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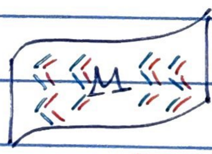
A Century of Frobenius Algebras:
IV. Ties to Topological Quantum Field Theories

TQFTs – formulated by Atiyah, Segal, Witten in the late 1980s...

Manifold M
$\shortparallel$
Space of a physical system



move with time

can represent the symmetric mon'l cat of topological states by the symmetric mon'l cat of vector spaces

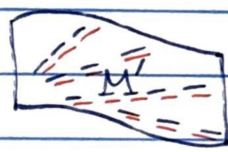
Vector Space V of quantum field states

Want to study how states evolve with time...

... captured by a linear map

$V \xrightarrow{f} V$

Take another M'



move with time

Disjoint union — Tensor Prod.

$M \amalg M' \rightsquigarrow V \otimes V'$

and get

$V' \xrightarrow{f'} V'$

Swap

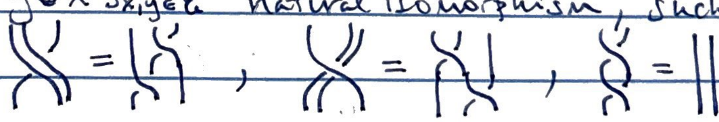
$M' \amalg M \rightsquigarrow V' \otimes V$

$\boxed{\text{Recall}}$: Symmetric monoidal category $= (\mathcal{C}, \otimes, \mathbb{1}, c)$:

- $\mathcal{C}$ category
- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ bifunctor
- $\mathbb{1} \in \mathcal{C}$

monoidal category (behaves like a monoid)

- $c = \{c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X\}_{X,Y \in \mathcal{C}}$ natural isomorphism, such that

for  , get 

c

$\boxed{\text{Example}}$ (algebraic) $\left(\mathrm{Vec}_{\mathbb{C}}, \otimes_{\mathbb{C}}, \mathbb{C}, \text{flip}: \begin{array}{c} V \otimes_{\mathbb{C}} W \longrightarrow W \otimes_{\mathbb{C}} V \\ v \otimes w \longmapsto w \otimes v \end{array}\right)$

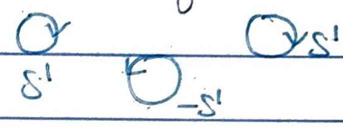
$\boxed{\text{Example}}$ (topological) $n \in \mathbb{N}_{\geq 1}$ $\boxed{\mathrm{Bord}_n := \mathrm{Bord}_{n,n-1}}$

Ex. $\boxed{\mathrm{Bord}_2}$

<u>objects</u>: closed, oriented $(n-1)$-dim'l mflds

disjoint union of oriented circles

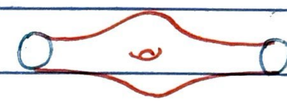
Ex.



<u>Morphisms</u>: n-dim'l bordisms $M \to N$ with boundary diffeomorphic to $-M \amalg N$

[omitting orientation in morphism pictures]

Ex.

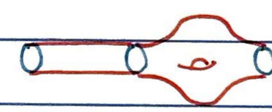


<u>identity morphism</u>: product mfld

Ex.

<u>composition</u>: gluing

Ex.

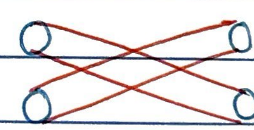


<u>$\otimes$</u>: disjoint union $\amalg$

<u>$\mathbb{1}$</u>: empty mfld $\emptyset$

<u>c</u>: swap.

Ex.



$\boxed{\text{Def'n}}$ Given a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, c)$, a <u>$\mathcal{C}$-valued n-dimensional Topological Quantum Field Theory</u> (n-TQFT$_{\mathcal{C}}$) is a symmetric monoidal functor:

$$\mathcal{Z}: \mathrm{Bord}_n \longrightarrow \mathcal{C}$$

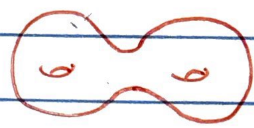
* traditionally $\mathrm{Vec}_{\mathbb{C}}$-valued.

$\boxed{\text{TQFTs produce topological invariants with data from } \mathcal{C}}$

Ex. $\qquad Z : \mathrm{Bord}_2 \longrightarrow \mathrm{Vec}_{\mathbb{C}}$

$$\emptyset = \mathbb{1}^{\mathrm{Bord}_2} \longmapsto \mathbb{1}^{\mathrm{Vec}_{\mathbb{C}}} = \mathbb{C}$$

$$[\emptyset \to \emptyset] \longmapsto [\mathbb{C} \to \mathbb{C}]$$

So,  $\longmapsto$ A scalar!

more later...

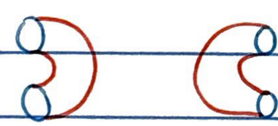
$\boxed{\text{TQFTs form a } \cancel{\text{category}} \text{ groupoid}}$

<u>objects</u>: n-TQFTs $\quad Z : \mathrm{Bord}_n \longrightarrow \mathcal{C}$

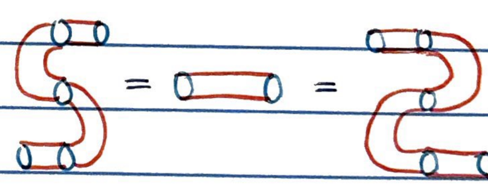
<u>morphisms</u>: monoidal natural transformations $\phi : Z \overset{\otimes}{\Rightarrow} \tilde{Z}$

But <u>observe</u>: Bord_n is <u>rigid</u> : $\quad ev_M : -M \amalg M \longrightarrow \emptyset$ $\quad$ satisfy

$\qquad\qquad coev_M : \emptyset \longrightarrow M \amalg -M$ $\quad$ rigidity axioms

Ex. $n=2$ 

$ev_M \qquad coev_M$

satisfy 

<u>Recall</u> Any monoidal nat'l transf
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "\tilde{F}"'] \arrow[r, phantom, "\Downarrow\otimes"] & \mathcal{C}' \end{tikzcd}$$
rigid

is a monoidal <u>nat'l isomorphism</u>

$\rightsquigarrow$ n-$\mathrm{TQFT}_{\mathcal{C}}$ is a groupoid!

Let's now examine 2-$\mathrm{TQFT}_{\mathcal{C}}$ in detail...

$\boxed{\text{Bord}_2 \text{ in detail}}$

⊗ generating objects

$\boxed{A \in \mathcal{C}}$

S^1 

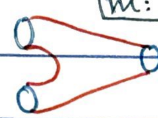
$\boxed{A^* \in \mathcal{C}}$

$-S^1$ 

⊗ generating morphisms

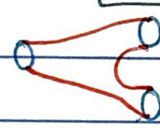
$\boxed{id_A}$ ✓

$\boxed{m: A \otimes A \to A}$ *



$\boxed{u: \mathbb{1} \to A}$ *

$\boxed{\Delta: A \to A \otimes A}$ *



$\boxed{\varepsilon: A \to \mathbb{1}}$ *

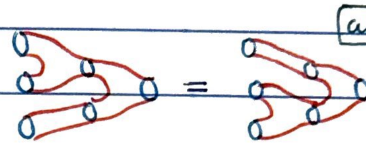
$\boxed{c: A \otimes A \to A \otimes A}$ ✓

✓ already in $\mathcal{C}$

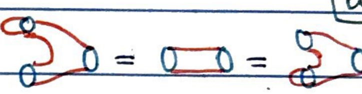
* added structure

morphism relations

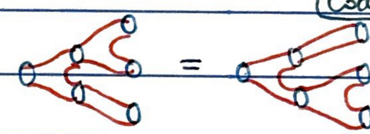
$\boxed{\text{associativity}}$



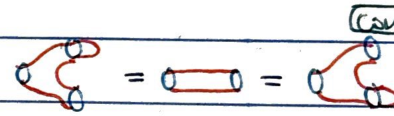
$\boxed{\text{unitality}}$



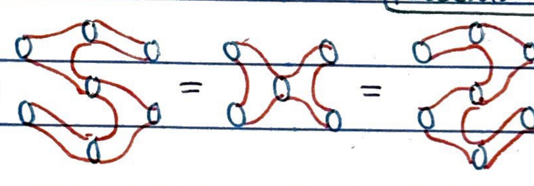
$\boxed{\text{coassociativity}}$



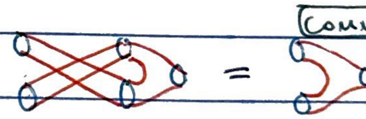
$\boxed{\text{counitality}}$



$\boxed{\text{Frobenius law}}$



$\boxed{\text{commutativity}}$



[Note:  $\neq \emptyset$]

Now for a 2-TQFT$_{\mathcal{C}}$: $\boxed{Z: \text{Bord}_2 \longrightarrow \mathcal{C} \text{ rigid monoidal}}$

$\downarrow$

$Z(S^1) =: A \in \text{ComFrobAlg}(\mathcal{C})$.

$\boxed{\text{Theorem}}$ [Abrams, Quinn, mid 1990s] We get an equivalence of cats:

$$\boxed{2\text{-TQFT}_{\mathcal{C}} \xrightarrow{\sim} \text{ComFrobAlg}(\mathcal{C})}$$

$$Z \longmapsto Z(S^1)$$

Note: Both categories are groupoids

=

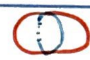
$\boxed{\text{Example}}$ Take $A := \mathbb{C}[x]/(x^2) \in \text{ComFrobAlg}(\text{FdVec}_{\mathbb{C}})$.

Here, $\Delta(1_A) = 1_A \otimes x + x \otimes 1_A$ $\qquad \Delta(x) = x \otimes x$

$\varepsilon(1_A) = 0$ $\qquad \varepsilon(x) = 1_{\mathbb{C}}$

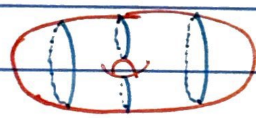
$\rightsquigarrow$ 2-TQFT$_{\mathbb{C}}$: $\quad Z : \text{Bord}_2 \longrightarrow \text{FdVec}_{\mathbb{C}}$

different mflds



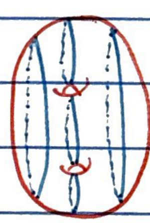
$\longmapsto \begin{bmatrix} \mathbb{C} \xrightarrow{u} A \xrightarrow{\varepsilon} \mathbb{C} \\ 1_{\mathbb{C}} \mapsto 1_A \longmapsto 0 \end{bmatrix}$

different scalars



$\longmapsto \begin{bmatrix} \mathbb{C} \xrightarrow{u} A \xrightarrow{\Delta} A \otimes A \xrightarrow{m} A \xrightarrow{\varepsilon} \mathbb{C} \\ 1_{\mathbb{C}} \mapsto 1_A \mapsto 1_A \otimes x + x \otimes 1_A \longmapsto 2x \longmapsto 2 \end{bmatrix}$

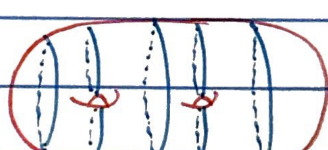
To tell apart, try another Frob. alg. like $\mathbb{C}[x]/(x^3)$...



$\longmapsto \begin{bmatrix} \mathbb{C} \xrightarrow{u} A \xrightarrow{\Delta^2} A \otimes A \otimes A \xrightarrow{m^2} A \xrightarrow{\varepsilon} \mathbb{C} \\ 1_{\mathbb{C}} \mapsto 1_A \mapsto 1_A \otimes x \otimes x + x \otimes 1_A \otimes x + x \otimes x \otimes 1_A \longmapsto 0_A \mapsto 0 \end{bmatrix}$

don't get a complete invariant

$\cong$



$\longmapsto \begin{bmatrix} \mathbb{C} \longrightarrow \mathbb{C} \\ 1_{\mathbb{C}} \mapsto 0 \text{ check!} \end{bmatrix}$

$$\boxed{\text{Altering: } 2\text{-TQFT}_{\mathcal{C}} \simeq \mathrm{ComFrobAlg}(\mathcal{C})}$$

Change the dimension

need to change the structural level...

Ex. $1\text{-TQFT}_{\mathbb{C}} \simeq$ A certain alteration of $\mathrm{FdVec}_{\mathbb{C}}$ ← not a groupoid

$3\text{-TQFT}_{\mathbb{C}} \longleftrightarrow$ "Modular Fusion Categories/$\mathbb{C}$"

= Braided, Rigid monoidal category that are abelian, $\mathbb{C}$-linear, finite, semisimple, subject to a 'nondegeneracy' & 'ribbonality' condition

Change the flavor of bordisms

need to change the flavor of Frob algebra

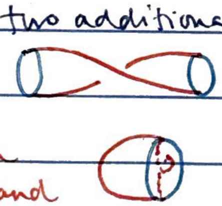
$\boxed{\text{Theorem}}$ [Turaev–Turner, 2006] $\exists$ equivalence of categories:

unoriented — "extended Frobenius algs"

$$\boxed{2\text{-TQFT}^{u}_{\mathcal{C}} \simeq \mathrm{ComExtFrobAlg}(\mathcal{C})}$$

$[Z: \mathrm{Bord}_2^u \to \mathcal{C}] \longmapsto Z(S^1) =: A, \mu, u, \Delta, \varepsilon$, with

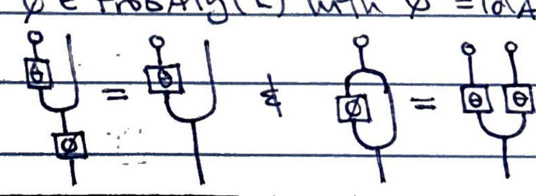
has two additional $\otimes$ gen. maps — two add'l structure maps



reversing orientation $\longmapsto \phi: A \to A$

gluing with Moebius band $\longmapsto \theta: \mathbb{1} \to A$

subject to relations $\longmapsto$ $\phi \in \mathrm{FrobAlg}(\mathcal{C})$ with $\phi^2 = \mathrm{id}_A$ &



$\boxed{\text{Extended Frobenius Algebras merit their own study: Work of A. Czenky – J. Kesten – A. Quinonez – W (2026)}}$

$\boxed{\text{Def'n}}$ For an extended Frobenius algebra $(A, m, u, \Delta, \varepsilon, \phi, \theta)$ in $\mathcal{C}$, we refer to (ϕ, θ) as the <u>extended structure</u> of $A \in \mathrm{FrobAlg}(\mathcal{C})$.

$\boxed{\text{Theorem}}$ The extended structures below are classified.

- $\mathbb{C}/\mathbb{C}$ (★)
- $\mathbb{C}/\mathbb{R}$ (★)
- $\mathbb{C}C_2$ (★)
- $\mathbb{C}C_3$
- $\mathbb{C}C_4$
- $\mathbb{C}(C_2 \times C_2)$
- $\mathbb{C}[x]/(x^n)$ (★) for n odd

(★ = all "trivial")

$\boxed{\text{Open}}$ Do more! Especially with the help of GPT Pro / LLMs, one could probably get a nice result for $\mathbb{C}C_n$ in general, ... & perhaps $\mathbb{C}G$ ← finite abelian in general ... & make a start for matrix algebras $\mathrm{Mat}_n(\mathbb{C})$ ...

<u>Fact</u> Finite-dimensional Hopf algebras/$\mathbb{C}$ are Frobenius, & ∃ a monoidal version of this fact involving integrals.

$\boxed{\text{Def'n–Prop'n}}$ We introduce the notion of an <u>extended Hopf alg in $\mathcal{C}$</u>, & show that it is extended Frobenius, functorially so.

$\boxed{\text{Def'n–Theorem}}$ We introduce the notion of an <u>extended Frobenius monoidal functor</u>; & show that they form a category (e.g. closed under ∘) and they preserve extended Frobenius algebras.

$\boxed{\text{Open}}$ Introduce "extended Frobenius monads" & "extended Frob. functors" & make connections with extended Frobenius algebras & extended Frobenius monoidal functors, as started in Lecture III and completed in Lecture V in the ordinary case.

$\boxed{\text{Not to be confused with extended TQFTs}}$

$\equiv$ Symmetric monoidal $(n-d)$-functors between $(n-d)$-categories $(d<n)$

$$Z : \mathrm{Bord}_{n,n-1,\dots,d} \longrightarrow \mathcal{C}$$

objects: d-mflds
$\vdots$
$(n-k)$-maps: k-mflds $k \in [d,n]$ ... forms a category $\boxed{(n,n-1,\dots,d)\text{-}\mathrm{TQFT}_{\mathcal{C}}}$

having/preserving orientation

Consider the 2-categ $\boxed{\mathrm{Bim}}$ $\left\{ \begin{array}{l} \text{objects} = \mathbb{C}\text{-algs} \\ \text{maps} = \text{bimodules/algs} \\ \text{2-maps} = \text{bimodule maps} \end{array} \right.$ ... forms a sym $\otimes$ 2-categ.

$\boxed{\text{Theorem}}$ [Schommer-Pries's thesis, 2009] $\exists$ 2-equivalence:

separable, symmetric Frobenius algebras / $\mathbb{C}$

$$\boxed{(2,1,0)\text{-}\mathrm{TQFT}_{\mathrm{Bim}} \simeq (\mathrm{SepSymFrobAlg}_{\mathbb{C}})_{\mathrm{mor}}}$$

up to Morita equiv.

$$Z \longmapsto Z(\text{point})$$

needed for extension to points $\longleftrightarrow$ separability

trivializing orientation $\longleftrightarrow$ symmetric

This story goes on, but I'll stop here.

Ref: This is covered in Chapter 8 of Symmetries of Algebras, vol 2 (in preparation)