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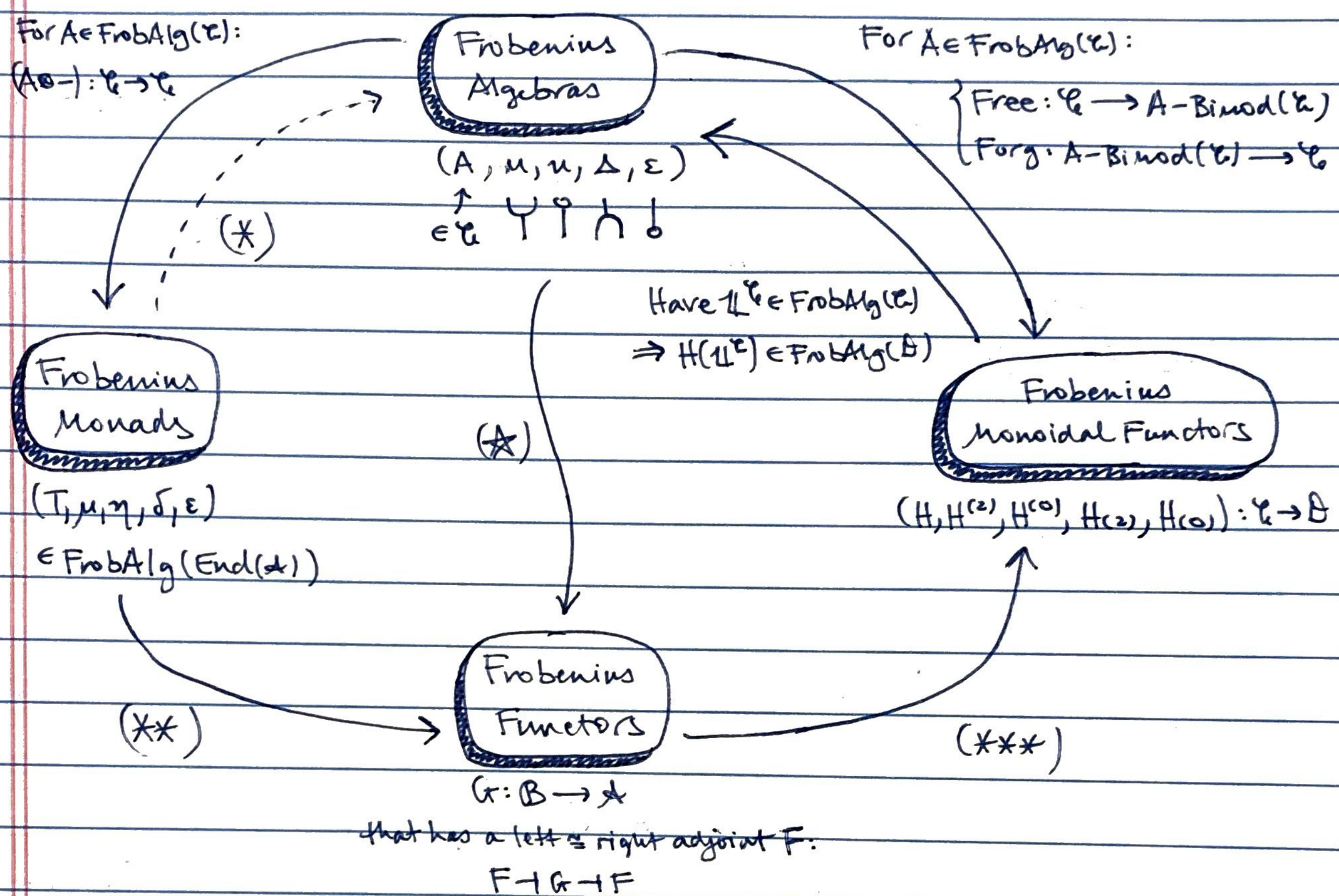
A Century of Frobenius Algebras: V. Connections between Frobenius structures

Picking up from the end of Lecture III: [$\mathcal{A}, \mathcal{B}$ ordinary categories; $\mathcal{C}, \mathcal{D}$ monoidal categories]



(*) $T \in \mathrm{FrobMonad}(\mathcal{C})$ & equipped with "strength" nat'l isom $\theta_{-,=}: T(-) \otimes = \xrightarrow{\sim} T(- \otimes =)$ $\Rightarrow$ $T \cong T(\mathbb{1}^{\mathcal{C}}) \otimes -$ via $\theta_{\mathbb{1},-}$ & $T(\mathbb{1}^{\mathcal{C}}) \in \mathrm{FrobAlg}(\mathcal{C})$ [Skipping details here]

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$\boxed{\text{Outline}}$: Will introduce Frobenius functors; cover (☆), (**), (***); & discuss some applications.

$\boxed{\text{Frobenius Functors}}$ (terminology coined by Caenepeel (1997))

$\boxed{\text{Defn}}$ A functor $G: \mathcal{B} \to \mathcal{A}$ is <u>Frobenius</u> if $\exists$ functor $F: \mathcal{A} \to \mathcal{B}$ that is both a left and right adjoint of G. That is, $\exists$ natural transformations:

For $F \dashv G$: $\quad \eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow GF, \quad \varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{B}}$,

For $G \dashv F$: $\quad \eta': \mathrm{Id}_{\mathcal{B}} \Rightarrow FG, \quad \varepsilon': GF \Rightarrow \mathrm{Id}_{\mathcal{A}}$,

such that the identities below hold, for $X \in \mathcal{A}$, $Y \in \mathcal{B}$:

$$\varepsilon_{F(X)} \circ F(\eta_X) = \mathrm{id}_{F(X)}, \quad G(\varepsilon_Y) \circ \eta_{G(Y)} = \mathrm{id}_{G(Y)},$$
$$\varepsilon'_{G(Y)} \circ G(\eta'_Y) = \mathrm{id}_{G(Y)}, \quad F(\varepsilon'_X) \circ \eta'_{F(X)} = \mathrm{id}_{F(X)}.$$

(★)

$A \in \mathrm{FrobAlg}(\mathcal{C})$

$\Downarrow$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\text{Free}"] \arrow[r, phantom, "\perp\top"] & A\text{-Mod}(\mathcal{C}) \arrow[l, bend left, "\text{Forg}"] \end{tikzcd}$$

$\boxed{\text{Proposition}}$ For $(A, m, u, \Delta, \varepsilon) \in \mathrm{FrobAlg}(\mathcal{C})$, we get that

$$G := \mathrm{Forg}: A\text{-Mod}(\mathcal{C}) \to \mathcal{C}, \quad (M, \triangleright: A \otimes M \to M) \mapsto M$$

is Frobenius, with left/right adjoint:

$$F := \mathrm{Free}: \mathcal{C} \to A\text{-Mod}(\mathcal{C}), \quad X \mapsto (A \otimes X, \triangleright := m \otimes \mathrm{id}_X).$$

Pf/ Define the components of the adjunction nat'l transformations as:

$$\eta_X: X \xrightarrow{u \otimes \mathrm{id}} A \otimes X, \qquad \varepsilon_{(M,\triangleright)}: A \otimes M \xrightarrow{\triangleright} M$$

$$\eta'_{(M,\triangleright)}: M \xrightarrow{u \otimes \mathrm{id}} A \otimes M \xrightarrow{\Delta \otimes \mathrm{id}} A \otimes A \otimes M \xrightarrow{\mathrm{id} \otimes \triangleright} A \otimes M, \qquad \varepsilon'_X: A \otimes X \xrightarrow{\varepsilon \otimes \mathrm{id}} X$$

for $X \in \mathcal{C}$ (=: "$\mathcal{A}$") & $(M, \triangleright) \in A\text{-Mod}(\mathcal{C})$ (=: "$\mathcal{B}$").

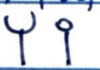
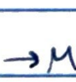
For instance,

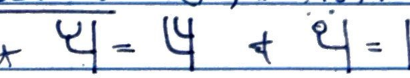
$$\begin{tikzcd} & A \otimes A \otimes X \arrow[dr, "\varepsilon_{F(X)} = \triangleright_{A \otimes X}", "m \otimes \mathrm{id}"'] & \\ F(X) = A \otimes X \arrow[ur, "F(\eta_X)", "\mathrm{id} \otimes u \otimes \mathrm{id}"'] \arrow[rr, bend right, "\mathrm{id}"'] \arrow[rr, phantom, "\text{[unitality of } A\text{]}"] & & A \otimes X = F(X) \end{tikzcd}$$

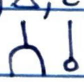
...

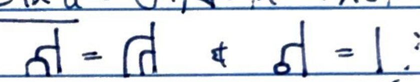
Towards (**)

Warm-up result

Given $(A, \mu, u) \in \mathrm{Alg}(\mathcal{C})$ , a <u>left A-module in $\mathcal{C}$</u> $= (M, \triangleright : A \otimes M \to M$ $)$

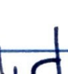
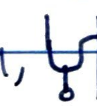
such that 

Given $(C, \Delta, \varepsilon) \in \mathrm{Coalg}(\mathcal{C})$ , a <u>left A-comodule in $\mathcal{C}$</u> $= (M, \triangleleft : M \to A \otimes M$ $)$

such that 

Proposition If $A \in \mathrm{FrobAlg}(\mathcal{C})$, then $A\text{-}\mathrm{Mod}(\mathcal{C}) \overset{\Gamma}{\cong} A\text{-}\mathrm{Comod}(\mathcal{C})$.

Here, $\Gamma \circ \mathrm{Free}^{\mathrm{Mod}} = \mathrm{Free}^{\mathrm{Comod}}$ & $\mathrm{Forg}^{\mathrm{Comod}} \circ \Gamma = \mathrm{Forg}^{\mathrm{Mod}}$.

Pf/ Define: $(M,$ $) \mapsto (M,$ $)$ & $(M,$ $) \mapsto (M,$ $)$ ... ///.

(Co-)Eilenberg Moore categories

also called "EM-algebras"

Def'n (1) Given $(T, \mu : T^2 \Rightarrow T, \eta : \mathrm{Id} \Rightarrow T) \in \mathrm{Monad}(\mathcal{A})$, an <u>EM-object</u> $= (Y, \xi : T(Y) \to Y)$ in $\mathcal{A}$ satisfying assoc, unitality conditions.

(like a "module" over T) $\rightsquigarrow$ category $\boxed{\mathcal{A}^T}$

"EM-coalgebras"

(2) Given $(T, \delta : T \Rightarrow T^2, \varepsilon : T \Rightarrow \mathrm{Id}) \in \mathrm{Comonad}(\mathcal{C})$, an <u>coEM object</u> $= (Y, \upsilon : Y \to T(Y))$ in $\mathcal{A}$ satisfying coassoc., counitality conditions.

(like a "comodule" over T) $\rightarrow$ category $\boxed{\mathcal{A}^{\mathrm{coT}}}$

Example (1) $\mathcal{C}^{(A \otimes -)} \cong A\text{-}\mathrm{Mod}(\mathcal{C})$ for $A \in \mathrm{Alg}(\mathcal{C})$.

(2) $\mathcal{C}^{(C \otimes -)} \cong C\text{-}\mathrm{Comod}(\mathcal{C})$ for $C \in \mathrm{Coalg}(\mathcal{C})$.

Proposition If $T \in \mathrm{FrobMonad}(\mathcal{A})$, then $\mathcal{A}^T \overset{\Gamma}{\cong} \mathcal{A}^{\mathrm{coT}}$.

Here, $\underset{\substack{\uparrow \\ Y \mapsto (T(Y), \mu_Y)}}{\Gamma \circ \mathrm{Free}^T} = \underset{\substack{\uparrow \\ Y \mapsto (T(Y), \delta_Y)}}{\mathrm{Free}^{\mathrm{coT}}}$ & $\underset{\substack{\uparrow \\ (Y, \upsilon) \mapsto Y}}{\mathrm{Forg}^{\mathrm{coT}} \circ \Gamma} = \underset{\substack{\uparrow \\ (Y, \xi) \mapsto Y}}{\mathrm{Forg}^T}$. ///.

(**)

$T \in \mathrm{FrobMonad}(\mathcal{A})$

$\Downarrow$

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\mathrm{Free}^T"] \arrow[r, phantom, "\perp\top"] & \mathcal{A}^T \arrow[l, bend left, "\mathrm{Forg}^T"] \end{tikzcd}$$

$\boxed{\text{Theorem}}$ [Street, 2004] If $(T, \mu, \eta, \delta, \varepsilon) \in \mathrm{FrobMonad}(\mathcal{A})$, then $\mathrm{Forg}^T : \mathcal{A}^T \to \mathcal{A}$ is a Frobenius functor, with left/right adjoint $\mathrm{Free}^T : \mathcal{A} \to \mathcal{A}^T$.

(converse holds as well)

Pf/ Take $F := \mathrm{Free}^T$, $G := \mathrm{Forg}^T$, $F' := \mathrm{Free}^{coT}$, $G' := \mathrm{Forg}^{coT}$.

Have $F \dashv G$ & $G' \dashv F'$ from theory of monads & comonads, resp.

To get $G \dashv F$, define

$$\eta^{G \dashv F} := \Gamma^{-1} \eta^{G' \dashv F'} \Gamma \quad \& \quad \varepsilon^{G \dashv F} := \varepsilon^{G' \dashv F'}.$$

Now adjunction triangle identities hold. For instance, for $y \in \mathcal{A}$:

$$\varepsilon_y^{G\dashv F} \circ G\left(\eta_{(y,\varsigma)}^{G \dashv F}\right) \overset{\text{def}}{=} \varepsilon_y^{G'\dashv F'} \circ G\left(\Gamma^{-1}\left(\eta_{\Gamma(y,\varsigma)}^{G'\dashv F'}\right)\right)$$

$$[G\Gamma^{-1} = G'] \overset{\text{Prop}}{=} \varepsilon_y^{G'\dashv F'} \circ G'\Big(\eta^{G'\dashv F'}_{\underbrace{\Gamma(y,\varsigma)}_{\in \mathcal{A}^{coT}}}\Big)$$

$$\overset{\text{triangle ident. for } G' \dashv F}{=} \mathrm{id}_y. \;///.$$

$\boxed{\text{Next, we turn our attention to Frobenius monoidal functors —}}$

(***)

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp\top"] & \mathcal{B} \arrow[l, bend left, "G"] \end{tikzcd}$$

$\Downarrow$

$\mathrm{Hom}(G,F) : \mathrm{End}(\mathcal{A}) \to \mathrm{End}(\mathcal{B})$ Frob. monoidal.

$\boxed{\text{Proposition}}$ [Flake–Laugwitz–Posur, 2025] If $G : \mathcal{B} \to \mathcal{A}$ is a Frob. functor, with left/right adjoint $F : \mathcal{A} \to \mathcal{B}$, then

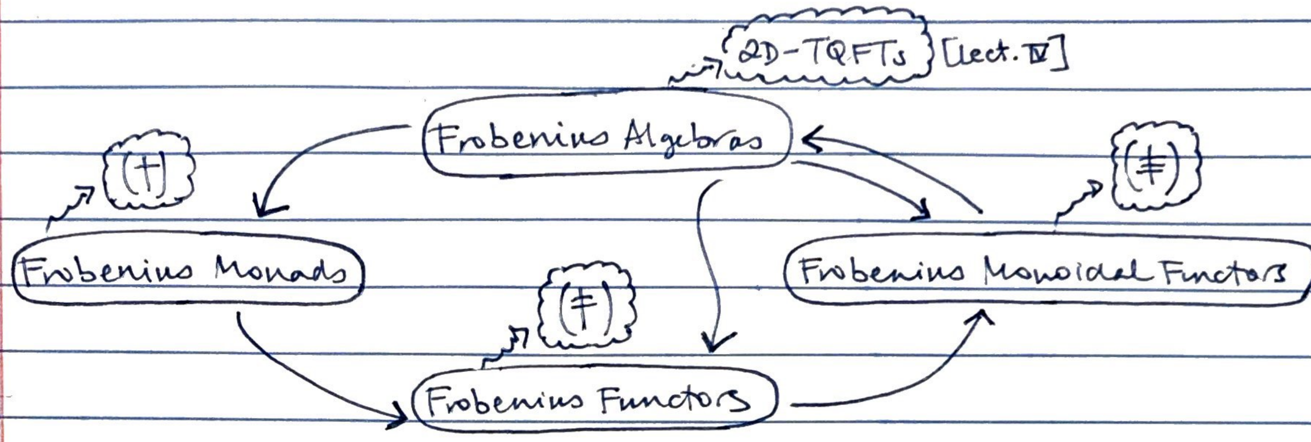
$$H := \mathrm{Hom}(G,F) : \mathrm{End}(\mathcal{A}) \longrightarrow \mathrm{End}(\mathcal{B}), \quad L \longmapsto FLG,$$

is Frobenius monoidal. Here:

$$\begin{cases} H^{(2)}_{L,L'} : H(L)\,H(L') = FLGFL'G \xrightarrow{FL\,\varepsilon^{G\dashv F} L'G} FLL'G = H(LL'), \\ H^{(0)} : \mathbb{1}^{\mathrm{End}(\mathcal{B})} = \mathrm{Id}_{\mathcal{B}} \xrightarrow{\eta^{G\dashv F}} FG = H(\mathrm{Id}_{\mathcal{A}}) = H(\mathbb{1}^{\mathrm{End}(\mathcal{A})}), \end{cases}$$

$$\begin{cases} H^{L,L'}_{(2)} : H(LL') = FLL'G \xrightarrow{FL\,\eta^{F\dashv G} L'G} FLGFL'G = H(L)\,H(L'), \\ H_{(0)} : H(\mathbb{1}^{\mathrm{End}(\mathcal{A})}) = H(\mathrm{Id}_{\mathcal{A}}) = FG \xrightarrow{\varepsilon^{F\dashv G}} \mathrm{Id}_{\mathcal{B}} = \mathbb{1}^{\mathrm{End}(\mathcal{B})} \end{cases}$$

This completes the diagram on page V.1. Now applications...



(†) Frobenius Monads in the Semantics of Programming Languages

Ref: Heunen–Karvonen's "Reversible Monadic Computing", Electronic Notes in Theoretical Computer Science, 2015.

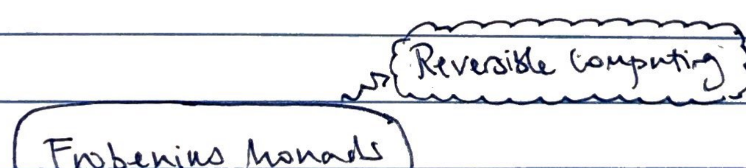
Monads (T, μ, η) are used to describe computational effects:

- $\mu : T^2 \Rightarrow T$: combines two back-to-back effects into one
- $\eta : \mathrm{Id} \Rightarrow T$: introduces a trivial effect.

When $T := (T, \mu, \eta, \delta, \varepsilon)$ is Frobenius, these moves can be undone:

- $\delta : T \Rightarrow T^2$: separates an effect into stages
- $\varepsilon : T \Rightarrow \mathrm{Id}$: removes the trivial effect.

In particular, the Frobenius law is a crucial coherence condition needed to properly model reversible computing.



(F) Frobenius Functors (twisted version) for Dualities in Geometry

Ref: Hoyois' "The six operations in equivariant motivic homotopy theory"
Advances in Mathematics, 2017

Here, a functor being Frobenius ~~and~~ having an "ambidextrous" adjunction.
Take a nice geometric map $f: X \to Y$, & consider its pullback f^*.
- Left adjoint $f_\#$ of f^* = "homological pushforward"
- Right adjoint f_* of f^* = "cohomological pushforward"

Ideal to get that $f_\# \cong f_*$ up to a twist by a "dualizing object".
This formalism captures:
- Poincaré duality
- Finite etale transfers
- Base-change formulas ... and much more.

Six functor formalism

Frobenius Functors

(F) Frobenius Monoidal Functors in Rational Conformal Field Theory

Ref: Kong-Runkel's "Cardy Algebras and Sewing constraints, I"
Communications in Mathematical Physics, 2009.

Take a modular fusion category $\mathcal{C}$ [page IV.6]
Then, the "Deligne product" functor $\underbrace{(\mathcal{C},c) \boxtimes (\mathcal{C},c^{-1})}_{=:\mathcal{E}} \longrightarrow \mathcal{C}$
has a right adjoint $R: \mathcal{C} \longrightarrow \mathcal{C} \boxtimes \overline{\mathcal{C}}$.

It is shown that R is not strong monoidal, but it is Frobenius monoidal & passes Frobenius algebras in $\mathcal{C}$ to those in $\mathcal{C} \boxtimes \mathcal{C}$. This is useful because Frobenius algebras model "sectors" in the physical theory of forming "world-sheets". The assignment $A \to R(A)$ is used to model sewing conditions..

Rational Conformal Field Theory

Frobenius Monoidal Functors

A Century of Frobenius Algebras

The Origins in Linear Algebra — 1903

Rep/Module Theoretic characterizations. — 1930s-1940s

Abstract Theory solidified — 1950s-1970s

Categorical Characteriz'ns — Connections to Topology/Physics — 1980s-1990s

More Applications to other fields & A Favorite Structure for many! — 2000s to now

1903 ---- 1930s-1940s ---- 1950s-1970s --- 1980s-1990s --- 2000s to now

Ref: C. Walton's "Symmetries of Algebras" textbook series

- Volume 1: Algs/$\mathbb{C}$, Cats, Mon. Cats, Algs in Mon. Cats
  [free copy out now, citations & library requests appreciated]

- Volume 2: Coalgs/$\mathbb{C}$, Coalgs in Mon Cats, FrobAlgs/$\mathbb{C}$, FrobAlgs in Mon Cats
  [expected: first half of 2027].