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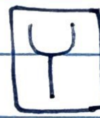
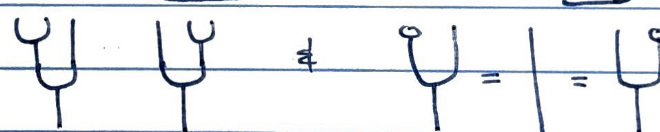
# A Century of Frobenius Algebras:
## III. Frobenius algebras in monoidal categories

**Outline:** Fix a strict monoidal category $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$

| Will cover... | Category | Examples | Properties | Characteriz'ns | Preserved by ... functor | Monadic setting |
|---|---|---|---|---|---|---|
| Algebras in $\mathcal{C}$ | * | * | * | | * | * |
| Coalgebras in $\mathcal{C}$ | * | * | * | | * | * |
| Frobenius algebras in $\mathcal{C}$ | * | * | * | * | * | * |

strings all labelled by "A"

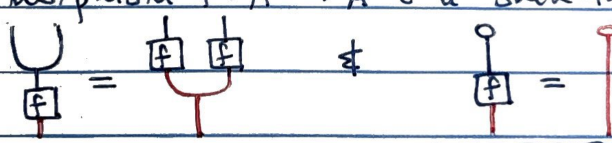
**Def'n** (1) An algebra in $\mathcal{C}$ is an object $A \in \mathcal{C}$ with maps $m: A \otimes A \to A$  & $u: \mathbb{1} \to A$  in $\mathcal{C}$

Such that 

[associativity] $m(m \otimes id) = m(id \otimes m)$ [unitality] $m(u \otimes id) = id = m(id \otimes u)$

red strings labelled by "A′"

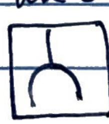
A morphism of algebras $(A, m, u) \longrightarrow (A', m', u')$ in $\mathcal{C}$ is a morphism $f: A \to A' \in \mathcal{C}$ such that

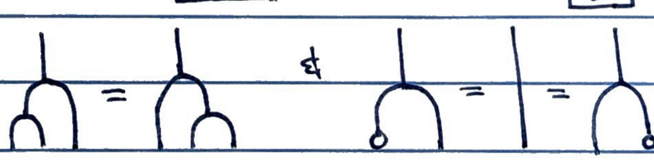


↝ Get the category $\boxed{\mathrm{Alg}(\mathcal{C})}$.

Ex. $\mathrm{Alg}(\mathrm{Vec}_{\mathbb{C}}) = \mathbb{C}\text{-Alg}$
$\mathrm{Alg}(\mathrm{Set}) = $ monoid
$\mathrm{Alg}(G\text{-mod}) = G$-module algs

strings = "C"

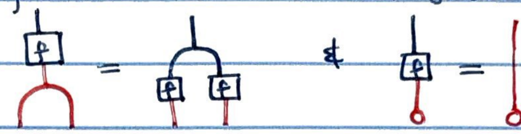
(2) A coalgebra in $\mathcal{C}$ is an object $C \in \mathcal{C}$ with maps $\Delta: C \to C \otimes C$  & $\varepsilon: C \to \mathbb{1}$  in $\mathcal{C}$

Such that 

[coassociativity] [counitality]

red strings = "C′"

A <u>morphism of coalgebras</u> $(C, \Delta, \varepsilon) \longrightarrow (C', \Delta', \varepsilon')$ in $\mathcal{C}$ is a morphism $f: C \longrightarrow C' \in \mathcal{C}$ such that




$\rightsquigarrow$ Get the <u>category</u> $\boxed{\mathrm{Coalg}(\mathcal{C})}$

We'll show later this is equivalent to the Frobenius algebras/$\mathbb{C}$ from Lecture I, when $\mathcal{C} = \mathrm{FdVec}_{\mathbb{C}}$

(3) A <u>Frobenius algebra in $\mathcal{C}$</u> is an object $A \in \mathcal{C}$ with maps in $\mathcal{C}$:

$m \qquad u \qquad \Delta \qquad \varepsilon$

such that $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$ & $(A, \Delta, \varepsilon) \in \mathrm{Coalg}(\mathcal{C})$

&


[Frobenius law]

$(\mathrm{id} \otimes m)(\Delta \otimes \mathrm{id}) = \Delta m = (m \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)$

A <u>morphism of Frobenius algebras</u> $(A, m, u, \Delta, \varepsilon) \rightarrow (A', m', u', \Delta', \varepsilon')$ in $\mathcal{C}$ is a morphism $f: A \rightarrow A'$ in $\mathcal{C}$ where $f \in \mathrm{Alg}(\mathcal{C}) \cap \mathrm{Coalg}(\mathcal{C})$.

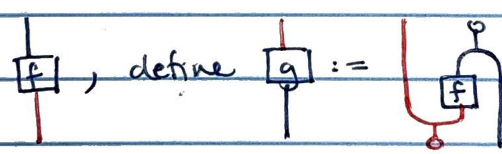
$\rightsquigarrow$ Get the category $\boxed{\mathrm{FrobAlg}(\mathcal{C})}$. {Ex. $\mathrm{FrobAlg}(\mathrm{Vec}_{\mathbb{C}}^{\mathrm{f.d.}}) = \mathbb{C}\text{-}\mathrm{FrobAlg}$}

//

$\boxed{\text{Proposition}}$ $\mathrm{FrobAlg}(\mathcal{C})$ is a groupoid.

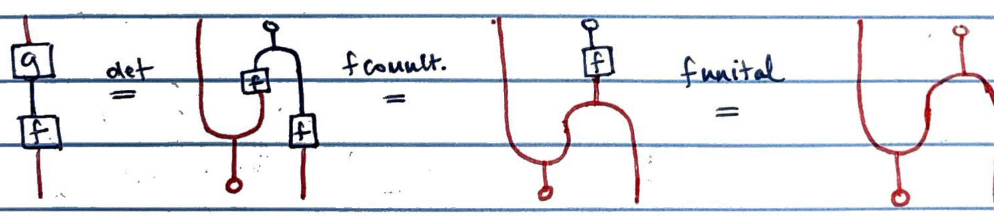
Strings: "A" "A′"

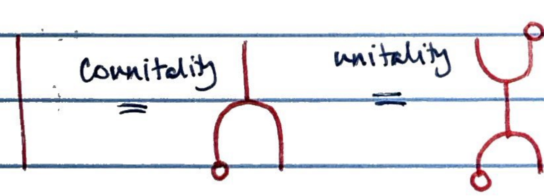
Pf/ Given [f], define [g] :=




. Then $fg = \mathrm{id}_{A'}$ & $gf = \mathrm{id}_A$.

For instance,

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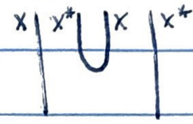
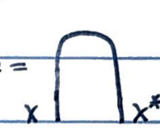
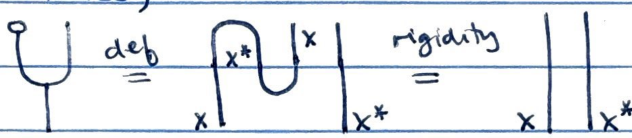
$X^* \cup X = ev_X^L$

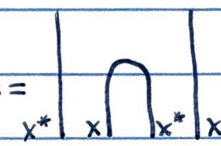
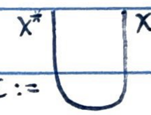
$X \cap X^* = coev_X^L$

$\boxed{\text{Examples}}$ Assume that $(\mathcal{C}, \otimes, \mathbb{1})$ is rigid monoidal, & take $X \in \mathcal{C}$.

(1) $X \otimes X^* \in Alg(\mathcal{C})$ with $m :=$  & $u :=$ 

For instance,




(2) $X^* \otimes X \in Coalg(\mathcal{C})$ with $\Delta :=$  & $\varepsilon :=$ 

E.g. $FdVec_{\mathbb{C}}$ is pivotal rigid & $V \otimes V^* \cong Mat_n(\mathbb{C})$ for $n = \dim_{\mathbb{C}} V$.

(3) If $X^{**} \cong X$ in $\mathcal{C}$ (e.g. if $\mathcal{C}$ is "pivotal rigid"), then $X \otimes X^* \in FrobAlg(\mathcal{C})$ with:

$m = id_X \otimes ev_X^L \otimes id_{X^*}$ $\qquad u = coev_X^L$

$\Delta = id_X \otimes coev_{X^*}^L \otimes id_{X^*}$ $\qquad \varepsilon = ev_{X^*}^L$

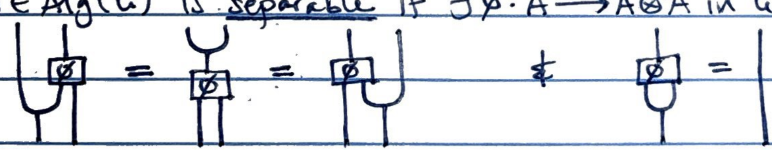
(0) [Boring] $\mathbb{1} \in Alg(\mathcal{C}), Coalg(\mathcal{C}), FrobAlg(\mathcal{C})$ as $m, u, \Delta, \varepsilon$ are all identity maps.

$\boxed{\text{Properties}}$

Recovers the same properties for structures/$\mathbb{C}$ when $\mathcal{C} = FdVec_{\mathbb{C}}$

(1) Back to $(\mathcal{C}, \otimes, \mathbb{1})$,

- $(A, m, u) \in Alg(\mathcal{C})$ is <u>separable</u> if $\exists \phi : A \to A \otimes A$ in $\mathcal{C}$ such that




$[m\phi = id]$

- $(C, \Delta, \varepsilon) \in Coalg(\mathcal{C})$ is <u>coseparable</u> if $\exists \phi : A \otimes A \to A$ in $\mathcal{C}$ such that


$[\phi\Delta = id]$

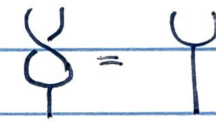
pivotal dim'n = v.s. dim'n when $\mathcal{C} = \mathrm{Vec}_{\mathbb{C}}$.

Ex. $\begin{cases} X \otimes X^* \in \mathrm{Alg}(\mathcal{C}) \text{ is separable} \\ X^* \otimes X \in \mathrm{Coalg}(\mathcal{C}) \text{ is coseparable} \end{cases}$ when $\mathcal{C}$ is pivotal & $\mathbb{1} \xrightarrow{\mathrm{coev}_X^L} X \otimes X^* \cong X^{**} \otimes X \xrightarrow{\mathrm{ev}_{X^*}^L} \mathbb{1}$ is invertible

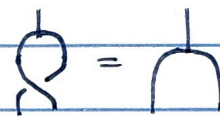
"piv. dim (X)"

(2) Take $(\mathcal{C}, \otimes, \mathbb{1}, c)$ to be braided (or symmetric) monoidal

- $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$ is <u>commutative</u> if $[m \circ c_{A,A} = m]$



- $(C, \Delta, \varepsilon) \in \mathrm{Coalg}(\mathcal{C})$ is <u>cocommutative</u> if $[c_{C,C} \circ \Delta = \Delta]$



(3) Take $(A, m, u, \Delta, \varepsilon) \in \mathrm{FrobAlg}(\mathrm{FdVec}_{\mathbb{C}})$. We say that $A$ is <u>symmetric</u> if $\varepsilon(ab) = \varepsilon(ba)$. (skipping sym Frob alg in $\mathcal{C}$)

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Recall the Brauer–Nakayama–Nesbitt Theorem.

$\boxed{\text{Theorem}}$ Take a <u>rigid</u>, <u>abelian</u> monoidal category.
Fix $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$. Then, the following are equivalent.

(1) $A \cong A^*$ as right $A$-modules in $\mathcal{C}$ [skipping def'n of module here]

(2) $A$ is equipped with a map $\nu : A \to \mathbb{1}$ such that $\ker(\nu)$ doesn't have a nonzero right ideal subobject. [skipping def'n of right ideal here].

(3) $A$ is equipped with maps $p : A \otimes A \to \mathbb{1}$ & $q : \mathbb{1} \to A \otimes A$ in $\mathcal{C}$ such that

(associativity) &

(nondegeneracy)

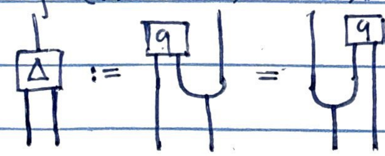
established by Abrams, Quinn (1996) (1995)
⋮
plays a key role in Lecture IV.

(4) $A$ is equipped with maps $\Delta : A \to A \otimes A$ & $\varepsilon : A \to \mathbb{1}$ in $\mathcal{C}$ such that $(A, m, u, \Delta, \varepsilon) \in \mathrm{FrobAlg}(\mathcal{C})$.

$\boxed{\text{Note}}$: When $\mathcal{C} = \mathrm{FdVec}_{\mathbb{C}}$, conditions (1), (2), (3) appear in BNN's Theorem that characterizes Frobenius algebras/$\mathbb{C}$.

Pf/ Will skip (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

(3) $\Rightarrow$ (4) :



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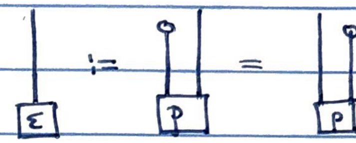
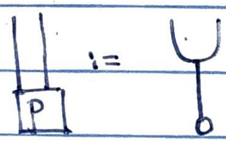


(4) $\Rightarrow$ (3) :



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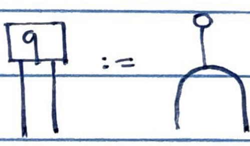

If bored, verify the axioms!

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$\boxed{\text{Preservation by certain monoidal functors}}$

(1) Take a monoidal functor $(F, F^{(2)}, F^{(0)}) : (\mathcal{C}, \otimes, \mathbb{1}) \longrightarrow (\mathcal{C}', \otimes', \mathbb{1}')$

$\left[ F^{(2)}_{X,Y} : F(X) \otimes' F(Y) \longrightarrow F(X \otimes Y), \quad F^{(0)} : \mathbb{1}' \longrightarrow F(\mathbb{1}) \right]$.

Then we get a functor:

$\boxed{\mathrm{Alg}(F)} : \mathrm{Alg}(\mathcal{C}) \longrightarrow \mathrm{Alg}(\mathcal{C}')$

$$(A, m_A, u_A) \longmapsto \left( F(A),\ \begin{array}{l} m_{F(A)} : F(A) \otimes F(A) \xrightarrow{F^{(2)}_{A,A}} F(A \otimes A) \xrightarrow{F(m_A)} F(A), \\ u_{F(A)} : \mathbb{1}' \xrightarrow{F^{(0)}} F(\mathbb{1}) \xrightarrow{F(u_A)} F(A) \end{array} \right)$$

(2) A <u>comonoidal functor</u> $(F, F_{(2)}, F_{(0)}) : (\mathcal{C}, \otimes, \mathbb{1}) \longrightarrow (\mathcal{C}', \otimes', \mathbb{1}')$ is defined likewise with
$$\begin{cases} F_{(2)}^{X,Y} : F(X \otimes Y) \longrightarrow F(X) \otimes' F(Y) \\ F_{(0)} : F(\mathbb{1}) \longrightarrow \mathbb{1}' \end{cases}$$

Then we get a functor:

$\boxed{\mathrm{Coalg}(F)} : \mathrm{Coalg}(\mathcal{C}) \longrightarrow \mathrm{Coalg}(\mathcal{C}')$

$$(C, \Delta_C, \varepsilon_C) \longmapsto \left( F(C),\ \begin{array}{l} \Delta_{F(C)} : F(C) \xrightarrow{F(\Delta_C)} F(C \otimes C) \xrightarrow{F_{(2)}^{C,C}} F(C) \otimes' F(C), \\ \varepsilon_{F(C)} : F(C) \xrightarrow{F(\varepsilon_C)} F(\mathbb{1}) \xrightarrow{F_{(0)}} \mathbb{1}' \end{array} \right)$$

(3) A <u>Frobenius monoidal functor</u>
$$(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}) : (\mathcal{C}, \otimes, \mathbb{1}) \longrightarrow (\mathcal{C}', \otimes', \mathbb{1}')$$
is defined with $(F, F^{(2)}, F^{(0)})$ monoidal, $(F, F_{(2)}, F_{(0)})$ comonoidal,

$$\begin{tikzcd} F(X\otimes Y)\otimes' F(Z) \arrow[r, "F^{(2)}_{X\otimes Y,Z}"] \arrow[d, "F^{X,Y}_{(2)}\otimes' id"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X\otimes Y\otimes Z) \arrow[d, "F^{X,Y\otimes Z}_{(2)}"] \\ F(X)\otimes' F(Y)\otimes' F(Z) \arrow[r, "id\otimes' F^{(2)}_{Y,Z}"] & F(X)\otimes' F(Y\otimes Z) \end{tikzcd} \quad \& \quad \begin{tikzcd} F(X)\otimes' F(Y\otimes Z) \arrow[r, "F^{(2)}_{X,Y\otimes Z}"] \arrow[d, "id\otimes' F^{Y,Z}_{(2)}"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X\otimes Y\otimes Z) \arrow[d, "F^{X\otimes Y,Z}_{(2)}"] \\ F(X)\otimes' F(Y)\otimes' F(Z) \arrow[r, "F^{(2)}_{X,Y}\otimes' id"] & F(X\otimes Y)\otimes' F(Z) \end{tikzcd}$$

We also get a functor $\boxed{\mathrm{FrobAlg}(F)} : \mathrm{FrobAlg}(\mathcal{C}) \longrightarrow \mathrm{FrobAlg}(\mathcal{C}')$.

$\boxed{\text{Example}}$ A strong monoidal functor is Frobenius monoidal.
Here $F_{(2)} := (F^{(2)})^{-1}$, $F_{(0)} := (F^{(0)})^{-1}$.

## $\boxed{\text{Non-strong Frobenius monoidal functors}}$

$\boxed{\text{Defn}}$ Given $A \in \mathrm{Alg}(\mathcal{C})$

(1) a left $A$-module in $\mathcal{C}$ is $(M, \triangleright : A\otimes M \to M)$ in $\mathcal{C}$ satisfying assoc, unitality axioms. $\rightsquigarrow$ category $\boxed{A\text{-Mod}(\mathcal{C})}$.

(2) Likewise, get category of right $A$-mods in $\mathcal{C}$: $\boxed{\text{Mod-}A(\mathcal{C})}$

(3) Also, get category of $A$-bimodules $(M, \triangleright, \triangleleft)$ in $\mathcal{C}$: $\boxed{A\text{-Bimod}(\mathcal{C})}$

$\boxed{\text{Proposition}}$ If $A := (A, m, u, \Delta, \varepsilon) \in \mathrm{FrobAlg}(\mathcal{C})$, then

(1) $\mathrm{Free} : \mathcal{C} \longrightarrow A\text{-Bimod}(\mathcal{C})$, $X \mapsto (A\otimes X\otimes A,\ \triangleright = m\otimes id_{X\otimes A},\ \triangleleft = id_{A\otimes X}\otimes m)$,

(2) $\mathrm{Forg} : A\text{-Bimod}(\mathcal{C}) \longrightarrow \mathcal{C}$, $(M, \triangleright, \triangleleft) \mapsto M$

are both Frobenius monoidal, but neither are strong monoidal. ///

$\boxed{\text{The monadic setting}}$

Take a category $\mathcal{A}$ & consider the monoidal category $(\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$.

A $\left\{\begin{array}{l} \underline{\text{monad}} \\ \underline{\text{comonad}} \\ \underline{\text{Frobenius monad}} \end{array}\right.$ on $\mathcal{A}$ is a(n) $\left\{\begin{array}{l} \text{algebra} \\ \text{coalgebra} \\ \text{Frobenius algebra} \end{array}\right.$ in $\mathrm{End}(\mathcal{A})$.

That is, a Frobenius monad on A consists of

- an endofunctor $T: \mathcal{A} \longrightarrow \mathcal{A}$
- natural transformations: $[T^n = \overbrace{T \circ \cdots \circ T}^{n}]$

$$\mu^T: T^2 \Rightarrow T, \quad \eta^T: \mathrm{Id}_{\mathcal{A}} \Rightarrow T, \quad \delta^T: T \Rightarrow T^2, \quad \varepsilon^T: T \Rightarrow \mathrm{Id}_{\mathcal{A}},$$

such that:

$$\mu_X \mu_{T(X)} = \mu_X T(\mu_X), \quad \mu_X \eta_{T(X)} = \mathrm{id}_{T(X)}, \quad \mu_X T(\eta_X) = \mathrm{id}_{T(X)},$$
$$\delta_{T(X)} \delta_X = T(\delta_X) \delta_X, \quad \varepsilon_{T(X)} \delta_X = \mathrm{id}_{T(X)}, \quad T(\varepsilon_X) \delta_X = \mathrm{id}_{T(X)}$$

&

$$\begin{tikzcd}
 & T^2(X) \arrow[dl, "\delta_{T(X)}"'] \arrow[d, "\mu_X"] \arrow[dr, "T(\delta_X)"] & \\
T^3(X) \arrow[r, phantom, "="] \arrow[dr, "T(\mu_X)"'] & T(X) \arrow[r, phantom, "="] \arrow[d, "\delta_X"] & T^3(X) \arrow[dl, "\mu_{T(X)}"] \\
 & T^2(X) &
\end{tikzcd}$$

$\boxed{\text{Example}}$ If $(A, m, u, \Delta, \varepsilon) \in \mathrm{FrobAlg}(\mathcal{C}, \otimes, \mathbb{1})$, then

$T := (A \otimes -) : \mathcal{C} \longrightarrow \mathcal{C}$ is a Frobenius monad on $\mathcal{C}$

with:

$$\mu^T_X := m \otimes \mathrm{id}_X, \quad \eta^T_X := u \otimes \mathrm{id}_X$$
$$\delta^T_X := \Delta \otimes \mathrm{id}_X, \quad \varepsilon^T_X := \varepsilon \otimes \mathrm{id}_X.$$

Several structures can be called "Frobenius" & we will connect them.

$$\begin{tikzcd}[column sep=large] & \boxed{\text{Frobenius Algebras}} \arrow[dl, bend right, "{\text{For } A \in \mathrm{FrobAlg}(\mathcal{C}):\ (A\otimes -): \mathcal{C} \to \mathcal{C}}"'] \arrow[dr, bend left, "{\text{For } A \in \mathrm{FrobAlg}(\mathcal{C})\ \left\{\begin{array}{l}\mathrm{Free}: \mathcal{C} \to A\text{-}\mathrm{Bimod}(\mathcal{C}) \\ \mathrm{Forg}: A\text{-}\mathrm{Bimod}(\mathcal{C}) \to \mathcal{C}\end{array}\right.}"] & \\ \boxed{\text{Frobenius Monads}} & & \boxed{\text{Frobenius Monoidal Functors}} \end{tikzcd}$$

$$\boxed{\text{“Frobenius Functors”}}$$

$\uparrow$

will define &
make many more connections
in Lecture V

Ref: This is covered in Chapter 4 of Symmetries of Algebras, Vol 1
& Chapters 6, 8 of Symmetries of Algebras, Vol 2 (in prep).