Frobenius.

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And then this game was put away for about 3 decades.

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Fast-forward to the mid-1930s: Tools of Abstract Alg. were developed ... Let's dust off that Frobenius game ...

$\boxed{\text{Theorem}}$ [Brauer-Nesbitt, Nakayama, 1937-1942]

Take a finite dim'l $\mathbb{C}$-alg A. The following are equivalent.

(1) A is Frobenius [$\exists\, \underline{\lambda} \in \mathbb{C}^{\dim_{\mathbb{C}} A}$ such that $P_{\underline{\lambda}}^{A}$ is invertible]

(2) $\rho_L : A \to \mathrm{End}_{\mathbb{C}}(A)$ & $\rho_{RT} : A \to \mathrm{End}_{\mathbb{C}}(A^*)$

$a \mapsto [b \mapsto ab]$ $\qquad$ $a \mapsto [f \mapsto f \circ \rho_R(a)]$

left regular rep'n of A $\qquad$ for $\rho_R : A \to \mathrm{End}_{\mathbb{C}}(A)$, $a \mapsto [b \mapsto ba]$

$\equiv$ are equivalent $\equiv$ $\qquad$ right regular rep'n of A

(3) $\exists$ isomorphism of right A-modules $\phi : A \to A^*$.

Here, $b \triangleleft a = ba$ & $(f \triangleleft a)(b) = f(ab)$

(4) $\exists$ $\mathbb{C}$-linear map $\nu : A \to \mathbb{C}$ such that $\ker(\nu)$ does not contain a nonzero right ideal of A. [ν = "Frobenius form" of A]

(5) $\exists$ nondegenerate, associative $\mathbb{C}$-linear map $p : A \otimes_{\mathbb{C}} A \to \mathbb{C}$

[p = "Frobenius pairing" on A, with "copairing" $q : \mathbb{C} \to A \otimes_{\mathbb{C}} A$]

<u>Remark</u>: There are other characterizations of a similar flavor, including replacing "right" with "left" in (3) & (4).

<u>Pf sketch</u>: (1) $\Leftrightarrow$ (2): $P_{\underline{\lambda}}^{A}$ is matrix that makes $\rho_L \sim \rho_{RT}$.

(2) $\Leftrightarrow$ (3): follows from the correspondence between modules & representations

(3) $\Rightarrow$ (4) : Define $\nu := \phi(1_A) \in A^*$.

(4) $\Rightarrow$ (5) : Define $p : A \otimes_{\mathbb{C}} A \xrightarrow{m} A \xrightarrow{\nu} \mathbb{C}$

(5) $\Rightarrow$ (3) : Define $\phi : A \to A^*$, $a \mapsto p(- \otimes_{\mathbb{C}} a)$. ///

$\boxed{\text{Examples}}$

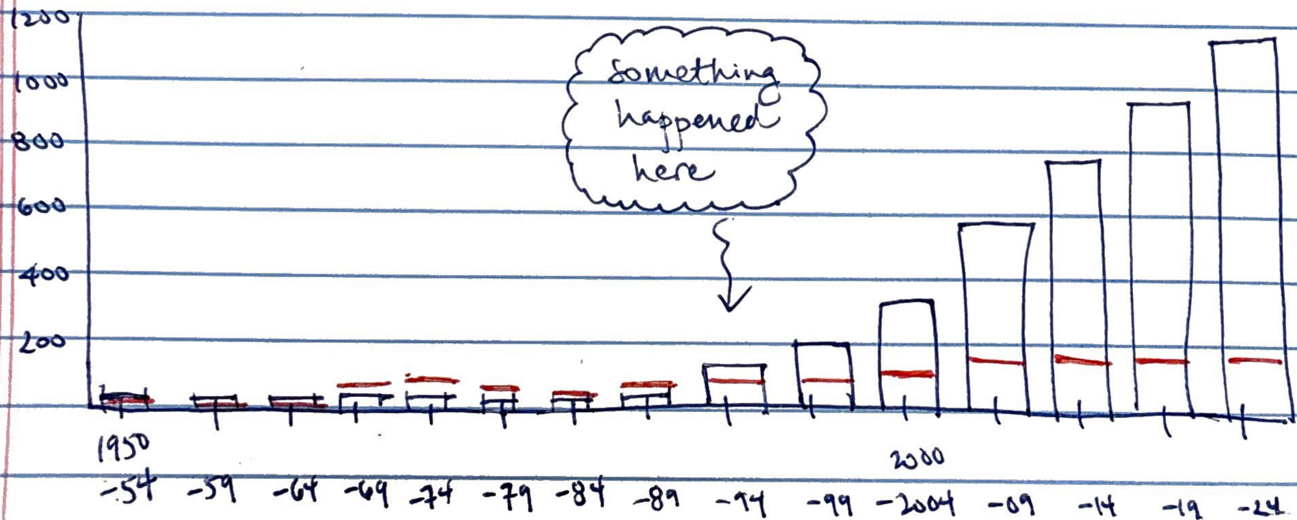
- Finite group algebras $\mathbb{C}G$, $|G| < \infty$ [via (1)/(2)/(3)]
- Matrix Algebras $Mat_n(\mathbb{C})$ [via (4) with ν = trace]
- Cohomology Algebras $H^*(X)$ [via (5) with Poincaré duality] for X = finite dim'l cpt oriented mfld

Fact: $\prod^{\text{finite}}$ (Frob algs) is Frobenius

Finite $\underbrace{\text{dim'l semisimple}}$ $\mathbb{C}$-algs are Frobenius ($\Leftrightarrow$ f.d. separable) [via the Artin-Wedderburn Theorem]

$\boxed{\text{Popularity throughout the years}}$

# of Google Scholar hits for "Frobenius algebra" ... for "separable algebra"



To explain what happened, we'll need to
- move to the setting of monoidal categories [Lecture II]
- discuss algebraic structures there including Frobenius algebras [Lecture III]
- make ties to Topological Quantum Field Thys [Lecture IV]

Then, we'll delve into other games for Frobenius Algebras in monoidal categories [Lecture V].

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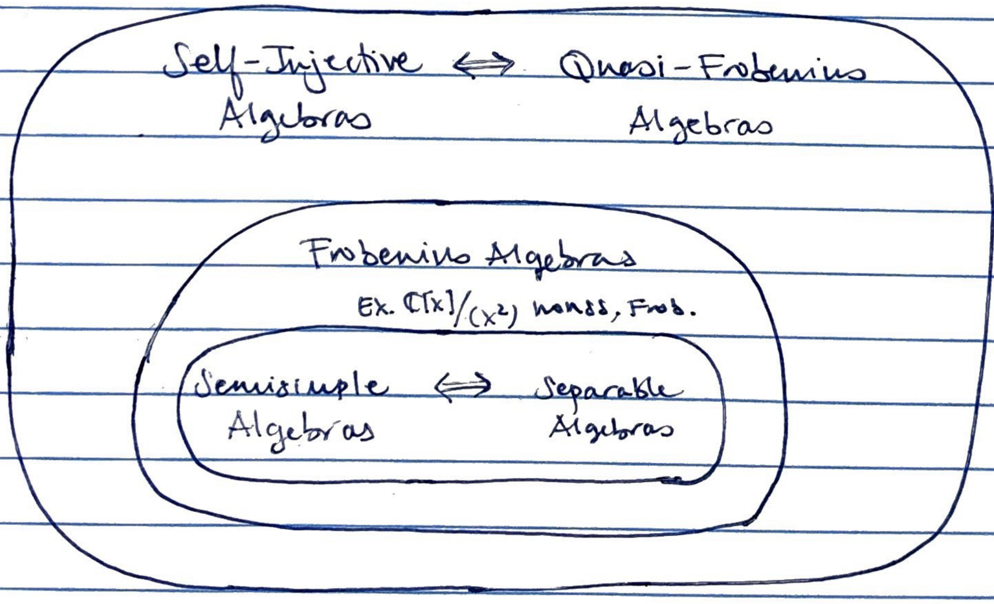
Finishing off with Frobenius Algebras over a field, let's discuss some important generalizations.

$\boxed{\text{Hierarchy of Finite-Dimensional } \mathbb{C}\text{-algebras}}$



- Definitions
- Examples
- Why Care?
- When Frobenius? ...

$\boxed{\text{Def'n}}$ A $\mathbb{C}$-algebra A is <u>self-injective</u> if $A \in A\text{-mod}$ is injective.

<u>Fact</u>: When A is finite dim'l, A is self-injective $\Leftrightarrow$ $A \in \text{Mod-}A$ is injective.

Towards "quasi-Frobenius"... Take a finite dim'l $\mathbb{C}$-algebra A.

- idempotents $e_i, e_j \in A$ are <u>orthogonal</u> if $e_i e_j = e_j e_i = \delta_{i,j} e_i$
- idempotent $e^{\neq 0} \in A$ is <u>primitive</u> if $e \neq e_i + e_j$ for e_i, e_j orthogonal
  - ↳ $eA \in \text{Mod-}A$ is indecomposable
- idempotents $\{e_1, \dots, e_n\} \subseteq A$ are <u>complete</u> if pairwise orthogonal & primitive & $1_A = \sum_{i=1}^{n} e_i$ [exists because A finite dim'l]
- idempotents $\{e_1, \dots, e_m\} \subseteq A$ are <u>basic</u> if pairwise orthogonal & primitive & $e_i A \not\cong e_j A$ in Mod-A $\forall i \neq j$
  - ↳ A is <u>basic</u> if $1_A = \sum_{i=1}^{m} e_i$ for $\{e_i\}_{i=1}^{m}$ basic idemp.
- can cut this down, removing e_j when $e_j A \cong e_i A$, to make basic; call $\{e_1, \dots, e_m\}$ "comp-basic" in this case

(★)
↳ $A \cong \bigoplus_{i=1}^{m} (e_i A)^{\oplus r_i}$ in Mod-A, for $r_i \in \mathbb{Z}_+$

decomposition into a direct sum of indecomposables.

unique by Krull-Schmidt Thm.

[$r_i = 1$ $\forall i$ if A is basic]

$\boxed{\text{Def'n}}$ Take the set-up (★) for a finite dim'l $\mathbb{C}$-algebra A. We say that A is <u>quasi-Frobenius</u> if:

$$A^* \cong \bigoplus_{i=1}^{m} (e_i A)^{\oplus s_i} \text{ in Mod-}A, \text{ for some } s_i \in \mathbb{Z}_+.$$

$\boxed{\text{Proposition}}$ A finite-dim'l $\mathbb{C}$-algebra A is <u>self-injective</u> $\Leftrightarrow$ A is <u>quasi-Frobenius</u>.

Pf/ ($\Leftarrow$) $A \in A$-Mod is projective $\Rightarrow$ $A^* \in$ Mod-A is injective.

$A^* \cong \bigoplus_{i=1}^{m} (e_i A)^{\oplus s_i}$ injective $\Rightarrow$ $e_i A \in$ Mod-A is inj $\forall i$

$\therefore A \cong \bigoplus_{i=1}^{m} (e_i A)^{\oplus r_i}$ is injective ✓

($\Rightarrow$) Fact: For a comp-basic set of idempotents $\{e_i\}_{i=1}^{m}$ of A, have {indecomp. projective right A-mods} $= \{e_i A\}_{i=1}^{m}$

$\hookrightarrow$ {indecomp. injective left A-mods} $= \{(e_i A)^*\}_{i=1}^{m}$

So, $A \cong \bigoplus_{i=1}^{m} ((e_i A)^*)^{\oplus t_i}$ in A-Mod, for some $t_i \in \mathbb{Z}_+$, by the injectivity of A & the Krull-Schmidt Theorem.

Now, $A^* \cong \bigoplus_{i=1}^{m} (e_i A)^{\oplus t_i}$ in Mod-A. $\therefore$ A is quasi-Frob. ///.

Versus Frobenius

Proposition: (1) Frobenius $\Rightarrow$ self-injective

(2) finite dim'l self-injective, basic $\Rightarrow$ Frobenius.

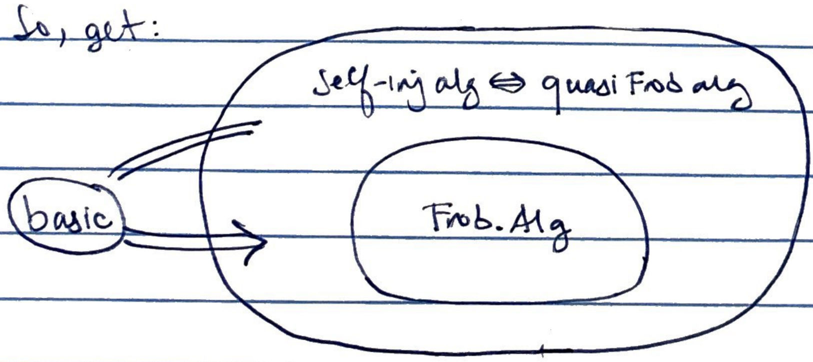
Pf/ (1) A Frobenius $\Rightarrow$ $A \cong A^*$ in Mod-A $\Rightarrow$ A is quasi-Frobenius.

(2) A basic $\Rightarrow$ $A \cong (e_1 A) \oplus \cdots \oplus (e_m A)$ in Mod-A for $\{e_i\}$ basic

A self-injective/qF $\Rightarrow$ $A^* \cong (e_1 A)^{\oplus t_1} \oplus \cdots \oplus (e_m A)^{\oplus t_m}$ in Mod-A.

But $\dim_{\mathbb{C}} A = \dim_{\mathbb{C}} A^*$. So, $t_i = 1$ $\forall i$. $\therefore$ $A \cong A^*$ in Mod-A ///.

So, get:



Main Examples

[Nakayama's original examples from 1939]

$B = \mathbb{C}Q / (ab, ba)$ for Q: $\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \end{tikzcd}$ (4-dimensional)

≡ Frobenius ≡

$$A = \frac{\mathbb{C}Q'}{\left(\begin{array}{c} ab, ba, a'b', b'a', ab', a'b \\ cd - e_1, \; dc - e_{1'} \\ ca' - a, \; da - a' \\ bc - b', \; b'd - b \end{array} \right)}$$ for Q': $\begin{tikzcd} 1 \arrow[rd, bend left, "a"] \arrow[d, bend right, "c"'] & \\ 1' \arrow[u, bend right, "d"'] \arrow[r, "a'"] & 2 \arrow[lu, "b"'] \arrow[l, bend left, "b'"] \end{tikzcd}$ (9-dimensional)

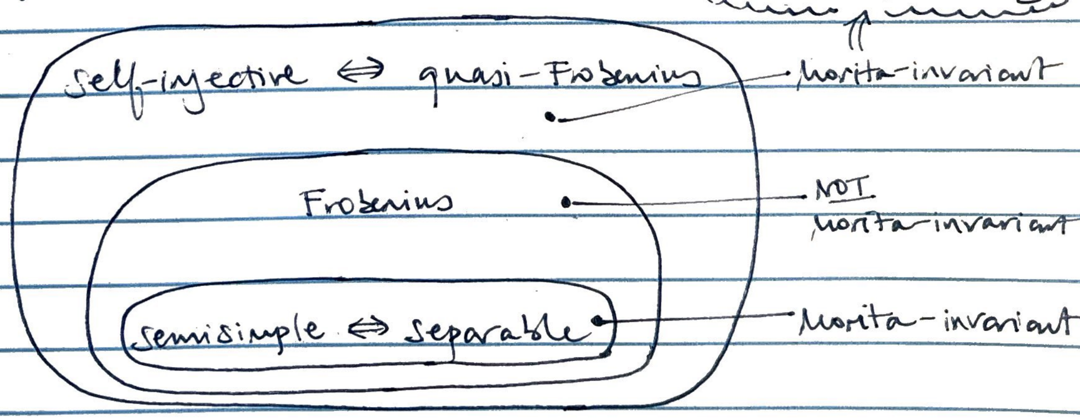
≡ is not Frobenius, is self-injective ≡

& A-Mod ≃ B-Mod !

Defn Algebraic Property P is Morita-invariant if $\underset{\mathbb{C}\text{-algs}}{A_1\text{-Mod} \simeq A_2\text{-Mod}} \Rightarrow$ A_1 & A_2 both sat. P —or— both don't.

The point of generalizing Frobenius algebras ...

Good for Rep'n Theory



Hierarchy of finite-dim'l $\mathbb{C}$-algebras

Ref: This is covered in Chapter 7 of "Symmetries of Algebras, Vol 2" (in prep).