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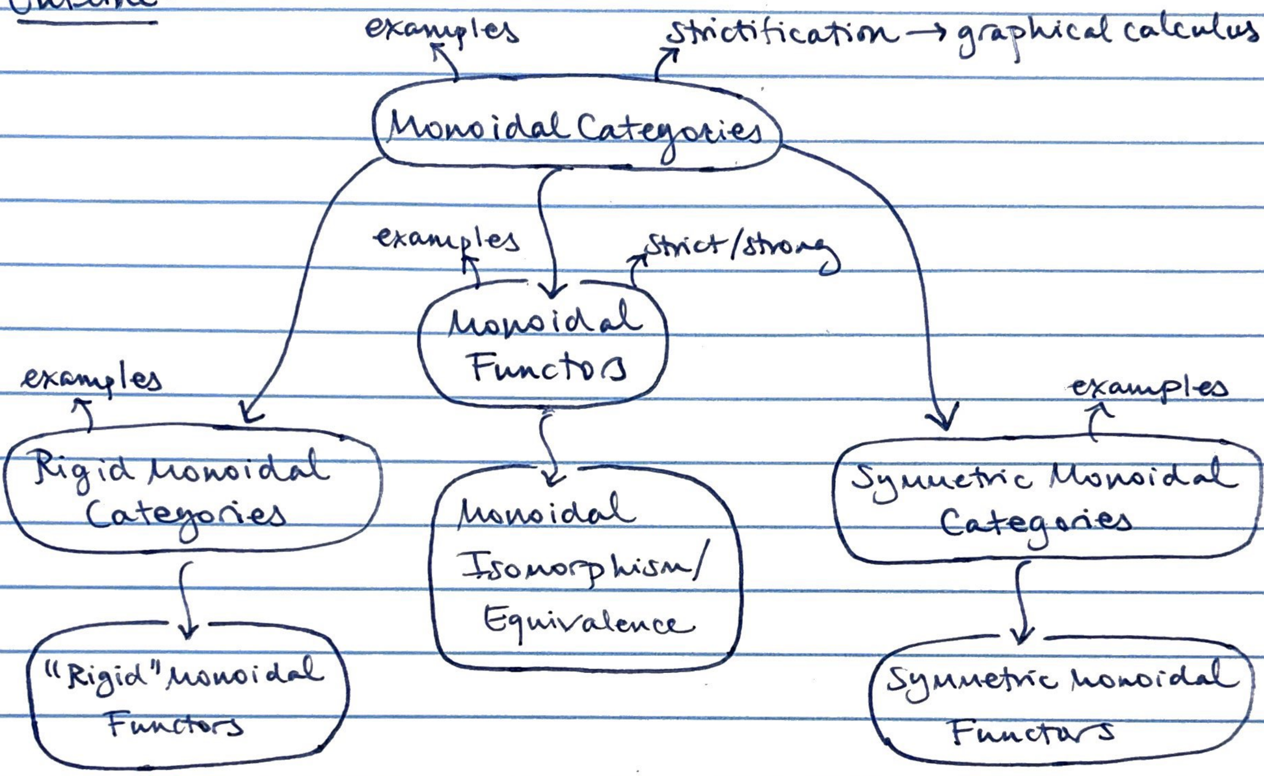
ICMS–Sofia

A Century of Frobenius Algebras:
II. Introduction to Monoidal Categories

Outline:



$\boxed{\text{Def'n}}$ A **monoidal category** consists of:

behaves like a monoid

- a category $\mathcal{C}$
- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
  [monoidal product]
- an object $\mathbb{1} \in \mathcal{C}$
  [monoidal unit]
- natural isomorphisms:

$a = \{a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)\}_{X,Y,Z \in \mathcal{C}}$
[associativity constraint]

$l = \{l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X\}_{X \in \mathcal{C}}$
$r = \{r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X\}_{X \in \mathcal{C}}$
[unitality constraints]

such that:

[pentagon axiom]

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes id"'] \arrow[dr, "a_{WX,Y,Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W,XY,Z}"'] & \circlearrowleft & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,YZ}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "id \otimes a_{W,X,Y}"'] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

[triangle axiom]

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}"] \arrow[dr, "l_X \otimes id"'] & \circlearrowleft & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "id \otimes r_Y"] \\
 & X \otimes Y &
\end{tikzcd}$$

$\boxed{\text{Examples}}$ [only discussing $(\mathcal{C}, \otimes, \mathbb{1})$]

(1) $(\mathrm{Vec}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C})$

category of $\mathbb{C}$-vector spaces

(2) $(\mathrm{Vec}_{\mathbb{C}}, \oplus, 0) =: \mathrm{Vec}_{\oplus}$

(3) $(\mathrm{FdVec}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C})$

category of finite dim'l $\mathbb{C}$-vs

(4) $\mathrm{FdVec}_{\oplus}$

(5) $(G\text{-Mod}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C})$

category of mods over a group G

- $(V, \triangleright) \otimes_{\mathbb{C}} (V', \triangleright') := (V \otimes_{\mathbb{C}} V', \blacktriangleright)$
  $g \blacktriangleright (v \otimes_{\mathbb{C}} v') := (g \triangleright v) \otimes_{\mathbb{C}} (g \triangleright' v')$
- $g \triangleright \lambda = \lambda$ for $\lambda \in \mathbb{C}$

(6) $(H\text{-Mod}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C})$

for a bialgebra $(H, m, u, \Delta, \varepsilon)$

- $h \blacktriangleright (v \otimes_{\mathbb{C}} v') = (h_1 \triangleright v) \otimes (h_2 \triangleright' v')$
  for $\Delta(h) := h_1 \otimes h_2$ [Sweedler]
- $h \triangleright \lambda = \varepsilon(h)\lambda$.

(7) $\underline{G}$, $\left\{\begin{array}{l}\text{objects of a group } G \\ \mathrm{Hom}_{\underline{G}}(g,h) = \begin{cases} \mathrm{id}_g & g = h \\ \emptyset & g \neq h \end{cases}\end{array}\right.$

- $g \otimes h := gh$
- $\mathbb{1} := e_G$

(8) $(A\text{-Bimod}_{\mathbb{C}}, \otimes_A, A)$

categ. of bimods over a $\mathbb{C}$-alg A

- $\otimes_A$ = tensor product over A
- A = regular A-bimodule.

(9) $(\mathrm{Set}, \times, \{\cdot\})$

(10) $(\mathrm{Set}, \amalg, \emptyset)$

(11) $(\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$

categ. of endofunctors of a cat. $\mathcal{A}$

(12) $(\mathrm{Aut}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$

categ. of autoequivalences of $\mathcal{A}$

$\boxed{\text{Defn}}$ $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ is <u>strict</u> if $\{a_{x,y,z}\,;\, l_x\,;\, r_x\}$ are identity maps.

Ex. (7), (11), (12) strict ; others are <u>not</u> strict.

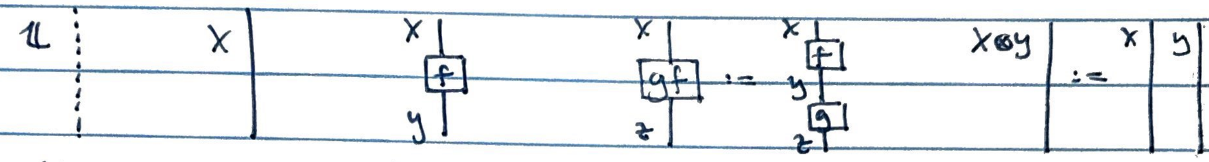
* defined later

$\boxed{\text{Fact}}$ Any monoidal cat. is monoidally equivalent* to a strict one.

[Strictification Theorem]

$\boxed{\text{Graphical calculus}}$

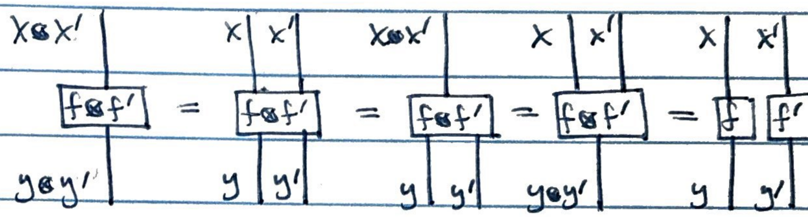
↳ nice pictures for objects & maps in strict monoidal cats $(\mathcal{C}, \otimes, \mathbb{1})$



$id_{\mathbb{1}}$ $\qquad$ id_X $\qquad$ $f: X \to y$ $\qquad$ $X \xrightarrow{f} y \xrightarrow{g} z$

Ex.

$X \otimes X' \xrightarrow{f \otimes f'} y \otimes y' :=$



$\boxed{\text{Def'n}}$ Take (strict) monoidal categories, $(\mathcal{C}, \otimes, \mathbb{1})$ & $(\mathcal{C}', \otimes', \mathbb{1}')$.

A <u>monoidal functor</u> from $(\mathcal{C}, \otimes, \mathbb{1})$ to $(\mathcal{C}', \otimes', \mathbb{1}')$ consists of:

- a functor $F: \mathcal{C} \longrightarrow \mathcal{C}'$,
- a nat'l transformation $F^{(2)} := \{F^{(2)}_{X,y}: F(X) \otimes' F(y) \longrightarrow F(X \otimes y)\}_{X,y \in \mathcal{C}}$, in $\mathcal{C}'$
- a morphism $F^{(0)}: \mathbb{1}' \longrightarrow F(\mathbb{1})$

Such that:

$$\begin{tikzcd} F(X) \otimes' F(y) \otimes' F(z) \arrow[r, "F^{(2)}_{X,y} \otimes id"] \arrow[d, "id \otimes F^{(2)}_{y,z}"'] \arrow[dr, phantom, "\circlearrowright"] & F(X \otimes y) \otimes' F(z) \arrow[d, "F^{(2)}_{X \otimes y, z}"] \\ F(X) \otimes' F(y \otimes z) \arrow[r, "F^{(2)}_{X, y \otimes z}"] & F(X \otimes y \otimes z) \end{tikzcd}$$

& $F^{(2)}_{\mathbb{1},X}\left(F^{(0)} \otimes' id_{F(X)}\right) = id_{F(X)}$, $\quad F^{(2)}_{X,\mathbb{1}}\left(id_{F(X)} \otimes' F^{(0)}\right) = id_{F(X)}$.

$\boxed{\text{Def'n}}$ $(F, F^{(2)}, F^{(0)}) : (\mathcal{C}, \otimes, \mathbb{1}) \longrightarrow (\mathcal{C}', \otimes', \mathbb{1}')$ is

- <u>strict</u> if $\{F^{(2)}_{X,y}\}$, $F^{(0)}$ are identity maps
- <u>strong</u> if $\{F^{(2)}_{X,y}\}$, $F^{(0)}$ are isomorphisms in $\mathcal{C}'$.

⚠ Sometimes <u>strong</u> is assumed in the literature

$\boxed{\text{Example}}$ The <u>regular action of $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ on itself</u> $\equiv$ the strong monoidal functor:

$$\rho : \mathcal{C} \longrightarrow \mathrm{End}(\mathcal{C})$$
$$X \longmapsto \left[(X \otimes -) : \mathcal{C} \to \mathcal{C},\ W \mapsto X \otimes W\right]$$

with:

$$\rho^{(2)}_{X,Y} : \underset{(X\otimes -)\circ(Y\otimes -)}{\rho(X) \circ \rho(Y)} \longrightarrow \underset{((X\otimes Y)\otimes -)}{\rho(X \otimes Y)}$$

where:

$$\left(\rho^{(2)}_{X,Y}\right)(Z) : X \otimes (Y \otimes Z) \xrightarrow{a^{-1}_{X,Y,Z}} (X \otimes Y) \otimes Z$$

& $\rho^{(0)} : \mathrm{Id}_{\mathcal{C}} \longrightarrow (\mathbb{1} \otimes -)$ in $\mathrm{End}(\mathcal{C})$

where

$$\rho^{(0)}(Z) : Z \xrightarrow{l^{-1}_Z} \mathbb{1} \otimes Z$$

$\boxed{\text{Def}^{\text{n}}}$ Two monoidal categories $\mathcal{C}, \mathcal{C}'$ are <u>isomorphic</u> (resp. <u>equivalent</u>) <u>as monoidal categories</u> if $\exists$ strong monoidal functor

$$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{C}'$$

such that F is a category isomorphism (resp. equivalence).

Write: $\mathcal{C} \overset{\otimes}{\cong} \mathcal{C}'$ (resp., $\mathcal{C} \overset{\otimes}{\simeq} \mathcal{C}'$).

$\boxed{\text{Example}}$ For a $\mathbb{C}$-algebra A, we get

$$(A\text{-Bimod}, \otimes_A, A) \overset{\otimes}{\simeq} (\mathrm{End}(\underbrace{A\text{-Mod}}_{\text{not nec. monoidal}}), \circ, \mathrm{Id}_{A\text{-Mod}})$$

via $\quad V \overset{\rho}{\longmapsto} V \otimes_A - : A\text{-Mod} \to A\text{-Mod}$

with

$$\left(\rho^{(2)}_{V,W}\right)(Z) := (a^A)^{-1}_{V,W,Z} \quad \& \quad \rho^{(0)}(Z) := (l^A_Z)^{-1}.$$

Note
$\mathcal{C} \overset{\otimes}{\simeq} \mathcal{C}'$ $\Leftrightarrow$ $\exists$ monoidal functors
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "{(F, F^{(2)}, F^{(0)})}"] & \mathcal{C}' \arrow[l, bend left, "{(G, G^{(2)}, G^{(0)})}"] \end{tikzcd}$$
such that ... $GF \overset{\otimes}{=} \mathrm{Id}_{\mathcal{C}}$, $FG \overset{\otimes}{=} \mathrm{Id}_{\mathcal{C}'}$

$\mathcal{C} \overset{\otimes}{\simeq} \mathcal{C}'$ $\Leftrightarrow$ ... $GF \overset{\otimes}{\cong} \mathrm{Id}_{\mathcal{C}}$, $FG \overset{\otimes}{\cong} \mathrm{Id}_{\mathcal{C}'}$

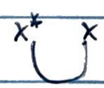
Def'n A <u>monoidal nat'l isomorphism (resp. transformation)</u> between $(F, F^{(2)}, F^{(0)})$ & $(\tilde{F}, \tilde{F}^{(2)}, \tilde{F}^{(0)}) : \mathcal{C} \to \mathcal{C}'$ is a natural isomorphism (resp. transformation) $\phi : F \Rightarrow F'$ such that:

$$\begin{tikzcd} F(X) \otimes' F(Y) \arrow[r, "F^{(2)}_{X,Y}"] \arrow[d, "\phi_X \otimes' \phi_Y"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X \otimes Y) \arrow[d, "\phi_{X \otimes Y}"] \\ \tilde{F}(X) \otimes' \tilde{F}(Y) \arrow[r, "\tilde{F}^{(2)}_{X,Y}"] & \tilde{F}(X \otimes Y) \end{tikzcd} \quad \& \quad \begin{tikzcd} & F(\mathbb{1}) \arrow[d, "\phi_{\mathbb{1}}"] \\ \mathbb{1}' \arrow[ur, "F^{(0)}"] \arrow[r, phantom, "\circlearrowleft"] \arrow[dr, "\tilde{F}^{(0)}"'] & {} \\ & \tilde{F}(\mathbb{1}) \end{tikzcd}$$

Next rigidity ... monoidal categories with "dual" objects & maps:

Def'n Fix a strict monoidal category $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$.

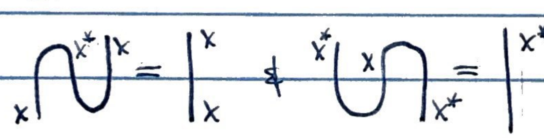
(1) A <u>left dual</u> of $X \in \mathcal{C}$ is.
- an object $X^* \in \mathcal{C}$
- with morphisms:

$ev^L_X : X^* \otimes X \to \mathbb{1}$ 

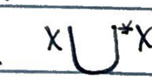
$coev^L_X : \mathbb{1} \to X \otimes X^*$ 

[left co/evaluation]

such that:



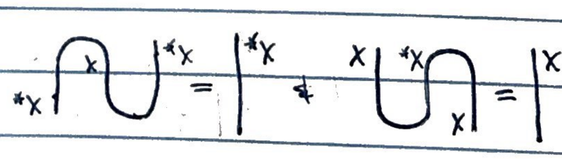
(2) A <u>right dual</u> of $X \in \mathcal{C}$ is
- an object ${}^*X \in \mathcal{C}$
- with morphisms:

$ev^R_X : X \otimes {}^*X \to \mathbb{1}$ 

$coev^R_X : \mathbb{1} \to {}^*X \otimes X$ 

[right co/evaluation]

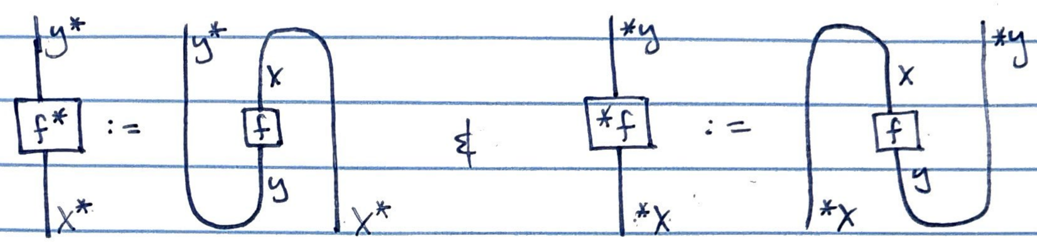
such that



(3) $\mathcal{C}$ is <u>rigid</u> if every object $X \in \mathcal{C}$ has both a left & right dual.

<u>Note</u>: Get dual morphisms in rigid categories:

For $f: X \to Y \in \mathcal{C}$, define



Examples

Consider $(\mathrm{Vec}, \otimes_{\mathbb{C}}, \mathbb{C})$. Take $V \in \mathrm{Vec}$ with basis $\{b_i\}_i$.

Assume $\exists$ left dual $W \in \mathrm{Vec}$ of V. So, we get linear maps

$$ev_V^L : W \otimes_{\mathbb{C}} V \to \mathbb{C} \qquad \& \qquad coev_V^L : \mathbb{C} \to V \otimes_{\mathbb{C}} W$$

$$w \otimes v \mapsto w(v) \quad \text{[Notation]} \qquad\qquad 1_{\mathbb{C}} \mapsto \sum_{i=1}^{n} b_i \otimes w_i \quad (n \leftarrow \text{finite})$$

Such that:

$$V \cong \mathbb{C} \otimes_{\mathbb{C}} V \xrightarrow{coev_V^L \otimes id_V} V \otimes_{\mathbb{C}} W \otimes_{\mathbb{C}} V \xrightarrow{id_V \otimes ev_V^L} V \otimes_{\mathbb{C}} \mathbb{C} \cong V \quad = id_V$$

$$v \equiv 1_{\mathbb{C}} \otimes v \longmapsto \sum_{i=1}^{n} b_i \otimes w_i \otimes v \longmapsto \sum_{i=1}^{n} \sum_{j=1}^{m} \underbrace{\lambda_j}_{\in \mathbb{C}}\, b_i\, \underbrace{w_i(b_j)}_{\in \mathbb{C}}$$

$(= \sum_{j=1}^{m} \lambda_j b_j$, for $\lambda_j \in \mathbb{C})$ ($m \leftarrow$ finite)

But $\dim_{\mathbb{C}}(\text{image of } \uparrow) < \infty$ ...... $\#$

	Not Rigid	**Rigid**
(1)	Vec	FdVec
(2)	G-Mod	G-FdMod : $(g \triangleright v^*)(w) := v^*(g^{-1} \triangleright w)$
(3)	bialgebra ↳ H-FdMod	Hopf alg. ↳ H-FdMod : $(h \triangleright v^*)(w) = v^*(S(h) \triangleright w)$
(4)	A-FdBimod in general	A-FdBimod for A finite dim'l & semisimple
	Ex. $A = \mathbb{C}[x]/(x^2)$	Ex. $A = \mathbb{C}[x]/(x^2-1)$

Rigidity is a property of monoidal categories, not a structure.
Can ask if monoidal functors preserve this property...

$F^{(-j)} := (F^{(j)})^{-1}$

$\boxed{\text{Fact}}$ Given a strong monoidal functor $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \to \mathcal{C}'$,
if $X \in \mathcal{C}$ has left dual $(X^*, ev^L_X : X^* \otimes X \to \mathbb{1},\ coev^L_X : \mathbb{1} \to X \otimes X^*)$,
then $F(X) \in \mathcal{C}'$ has left dual $F(X^*)$ with

$$\begin{cases} ev^L_{F(X)} : F(X^*) \otimes' F(X) \xrightarrow{F^{(2)}_{X^*,X}} F(X^* \otimes X) \xrightarrow{F(ev^L_X)} F(\mathbb{1}) \xrightarrow{F^{(-0)}} \mathbb{1}' \\ coev^L_{F(X)} : \mathbb{1}' \xrightarrow{F^{(0)}} F(\mathbb{1}) \xrightarrow{F(coev^L_X)} F(X \otimes X^*) \xrightarrow{F^{(-2)}_{X,X^*}} F(X) \otimes' F(X^*). \end{cases}$$

(Similar for right duals)

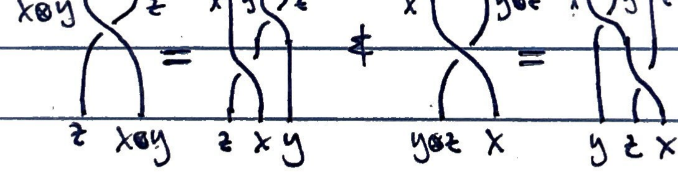
$\boxed{\text{Fact}}$ If
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "{(F, F^{(2)}, F^{(0)})}"] \arrow[r, bend right, "{(\tilde{F}, \tilde{F}^{(2)}, \tilde{F}^{(0)})}"'] \arrow[r, phantom, "\Downarrow \phi"] & \mathcal{C}' \end{tikzcd}$$
Rigid

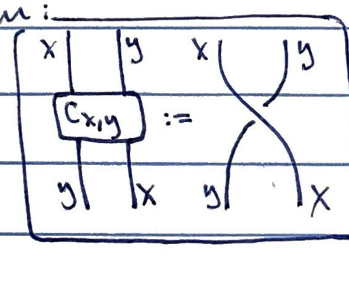
is a monoidal transformation,
then ϕ is a monoidal isomorphism.

↳ That is, the category $MonFun(\mathcal{C}, \mathcal{C}')$ is a groupoid when $\mathcal{C}$ is rigid.
[objects = monoidal functors
morphisms = monoidal nat'l transfs]

Now a structure on monoidal categories...

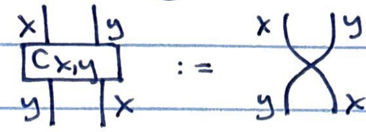
$\boxed{\text{Def'n}}$ A (strict) monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ is braided
if it is equipped with a natural isomorphism:
$$c := \{c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X\}_{X,Y \in \mathcal{C}}$$
such that





Further, $(\mathcal{C}, \otimes, \mathbb{1}, c)$ is <u>symmetric</u> if $c_{y,x} \circ c_{x,y} = \mathrm{id}_{x \otimes y}$ $\forall x, y$.

Here,



$\boxed{\text{Examples}}$

(1) $(\mathrm{Vec}, \otimes_{\mathbb{C}}, \mathbb{C}, \text{flip across } \otimes_{\mathbb{C}})$ — symmetric.

(2) $(G\text{-Mod}, \otimes_{\mathbb{C}}, \mathbb{C}, \text{flip across } \otimes_{\mathbb{C}})$ — symmetric
G abelian.

(3) $\left(H\text{-Mod}, \otimes_{\mathbb{C}}, \mathbb{C}, c_{x,y}(x \otimes y) := (R^2 \triangleright y) \otimes_{\mathbb{C}} (R^1 \triangleright x)\right)$ — braided
H quasitriangular bialgebra with R-matrix $R = R^1 \otimes R^2$

- symmetric $\Leftrightarrow$ (H, R) triangular.

$\boxed{\text{Def'n}}$ Given braided (resp. symmetric) monoidal cats $\mathcal{C}, \mathcal{C}'$, a <u>braided (resp. symmetric) monoidal functor</u> from $\mathcal{C}$ to $\mathcal{C}'$ is a strong monoidal functor $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \to \mathcal{C}'$ such that

$$\begin{tikzcd} F(x) \otimes' F(y) \arrow[r, "F^{(2)}_{x,y}"] \arrow[d, "c'_{F(x), F(y)}"'] \arrow[dr, phantom, "\circlearrowright"] & F(x \otimes y) \arrow[d, "F(c_{x,y})"] \\ F(y) \otimes' F(x) \arrow[r, "F^{(2)}_{y,x}"] & F(y \otimes x) \end{tikzcd}$$

Very important symmetric monoidal functors from <u>topological</u> symmetric monoidal categories will be introduced in Lecture IV.

Ref: This is covered in Chapter 3 of SoAv1 & Chapter 8 of SoAv2 (in prep)