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Functorial constructions of Frobenius algebras in the Drinfeld center

by

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Abstract

Frobenius algebras in vector spaces are classical algebraic structures. However, because of their discovered connections to various fields, including computer science and topological quantum field theories, there is a growing interest in exploring their generalizations within the framework of monoidal categories. Inspired by these connections, this thesis delves into the problem of functorially constructing 'nice' Frobenius algebra objects in such categories.

We introduce unimodular module categories and employ them to provide a functorial construction of Frobenius algebras in the Drinfeld center of a finite tensor category. We also classify unimodular module categories over the category of representations of a finite dimensional Hopf algebra.

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Contents

Chapter 1

Introduction

This thesis is inspired by algebraic structures that arise in the axiomatic study of Quantum Field Theories (QFTs). Quantum field theory is a fascinating and challenging area of modern physics, where the laws of quantum mechanics and special relativity converge. In layman's terms, a *field theory* is a mathematical theory describing the time evolution of a field in space, like the electric or magnetic field. Examples of field theories are Maxwell's theory of electromagnetism or Newton's theory of gravitation. These field theories, however, do not explain the phenomenon like special relativity and quantum mechanics. A field theory that incorporates these phenomenon is called a *quantum field theory* (QFT). Its intricate algebraic structures have attracted the attention of mathematicians for the last four decades, leading to a deep and fruitful cross-fertilization between physics and mathematics.

Although the previous discourse offers an informal guide of what a quantum field theory (QFT) should look like, still there is no universal axiomatic framework to define QFT after extensive attempts from mathematicians and physicists. To establish a reliable mathematical construct, one has to enforce supplementary constraints leading to various diverse QFTs. The most extensively studied amongst these happen to be the *topological quantum field theories* (TQFTs) and *conformal field theories* (CFTs).

The algebraic structures that arise in the context of axiomatic quantum field theories like 3-dimensional TQFTs and 2-dimensional CFTs have been the driving force behind this thesis. While TQFTs and CFTs do not represent the full complexity of QFTs, they are crucial models that provide invaluable insights into the overarching framework.

Topological Quantum Field Theories

TQFTs are a class of QFTs that possess the desirable property of being independent of spacetime metrics – in other words, they only depend on the underlying manifold topology. Notably, this theory becomes particularly intriguing when restricted to

3-dimensional contexts, largely due to its fascinating topological properties. Roughly, a 3D-TQFT is defined as a certain functor, from a category of 3D manifolds to the category of vector spaces, that behaves nicely with respect to gluing and cutting manifolds. This connection between topology and algebra, as formulated by the definition of a TQFT, has served as a catalyst for developing invariants of knots and 3-dimensional manifolds, such as the Reshetikhin-Turaev invariants [RT91].

Conformal Field Theories

In this work, by a CFT, we will mean an axiomatic model of QFT which is invariant under conformal transformations of the spacetime. As condensed matter systems are often conformally invariant at their critical points, CFTs have important applications to condensed matter physics and string theory. Due to their complexity, the 2-dimensional case is currently the most mathematically interesting one. In particular, the study of 2D-CFTs has led to the development of a rich mathematical theory, which has been used to study various mathematical objects such as vertex operator algebras [Hua97].

Modular tensor categories (MTCs)

3D-TQFTs and 2D-CFTs are mathematically described by certain algebraic objects called *modular tensor categories*, which are fascinating objects of study in their own right. On the one hand, major works in the last three decades have established that the main source of producing 3D-TQFTs are *modular tensor categories* [Tur92, KL01, BDSPV15, TV17, DRGG+22]. Hence, to produce new rich new invariants of manifolds, one needs new examples of TQFTs and this in turn is achieved by finding new MTCs. On the other hand, MTCs are important models for the mathematical study of 2-dimensional rational Conformal Field Theories as well [FRS02]. In addition, MTCs also play an important role in quantum computing [Wan13].

1.1 Problem description

The overarching objective of this thesis is to enhance our understanding of modular tensor categories. One way to do so is by analyzing already known constructions of producing MTCs.

In the literature, there are numerous techniques of producing new MTCs from a given MTC. We list the major ones below:

- taking Deligne product of two MTCs,
- by gauging symmetry of MTCs [CGPW16],
- by zesting [DGP+21], or
- by forming the category of local modules over a 'nice' Frobenius algebra object in a MTC [Par95, Sch01, KJO02, LW22a].

To narrow down our problem, we focus on the last technique of constructing new MTCs by using local modules over a Frobenius algebra object. Because of this, we focus on the role of Frobenius algebras in MTCs. Thus, the above discussion raises the following important problem.

Problem. *Provide a construction of Frobenius algebras in a modular tensor categories.*

Our investigation started with this mathematical question. While previous works such as [FFRS06] have addressed the construction of Frobenius algebra objects in semisimple MTCs, recent research on non-semisimple CFTs [FS21a] and TQFTs [DRGG+22] has created a need for new constructions of Frobenius algebra objects in general MTCs.

There are various sources of MTCs. Below we list the major ones.

- category of representations categories of quantum groups [KJO02, LW22b],
- category of representations of vertex operator algebras [Hua05], and
- Drinfeld centers of spherical finite tensor categories [Müg03, Shi17].

Our work aims to address the problem mentioned above for MTCs that are obtained as Drinfeld centers of spherical tensor categories. Moreover, we go beyond this specific case to address the general problem of constructing 'nice' Frobenius algebra objects in the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of any finite tensor category. In this context, we use the term 'nice' to refer to Frobenius algebra objects that are suitable for constructing new MTCs using local modules. To qualify as 'nice', these Frobenius algebra objects must satisfy additional conditions, such as being *special*, *symmetric*, *commutative*, and so on.

Moreover, we aim to present constructions that are amenable to algebraic manipulations, and to do so, we need functorial constructions. Here, by functorial construction, we mean a construction that involves building the Frobenius algebra object as the image of certain functors, called *Frobenius monoidal functors*. These functors allow for the transfer of Frobenius algebras from the source to the target category. If a Frobenius monoidal functor preserves additional properties of the Frobenius algebra such as being special or symmetric, we refer to it as a special or pivotal Frobenius monoidal functor, respectively. The study of these functors is a central focus of this thesis.

Thus, we have narrowed down our problem of the following goal.

Goal. *Construct 'nice' Frobenius monoidal functors whose target categories are Drinfeld centers of finite tensor categories.*

1.2 Main results

To achieve our goal of constructing Frobenius monoidal functors whose target categories are Drinfeld centers of finite tensor categories, we utilize Balan's findings. In her publication [Bal17], she provided necessary conditions under which the left adjoint of a strong monoidal functor is a Frobenius monoidal functor. By, slightly rephrasing her results, we obtain the following result.

Theorem 1.1. (Theorem 3.5) *Let $U : \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor between abelian monoidal categories admitting a right adjoint R, such that R is exact, faithful and the adjunction $U \dashv R$ is coHopf. Then, we get that R is a Frobenius monoidal functor if and only if $R(\mathbb{1}_{\mathcal{D}})$ is a Frobenius algebra in $\mathcal{C}$.*

This result provides us the following heuristic for constructing Frobenius monoidal functors.

(a) Start with a strong monoidal functor $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{R}$ where $\mathcal{Z}(\mathcal{C})$ is the Drinfeld center of a finite tensor category $\mathcal{C}$ and $\mathcal{R}$ is any abelian monoidal category.

(b) Find a right adjoint R to U such that the adjunction $U \dashv R$ is coHopf and R is exact and faithful.

(c) Find the appropriate conditions (†) on U, $\mathcal{C}$ and $\mathcal{R}$ such that $R(\mathbb{1}_{\mathcal{R}})$ is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$.

Once the above mentioned steps are completed, we get that if U, $\mathcal{C}$ and $\mathcal{R}$ satisfy the conditions (†), then the right adjoint of, namely $U^{\mathsf{ra}} : \mathcal{R} \to \mathcal{Z}(\mathcal{C})$, is a Frobenius monoidal functor.

One good candidate for the functor U in step (a) is the following forgetful functor from the Drinfeld center $\mathcal{Z}(\mathcal{C})$ to $\mathcal{C}$, which forgets the half-braiding on the objects of $\mathcal{Z}(\mathcal{C})$.

$$U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}, \qquad (X, \sigma) \mapsto X$$

When $\mathcal{C}$ is a finite tensor category, the right adjoint R of U satisfies all the conditions required for step (b). Therefore, R is a Frobenius monoidal functor if and only if $R(\mathbb{1}_{\mathcal{C}})$ is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$.

A finite tensor category $\mathcal{C}$ is called *unimodular* if its distinguished invertible object (defined as the socle of projective cover of $\mathbb{1}_{\mathcal{C}}$) is isomorphism to the unit object $\mathbb{1}_{\mathcal{C}}$ of $\mathcal{C}$. Now, we can use the following result of Shimizu to complete step (c) and achieve our goal.

Theorem 1.2. ([Shi16, Corollary 5.10]) *Let $\mathcal{C}$ be a finite tensor category. Let $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ denote the forgetful functor from the Drinfeld center to $\mathcal{C}$ and R its right adjoint. Then, $\mathcal{C}$ is unimodular if and only if $R(\mathbb{1})$ is a Frobenius algebra in $\mathcal{C}$.*

Thus, the condition (†) is that $\mathcal{C}$ is unimodular. By combining Theorems 1.1 and 1.2, we obtain the following result.

Theorem 1.3. *Let $\mathcal{C}$ be a finite tensor category. Let $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ denote the forgetful functor from the Drinfeld center to $\mathcal{C}$ and R its right adjoint. Then, $\mathcal{C}$ is unimodular if and only if R is a Frobenius monoidal functor.*

This is the blueprint that we follow to construct Frobenius monoidal functors to the Drinfeld center of a finite tensor category. In this thesis, we generalize Theorem 1.3 in two directions.

First, for constructing local modules, we need special and symmetric Frobenius algebras. Thus, we need to construct special and pivotal Frobenius monoidal functors with target $\mathcal{Z}(\mathcal{C})$. To do so, we establish the following generalization of Theorem 1.1.

Theorem 1.4. *Let $U : \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor between abelian monoidal categories admitting a right adjoint R, such that R is exact, faithful and the adjunction $U \dashv R$ is coHopf. Then, we get that R is a $\circledast$ monoidal functor if and only if $R(\mathbb{1}_{\mathcal{D}})$ is a $⋇$ algebra in $\mathcal{C}$, as summarized in the table below.*

Reference	Input			$R(\mathbb{1}_{\mathcal{D}})$	R
	$\mathcal{C}$	$\mathcal{D}$	U	⋇	⊛
Thm. 3.9	$\otimes$	$\otimes$	*strong* $\otimes$	*separable Frob.*	*separable Frob.*
Thm. 3.9	$\otimes$	$\otimes$	*strong* $\otimes$	*special Frob.*	*special Frob.*
Thm. 3.11	*pivotal*	*pivotal*	*pivotal, strong* $\otimes$	*symmetric Frob.*	*pivotal Frob.*
Thm. 3.13	*ribbon*	*ribbon*	*ribbon*	*symmetric Frob.*	*ribbon Frob.*

Second, we consider a generalization of the forgetful functor $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. More precisely, we consider the following functor which depends on an exact left $\mathcal{C}$-module category $(\mathcal{M}, \triangleright)$.

$$\Psi := \Psi_{\mathcal{M}} : \mathcal{Z}(\mathcal{C}) \to \mathsf{Rex}_{\mathcal{C}}(\mathcal{M}), \qquad (c, \sigma) \mapsto (c \triangleright -, s^{\sigma}).$$

This functor was introduced in [Shi20]. When we take $\mathcal{M} = \mathcal{C}$, we recover the forgetful functor U from the Drinfeld center $\mathcal{Z}(\mathcal{C})$ to $\mathcal{C}$. In Proposition 4.14, we show that when $\mathcal{M}$ is an indecomposable exact left $\mathcal{C}$-module category, the right adjoint Ψ^{ra} of the functor Ψ is exact, faithful and the adjunction $\Psi \dashv \Psi^{\mathsf{ra}}$ is coHopf. Thus, by applying Theorem 1.4 to the adjunctions $\Psi \dashv \Psi^{\mathsf{ra}}$, we get the following result.

Theorem 1.5. *Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ be an indecomposable, exact left $\mathcal{C}$-module category. Then the following statements hold.*

(a) *Ψ^{ra} is a (resp., separable, special) Frobenius monoidal functor if and only if $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is (resp., separable, special) Frobenius algebra in $\mathcal{Z}(\mathcal{C})$.*

(b) *If $\mathcal{C}$ is pivotal and $\mathcal{M}$ is a pivotal $\mathcal{C}$-module category, then Ψ^{ra} is a pivotal functor if and only if $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is a symmetric Frobenius algebra in $\mathcal{Z}(\mathcal{C})$.*

This theorem highlights the importance of the algebra $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ for constructing Frobenius monoidal functors to the Drinfeld center. In particular, we are interested in the question of when this algebra is (symmetric) Frobenius.

One of the major contributions of this thesis is the following notion of unimodularity for exact left $\mathcal{C}$-module categories. The following definition is inspired by [FSS20, Remark 4.27].

Definition 1.6. An exact left $\mathcal{C}$-module category $\mathcal{M}$ is called *unimodular* if the multitensor category $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ of right exact, left $\mathcal{C}$-module endofunctors of $\mathcal{M}$ is unimodular.

Using this notion, we are able to answer the question of when the algebra $\mathbf{A}_{\mathcal{M}}$ is Frobenius. In particular, we obtain the following result.

Theorem 1.7. (Theorem 4.15) *Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ an indecomposable, exact, left $\mathcal{C}$-module category. Then, the following are equivalent.*

(a) *$\mathcal{M}$ is a unimodular module category.*
(b) *$\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ is a unimodular tensor category.*
(c) *$\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$.*
(d) *Ψ^{ra} is a Frobenius monoidal functor.*

Here, for any functor F, F^{la} (resp., F^{ra}) denotes the left (resp., right) adjoint of F. Part (d) yields a supply of Frobenius algebras in the Drinfeld center. Furthermore, the following result describes when the functor Ψ^{ra} and the algebra obtained using it are separable or special.

Theorem 1.8. (Theorem 4.16) *Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ be an indecomposable, unimodular left $\mathcal{C}$-module category. Then, Ψ^{ra} is a separable (resp. special) Frobenius monoidal functor if and only if the Frobenius algebra $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ in $\mathcal{Z}(\mathcal{C})$ is separable (resp. special).*

Recall that our goal is to construct special, symmetric Frobenius algebras. However, to discuss symmetric Frobenius algebras, we have to move to the pivotal setting. When $\mathcal{C}$ is pivotal and $\mathcal{M}$ is a pivotal left $\mathcal{C}$-module category, then the categories $\mathcal{Z}(\mathcal{C})$ and $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ are pivotal [Sch15, Shi19], and Ψ is a pivotal functor. Then, the following result describes sufficient conditions needed to ensure that the functor Ψ^{ra} is a pivotal functor. When these conditions are satisfied, Ψ^{ra} becomes a tool of producing special, symmetric Frobenius algebras in $\mathcal{Z}(\mathcal{C})$.

Theorem 1.9. (Theorems 4.21, 4.22) *Let $\mathcal{C}$ be a pivotal finite tensor category and $\mathcal{M}$ an indecomposable, unimodular, pivotal left $\mathcal{C}$-module category. Then,*

(a) *Ψ^{ra} is a pivotal Frobenius monoidal functor.*
(b) *Furthermore, Ψ^{ra} is a special pivotal Frobenius monoidal functor if and only if $\dim(\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})) \neq 0$.*

To provide a concrete example, we consider the case when $\mathcal{C} := \mathsf{Rep}(H)$ is the category of finite-dimensional representations of a finite-dimensional Hopf algebra H.

In this case, every exact left $\mathcal{C}$-module category $\mathcal{M}$ is of the form $\mathsf{Rep}(A)$ for A a left H-comodule algebra. We employ results from [Shi19, SS21, FGJS22] to calculate the distinguished invertible object $D_{\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})}$ of $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$. Additionally, to understand when $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ is unimodular, that is, when $D_{\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})}$ and $\mathsf{Id}_{\mathcal{M}}$ are isomorphic, we introduce *unimodular elements* of an exact H-comodule algebras (Definition 6.2) and obtain the following result.

Theorem 1.10. (Theorem 6.3) *Let H be a finite-dimensional Hopf algebra and A an exact left H-comodule algebra. Then the left $\mathsf{Rep}(H)$-module category $\mathsf{Rep}(A)$ is unimodular if and only if A admits a unimodular element.*

The question of unimodularity of $\mathcal{M}$ (which, by definition, is equivalent to the unimodularity of $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$) has also recently been investigated in [Shi22] in the case when the algebra A admits a grouplike cointegral. Theorem 1.10 provides explicit conditions for when the module category $\mathsf{Rep}(A)$ is unimodular without any such assumption, thereby answering [Shi22, Question 7.25]. Furthermore, Theorem 1.10 reduces to Shimizu's result [Shi22, Corollary 7.10], when the comodule algebra A admits a grouplike cointegral; see Corollary 6.6.

A natural question to ask is, whether every finite tensor category admits a unimodular module category. We obtain the following result which provides a negative answer to this question.

Theorem 1.11. (Theorem 6.7) *Let $T(\omega)$ denote the Taft algebra. Then, the category $\mathsf{Rep}(T(\omega))$ does not admit a unimodular module category.*

The results above have first appeared in our preprints [Yad22] and [Yad23].

1.3 Structure of the thesis

This thesis is organized as follows. In Chapter 2, we recall the background material on tensor categories and modules categories. Chapter 3 is dedicated to establishing general results on when the right adjoint of a strong monoidal functor is pivotal, Frobenius monoidal. In particular, we prove Theorem 1.4 in Chapter 3.

In Chapter 4, we study unimodular module categories and use them to achieve our goal of constructing Frobenius monoidal functors to $\mathcal{Z}(\mathcal{C})$. Namely, we prove Theorems 1.7, 1.8 and 1.9 in Chapter 4. After introducing background material on Hopf algebras H in Chapter 5, we classify unimodular module categories over $\mathcal{C} = \mathsf{Rep}(H)$ in Chapter 6 and prove Theorem 1.10.

We end this thesis with some remarks and open questions in Section 6.5.

Chapter 2

Background

In this chapter, we introduce the background material on tensor categories. We review monoidal categories in Section 2.1, module categories in Section 2.2, rigid and pivotal categories in Section 2.3, algebras in monoidal categories in Section 2.4, adjunctions in Section 2.5, and finite tensor categories and their module categories in Section 2.6. We refer the reader to the textbooks [ML13], [EGNO16] and [TV17] for further details.

Convention 2.1. Unless otherwise specified, throughout this work, $\Bbbk$ will denote an algebraically closed field, algebraic structures will be over $\Bbbk$. By $\Bbbk^{\times}$, we will denote the multiplicative set $\Bbbk \setminus \{0\}$.

2.1 Monoidal categories

We begin by presenting the algebraic structures that will play a key role in our study, namely monoidal categories. Monoidal categories are categories that possess a tensor product, similar to the way in which we can construct a tensor product of $\Bbbk$-vector spaces.

Definition 2.2 $(\mathcal{C}, \otimes, \mathbb{1})$**.** A category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called the *tensor product*), an object $\mathbb{1} \in \mathcal{C}$ (called the *unit object*) and natural isomorphisms

$$X \otimes (Y \otimes Z) \;\cong (Z \otimes Y) \otimes Z, \tag{2.1}$$

$$X \otimes \mathbb{1} \;\cong X \cong\; \mathbb{1} \otimes X, \tag{2.2}$$

for all $X, Y, Z \in \mathcal{C}$, is called a *monoidal category* if the isomorphisms in (2.1) satisfy the pentagon and the triangle axioms. If these isomorphisms are identities, we call $\mathcal{C}$ a *strict* monoidal category.

Remark 2.3. By Mac Lane's coherence theorem [ML13, VII.2], we can and will assume that all monoidal categories are strict. We let $\mathcal{C}^{\mathrm{rev}}$ denote the category $\mathcal{C}$ with the opposite tensor product $\otimes^{\mathrm{rev}}$, that is, $X \otimes^{\mathrm{rev}} Y := Y \otimes X$. We denote the opposite category of $\mathcal{C}$ as $\mathcal{C}^{\mathrm{op}}$. Then both $(\mathcal{C}^{\mathrm{op}}, \otimes, \mathbb{1})$ and $(\mathcal{C}^{\mathrm{rev}}, \otimes^{\mathrm{rev}}, \mathbb{1})$ are monoidal categories.

Given two vector spaces V and W, we have a flip map $V \otimes W \to W \otimes V$. Next, we define monoidal categories that come equipped with such maps.

Definition 2.4 $(\mathcal{C}, c)$**.** A *braided monoidal category* is a monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ equipped with a natural isomorphism

$$c = \{c_{X,Y} : X \otimes Y \to Y \otimes X\}_{X,Y \in \mathcal{C}}$$

(called a *braiding*) satisfying the hexagon axiom.

Remark 2.5. The *mirror* c' of a braiding c on $\mathcal{C}$ is defined by $c'_{X,Y} := c^{-1}_{Y,X}$. We will let $\mathcal{C}^{\mathrm{mir}}$ denote the braided monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, c')$.

2.1.1 Monoidal functors

Next, we define functors that preserve monoidal structure. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ be two monoidal categories.

Definition 2.6 (F, F_2, F_0)**.** A *monoidal functor* $(F, F_2, F_0) : \mathcal{C} \to \mathcal{D}$ consists of the following:

- a functor $F : \mathcal{C} \to \mathcal{D}$,
- a natural transformation
$$F_2 = \{F_2(X, X') : F(X) \otimes_{\mathcal{D}} F(X') \to F(X \otimes_{\mathcal{C}} X')\}_{X,X' \in \mathcal{C}},$$
- a morphism $F_0 : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$ in $\mathcal{D}$,

that satisfy the following associativity and unitality constraints, for $X, X', X'' \in \mathcal{C}$:

$$F_2(X, X' \otimes_{\mathcal{C}} X'')(\mathsf{Id}_{F(X)} \otimes_{\mathcal{D}} F_2(X', X'')) = F_2(X \otimes_{\mathcal{C}} X', X'')(F_2(X, X') \otimes_{\mathcal{D}} \mathsf{Id}_{F(X'')}),$$
$$F_2(\mathbb{1}_{\mathcal{C}}, X)(F_0 \otimes_{\mathcal{D}} \mathsf{Id}_{F(X)}) = \mathsf{Id}_{F(X)},$$
$$F_2(X, \mathbb{1}_{\mathcal{C}})(\mathsf{Id}_{F(X)} \otimes_{\mathcal{D}} F_0) = \mathsf{Id}_{F(X)}.$$

We call a monoidal functor (F, F_2, F_0) *strong* (resp., *strict*) if F_2 and F_0 are isomorphisms (resp., identity maps) in $\mathcal{D}$. If F is strong monoidal and an equivalence between the underlying categories, we call it a *monoidal equivalence*.

Definition 2.7 (F, F^2, F^0)**.** A *comonoidal functor* $(F, F^2, F^0) : \mathcal{C} \to \mathcal{D}$ consists of

- a functor $F : \mathcal{C} \to \mathcal{D}$,
- a natural transformation $F^2 = \{F^2(X, Y) : F(X \otimes_{\mathcal{C}} Y) \to F(X) \otimes_{\mathcal{D}} F(Y)\}_{X,Y \in \mathcal{C}}$, and

- a morphism $F^0 : F(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$

such that $(F^{\text{op}}, (F^2)^{\text{op}}, (F^0)^{\text{op}}) : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ is a monoidal functor.

Next we introduce a weaker version of strong monoidal functors which will be crucial for our purposes.

Definition 2.8 (F, F_2, F_0, F^2, F^0)**.** A *Frobenius monoidal functor* [DP08, Definition 1] is a tuple (F, F_0, F_2, F^2, F^0) where $(F, F_2, F_0) : \mathcal{C} \to \mathcal{D}$ is a monoidal functor, $(F, F^2, F^0) : \mathcal{C} \to \mathcal{D}$ is a comonoidal functor and for all $X, Y, Z \in \mathcal{C}$, the following holds:

$$\begin{aligned}
(\mathsf{Id}_{F(X)} \otimes_{\mathcal{D}} F_2(Y,Z))\,(F^2(X,Y) \otimes_{\mathcal{D}} \mathsf{Id}_{F(Z)}) &= F^2(X, Y \otimes_{\mathcal{C}} Z) F_2(X \otimes_{\mathcal{C}} Y, Z),\\
(F_2(X,Y) \otimes_{\mathcal{D}} \mathsf{Id}_{F(Z)})\,(\mathsf{Id}_{F(X)} \otimes_{\mathcal{D}} F^2(Y,Z)) &= F^2(X \otimes_{\mathcal{C}} Y, Z) F_2(X, Y \otimes_{\mathcal{C}} Z).
\end{aligned}$$

Remark 2.9. From Definition 2.8, it is clear that, a functor F is Frobenius monoidal if and only if F^{op} is.

Before we introduce the next definition, we need to introduce $\Bbbk$-linear categories.

Definition 2.10. A category $\mathcal{C}$ is called $\Bbbk$-*linear* if the hom spaces $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ are $\Bbbk$-vector spaces for all $X, Y \in \mathcal{C}$ and the composition map

$$\operatorname{Hom}_{\mathcal{C}}(Y,Z) \times \operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{C}}(X,Z) \qquad (X, Y, Z \in \mathcal{C})$$

is a $\Bbbk$-linear map.

Now consider the following definition.

Definition 2.11 (β_2, β_0)**.** Let (F, F_2, F_0, F^2, F^0) be a Frobenius monoidal functor between $\Bbbk$-linear monoidal categories $\mathcal{C}$, $\mathcal{D}$. Then,

- We call F *separable* if it satisfies $F_2(X,Y) \circ F^2(X,Y) = \beta_2 \,\mathsf{Id}_{F(X \otimes Y)}$ for some $\beta_2 \in \Bbbk^{\times}$.
- If in addition, $F^0 \circ F_0 = \beta_0 \,\mathsf{Id}_{\mathbb{1}}$ holds for some $\beta_0 \in \Bbbk^{\times}$, we call F *special*.

Next, we consider functors between braided categories.

Definition 2.12. A monoidal functor (F, F_2, F_0) between braided categories $(\mathcal{C}, c)$ and $(\mathcal{D}, d)$ is called *braided* if it satisfies

$$F_2(Y,X) \circ d_{F(X),F(Y)} = F(c_{X,Y}) \circ F_2(X,Y) \qquad (X, Y \in \mathcal{C}).$$

A braided functor that is also a monoidal equivalence is called a *braided equivalence*. A comonoidal functor (F, F^2, F^0) between braided categories $(\mathcal{C}, c)$ and $(\mathcal{D}, d)$ is called *cobraided* if $(F^{\mathrm{op}}, (F^2)^{\mathrm{op}}, (F^0)^{\mathrm{op}})$ is braided.

The following lemma is straightforward.

Lemma 2.13. *A braided strong monoidal functor is cobraided as well.* □

2.1.2 Monoidal natural transformations

Next, we introduce natural transformation between monoidal functors.

Definition 2.14. Let $F, G : \mathcal{C} \to \mathcal{D}$ be monoidal functors. Then, we call a natural transformations $\alpha : F \Rightarrow G$, a *monoidal natural transformation* if the following condition is satisfied

$$G_2(X, Y) \circ (\alpha_X \otimes_{\mathcal{D}} \alpha_Y) = \alpha_{X \otimes_{\mathcal{C}} Y} \circ F_2(X, Y), \qquad \alpha_{\mathbb{1}} \circ F_0 = G_0.$$

Definition 2.15. Let $F, G : \mathcal{C} \to \mathcal{D}$ be comonoidal functors. A *comonoidal natural transformation* $\alpha : F \to G$ is a natural transformation α such that α^{op} (defined as $\alpha^{\mathrm{op}}_X = (\alpha_X)^{\mathrm{op}}$) is a monoidal natural transformation.

2.2 Module categories

Similar to the way we define modules over a $\Bbbk$-algebras, one can define module categories over monoidal categories. In this section, we recall some basic facts about module categories.

Let $\mathcal{C}$ be a monoidal category.

Definition 2.16 $(\mathcal{M}, \triangleright)$**.** A category $\mathcal{M}$ equipped with a functor $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ (called the *action* of $\mathcal{C}$) and natural isomorphisms

$$(X \otimes Y) \triangleright M \cong X \triangleright (Y \triangleright M), \tag{2.3}$$

$$\mathbb{1} \triangleright M \cong M, \tag{2.4}$$

for all $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$, is called a *left* $\mathcal{C}$*-module category* if the isomorphisms above satisfy certain coherence conditions.

Analogously, one can define right $\mathcal{C}$-module categories and bimodule categories. By a variant of Mac Lane's coherence theorem (see [EGNO16, Remark 7.2.4]), we can (and will) assume that the isomorphisms (2.3) are identity maps.

Example 2.17 $({}_F\mathcal{M}, \triangleright_F)$**.** Let $\mathcal{C}$, $\mathcal{D}$ be monoidal categories and $(\mathcal{M}, \triangleright)$ a left $\mathcal{D}$-module category. Given a strong monoidal functor $F : \mathcal{C} \to \mathcal{D}$, we will denote by $({}_F\mathcal{M}, \triangleright_F)$ the category $\mathcal{M}$ with $\mathcal{C}$-action given by

$$X \triangleright_F M := F(X) \triangleright M, \qquad (X \in \mathcal{C}, M \in \mathcal{M}).$$

Then $({}_F\mathcal{M}, \triangleright_F)$ is a left $\mathcal{C}$-module category. In this case, we call $\mathcal{M}$ a *(F-)twisted left $\mathcal{C}$-module category.*

2.2.1 $\mathcal{C}$-module functors and natural transformations

Next, we define functors and natural transformations that preserve the $\mathcal{C}$-action. Let $(\mathcal{M}, \triangleright_{\mathcal{M}})$ and $(\mathcal{N}, \triangleright_{\mathcal{N}})$ be left $\mathcal{C}$-module categories.

Definition 2.18 (F, s)**.** A *left $\mathcal{C}$-module functor* from $(\mathcal{M}, \triangleright_{\mathcal{M}})$ to $(\mathcal{N}, \triangleright_{\mathcal{N}})$ is a tuple (F, s) where:

- $F : \mathcal{M} \to \mathcal{N}$ is a functor, and
- $s = \{s_{X,M} : F(X \triangleright_{\mathcal{M}} M) \to X \triangleright_{\mathcal{N}} F(M)\}_{X \in \mathcal{C}, M \in \mathcal{M}}$ is a natural isomorphism satisfying

$$s_{X \otimes Y, M} = (\mathsf{Id}_X \triangleright_{\mathcal{N}} s_{Y,M}) \circ s_{X, Y \triangleright M}, \quad s_{\mathbb{1}, M} = \mathsf{Id}_{F(M)} \qquad (X, Y \in \mathcal{C}, M \in \mathcal{M}).$$

One can define right $\mathcal{C}$-module functors and $\mathcal{C}$-bimodule functors in a similar manner.

Definition 2.19. Let $(F, s^F), (G, s^G) : (\mathcal{M}, \triangleright_{\mathcal{M}}) \to (\mathcal{N}, \triangleright_{\mathcal{N}})$ be left $\mathcal{C}$-module functors. A *left $\mathcal{C}$-module natural transformation* is a natural transformation $\eta : F \Rightarrow G$ satisfying

$$(\mathsf{Id}_X \triangleright_{\mathcal{N}} \eta_M) \circ s^F_{X,M} = s^G_{X,M} \circ \eta_{X \triangleright_{\mathcal{M}} M} \qquad (\forall\ X \in \mathcal{C}, M \in \mathcal{M}).$$

2.3 Duality in monoidal categories

In the category of vector spaces, one can define the dual vector space of any vector space. Next, we define a similar notion for monoidal categories.

Definition 2.20 $({}^\vee X, X^\vee, \mathrm{ev}, \mathrm{coev})$**.** A monoidal category is called *rigid* if every object X in $\mathcal{C}$ comes equipped with a left and right dual, i.e., there exist an object ${}^\vee X$ (*left dual*) along with co/evaluation maps

$$\mathrm{ev}_X : {}^\vee X \otimes X \to \mathbb{1}, \quad \mathrm{coev}_X : \mathbb{1} \to X \otimes {}^\vee X$$

and an object $X^\vee$ (*right dual*) with co/evaluation maps

$$\widetilde{\mathrm{ev}}_X : X \otimes X^\vee \to \mathbb{1}, \quad \widetilde{\mathrm{coev}}_X : \mathbb{1} \to X^\vee \otimes X$$

satisfying the following relations:

$$(\mathsf{Id}_X \otimes \mathrm{ev}_X)(\mathrm{coev}_X \otimes \mathsf{Id}_X) = \mathsf{Id}_X = (\widetilde{\mathrm{ev}}_X \otimes \mathsf{Id}_X)(\mathsf{Id}_X \otimes \widetilde{\mathrm{coev}}_X),$$
$$(\mathrm{ev}_X \otimes \mathsf{Id}_{{}^\vee X})(\mathsf{Id}_{{}^\vee X} \otimes \mathrm{coev}_X) = \mathsf{Id}_{{}^\vee X} \qquad (\mathsf{Id}_{X^\vee} \otimes \widetilde{\mathrm{ev}}_X)(\widetilde{\mathrm{coev}}_X \otimes \mathsf{Id}_{X^\vee}) = \mathsf{Id}_{X^\vee}.$$

Remark 2.21. The maps $X \mapsto {}^\vee X$ and $X \mapsto X^\vee$ extend to monoidal equivalences from $\mathcal{C}^{\mathrm{rev}}$ to $\mathcal{C}^{\mathrm{op}}$. We can and will replace $\mathcal{C}$ by an equivalent monoidal category and choose duals in a suitable way to ensure that ${}^\vee(-)$ and $(-)^\vee$ are strict monoidal and mutually inverse to each other (see [Shi15, Lemma 5.4] for further details).

2.3.1 Frobenius monoidal functors and duality

In this section, we see that Frobenius monoidal functors preserve duality.

Say we have a Frobenius monoidal functor (F, F_2, F_0, F^2, F^0) between rigid categories $\mathcal{C}$, $\mathcal{D}$. Then by [DP08, Theorem 2],

$$(F({}^\vee X),\ \overline{\mathrm{ev}}_{F(X)} = F^0 \circ F(\mathrm{ev}_X) \circ F_2({}^\vee X, X),\ \overline{\mathrm{coev}}_{F(X)} = F^2(X, {}^\vee X) \circ F(\mathrm{coev}_X) \circ F_0)$$

is a left dual of $F(X)$ for all $X \in \mathcal{C}$. Thus, by uniqueness of dual objects, we have a unique family of natural isomorphisms

$$\zeta_X^F : F({}^\vee X) \to {}^\vee F(X)$$

called the *duality transformation* of F (see [Shi15, Section 3.1], [NS07, Section 1]). Explicitly, ζ_X^F and its inverse are given by

$$\begin{aligned} \zeta_X^F &= (\overline{\mathrm{ev}}_{F(X)} \otimes \mathsf{Id}_{{}^\vee F(X)}) \circ (\mathsf{Id}_{F({}^\vee X)} \otimes \mathrm{coev}_{F(X)}), \\ (\zeta_X^F)^{-1} &= (\mathrm{ev}_{F(X)} \otimes F({}^\vee X)) \circ (\mathsf{Id}_{{}^\vee F(X)} \otimes \overline{\mathrm{coev}}_{F(X)}). \end{aligned} \tag{2.5}$$

Also, define the natural isomorphism

$$\xi_X^F := {}^\vee((\zeta_X^F)^{-1}) \circ \zeta_{{}^\vee X}^F : F({}^{\vee\vee} X) \to {}^{\vee\vee} F(X). \tag{2.6}$$

Then, we obtain the following result.

Lemma 2.22. *[Shi15, Lemma 3.1] Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be a sequence of Frobenius monoidal functors between rigid monoidal categories. Then, we have that*

$$\zeta_X^{GF} = \zeta_{F(X)}^G \circ G(\zeta_X^F) \quad \textit{and} \quad \xi_X^{GF} = \xi_{F(X)}^G \circ G(\xi_X^F). \tag{2.7}$$

□

2.3.2 Pivotal categories

Given any vector space V, we have a canonical isomorphism $\phi_V : V \to V^{**}$, where V^{**} is the dual of the dual of V. This isomorphism is given be sending $v \in V$ to the evaluation map at v, namely $\phi_V(v)(f) := f(v)$. In this section, we introduce monoidal categories which come equipped with a similar natural isomorphism.

Definition 2.23 ($\mathfrak{p}$)**.** Let $\mathcal{C}$ be a monoidal category equipped with left duals. We call $\mathcal{C}$ *pivotal* if it is further equipped with a natural isomorphism

$$\mathfrak{p} = \{\mathfrak{p}_X : X \to {}^{\vee\vee}X\}_{X \in \mathcal{C}}$$

satisfying

$$\mathfrak{p}_{X \otimes Y} = \mathfrak{p}_X \otimes \mathfrak{p}_Y$$

for all $X, Y \in \mathcal{C}$.

An immediate consequence of the above definition is that in pivotal categories, we have $\mathfrak{p}^{-1}_{{}^\vee X} = {}^\vee\mathfrak{p}_X$. Also, for each object X, ${}^\vee X$ is also a right dual to X with co/evaluation maps

$$\widetilde{\mathrm{coev}}_X := (\mathsf{Id}_{{}^\vee X} \otimes \mathfrak{p}_X^{-1})\mathrm{coev}_{{}^\vee X} \quad \text{and} \quad \widetilde{\mathrm{ev}}_X := \mathrm{ev}_{{}^\vee X}(\mathfrak{p}_X \otimes \mathsf{Id}_{{}^\vee X}).$$

Definition 2.24. The (quantum) dimension of an object $X \in \mathcal{C}$ (with respect to a pivotal structure $\mathfrak{p}$) is defined as the following endomorphism of the unit object.

$$\dim(X) := \dim^{\mathfrak{p}}_{\mathcal{C}}(X) = \widetilde{\mathrm{ev}}_X \circ \mathrm{coev}_X \in \mathrm{End}(\mathbb{1}_{\mathcal{C}}).$$

If $\mathcal{C}$ is $\Bbbk$-linear and $\mathrm{End}(\mathbb{1}_{\mathcal{C}}) \cong \Bbbk$, then $\dim(X)$ is a scalar.

2.3.3 Pivotal functors

Next we introduce functors which preserve pivotal structures. In [NS07], the notion of a strong monoidal functor preserving pivotal structure was introduced. Generalizing it, we give the following definition.

Definition 2.25. A Frobenius monoidal functor $F : (\mathcal{C}, \mathfrak{p}^{\mathcal{C}}) \to (\mathcal{D}, \mathfrak{p}^{\mathcal{D}})$ between pivotal categories is said to *pivotal* if it satisfies

$$\mathfrak{p}^{\mathcal{D}}_{F(X)} = \xi^F_X \circ F(\mathfrak{p}^{\mathcal{C}}_X) \qquad (X \in \mathcal{C}). \tag{2.8}$$

Consider the following result about composition of Frobenius monoidal functors.

Lemma 2.26. *Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be a sequence of Frobenius monoidal functors between rigid monoidal categories. If $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ are pivotal and F, G are pivotal functors, then so is their composition $G \circ F$.*

Proof. By, [DP08, Proposition 4] $G \circ F$ is Frobenius monoidal. Also,

$$\mathfrak{p}^{\mathcal{E}}_{GF(X)} \overset{(2.8)}{=} \xi^{G}_{F(X)} \circ G(\mathfrak{p}^{\mathcal{D}}_{F(X)}) \overset{(2.8)}{=} \xi^{G}_{F(X)} \circ G(\xi^{F}_{X} \circ F(\mathfrak{p}^{\mathcal{C}}_{X})) \overset{(2.7)}{=} \xi^{GF}_{X} \circ GF(\mathfrak{p}^{\mathcal{C}}_{X}).$$

Hence, GF is a pivotal Frobenius functor. □

Next, we see how the dimension of an object X in $\mathcal{C}$ is related to the dimension of $F(X)$ in $\mathcal{D}$ when F is a pivotal Frobenius monoidal functor.

Lemma 2.27. *Let $F : (\mathcal{C}, \mathfrak{p}) \to (\mathcal{D}, \mathfrak{q})$ be a pivotal Frobenius monoidal functor. Then*

$$\dim^{\mathfrak{q}}(F(X)) = F^0\ F(\widetilde{\mathrm{ev}}_X)\ F_2(X, {}^{\vee}X)\ F^2(X, {}^{\vee}X)\ F(\mathrm{coev}_X)\ F_0. \tag{2.9}$$

Proof. This follows from a straightforward calculation using the definition of ζ^F_X. □

Lemma 2.28. *Let $(F, F_2, F_0, F^2, F^0) : \mathcal{C} \to \mathcal{D}$ be a pivotal, special, Frobenius monoidal functor (with constants β_0, β_2) between pivotal monoidal categories $(\mathcal{C}, \mathfrak{p})$ and $(\mathcal{D}, \mathfrak{q})$. If $\mathrm{End}(\mathbb{1}_{\mathcal{C}}) \cong \mathbb{k}$, then,*

$$\dim^{\mathfrak{q}}(F(X)) = \beta_2 \beta_0 \dim^{\mathfrak{p}}(X)\, \mathsf{Id}_{\mathbb{1}_{\mathcal{D}}}.$$

Proof. The following calculation proves the result.

$$\begin{aligned}
\dim^{\mathfrak{q}}(F(X)) &\overset{(2.9)}{=} F^0\ F(\widetilde{\mathrm{ev}}_X)\ F_2(X, {}^{\vee}X)\ F^2(X, {}^{\vee}X)\ F(\mathrm{coev}_X)\ F_0 \\
&= F^0\ F(\widetilde{\mathrm{ev}}_X)\ (\beta_2 \mathsf{Id}_{F(X \otimes {}^{\vee}X)})\ F(\mathrm{coev}_X)\ F_0 \\
&= \beta_2\ F^0\ F(\widetilde{\mathrm{ev}}_X \circ \mathrm{coev}_X)\ F_0 \\
&= \beta_2 \dim^{\mathfrak{p}}(X)\ F^0\ F_0\, \mathsf{Id}_{\mathbb{1}_{\mathcal{D}}} \\
&= \beta_2 \beta_0\ \dim^{\mathfrak{p}}(X)\, \mathsf{Id}_{\mathbb{1}_{\mathcal{D}}}.
\end{aligned}$$

□

2.3.4 Ribbon categories and functors

In this section, we introduce the notion of a ribbon category and a ribbon functor.

Definition 2.29 (θ^l, θ^r)**.** A *ribbon category* $\mathcal{C}$ is a braided pivotal category such that the left and right twists coincide, that is, $\theta^l_X = \theta^r_X$ ($:= \theta_X$) for any object X of C, where

$$\theta^l_X := (\mathrm{ev}_X \otimes \mathsf{Id}_X)(\mathsf{Id}_{{}^{\vee}X} \otimes c_{X,X})(\widetilde{\mathrm{coev}}_X \otimes \mathsf{Id}_X),$$

$$\theta^r_X := (\mathsf{Id}_X \otimes \widetilde{\mathrm{ev}}_X)(c_{X,X} \otimes \mathsf{Id}_{^\vee X})(\mathsf{Id}_X \otimes \mathrm{coev}_X).$$

Definition 2.30. A braided functor $F : \mathcal{C} \to \mathcal{D}$ between ribbon categories is called a *ribbon functor* if it preserves twists, that is, $F(\theta^{\mathcal{C}}_X) = \theta^{\mathcal{D}}_{F(X)}$ for all X in $\mathcal{C}$.

We will need the following result later.

Proposition 2.31. *[Mul22, Proposition 4.4] A braided Frobenius functor $F : \mathcal{C} \to \mathcal{D}$ between ribbon categories is ribbon if and only if it is pivotal and cobraided.* $\square$

2.4 Algebras in monoidal categories

Next, we recall the notion of an algebra in a monoidal category.

2.4.1 Algebras, coalgebras and Frobenius algebras

Definition 2.32 $(m, u, \mathsf{Alg}(\mathcal{C}), \Delta, \nu, \mathsf{Frob}(\mathcal{C}))$**.** Take $\mathcal{C}$ to be a monoidal category, and consider the (categories of) algebraic structures below.

(a) An *algebra* in $\mathcal{C}$ is a triple (A, m, u) consisting of an object $A \in \mathcal{C}$, and morphisms $m : A \otimes A \to A$, $u : \mathbb{1} \to A$ in $\mathcal{C}$, satisfying associativity and unitality constraints: $m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m)$, and $m(u \otimes \mathrm{id}_A) = \mathrm{id}_A = m(\mathrm{id}_A \otimes u)$. A *morphism* of algebras (A, m_A, u_A) and (B, m_B, u_B) is a morphism $f : A \to B$ in $\mathcal{C}$ so that $fm_A = m_B(f \otimes f)$ and $fu_A = u_B$. Algebras and their morphisms in $\mathcal{C}$ form a category, which we denote by $\mathsf{Alg}(\mathcal{C})$.

(b) A *coalgebra* in $\mathcal{C}$ is a triple (A, Δ, ν) consisting of an object $A \in \mathcal{C}$, and morphisms $\Delta : A \to A \otimes A$, $\nu : A \to \mathbb{1}$ in $\mathcal{C}$, satisfying coassociativity and counitality constraints: $(\Delta \otimes \mathrm{id}_A)\Delta = (\mathrm{id}_A \otimes \Delta)\Delta$ and $(\nu \otimes \mathrm{id}_A)\Delta = \mathrm{id}_A = (\mathrm{id}_A \otimes \nu)\Delta$. A *morphism* of coalgebras (A, Δ_A, ν_A) and (B, Δ_B, ν_B) is a morphism $f : A \to B$ in $\mathcal{C}$ so that $\Delta_B f = (f \otimes f)\Delta_A$ and $\nu_B f = \nu_A$. Coalgebras and their morphisms in $\mathcal{C}$ form a category, which we denote by $\mathsf{Coalg}(\mathcal{C})$.

(c) A *Frobenius algebra* in $\mathcal{C}$ is a 5-tuple (A, m, u, Δ, ν) where $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$ and $(A, \Delta, \nu) \in \mathsf{Coalg}(\mathcal{C})$ so that $(m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta) = \Delta m = (\mathrm{id}_A \otimes m)(\Delta \otimes \mathrm{id}_A)$. A *morphism* of Frobenius algebras $f : A \to B$ is a map in $\mathsf{Alg}(\mathcal{C}) \cap \mathsf{Coalg}(\mathcal{C})$. Frobenius algebras and their morphisms in $\mathcal{C}$ form a category, which we denote by $\mathsf{Frob}(\mathcal{C})$.

Definition 2.33. Let $\mathcal{C}$ be a braided monoidal category. Then an algebra $(A, m, u) \in \mathcal{C}$ is called *(braided) commutative* if it satisfies $m\ c_{A,A} = m$. We call a coalgebra (C, Δ, ν) in a $\mathcal{C}$ *cocommutative* is it satisfies $c_{C,C}\ \Delta = \Delta$.

2.4.2 Symmetric, separable and special Frobenius algebras

Definition 2.34. ([FFRS06, Definition 2.22]) A Frobenius algebra (A, m, u, Δ, ν) in a pivotal monoidal category $(\mathcal{C}, \mathfrak{p})$ is called *symmetric* if it satisfies

$$(\nu\ m \otimes \mathsf{Id}_{^\vee A})(\mathsf{Id}_A \otimes \mathrm{coev}_A) = (\mathsf{Id}_A \otimes \nu\ m)(\mathsf{Id}_{^\vee A} \otimes \mathfrak{p}_A^{-1} \otimes \mathsf{Id}_A)(\mathrm{coev}_{^\vee A} \otimes \mathsf{Id}_A). \quad (2.10)$$

Next, we consider some basic results that will be used in the sequel.

Lemma 2.35. *Let (A, m, u, Δ, ν) be a Frobenius algebra in a pivotal category $\mathcal{C}$. Then A is symmetric if and only if the following holds*

$$(\mathrm{ev}_A \otimes \mathsf{Id}_A)(\mathsf{Id}_{^\vee A} \otimes \Delta\ u) = (\mathsf{Id}_A \otimes \mathrm{ev}_{^\vee A})(\mathsf{Id}_A \otimes \mathfrak{p}_A \otimes \mathsf{Id}_{^\vee A})(\Delta\ u \otimes \mathsf{Id}_{^\vee A}).$$

Proof. Observe that, $(\mathsf{Id}_A \otimes \mathrm{ev}_{^\vee A})(\mathsf{Id}_A \otimes \mathfrak{p}_A \otimes \mathsf{Id}_{^\vee A})(\Delta\ u \otimes \mathsf{Id}_{^\vee A})$ is the inverse of $(\mathsf{Id}_A \otimes \nu\ m)(\mathsf{Id}_{^\vee A} \otimes \mathfrak{p}_A^{-1} \otimes \mathsf{Id}_A)(\mathrm{coev}_{^\vee A} \otimes \mathsf{Id}_A)$. Similarly, $(\mathrm{ev}_A \otimes \mathsf{Id}_A)(\mathsf{Id}_{^\vee A} \otimes \Delta\ u)$ is the inverse of $(\nu\ m \otimes \mathsf{Id}_{^\vee A})(\mathsf{Id}_A \otimes \mathrm{coev}_A)$. Thus by (2.10), the claim follows. $\square$

From here on, we assume that our monoidal category $\mathcal{C}$ is $\Bbbk$-linear.

Definition 2.36 $(\beta_A, \beta_{\mathbb{1}})$**.** [FFRS06] Consider a Frobenius algebra (A, m, u, Δ, ν) in a $\Bbbk$-linear monoidal category $\mathcal{C}$.

- If $m \circ \Delta = \beta_A \mathsf{Id}_A$ holds for some $\beta_A \in \Bbbk$, we call the Frobenius algebra *separable*.
- If furthermore, $\nu \circ u = \beta_{\mathbb{1}} \mathsf{Id}_{\Bbbk}$ for some $\beta_{\mathbb{1}} \in \Bbbk$, we call it *special*.

Lemma 2.37. *Let $\mathcal{C}$ be a pivotal monoidal category. Then, $A = X \otimes {^\vee X}$ is a symmetric Frobenius algebra for any object $X \in \mathcal{C}$. Furthermore, A is separable if and only if $\dim({^\vee X}) \neq 0$. If in addition, $\dim(X) \neq 0$, then A is a special Frobenius algebra in $\mathcal{C}$.*

Proof. We define

$$m = \mathsf{Id}_{^\vee X} \otimes \mathrm{ev}_X \otimes \mathsf{Id}_X, \quad u = \mathrm{coev}_X, \quad \Delta = \mathsf{Id}_X \otimes \widetilde{\mathrm{coev}}_X \otimes \mathsf{Id}_{^\vee X}, \quad \nu = \widetilde{\mathrm{ev}}_X$$

With the above maps, the it is straightforward to check the claim. $\square$

Lemma 2.38. *Let (A, m, u, Δ, ν) be a connected, Frobenius algebra in a $\Bbbk$-linear pivotal category $\mathcal{C}$. If $\dim(A) \neq 0$, then A is a special Frobenius algebra in $\mathcal{C}$.*

Proof. As Frobenius algebras are self dual, we have that $\dim(A) = \nu\, m\, \Delta\, u$. Also, m, Δ are maps of left A-modules in $\mathcal{C}$. Thus, $m\, \Delta \in \mathrm{Hom}_A(A, A) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) \cong \Bbbk$. If $m\, \Delta = 0$, then we will have that $\dim(A) = 0$. Therefore, we get that $m\Delta = \beta_2 \mathsf{Id}_A$ for some $\beta_2 \in \Bbbk^{\times}$. Hence, $\dim(A) = \beta_2 \nu u$. Since $\dim(A) \neq 0$, we have that $\nu u = \beta_0 \mathsf{Id}_{\mathbb{1}}$ for some $\beta_0 \in \Bbbk^{\times}$. Thus, A is a special Frobenius algebra. □

2.4.3 Frobenius monoidal functors preserve Frobenius algebras

It is well known that (co)monoidal functors preserve (co)algebras. In this section, we will show that Frobenius monoidal functors preserve Frobenius algebras. The following lemma summarizes the relationship between the various functors and Frobenius algebras that we have defined above.

Lemma 2.39. *Let (F, F_2, F_0, F^2, F^0) be a Frobenius monoidal functor between monoidal categories $\mathcal{C}, \mathcal{D}$. Take (A, m, u, Δ, ν) a Frobenius algebra in $\mathcal{C}$. Then the following statements hold.*

(a) $(F(A),\ F(m)F_2(A, A),\ F(u)F_0,\ F^2(A, A)F(\Delta),\ F^0F(\nu))$ *is a Frobenius algebra in $\mathcal{D}$.*
(b) *If F is separable (resp., special) and A is separable (resp., special) Frobenius algebra, then $F(A)$ is a separable (resp., special) Frobenius algebra.*
(c) *If $\mathcal{C}, \mathcal{D}$ are pivotal categories, F is a pivotal monoidal functor and A is a symmetric Frobenius algebra, then $F(A)$ is symmetric Frobenius as well.*

Proof. Part (a) is [DP08, Corollary 5], part (b) is straightforward, and part (c) is [Mul22, Proposition 4.8]. □

Is it straightforward to see that if $F : \mathcal{C} \to \mathcal{D}$ is a braided (resp., ribbon) functor between $\mathcal{C}, \mathcal{D}$ are braided (resp., ribbon) categories and A is commutative algebra, then $F(A)$ is a commutative algebra.

Lemma 2.40. *Let (F, F_2, F_0) be a braided monoidal functor between braided monoidal categories $\mathcal{C}, \mathcal{D}$. Take (A, m, u) a commutative algebra in $\mathcal{C}$. Then the algebra $(F(A),\ F(m)F_2(A, A),\ F(u)F_0)$ is a commutative algebra in $\mathcal{D}$.* □

2.4.4 Frobenius algebras in abelian rigid monoidal categories

Next, we recall from [WY22] an alternate characterization of Frobenius algebras in multitensor categories as we will soon need it. Let $\mathcal{C}$ be a multitensor category and A

an algebra in $\mathcal{C}$. Then, a *left ideal* I of A is a tuple (I, λ, ϕ) where $\lambda : A \otimes I \to I$ is morphism that satisfies the relations $\lambda(m \otimes \mathsf{Id}_I) = \lambda(\mathsf{Id}_A \otimes \lambda)$, $\lambda(u \otimes \mathsf{Id}_I) = \mathsf{Id}_I$ and $\phi : I \to A$ is a monic map satisfying $\phi\lambda_I = m(\mathsf{Id}_A \otimes \lambda)$. Now, consider the following result.

Theorem 2.41. *[WY22, Theorem 5.3] Let $\mathcal{C}$ be an abelian rigid monoidal category. An algebra (A, m, u) is a Frobenius algebra in $\mathcal{C}$ if and only if there exists a morphism $\nu : A \to \mathbb{1}$ so that if a left or right ideal (I, λ, ϕ) of A factors through* $\ker(\nu)$*, then ϕ is a zero morphism in $\mathcal{C}$.* □

Now, let $\mathcal{C}$ be an abelian rigid monoidal category. Recall that an abelian category is additive, and hence it admits direct sums of objects. This means for any two objects $X_1, X_2 \in \mathcal{C}$, there exists an object $X_1 \oplus X_2 \in \mathcal{C}$ along with maps $\iota_k : X_k \to X_1 \oplus X_2$ and $p_k : X_1 \oplus X_2 \to X_k$ where $k \in \{1, 2\}$ such that the following relations hold,

$$\iota_1 p_1 + \iota_2 p_2 = \mathsf{Id}_{X_1 \oplus X_2} \text{ and } p_j \iota_k = \delta_{j,k} \mathsf{Id}_{X_k} \text{ for } j, k \in \{1, 2\}.$$

It is well known that if (A_i, m_i, u_i) $(i \in 1, 2)$ are algebras in $\mathcal{C}$, then so is $(A_1 \oplus A_2, m_{12}, u_{12})$ with

$$m_{12} = \sum_{k \in \{1,2\}} \iota_k m_k (p_k \otimes p_k) \text{ and } u_{12} = \sum_{k \in \{1,2\}} \iota_k u_k$$

Now consider the following result which strengthens [FRS02, Proposition 3.21].

Proposition 2.42. *Let $\{(A_i, m_i, u_i)\}_{i \in I}$ be a finite set of algebras in a multitensor category $\mathcal{C}$. Then, $A = \oplus_{i \in I} A_i$ is a Frobenius algebra in $\mathcal{C}$ if and only if each A_i is a Frobenius algebra in $\mathcal{C}$.*

Proof. It suffices to prove the result when $|I| = 2$, that is, when $A = A_1 \oplus A_2$.

($\Leftarrow$) If $(A_k, m_k, u_k, \Delta_k, \nu_k) \in \mathsf{Frob}(\mathcal{C})$ for $k \in \{1, 2\}$, then by [FRS02, Proposition 3.21], we have that the algebra $(A_1 \oplus A_2, m_{12}, u_{12}, \Delta_{12}, \nu_{12}) \in \mathsf{Frob}(\mathcal{C})$ with

$$\Delta_{12} = \sum_{k \in \{1,2\}} (\iota_k \otimes \iota_k) \Delta_k p_k \text{ and } \nu_{12} = \sum_{k \in \{1,2\}} \nu_k p_k.$$

($\Rightarrow$) By assumption, $(A = A_1 \oplus A_2, m_{12}, u_{12})$ is a Frobenius algebra in $\mathcal{C}$. Then, by Theorem 2.41, there exists a morphism $\nu : A \to \mathbb{1}$ such that $\ker(\nu)$ contains no left or right ideal of A. Set $\nu_k = \nu \circ \iota_k$ for $k \in \{1, 2\}$. In order to prove that A_1 and A_2 are Frobenius, we will show that $\ker(\nu_k)$ contains no left or right ideal of A_k for $k \in \{1, 2\}$.

Suppose that the algebra A_1 admits a left ideal (I, λ_1, ϕ_1) that factors through $\ker(\nu_1)$. Let $\iota : \ker(\nu_1) \to A_1$ denote the inclusion map. Then, there exists a map $f : I \to \ker(\nu_1)$ such that the lower triangle (†) in the following diagram commutes.

$$\begin{tikzcd}
& \ker(\nu) \arrow[dr] & & \\
\ker(\nu_1) \arrow[r, "\iota"] \arrow[ur, dotted, "g"] \arrow[dr, phantom, "(\dagger)"] & A_1 \arrow[r, "\iota_1"] \arrow[u, phantom, "(\ddagger)"] & A_1 \oplus A_2 \arrow[r, "\nu"] & \mathbb{1} \\
I \arrow[u, "f"] \arrow[ur, "\phi"'] & & &
\end{tikzcd}$$

As the composition $\nu\, \iota_1\, \iota$ is equal to the zero map, by the universal property of the kernel $\ker(\nu)$, there exists a unique map g making the diagram (‡) commute. Now consider the tuple $(I, (\iota_1 \otimes \mathsf{Id}_I)\lambda, \iota_1\phi)$. As both ι_1 and ϕ are monic, their composition $\iota_1\, \phi$ is monic as well. A straightforward check shows that $(I, (\iota_1 \otimes \mathsf{Id}_I)\lambda, \iota_1\phi)$ is a left ideal of $A_1 \oplus A_2$ that factors through $\ker(\nu)$. This contradicts the assumption that $A_1 \oplus A_2$ is Frobenius by Theorem 2.41.

In a similar manner we can show that A_1 does not admit a right ideal that factors through $\ker(\nu_1)$. Repeating the same argument for A_2 we conclude that both A_1 and A_2 are Frobenius. □

2.5 Adjunctions

This section is dedicated to recalling adjoint functors and their generalizations to monoidal categories.

2.5.1 Adjoint functors

We will use the following notations going forward.

Notation 2.43. Given a functor $F : \mathcal{C} \to \mathcal{D}$, $1_F : F \Rightarrow F$ will denote the identity natural transformation. We denote horizontal composition of natural transformations $\alpha : F \Rightarrow G$ and $\beta : F' \Rightarrow G'$ as $\beta \circ \alpha : F' \circ F \Rightarrow G' \circ G$. Moreover, the vertical composition of $\alpha : F \Rightarrow G$ and $\alpha' : G \Rightarrow H$ is denoted as $\alpha' \cdot \alpha$.

Consider the following definition.

Definition 2.44 $(L, R, \dashv, (-)^{\mathsf{ra}}, (-)^{\mathsf{la}})$**.** An *adjoint pair* $(L, R, \eta, \varepsilon)$ is a pair of functors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ between categories $\mathcal{C}$ and $\mathcal{D}$ along with natural transformations $\eta = \{\eta_X : X \to RL(X)\}_{X \in \mathcal{C}}$ and $\varepsilon = \{\varepsilon_Y : LR(Y) \to Y\}_{Y \in \mathcal{D}}$ such that the following equations are satisfied.

$$R(\varepsilon_Y)\,\eta_{R(Y)} = \mathsf{Id}_{R(Y)} \qquad \varepsilon_{L(X)}\,L(\eta_X) = \mathsf{Id}_{L(X)}. \tag{2.11}$$

In this situation, we call L the *left adjoint* of R and denote it as R^{la}. Similarly, we call R the *right adjoint* of L and denote it as L^{ra}. We will also use the notation $L \dashv R$ to mean the L, R form an adjoint pair with L as the left adjoint.

Remark 2.45. We have that $L \dashv R$ if and only if $R^{\mathrm{op}} \dashv L^{\mathrm{op}}$.

2.5.2 (Co)Monoidal adjunctions

Now, suppose that $\mathcal{C}$, $\mathcal{D}$ are monoidal categories. The, we obtain the following notion.

Definition 2.46. [BLV11, Section 2.5] A *(co)monoidal adjunction* is an adjunction $L \dashv R$, between monoidal categories such that L, R are (co)monoidal functors, and the unit, counit of the adjunction are (co)monoidal natural transformations.

2.5.3 (co)Hopf adjunctions

We first recall the notion of (co)Hopf adjunctions from [BLV11, Section 2.8]. Suppose that we have two functors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ between monoidal categories $\mathcal{C}$ and $\mathcal{D}$.

Definition 2.47. A *Hopf adjunction* is a comonoidal adjunction $L \dashv R$ such that the following *Hopf operator morphisms* are invertible for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

$$\begin{aligned} H^l_{X,Y} &: L(X \otimes R(Y)) \xrightarrow{L^2(X,R(Y))} L(X) \otimes LR(Y) \xrightarrow{\mathsf{Id}_{L(X)} \otimes \varepsilon_Y} L(X) \otimes Y, \\ H^r_{Y,X} &: L(R(Y) \otimes X) \xrightarrow{L^2(R(Y),X)} LR(Y) \otimes L(X) \xrightarrow{\varepsilon_Y \otimes \mathsf{Id}_{L(X)}} Y \otimes L(X). \end{aligned} \tag{2.12}$$

In this thesis, we will primarily work with the following dual notion.

Definition 2.48. A *coHopf adjunction* is a monoidal adjunction $L \dashv R$ such that the following *coHopf operator morphisms* are invertible for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

$$\begin{aligned} h^l_{Y,X} &: R(Y) \otimes X \xrightarrow{\mathsf{Id}_{R(Y)} \otimes \eta_X} R(Y) \otimes RL(X) \xrightarrow{R_2(Y,L(X))} R(Y \otimes L(X)), \\ h^r_{X,Y} &: X \otimes R(Y) \xrightarrow{\eta_X \otimes \mathsf{Id}_{R(Y)}} RL(X) \otimes R(Y) \xrightarrow{R_2(L(Y),X)} R(L(X) \otimes Y). \end{aligned} \tag{2.13}$$

Remark 2.49. From the definitions, it is clear that $L \dashv R$ is a coHopf adjunction if and only if $R^{\mathrm{op}} \dashv L^{\mathrm{op}}$ is a Hopf adjunction.

Now consider the following results.

Theorem 2.50. *Let $L : \mathcal{C} \to \mathcal{D}$ be left adjoint to $R : \mathcal{D} \to \mathcal{C}$ with unit η and counit ε. Then the following hold:*

(a) *If L is strong monoidal with structure maps L^2, L^0, then R admits the following monoidal structure making the adjunction $L \dashv R$ monoidal:*

$$R_2(Y, Y') = R(\varepsilon_Y \otimes \varepsilon_{Y'}) \circ R(L^2(R(Y), R(Y'))) \circ \eta_{R(Y) \otimes R(Y')}, \quad R_0 = R(L^0) \circ \eta_{\mathbb{1}_{\mathcal{C}}}.$$

(b) *If $\mathcal{C}, \mathcal{D}$ are rigid, then any comonoidal adjunction between them is Hopf and any monoidal adjunction is coHopf.*

Proof. Part (a) is a classical result of Kelly [Kel74, Section 2.1]. Part (b) follows from [BLV11, Proposition 3.5]. □

2.6 Finite multitensor categories and their module categories

In this section, we introduce multitensor categories and module categories over them.

Notation 2.51. For a $\Bbbk$-algebra A, $\mathsf{Rep}(A)$ will denote the category of finite dimensional left A-modules over $\Bbbk$.

A *finite abelian category* is a $\Bbbk$-linear category that is equivalent to $\mathsf{Rep}(A)$ for some finite dimensional $\Bbbk$-algebra A.

Definition 2.52. A *finite multitensor category* is a rigid monoidal category that is finite abelian and the tensor product functor $\otimes$ is $\Bbbk$-linear in each variable. If further the unit object $\mathbb{1}$ is simple, we call it a *finite tensor category.* A *tensor functor* is an exact, faithful, $\Bbbk$-linear, strong monoidal functor between finite multitensor categories.

Definition 2.53. Let $\mathcal{C}$ be a finite multitensor category. A *finite left $\mathcal{C}$-module category* is a left $\mathcal{C}$-module category $\mathcal{M}$ such that $\mathcal{M}$ is a finite abelian category and the action of $\mathcal{C}$ on $\mathcal{M}$ is $\Bbbk$-bilinear and right exact in each variable. Such a category is called *exact* if for all objects $M \in \mathcal{M}$ and all projective objects $P \in \mathcal{C}$, $P \triangleright M$ is projective in $\mathcal{M}$. A module category is called *indecomposable* if it is not equivalent to a direct sum of two non-trivial module categories.

Notation 2.54. We will use the following notations going forward.

(a) For $\mathcal{M}, \mathcal{N}$ two finite left $\mathcal{C}$-module categories, $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ will denote the category of right exact $\mathcal{C}$-module functors from $\mathcal{M}$ to $\mathcal{N}$. If $\mathcal{M} = \mathcal{N}$, we call it $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$.

(b) Given $(F, s^F), (G, s^G)$ two left $\mathcal{C}$-module functors, we will use the notation $F \cong_{\mathcal{C}} G$ to mean that F and G are isomorphic as left $\mathcal{C}$-module functors.

(c) For $X \in \mathcal{C}$ and $F \in \mathsf{Rex}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$, we use the notation $X \triangleright F$ to denote the functor from $\mathcal{M}$ to $\mathcal{N}$ defined as follows: $(X \triangleright F)(M) = X \triangleright F(M)$.

2.6.1 Internal Homs

We refer the reader to [EGNO16, Section 7.4] for a more detailed exposition. Let $\mathcal{C}$ be a monoidal category and $(\mathcal{M}, \triangleright)$ be a left $\mathcal{C}$-module category. Consider the functor

$$Y_M : \mathcal{C} \to \mathcal{M}, \qquad X \mapsto X \triangleright M.$$

If Y_M admits a right adjoint, we will denote it by

$$\underline{\mathrm{Hom}}(M, -) := \underline{\mathrm{Hom}}^{\mathcal{C}}_{\mathcal{M}}(M, -) : \mathcal{M} \to \mathcal{C}$$

and call $\underline{\mathrm{Hom}}(M, N)$ as the *internal Hom* of M and N. In this case, by definition of adjoint functors, we get the following isomorphism of hom spaces:

$$\mathrm{Hom}_{\mathcal{M}}(X \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M, N)) \tag{2.14}$$

For $\mathcal{C}$ a finite multitensor category and $\mathcal{M}$ a finite left $\mathcal{C}$-module category, the functor Y_M is a right exact functor between finite abelian categories. Hence, it admits a right adjoint, that is, internal Homs exist. In fact, the internal Hom extends to a functor

$$\underline{\mathrm{Hom}}(-, -) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{C}$$

in such a way that the isomorphism (2.14) is natural in both M and N [Shi20, §2.4].

2.6.2 Canonical Vec action

Let Vec denote the category of finite dimensional $\Bbbk$-vector spaces. Every finite abelian category $\mathcal{M}$ comes equipped with a canonical action of Vec given by

$$\blacktriangleright : \mathsf{Vec} \times \mathcal{M} \to \mathcal{M}$$

and defined via the following isomorphism.

$$\mathrm{Hom}_{\mathcal{M}}(V \blacktriangleright M, N) \cong \mathrm{Hom}_{\Bbbk}(V, \mathrm{Hom}_{\mathcal{M}}(M, N))$$

With this action, every finite abelian category becomes a left Vec-module category. From here on, we will use the notation $\blacktriangleright$ to denote this action of Vec on any finite abelian category.

2.6.3 Relative Serre functors

Let $\mathcal{C}$ be a finite multitensor category and $\mathcal{M}$ a finite left $\mathcal{C}$-module category. A *relative (right) Serre functor* [FSS20, Definition 4.22] of $\mathcal{M}$ is a pair $(\mathbb{S}, \phi)$ where

$$\mathbb{S} := \mathbb{S}^{\mathcal{C}}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$$

is a functor and

$$\phi := \phi^r = \{\phi^r_{M,M'} : {}^{\vee}\underline{\mathrm{Hom}}(M, M') \to \underline{\mathrm{Hom}}(M', \mathbb{S}(M))\}_{M,M' \in \mathcal{M}}$$

is a natural isomorphism. If we want to emphasize the categories $\mathcal{C}, \mathcal{M}$, we will write $\mathbb{S}^{\mathcal{C}}_{\mathcal{M}}$ instead of $\mathbb{S}$. Similarly, a *relative (left) Serre functor* is an endofunctor $\overline{\mathbb{S}}$ of $\mathcal{M}$ together with a natural isomorphism

$$\phi^l = \{\phi^l_{M,M'} : \underline{\mathrm{Hom}}(M, M')^{\vee} \to \underline{\mathrm{Hom}}(\overline{\mathbb{S}}(M'), M)\}_{M,M' \in \mathcal{M}}.$$

We refer the reader to the works [Sch15, FSS20, Shi19] for further details. Below, we recall some important results about Serre functors.

Theorem 2.55. *[FSS20, Lemma 4.23, Proposition 4.24] Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ a finite left $\mathcal{C}$-module category.*

(a) *$\mathcal{M}$ is exact if and only if a relative Serre functor $\mathbb{S}$ exists.*

(b) *If $\mathcal{M}$ is exact, $\mathbb{S}$ is a category equivalence with quasi-inverse $\overline{\mathbb{S}}$.*

(c) *If $\mathcal{M}$ is exact, $\mathbb{S}$ is a $\mathcal{C}$-module functor from $\mathcal{M} \to {}_{\vee\vee(-)}\mathcal{M}$, that is, we have natural isomorphisms $\mathfrak{s}_{X,M} : \mathbb{S}(X \triangleright M) \xrightarrow{\sim} {}^{\vee\vee}X \triangleright \mathbb{S}(M)$ satisfying the module compatibility conditions.* □

2.6.4 Nakayama functors

We assume that the reader is familiar with ends and coends, see [ML13, IX] for details. For $\mathcal{M}$ a finite abelian category, define the *left, right Nakayama functors* [FSS20, Definition 3.14] to be the endofunctors $\overline{\mathbb{N}}, \mathbb{N} : \mathcal{M} \to \mathcal{M}$, respectively, given by

$$\overline{\mathbb{N}}_{\mathcal{M}}(M) = \int_{M' \in \mathcal{M}} \mathrm{Hom}_{\mathcal{M}}(M', M) \blacktriangleright M', \qquad \mathbb{N}_{\mathcal{M}}(M) = \int^{M' \in \mathcal{M}} \mathrm{Hom}_{\mathcal{M}}(M, M')^* \blacktriangleright M'$$

If the category $\mathcal{M}$ is clear from the context, we will often drop it from the subscript. Below we collect some properties of the Nakayama functor that will be needed later.

Theorem 2.56. *[FSS20, Theorem 4.5, Corollary 4.7] Let $\mathcal{C}$ be a finite multitensor category and $\mathcal{M}$ a finite left $\mathcal{C}$-module category. Then,*

(a) *$\mathbb{N}$ is a left $\mathcal{C}$-module functor from $\mathcal{M} \to {}_{(-)^{\vee\vee}}\mathcal{M}$, that is, we have natural isomorphisms $\mathfrak{n}_{X,M} : \mathbb{N}(X \triangleright M) \xrightarrow{\sim} X^{\vee\vee} \triangleright \mathbb{N}(M)$ satisfying the module compatibility conditions.*

(b) *If $\mathcal{M}$ is exact, then $\mathbb{N}$ is a category equivalence with quasi-inverse $\overline{\mathbb{N}}$.* □

Remark 2.57. Theorems 2.55 and 2.56 were proved in [FSS20] under the assumption that $\mathcal{C}$ is a finite tensor category. However, the same arguments work for multitensor categories.

2.6.5 Unimodular multitensor categories

Let $\mathcal{C}$ be a finite multitensor category over $\Bbbk$.

Definition 2.58. The *distinguished invertible object* of $\mathcal{C}$, denoted $D_{\mathcal{C}}$, is defined as the object $\overline{\mathbb{N}}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}) \in \mathcal{C}$. We call $\mathcal{C}$ *unimodular* if $D_{\mathcal{C}} \cong \mathbb{1}$.

Remark 2.59. By assumption, $\Bbbk$ is algebraically closed, and hence perfect. Thus, we can employ results from [Shi16] about the distinguished invertible object. In particular, by [Shi16, Lemma 5.1], we get that the above definition of the object $D_{\mathcal{C}}$ matches the one given in [ENO04, EGNO16].

As $\mathcal{C}$ is a multitensor category, following [ENO04, §4.3], we can write $\mathbb{1} = \oplus_{i \in I} \mathbb{1}_i$ for some finite set I, where the objects $\mathbb{1}_i$ are simple and pairwise non-isomorphic. The categories $\mathcal{C}_{ij} := \mathbb{1}_i \otimes \mathcal{C} \otimes \mathbb{1}_j$ are called as the *component subcategories* of $\mathcal{C}$ and we have that $\mathcal{C} = \oplus_{i,j \in I} \mathcal{C}_{ij}$. Here, for all $i \in I$, $\mathcal{C}_{ii}$ is a finite tensor category with unit object $\mathbb{1}_i$. Now, consider the following result.

Lemma 2.60. *We have that $\mathcal{C}$ is unimodular if and only if all its component subcategories $\mathcal{C}_{ii}$ are unimodular.*

Proof. By [Shi16, Theorem 5.3], $D_{\mathcal{C}} = \oplus_{i \in I} D_{\mathcal{C}_{ii}}$. If each $\mathcal{C}_{ii}$ is unimodular, then $\mathcal{D}_{\mathcal{C}_{ii}} \cong \mathbb{1}_i$. Hence,

$$\mathbb{1}_{\mathcal{C}} = \oplus_{i \in I} \mathbb{1}_i = \oplus_{i \in I} D_{\mathcal{C}_{ii}} \cong D_{\mathcal{C}}.$$

Thus, $\mathcal{C}$ is unimodular. Conversely, if $\mathcal{C}$ is unimodular, then we get that $\oplus_{i \in I} \mathbb{1}_i = \oplus_{i \in I} D_{\mathcal{C}_{ii}}$. Tensoring both sides with $\mathbb{1}_j$, we get that $D_{\mathcal{C}_{jj}} \cong \mathbb{1}_j$. Hence, each one of the categories $\mathcal{C}_{ii}$ is unimodular. □

Every finite tensor category $\mathcal{C}$ comes equipped with a natural isomorphism

$$\mathfrak{R} = \{\mathfrak{R}_X : {}^{\vee\vee}X \otimes D \to D \otimes X^{\vee\vee}\}_{X\in\mathcal{C}} \tag{2.15}$$

called the *Radford isomorphism* of $\mathcal{C}$ [ENO04]. The following results provides a relation between the relative Serre functor and the Nakayama functor of an exact $\mathcal{C}$-module category $\mathcal{M}$.

Theorem 2.61. [FSS20, Theorem 4.26] *Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ be an exact left $\mathcal{C}$-module category. Then, $\mathbb{S}_{\mathcal{M}} \cong_{\mathcal{C}} D \triangleright \mathbb{N}_{\mathcal{M}}$ as a twisted $\mathcal{C}$-module functor and its $\mathcal{C}$-module constraints are given by*

$$\mathfrak{s}_{X,M} := D \triangleright \mathbb{N}(X \triangleright M) \xrightarrow{\mathsf{Id} \triangleright \mathfrak{n}_{X,M}} D \triangleright (X^{\vee\vee} \triangleright \mathbb{N}(M)) \xrightarrow{\mathfrak{R}_X^{-1} \triangleright \mathsf{Id}} {}^{\vee\vee}X \triangleright (D \triangleright \mathbb{N}(M))$$

for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$. □

2.6.6 Drinfeld centers

Let $\mathcal{C}$ be a monoidal category. Then the *Drinfeld center* of $\mathcal{C}$, denoted $\mathcal{Z}(\mathcal{C})$, is defined as the category with objects as pairs (X, σ), where X is an object in $\mathcal{C}$, and

$$\sigma = \{\sigma_Y : Y \otimes X \to X \otimes Y\}_{Y\in\mathcal{C}}$$

is a natural isomorphism (called a *half-braiding*) satisfying

$$\sigma_{Y\otimes Z} = (\sigma_Y \otimes \mathsf{Id}_Z)(\mathsf{Id}_Y \otimes \sigma_Z), \qquad (Y, Z \in \mathcal{C}).$$

Morphisms $(X, \sigma) \to (Y, \sigma')$ are given by $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ satisfying

$$(f \otimes \mathsf{Id}_Z)\sigma_Z = \sigma'_Z(\mathsf{Id}_Z \otimes f).$$

The monoidal product is given by $(X, \sigma) \otimes (Y, \sigma') = (X \otimes Y, \gamma)$ where

$$\gamma_Z := (\mathsf{Id}_X \otimes \sigma'_Z)(\sigma_Z \otimes \mathsf{Id}_Y).$$

We have the forgetful functor

$$U := U_{\mathcal{C}} : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}, \qquad (X, \sigma) \mapsto X,$$

which is strong monoidal. Drinfeld centers are important because they are braided monoidal categories with braiding $c_{(X,\sigma),(Y,\sigma')} := \sigma'_X$. Furthermore, if $\mathcal{C}$ is a (finite) tensor category, then $\mathcal{Z}(\mathcal{C})$ is a braided (finite) tensor category and $U_{\mathcal{C}}$ is a $\Bbbk$-linear, exact functor. The map $(X, \sigma) \mapsto (X, \sigma^{-1}) : \mathcal{Z}(\mathcal{C}^{\mathrm{rev}}) \to \mathcal{Z}(\mathcal{C})$ is a canonical equivalence of monoidal categories [EGNO16, Exercise 7.13.5]. If we take the braidings into account as well, then this is a braided equivalence $\mathcal{Z}(\mathcal{C}^{\mathrm{rev}}) \to \mathcal{Z}(\mathcal{C})^{\mathrm{mir}}$ [EGNO16, Exercise 8.5.2].

Let $\mathcal{C}$ be a finite tensor category. Then, the forgetful functor $U_{\mathcal{C}}$ admits a right adjoint $R_{\mathcal{C}}$, which is monoidal. The following theorem collects important known results which we will need later.

Theorem 2.62. *Let $\mathcal{C}$ be a finite tensor category. Then, we get the following results.*

(a) *The algebra $R_{\mathcal{C}}(\mathbb{1})$ in $\mathcal{Z}(\mathcal{C})$ is commutative.*
(b) *The algebra $R_{\mathcal{C}}(\mathbb{1})$ is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$ if and only if $\mathcal{C}$ is unimodular.*
(c) *If $\mathcal{C}$ is pivotal and unimodular, then $R_{\mathcal{C}}(\mathbb{1})$ is a symmetric Frobenius algebra in $\mathcal{Z}(\mathcal{C})$.*

Proof. Parts (a) follow from [BN11, Proposition 6.1] and part (b) from [Shi16, Theorem 5.6(3)]. Part (c) is [SW22, Lemma 7.1]. $\square$

Chapter 3

Frobenius monoidal functors from coHopf adjunctions

The chapter is based on our article [Yad22]. The goal of this chapter is to give the right adjoint functor R in a coHopf adjunctions $U \dashv R$ the structure of a separable, special, pivotal, ribbon Frobenius monoidal functors. To do this, we start by recalling results from [Bal17] which established sufficient conditions under which the right adjoint R is Frobenius monoidal (see Theorem 3.5). This and some preliminaries are discussed in Section 3.1. After that, Balan's result is generalized to the separable and special Frobenius setting in Section 3.2, to the pivotal setting in Section 3.3, and to the ribbon setting in Section 3.4.

3.1 Preliminaries

In this following, we will often replace $\otimes$ by $\cdot$, in order to fit equations. Equalities marked as (N) will commute because of naturality. Consider the following conditions on a functor $U : \mathcal{C} \to \mathcal{D}$, which will be used throughout the rest of this section.

Condition 3.1. $(U : \mathcal{C} \to \mathcal{D}, U^2, U^0)$ is a strong monoidal functor between abelian monoidal categories admitting a right adjoint R (with unit η^r, counit ε^r) such that:

(a) $U \dashv R$ is a coHopf adjunction,
(b) R is exact, and
(c) R is faithful.

Remark 3.2. Under these conditions, R is monoidal with structure maps R_2, R_0 in Theorem 2.50(a) making $U \dashv R$ a monoidal adjunction. Hence, condition (a) is satisfied when $\mathcal{C}, \mathcal{D}$ are rigid [Theorem 2.50(b)]. In other words, the left and right coHopf operators h^l, h^r of (2.13) are invertible in this case.

Next, we collect a few consequences of Condition 3.1 which we will need in the following.

Lemma 3.3. *Suppose that we have functors $U : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ between monoidal categories. Then, we get the following results.*

(a) *If $R \dashv U$ is a Hopf adjunction, then $((R(\mathbb{1}_\mathcal{D}), \sigma), R^2(\mathbb{1}, \mathbb{1}), R^0)$ is a cocommutative coalgebra in $\mathcal{Z}(\mathcal{C})$ with half-braiding $\rho_X = H^l_{\mathbb{1},X}(H^r_{X,\mathbb{1}})^{-1} : X \otimes R(\mathbb{1}) \to R(\mathbb{1}) \otimes X$.*

(b) *If $U \dashv R$ is a coHopf adjunction, then $((R(\mathbb{1}_\mathcal{D}), \sigma), R_2(\mathbb{1}, \mathbb{1}), R_0)$ is a commutative algebra in $\mathcal{Z}(\mathcal{C})$ with half-braiding $\sigma_X = (h^l_{\mathbb{1},X})^{-1} h^r_{X,\mathbb{1}} : X \otimes R(\mathbb{1}) \to R(\mathbb{1}) \otimes X$.*

Proof. Part (a) follows from [BLV11, Corollary 6.7]. For part (b), apply part (a) to the Hopf adjunction $R^{\mathrm{op}} \dashv U^{\mathrm{op}}$. $\square$

Consider an adjoint pair $L \dashv L^{\mathsf{ra}}$ with $L : \mathcal{A} \to \mathcal{B}$. Let ε be the counit and $T = L^{\mathsf{ra}} L$ be the corresponding monad on $\mathcal{A}$ [ML13, Chapter 6]. Then, there is a unique functor $K : \mathcal{B} \to \mathcal{A}^T$, where $\mathcal{A}^T$ is the category of T-modules. The adjunction $L \dashv L^{\mathsf{ra}}$ is called *premonadic* if the functor K is full and faithful. Dually, for a functor U, an adjunction $U^{\mathsf{la}} \dashv U$ is called *precomonadic* if $U^{\mathrm{op}} \dashv (U^{\mathsf{la}})^{\mathrm{op}}$ is premonadic. By [BW00, Corollary 3.9 and Theorem 3.11], the following are equivalent:

(a) $L \dashv L^{\mathsf{ra}}$ is precomonadic;

(b) ε_X is the cokernel of some parallel pair of morphisms $\forall\ X \in \mathcal{B}$;

(c) $$LL^{\mathsf{ra}}LL^{\mathsf{ra}}(X) \underset{\varepsilon_{LL^{\mathsf{ra}}(X)}}{\overset{LL^{\mathsf{ra}}(\varepsilon_X)}{\rightrightarrows}} LL^{\mathsf{ra}}(X) \xrightarrow{\varepsilon_X} X, \text{ is a coequalizer } \forall\ X \in \mathcal{B}. \tag{3.1}$$

The following lemma is probably well-known, but we could not find a proof.

Lemma 3.4. *Adjunctions $L \dashv R : \mathcal{B} \to \mathcal{A}$ between abelian categories satisfy that:*

$$L \dashv R \textit{ is precomonadic} \iff \varepsilon_X \textit{ is epic } \forall\ X \in \mathcal{B} \iff R \textit{ is faithful.}$$

Proof. By the definition of an abelian category, a morphism f in it is an epimorphism if and only if it is a cokernel of some parallel pair of morphisms. Thus, using the equivalent definition of being precomonadic given in (3.1), we get that ε_X is an epimorphism if and only if $L \dashv R$ is precomonadic. Lastly, ε_X is epic if and only if R is faithful [ML13, Chapter 4]. Thus, the proof is finished. $\square$

Now consider the following result connecting the Frobenius properties of the right adjoint R and the algebra $R(\mathbb{1})$.

Theorem 3.5. *Suppose that Condition 3.1 is satisfied. Then, R is Frobenius monoidal if and only if the algebra $R(\mathbb{1})$ is a Frobenius algebra in $\mathcal{C}$.*

In particular, when $R(\mathbb{1})$ is Frobenius with comultiplication Δ and counit ν, then we get the following results:

(a) *R is Frobenius monoidal with the following comonoidal constraints:*
$$R^2(X,Y) = (h^l_{X,R(Y)})^{-1} \circ R(\mathsf{Id}_X \otimes \eta^l_Y), \qquad R^0 = \nu.$$

(b) *$R \dashv U$ with the counit ε^l defined as the following composition:*
$$\varepsilon^l(X) : RU(X) = R(\mathbb{1} \otimes U(X)) \xrightarrow{(h^l_{\mathbb{1},X})^{-1}} R(\mathbb{1}) \otimes X \xrightarrow{\nu \otimes \mathsf{Id}_X} X,$$
and the unit $\eta^l : X \to UR(X)$ as the unique morphism making the following equation hold:
$$\eta^l_X \circ \varepsilon^r_X = UR(\varepsilon^r_X) \circ \varepsilon^r_{URUR(X)} \circ U(h^l_{\mathbb{1},RUR(X)}) \circ U(\mathsf{Id}_{R(\mathbb{1})} \cdot h^l_{\mathbb{1},R(X)}) \circ U(\Delta u \cdot \mathsf{Id}_{R(X)}). \tag{3.2}$$

Also, the adjunction $(R \dashv U, \eta^l, \varepsilon^l)$ is a Hopf adjunction.

Proof. ($\Rightarrow$): Suppose that R is Frobenius monoidal. Then, since $\mathbb{1}$ is Frobenius, $R(\mathbb{1})$ is Frobenius by Lemma 2.39(a).

($\Leftarrow$): By Condition 3.1(b), $U \dashv R$ is a coHopf adjunction, which implies that $R^{\mathrm{op}} \dashv U^{\mathrm{op}}$ is Hopf adjunction. Also, $R(\mathbb{1})$ is a Frobenius algebra in $\mathcal{C}$; this implies that $R^{\mathrm{op}}(\mathbb{1})$ is a Frobenius algebra in $\mathcal{C}^{\mathrm{op}}$. As R is faithful, by Lemma 3.4, we get that R is premonadic. Hence, R^{op} is precomonadic. Thus, we can apply [Bal17, Proposition 4.5] to the adjunction $R^{\mathrm{op}} \dashv U^{\mathrm{op}}$ to get that R^{op} is Frobenius monoidal. Hence, R is Frobenius monoidal.

The claim about the expressions for $\varepsilon^l, \eta^l, R^2, R^0$ follows by translating the expressions in [Bal17] into our setting. The claim about $R \dashv U$ being a Hopf adjunction follows from [Bal17, Theorem 4.4]. □

The next properties of the coHopf operators h^l of (2.13) will be used in the following sections.

Lemma 3.6. *The coHopf operate h^l satisfies the following relations:*

(a) $R_2(X, Y \otimes UR(Z)) \circ (\mathsf{Id}_{R(X)} \otimes h^l_{Y,R(Z)}) = h^l_{X\otimes Y,R(Z)} \circ (R_2(X,Y) \otimes \mathsf{Id}_{R(Z)})$,

(b) $\varepsilon^r_{X\otimes UR(Y)} \circ U(h^l_{X,R(Y)}) = (\varepsilon^r_X \otimes \mathsf{Id}_{UR(Y)}) \circ U_2^{-1}(R(X), R(Y))$,

(c) $R(\mathsf{Id}_X \otimes \varepsilon^r_Y) \circ h^l_{X,R(Y)} = R_2(X,Y)$,

(d) $R(\mathsf{Id}_X \otimes U_2^{-1}(Y,Z)) \circ h^l_{X,Y\otimes Z} = h^l_{X\otimes U(Y),Z} \circ (h^l_{X,Y} \otimes \mathsf{Id}_Z)$.

Proof. We prove part (a) here, parts (b), (c) and (d) are proved in a similar manner.

$$
\begin{aligned}
\text{LHS} &= R_2(X, Y \otimes UR(Z)) \circ (\mathsf{Id}_{R(X)} \otimes h^l_{Y,R(Z)}) \\
&\overset{(2.13)}{=} R_2(X, Y \otimes UR(Z)) \circ (\mathsf{Id}_{R(X)} \otimes R_2(Y, UR(Z))) \circ (\mathsf{Id}_{R(X)\otimes R(Y)} \otimes \eta^r_{R(Z)}) \\
&\overset{(\heartsuit)}{=} R_2(X \otimes Y, UR(Z)) \circ (R_2(X,Y) \otimes \mathsf{Id}_{RUR(Z)}) \circ (\mathsf{Id}_{R(X)\otimes R(Y)} \otimes \eta^r_{R(Z)}) \\
&\overset{(N)}{=} R_2(X \otimes Y, UR(Z)) \circ (\mathsf{Id}_{R(X\otimes Y)} \otimes \eta^r_{R(Z)}) \circ (R_2(X,Y) \otimes \mathsf{Id}_{R(Z)}) \\
&\overset{(2.13)}{=} h^l_{X\otimes Y, R(Z)} \circ (R_2(X,Y) \otimes \mathsf{Id}_{R(Z)}).
\end{aligned}
$$

Here, the equality $(\heartsuit)$ holds because R is a monoidal functor. $\square$

Lemma 3.7. *Suppose that Condition 3.1 is satisfied and $R(\mathbb{1})$ is a Frobenius algebra in $\mathcal{C}$ with comultiplication Δ and counit ν. Then, $R^2(\mathbb{1},\mathbb{1}) = \Delta$.*

Proof. By Theorem 3.5, $R^2(\mathbb{1},\mathbb{1}) = (h^l_{\mathbb{1},R(\mathbb{1})})^{-1} \circ R(\mathsf{Id}_{\mathbb{1}} \otimes \eta^l_{\mathbb{1}})$. As $\varepsilon^r_{\mathbb{1}}$ is epic and R is exact, $R(\varepsilon^r_{\mathbb{1}})$ is epic. Thus, to prove the claim, it suffices to show

$$(h^l_{\mathbb{1},R(\mathbb{1})})^{-1} \circ R(\eta^l_{\mathbb{1}}) \circ R(\varepsilon^r_{\mathbb{1}}) = \Delta \circ R(\varepsilon^r_{\mathbb{1}}).$$

This in turn is equivalent to showing that $h^l_{\mathbb{1},R(\mathbb{1})} \circ \Delta \circ R(\varepsilon^r_{\mathbb{1}}) = R(\eta^l_{\mathbb{1}} \circ \varepsilon^r_{\mathbb{1}})$. Now, consider the following diagram.

$$
\begin{tikzcd}
RUR(\mathbb{1}) \arrow[r, "(h^l_{\mathbb{1},R(\mathbb{1})})^{-1}"] \arrow[d, "RU(u\cdot\mathsf{Id})"'] \arrow[rrr, bend left, "R(\varepsilon^r_{\mathbb{1}})"] \arrow[dr, phantom, "(N)"] & R(\mathbb{1})R(\mathbb{1}) \arrow[r, equal] \arrow[d, "\mathsf{Id}\cdot u\cdot\mathsf{Id}"] \arrow[dr, phantom, "(\diamondsuit)"] \arrow[r, phantom, bend left, "(3.6(c))"] & R(\mathbb{1})R(\mathbb{1}) \arrow[r, "m"] \arrow[d, equal] & R(\mathbb{1}) \arrow[dd, "\Delta"] \\
RU(R(\mathbb{1})R(\mathbb{1})) \arrow[r, "(h^l_{\mathbb{1},R(\mathbb{1})R(\mathbb{1})})^{-1}"'] \arrow[d, "RU(\Delta\cdot\mathsf{Id})"'] \arrow[dr, phantom, "(N)"] & R(\mathbb{1})R(\mathbb{1})R(\mathbb{1}) \arrow[r, "\mathsf{Id}\cdot m"'] \arrow[d, "\mathsf{Id}\cdot\Delta\cdot\mathsf{Id}"] \arrow[dr, phantom, "(\heartsuit)"] & R(\mathbb{1})R(\mathbb{1}) \arrow[d, "\mathsf{Id}\cdot\Delta"] \arrow[r, phantom, "(\heartsuit)"] & {} \\
RU(R(\mathbb{1})R(\mathbb{1})R(\mathbb{1})) \arrow[r, "(h^l_{\mathbb{1},R(\mathbb{1})R(\mathbb{1})R(\mathbb{1})})^{-1}"] \arrow[d, "RU(\mathsf{Id}\cdot h^l_{\mathbb{1},R(\mathbb{1})})"'] \arrow[dr, "RU(\mathsf{Id}\cdot m)"] & R(\mathbb{1})R(\mathbb{1})R(\mathbb{1})R(\mathbb{1}) \arrow[r, "\mathsf{Id}\cdot\mathsf{Id}\cdot m"] \arrow[d, phantom, "(N)"] & R(\mathbb{1})R(\mathbb{1})R(\mathbb{1}) \arrow[r, "m\cdot\mathsf{Id}"] \arrow[d, "h^l_{\mathbb{1},R(\mathbb{1})}\cdot\mathsf{Id}"] \arrow[dr, phantom, "(3.6(c))"] & R(\mathbb{1})R(\mathbb{1}) \arrow[d, equal] \\
RU(R(\mathbb{1})RUR(\mathbb{1})) \arrow[r, "RU(\mathsf{Id}\cdot R(\varepsilon^r_{\mathbb{1}}))"'] \arrow[d, equal] \arrow[ur, phantom, "(3.6(c))"] \arrow[dr, phantom, "(N)"] & RU(R(\mathbb{1})R(\mathbb{1})) \arrow[ur, "(h^l_{\mathbb{1},R(\mathbb{1})R(\mathbb{1})})^{-1}"] \arrow[dr, "R(U_2^{-1})"] \arrow[r, phantom, "(3.6(d))"] & RUR(\mathbb{1})R(\mathbb{1}) \arrow[r, "R(\varepsilon^r_{\mathbb{1}})\cdot\mathsf{Id}"'] \arrow[d, "h^l_{UR(\mathbb{1}),R(\mathbb{1})}"] \arrow[dr, phantom, "(N)"] & R(\mathbb{1})R(\mathbb{1}) \arrow[d, "h^l_{\mathbb{1},R(\mathbb{1})}"] \\
RU(R(\mathbb{1})RUR(\mathbb{1})) \arrow[r, "R(U_2^{-1})"] \arrow[d, "RU(h^l_{\mathbb{1},RUR(\mathbb{1})})"'] \arrow[dr, phantom, "(3.6(b))"] & R(UR(\mathbb{1})URUR(\mathbb{1})) \arrow[r, "R(\mathsf{Id}\cdot UR(\varepsilon^r_{\mathbb{1}}))"] \arrow[d, "R(\varepsilon^r_{\mathbb{1}}\cdot\mathsf{Id})"] \arrow[drr, phantom, "(N)"] & R(UR(\mathbb{1})UR(\mathbb{1})) \arrow[r, "R(\varepsilon^r_{\mathbb{1}}\cdot\mathsf{Id})"] & RUR(\mathbb{1}) \arrow[d, equal] \\
RURURUR(\mathbb{1}) \arrow[r, "R(\varepsilon^r_{URUR(\mathbb{1})})"'] & RURUR(\mathbb{1}) \arrow[rr, "RUR(\varepsilon^r_{\mathbb{1}})"'] & & RUR(\mathbb{1})
\end{tikzcd}
$$

Using (3.2), the bottom path of this above diagram equals $R(\eta^l_X \circ \varepsilon^r_X)$. Also, the top path reads $h^l_{\mathbb{1},R(\mathbb{1})} \circ \Delta \circ R(\varepsilon^r_X)$. Thus, we get that $R^2(\mathbb{1},\mathbb{1}) = \Delta$. $\square$

Proposition 3.8. *Suppose that Condition (3.1) is satisfied and $R(\mathbb{1})$ is a Frobenius algebra in $\mathcal{C}$ with comultiplication Δ and counit ν. Then,*

$$((R(\mathbb{1}),\sigma),\ R_2(\mathbb{1},\mathbb{1}),\ R_0,\ \Delta,\ \nu)$$

is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$. Here $\sigma_X = (h^l_{\mathbb{1},X})^{-1} h^r_{X,\mathbb{1}}$ is the half-braiding on the object $R(\mathbb{1})$.

Proof. By assumption, $U \dashv R$ is a coHopf adjunction. Thus, by Lemma 3.3, $((R(\mathbb{1}),\sigma), R_2(\mathbb{1},\mathbb{1}), R_0)$ is an algebra in $\mathcal{Z}(\mathcal{C})$ where $\sigma_X = (h^l_{\mathbb{1},X})^{-1} h^r_{X,\mathbb{1}}$.

Also, by Theorem 3.5, $R \dashv U$ is a Hopf adjunction. Thus, by Lemma 3.3, $((R(\mathbb{1}),\rho), R^2(\mathbb{1},\mathbb{1}), R^0)$ is a coalgebra in $\mathcal{Z}(\mathcal{C})$ where $\rho_X = H^l_{\mathbb{1},X}(H^r_{X,\mathbb{1}})^{-1}$. But, by Lemma 3.7, $R^2(\mathbb{1},\mathbb{1}) = \Delta$. Furthermore, $R^0 = \nu$.

By [Bal17, Remark 4.3(3)], $H^r_{X,Y} = (h^r_{X,Y})^{-1}$ and $H^l_{X,Y} = (h^l_{X,Y})^{-1}$ for all $X, Y \in \mathcal{C}$. Thus, it is clear that $\sigma_X = \rho_X$ for all $X \in \mathcal{C}$.

Thus, $((R(\mathbb{1}),\sigma), R_2(\mathbb{1},\mathbb{1}), R_0, \Delta, \nu)$ is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$. $\square$

3.2 Separable and special Frobenius case

Now consider the following result.

Theorem 3.9. *Assume that Condition 3.1 satisfied. Then, $(R(\mathbb{1}), m, u, \Delta, \nu)$ is a separable (resp., special) Frobenius algebra in $\mathcal{C}$ if and only if R is a separable (resp., special) Frobenius monoidal functor.*

Proof. ($\Rightarrow$): Suppose that $R(\mathbb{1})$ is a separable Frobenius algebra. To start, consider the following commutative diagram.

$$\begin{tikzcd}
 & & R(X)\cdot R(Y) \arrow[r, "\eta^r_{R(X)\cdot R(Y)}"] \arrow[d, phantom, "(N)"] & RU(R(X)\cdot R(Y)) \arrow[d, "R(U_2^{-1})"] \\
R(X\cdot Y) \arrow[r, "R(\mathsf{Id}\cdot\eta^l_Y)"] & R(X\cdot UR(Y)) \arrow[ur, "(h^l_{X,R(Y)})^{-1}"] \arrow[r, "\eta^r_{X\cdot UR(Y)}"] \arrow[dr, "\mathsf{Id}_{R(X\cdot UR(Y))}"'] \arrow[dr, phantom, bend left, "(2.11)"] & RUR(X\cdot UR(Y)) \arrow[ur, "RU((h^l_{X,R(Y)})^{-1})"] \arrow[d, "R(\varepsilon^r_{R(X\cdot UR(Y))})"] \arrow[r, phantom, "(3.6(b))"] & R(UR(X)\cdot UR(Y)) \arrow[dl, "R(\varepsilon^r_X\cdot\mathsf{Id}_{UR(Y)})"'] \arrow[d, "R(\varepsilon^r_X\cdot\varepsilon^r_Y)"] \arrow[dl, phantom, bend left, "(N)"] \\
 & & R(X\cdot UR(Y)) \arrow[r, "R(\mathsf{Id}_X\cdot\varepsilon^r_Y)"'] & R(X\cdot Y)
\end{tikzcd}$$

Using Theorems 2.50(a) and 3.5(a) we get that the compositions along the top of the diagram is equal to $R_2(X,Y)\ R^2(X,Y)$. Thus, we get that

$$R_2(X,Y)\ R^2(X,Y) = R(\mathsf{Id}_X \otimes \varepsilon^r_Y\ \eta^l_Y).$$

Now, observe that $\varepsilon^r_Y\ \eta^l_Y\ \varepsilon^r_Y$ is equal to

$$\begin{aligned}
&\overset{(3.5(b))}{=} \varepsilon^r_Y \circ UR(\varepsilon^r_Y) \circ \varepsilon^r_{URUR(Y)} \circ U(h^l_{\mathbb{1},RUR(Y)}) \circ U(\mathsf{Id}_{R(\mathbb{1})} \cdot h^l_{\mathbb{1},R(Y)}) \circ U(\Delta u \cdot \mathsf{Id}_{R(Y)})\\
&\overset{(3.1)}{=} \varepsilon^r_Y \circ \varepsilon^r_{UR(Y)} \circ \varepsilon^r_{URUR(Y)} \circ U(h^l_{\mathbb{1},RUR(Y)}) \circ U(\mathsf{Id}_{R(\mathbb{1})} \cdot h^l_{\mathbb{1},R(Y)}) \circ U(\Delta u \cdot \mathsf{Id}_{R(Y)})\\
&\overset{(3.1)}{=} \varepsilon^r_Y \circ \varepsilon^r_{UR(Y)} \circ UR(\varepsilon^r_{UR(Y)}) \circ U(h^l_{\mathbb{1},RUR(Y)}) \circ U(\mathsf{Id}_{R(\mathbb{1})} \cdot h^l_{\mathbb{1},R(Y)}) \circ U(\Delta u \cdot \mathsf{Id}_{R(Y)})\\
&\overset{(3.1)}{=} \varepsilon^r_Y \circ UR(\varepsilon^r_Y) \circ UR(\varepsilon^r_{UR(Y)}) \circ U(h^l_{\mathbb{1},RUR(Y)}) \circ U(\mathsf{Id}_{R(\mathbb{1})} \cdot h^l_{\mathbb{1},R(Y)}) \circ U(\Delta u \cdot \mathsf{Id}_{R(Y)})\\
&= \varepsilon^r_Y \circ UR(\varepsilon^r_Y) \circ U[R(\mathsf{Id}_{\mathbb{1}} \cdot \varepsilon^r_{UR(Y)})(h^l_{\mathbb{1},RUR(Y)})] \circ U(\mathsf{Id}_{R(\mathbb{1})} \cdot h^l_{\mathbb{1},R(Y)})\\
&\qquad \circ U(\Delta u \cdot \mathsf{Id}_{R(Y)})\\
&\overset{(3.6(c))}{=} \varepsilon^r_Y \circ UR(\varepsilon^r_Y) \circ U(R_2(\mathbb{1}, UR(Y))) \circ U(\mathsf{Id}_{R(\mathbb{1})} \cdot h^l_{\mathbb{1},R(Y)}) \circ U(\Delta u \cdot \mathsf{Id}_{R(Y)})\\
&\overset{(3.6(a))}{=} \varepsilon^r_Y \circ UR(\varepsilon^r_Y) \circ U(h^l_{\mathbb{1},R(Y)}) \circ U(R_2(\mathbb{1},\mathbb{1}) \cdot \mathsf{Id}_{R(Y)}) \circ U(\Delta u \cdot \mathsf{Id}_{R(Y)})\\
&\overset{(3.6(c))}{=} \varepsilon^r_Y \circ U(R_2(\mathbb{1}, Y)) \circ U(R_2(\mathbb{1},\mathbb{1}) \cdot \mathsf{Id}_{R(Y)}) \circ U(\Delta u \cdot \mathsf{Id}_{R(Y)})\\
&= \varepsilon^r_Y \circ U(R_2(\mathbb{1}, Y)) \circ U((R_2(\mathbb{1},\mathbb{1}) \circ \Delta) \cdot \mathsf{Id}_{R(Y)}) \circ U(u \cdot \mathsf{Id}_{R(Y)})\\
&\overset{(\diamondsuit)}{=} \varepsilon^r_Y \circ U(R_2(\mathbb{1}, Y)) \circ U(u \cdot \mathsf{Id}_{R(Y)})\\
&\overset{u=R_0}{=} \varepsilon^r_Y \circ U(R_2(\mathbb{1}, Y) \circ (R_0 \cdot \mathsf{Id}_{R(Y)})) \overset{(\spadesuit)}{=} \varepsilon^r_Y.
\end{aligned}$$

Here, the equality ($\diamondsuit$) holds because $R(\mathbb{1})$ is separable and the equality ($\spadesuit$) holds because R is a monoidal functor. By Condition 3.1, ε^r_Y is an coequalizer, and therefore, it is epic. Hence, we get that $\varepsilon^r_Y\ \eta^l_Y = \mathsf{Id}_Y$, thereby proving that $R_2(X,Y)\ R^2(X,Y) = \mathsf{Id}_{R(X\otimes Y)}$. Hence, R is separable Frobenius.

Lastly, observe that $R^0\ R_0 = \nu\ u = \mathsf{Id}_{\mathbb{1}}$ is equal to the identity map on $\mathbb{1}$ if and only if $R(\mathbb{1})$ is special Frobenius. Hence, this direction of the proof is finished.
($\Leftarrow$): As the unit object is separable (resp., special) Frobenius, by Lemma 2.39(b), it follows that $R(\mathbb{1})$ is separable (resp., special) Frobenius. $\square$

3.3 Pivotal case

Recall the definitions of pivotal categories and functors from Section 2.3.

Proposition 3.10. *Let $\mathcal{C}$ be a monoidal category.*

(a) *Suppose that $((A,\sigma), m, u, \Delta, \nu) \in \mathsf{Frob}(\mathcal{Z}(\mathcal{C}))$, then $F = A \otimes - : \mathcal{C} \to \mathcal{C}$ is a*

Frobenius monoidal functor with structure maps:

$$\begin{gathered}
F_2(X,Y) := (m \otimes \mathsf{Id}_X \otimes \mathsf{Id}_Y)(\mathsf{Id}_A \otimes \sigma_X \otimes \mathsf{Id}_Y) : A \otimes X \otimes A \otimes Y \to A \otimes X \otimes Y \\
F^2(X,Y) := (\mathsf{Id}_A \otimes \sigma_X^{-1} \otimes \mathsf{Id}_Y)(\Delta \otimes \mathsf{Id}_X \otimes \mathsf{Id}_Y) : A \otimes X \otimes Y \to A \otimes X \otimes A \otimes Y \\
F_0 := u : \mathbb{1} \to A \qquad\qquad F^0 := \nu : A \to \mathbb{1}.
\end{gathered}$$

(b) *If $\mathcal{C}$ is pivotal and (A, m, u, Δ, ν) is a symmetric Frobenius algebra in $\mathcal{Z}(\mathcal{C})$, then $A \otimes - : \mathcal{C} \to \mathcal{C}$ is a pivotal functor with the above structure maps.*

Proof. The proof of (a) is straightforward, so we will only prove (b). Let $\mathfrak{p}$ denote the pivotal structure of $\mathcal{C}$. By the definition of a pivotal functor (Definition 2.25), we need to prove that

$$\xi_X^{A\otimes -} \circ (\mathsf{Id}_A \otimes \mathfrak{p}_X) = \mathfrak{p}_{A\otimes X} : A \otimes {}^{\vee\vee}X \to {}^{\vee\vee}(A \otimes X) = {}^{\vee\vee}A \otimes {}^{\vee\vee}X.$$

where $\xi_X^{A\otimes -} \overset{(2.6)}{=} {}^{\vee}((\zeta_X^{A\otimes -})^{-1}) \circ \zeta_{{}^{\vee}X}^{A\otimes -}$. We first calculate that

$$\begin{aligned}
\zeta_X^{A\otimes -} &\overset{(2.5)}{=} ([F^0 \circ F(\mathrm{ev}_X) \circ F_2({}^{\vee}X, X)] \otimes \mathsf{Id}_{{}^{\vee}F(X)})\ (\mathsf{Id}_{F({}^{\vee}X)} \otimes \mathrm{coev}_{F(X)}) \\
&\overset{(\spadesuit)}{=} (\nu\ m \otimes \mathrm{ev}_X \otimes \mathsf{Id}_{{}^{\vee}(A\otimes X)})(\mathsf{Id}_A \otimes \sigma_{{}^{\vee}X} \otimes_{X\otimes {}^{\vee}(A\otimes X)})(\mathsf{Id}_{A\otimes {}^{\vee}X} \otimes \mathrm{coev}_{A\otimes X}) \\
&\overset{(\diamondsuit)}{=} (\nu\ m \otimes \mathsf{Id}_{{}^{\vee}X\otimes {}^{\vee}A})(\mathsf{Id}_A \otimes \sigma_{{}^{\vee}X} \otimes \mathsf{Id}_{{}^{\vee}A})(\mathsf{Id}_{A\otimes {}^{\vee}X} \otimes \mathrm{coev}_A).
\end{aligned}$$

Here, the equality ($\spadesuit$) is obtained by plugging in the description of F^2, F_2, F^0, F_0 from part (a), and the equality ($\diamondsuit$) using that $\mathrm{coev}_{A\cdot X} = (\mathsf{Id}_A \otimes \mathrm{coev}_X \otimes \mathsf{Id}_{{}^{\vee}A})\mathrm{coev}_A$ and the snake relation. A similar calculation shows that

$$(\zeta_X^{A\otimes -})^{-1} = \sigma_{{}^{\vee}X}\ (\mathsf{Id}_{{}^{\vee}X} \otimes \mathrm{ev}_A \otimes \mathsf{Id}_A)(\mathsf{Id}_{{}^{\vee}X\cdot {}^{\vee}A} \otimes u\ \Delta).$$

Using the descriptions of $\zeta_X^{A\otimes -}$ and $(\zeta_X^{A\otimes -})^{-1}$ above, we get an expression for $\xi_X^{A\otimes -}$. Using the naturality of σ and the snake relation one can simplify further to get that $\xi_X^{A\otimes -} = \kappa_A \otimes \mathsf{Id}_{{}^{\vee\vee}X}$, where $\kappa_A : A \to {}^{\vee\vee}A$ is the following morphism

$$\begin{aligned}
&= (\nu m \otimes \mathsf{Id}_{{}^{\vee\vee}A})(\mathsf{Id}_A \otimes \mathrm{ev}_A \otimes \mathsf{Id}_A \otimes \mathsf{Id}_{{}^{\vee\vee}A})(\mathsf{Id}_A \otimes \mathsf{Id}_{{}^{\vee}A} \otimes \Delta u \otimes \mathsf{Id}_{{}^{\vee\vee}A})(\mathsf{Id}_A \otimes \mathrm{coev}_{{}^{\vee}A}) \\
&\overset{(2.35)}{=} (\nu m \otimes \mathsf{Id}_{{}^{\vee\vee}A})(\mathsf{Id}_A \otimes \mathsf{Id}_A \otimes \mathrm{ev}_{{}^{\vee}A}(\mathfrak{p}_A \otimes \mathsf{Id}_{{}^{\vee}A}) \otimes \mathsf{Id}_{{}^{\vee\vee}A})(\mathsf{Id}_A \otimes \Delta u \otimes \mathrm{coev}_{{}^{\vee}A}) \\
&\overset{(N)}{=} (\mathrm{ev}_{{}^{\vee}A} \otimes \mathsf{Id}_{{}^{\vee\vee}A})(\mathsf{Id}_{{}^{\vee\vee}A} \otimes \mathrm{coev}_{{}^{\vee}A})\mathfrak{p}_A(\nu m \otimes \mathsf{Id}_A)(\mathsf{Id}_A \otimes \Delta u) \\
&\overset{(\heartsuit)}{=} (\mathrm{ev}_{{}^{\vee}A} \otimes \mathsf{Id}_{{}^{\vee\vee}A})(\mathsf{Id}_{{}^{\vee\vee}A} \otimes \mathrm{coev}_{{}^{\vee}A})\mathfrak{p}_A \quad = \quad \mathfrak{p}_A.
\end{aligned}$$

Here, the equality ($\heartsuit$) holds because A is a Frobenius algebra. Finally,

$$\xi_X^{A\otimes -} \circ (\mathsf{Id}_A \otimes \mathfrak{p}_X) = (\mathfrak{p}_A \otimes \mathsf{Id}_{{}^{\vee\vee}X})(\mathsf{Id}_A \otimes \mathfrak{p}_X) = (\mathfrak{p}_A \otimes \mathfrak{p}_X) = \mathfrak{p}_{A\otimes X}.$$

Hence, the proof is finished. ◻

Now, we are ready to prove the main result of this section.

Theorem 3.11. *Suppose that Condition 3.1 is satisfied with $U : \mathcal{C} \to \mathcal{D}$ a pivotal functor (thus, $\mathcal{C}$ is pivotal). Then, $R(\mathbb{1})$ is a symmetric Frobenius algebra in $\mathcal{C}$ if and only if R is a pivotal Frobenius functor.*

Proof. We will prove the forward direction; the converse follows by Lemma 2.39(c). By Proposition 3.8, $R(\mathbb{1})$ is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$. Thus, in particular, $R(\mathbb{1})$ is a symmetric Frobenius algebra in $\mathcal{Z}(\mathcal{C})$. Hence, by Proposition 3.10(b), $R(\mathbb{1}) \otimes -$ is a pivotal functor. Since $U \dashv R$ is a coHopf adjunction by assumption, we get that $h^l_{\mathbb{1},X} : R(\mathbb{1}) \otimes X \to R(\mathbb{1} \otimes U(X)) = RU(X)$ is a monoidal natural isomorphism between the functors $R(\mathbb{1}) \otimes -$ and RU. Therefore, we get that RU is pivotal. Now, observe that

$$\begin{aligned}
\xi^R_X \circ R(\mathfrak{p}^{\mathcal{D}}_X) \circ R(\varepsilon^r_X) &\overset{(N)}{=} \xi^R_X \circ R({}^{\vee\vee}\varepsilon^r_X) \circ R(\mathfrak{p}^{\mathcal{D}}_{UR(X)}) \\
&\overset{(N)}{=} {}^{\vee\vee}R(\varepsilon^r_X) \circ \xi^R_{UR(X)} \circ R(\mathfrak{p}^{\mathcal{D}}_{UR(X)}) \\
&\overset{(2.8)}{=} {}^{\vee\vee}R(\varepsilon^r_X) \circ \xi^R_{UR(X)} \circ R(\xi^U_{R(X)}) \circ RU(\mathfrak{p}^{\mathcal{C}}_{R(X)}) \\
&\overset{(2.22)}{=} {}^{\vee\vee}R(\varepsilon^r_X) \circ \xi^{RU}_{R(X)} \circ RU(\mathfrak{p}^{\mathcal{C}}_{R(X)}) \\
&\overset{(2.8)}{=} {}^{\vee\vee}R(\varepsilon^r_X) \circ \mathfrak{p}^{\mathcal{C}}_{RUR(X)} \\
&\overset{(N)}{=} \mathfrak{p}^{\mathcal{C}}_{R(X)} \circ R(\varepsilon^r_X).
\end{aligned}$$

As ε^r_X is epic and R is exact, $R(\varepsilon^r_X)$ is epic. Consequently, $\xi^R_X \circ R(\mathfrak{p}^{\mathcal{D}}_X) = \mathfrak{p}^{\mathcal{C}}_{R(X)}$, thereby proving that R is pivotal. □

3.4 Ribbon case

In this section, we equip our categories with braidings and further strengthen the results obtained in previous sections.

Lemma 3.12. *Let $U : (\mathcal{C}, c) \to (\mathcal{D}, d)$ be a braided strong monoidal functor between braided monoidal categories. Then, U^{ra} is braided and U^{la} is cobraided.*

Proof. Let $R := U^{\mathsf{ra}}$. Observe that $R(d_{X,Y}) \circ R_2(X, Y)$ is

$$\begin{aligned}
&\overset{(2.50(a))}{=} R(d_{X,Y}) \circ R(\varepsilon_X \cdot \varepsilon_Y) \circ R(U_2^{-1}(R(X), R(Y))) \circ \eta_{R(X)\cdot R(Y)} \\
&\overset{(N)}{=} R(\varepsilon_Y \cdot \varepsilon_X) \circ R(d_{UR(X),UR(Y)}) \circ R(U_2^{-1}(R(X), R(Y))) \circ \eta_{R(X)\cdot R(Y)} \\
&\overset{(\diamondsuit)}{=} R(\varepsilon_Y \cdot \varepsilon_X) \circ R(U_2^{-1}(R(Y), R(X))) \circ RU(c_{R(X),R(Y)}) \circ \eta_{R(X)\cdot R(Y)} \\
&\overset{(N)}{=} R(\varepsilon_Y \cdot \varepsilon_X) \circ R(U_2^{-1}(R(Y), R(X))) \circ \eta_{R(Y)\cdot R(X)} \circ c_{R(X),R(Y)} \\
&\overset{(2.50(a))}{=} R_2(Y, X) \circ c_{R(X),R(Y)}.
\end{aligned}$$

Here, the equality ($\diamondsuit$) holds because U is braided. Thus, R is braided.

By Lemma 2.13, we know that U is cobraided. Hence, U^{op} is a braided strong monoidal functor. Then, the claim about $L := U^{\mathsf{la}}$ follows by applying the above result to the adjunction $U^{\mathrm{op}} \dashv L^{\mathrm{op}}$. $\square$

Theorem 3.13. *Let $U : \mathcal{C} \to \mathcal{D}$ be a ribbon functor between ribbon categories satisfying Condition 3.1. Then $R(\mathbb{1})$ is a symmetric Frobenius algebra in $\mathcal{C}$ if and only if R is a ribbon Frobenius functor.*

Proof. By Theorem 3.5, $U^{\mathsf{la}} = U^{\mathsf{ra}} = R$. Since U is braided, by Lemma 3.12, we get that $U^{\mathsf{la}} = R$ is cobraided and $U^{\mathsf{ra}} = R$ is braided. Finally, as $R(\mathbb{1})$ is symmetric Frobenius, we obtain that R is pivotal and Frobenius by Theorem 3.11. Now, by Proposition 2.31, R is a ribbon Frobenius functor. The converse is straightforward. $\square$

Chapter 4

Unimodular module categories

Unimodularity is a classical notion, with roots in linear algebra. Building upon the definition of a unimodular matrix (determinant $= \pm 1$), unimodularity of lattices, bilinear forms, topological groups, Hopf algebras, Poisson algebras, tensor categories, etc. is defined. In this thesis, we are interested in unimodular tensor categories. Research in this direction began with work on locally compact topological groups. Such a group is equipped with a left and a right invariant Haar measure, and when the left invariant Haar measure is also right invariant, we call the group *unimodular*. Generalizing this, Sweedler [Swe69] introduced the notions of left and right integrals for Hopf algebras. Then, a finite dimensional Hopf algebra is said be unimodular if its *distinguished character*, which measures how far a left integral is from being a right integral, is identically 1 [LS69]. Etingof and Ostrik [EO04] defined an analogue of the distinguished character called the *distinguished invertible object*, and denoted as D, for any tensor category $\mathcal{C}$. This section is devoted to studying unimodular module categories.

Unimodularity of a tensor category is a crucial property for topological applications. Non-semisimple generalizations of the Reshetikhin-Turaev invariants [RT91], which are defined using certain tensor categories as input, require the input category to be unimodular. Another important class of invariants, the Turaev-Viro invariants [TV92], are built using spherical fusion categories as input. However, in the non-semisimple setting, the definition of a spherical tensor category [DSPS18] requires the underlying category to be unimodular. Recent works like [BDR22] have also employed unimodular (ribbon) tensor categories to construct invariants of 4-dimensional manifolds.

In this chapter, we introduce unimodular module categories. We discuss their basic properties and collect various characterizations in Section 4.1. In Section 4.2, we employ unimodular module categories to construct Frobenius algebras in the Drinfeld center. In Section 4.3, we construct symmetric Frobenius algebras using pivotal categories. This chapter is based on our article [Yad23].

4.1 Definition and basic properties

Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ an exact left $\mathcal{C}$-module category. Recall that $\mathbb{S}$ and $\mathbb{N}$ are left $\mathcal{C}$-module functors with module constraints $\mathfrak{s}$ and $\mathfrak{n}$ given in Theorem 2.55(c) and Theorem 2.56(a), respectively. Thus, $\mathbb{S}\,\mathbb{N}$ is left $\mathcal{C}$-module endofunctor of $\mathcal{M}$ with module constraints

$$\mathfrak{d}_{X,M} : \mathbb{S}\,\mathbb{N}(X \triangleright M) \xrightarrow{\mathbb{S}(\mathfrak{n}_{X,M})} \mathbb{S}(X^{\vee\vee} \triangleright \mathbb{N}(M)) \xrightarrow{\mathfrak{s}_{X^{\vee\vee},\mathbb{N}(M)}} X \triangleright \mathbb{S}\,\mathbb{N}(M)$$

Similarly, $\overline{\mathbb{N}}\,\overline{\mathbb{S}} \in \mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$. Using Theorems 2.56(b) and 2.55(b), we get that $\mathbb{S}\,\mathbb{N} = (\overline{\mathbb{N}}\,\overline{\mathbb{S}})^{-1}$ in $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$. It was shown in [FGJS22, Proposition 4.13] that $D_{\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})} \cong_{\mathcal{C}} \overline{\mathbb{N}}\,\overline{\mathbb{S}}$. Thus,

$$\mathsf{Rex}_{\mathcal{C}}(\mathcal{M}) \text{ is unimodular } \iff \overline{\mathbb{N}}\,\overline{\mathbb{S}} \cong_{\mathcal{C}} \mathsf{Id}_{\mathcal{M}} \iff \mathbb{S}\,\mathbb{N} \cong_{\mathcal{C}} \mathsf{Id}_{\mathcal{M}}. \tag{4.1}$$

The above discussion motivates the following definition.

Definition 4.1. A *unimodular structure* on an exact left $\mathcal{C}$-module category $\mathcal{M}$ is a $\mathcal{C}$-module natural isomorphism $\mathfrak{u} : \mathsf{Id}_{\mathcal{M}} \to \mathbb{S}\ \mathbb{N}$ such that the following diagram commutes for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$.

$$\begin{tikzcd} X \triangleright M \arrow[r, "\mathsf{Id}_X \triangleright \mathfrak{u}_M"] \arrow[d, "\mathfrak{u}_{X \triangleright M}"'] & X \triangleright \mathbb{S}\ \mathbb{N}(M) \\ \mathbb{S}\ \mathbb{N}(X \triangleright M) \arrow[r, "\mathbb{S}(\mathfrak{n}_{X,M})"'] & \mathbb{S}(X^{\vee\vee} \triangleright \mathbb{N}(M)) \arrow[u, "\mathfrak{s}_{X^{\vee\vee},\, \mathbb{N}(M)}"'] \end{tikzcd} \tag{4.2}$$

An exact, left $\mathcal{C}$-module category is called *unimodular* if it admits a unimodular structure $\mathfrak{u}$.

Lemma 4.2. *Definitions 1.6 and 4.1 of a unimodular module category are equivalent.*

Proof. According to Definition 1.6, $\mathcal{M}$ is unimodular if and only if $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ is unimodular. By equation (4.1), $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ is unimodular if and only if there exists a $\mathcal{C}$-module natural isomorphism $\mathfrak{u} : \mathsf{Id}_{\mathcal{M}} \to \mathbb{S}\,\mathbb{N}$, that is a unimodular structure on $\mathcal{M}$. Thus, the claim follows. □

Remark 4.3. Since $\mathbb{N}$ is an equivalence with quasi-inverse $\overline{\mathbb{N}}$ by Theorem 2.56(b), $\mathcal{M}$ is unimodular if and only if there exists a natural isomorphism $\overline{\mathbb{N}} \cong \mathbb{S}$ of $\mathcal{C}$-module functors. Thus, Definition 4.1 is the same as the one suggested in [FSS20, Remark 4.27].

Next, we present some examples of unimodular module categories.

Example 4.4. (i) Let $\mathcal{C}$ be a finite tensor category. Then, $\mathcal{M} = \mathcal{C}$ considered as a left $\mathcal{C}$-module category is unimodular if and only if $\mathcal{C}$ is a unimodular tensor category.

(ii) Let $\mathcal{C}$ be a nondegenerate (*i.e.* global dimension of $\mathcal{C}$ is nonzero) fusion tensor category. For instance, the category $\mathsf{Rep}(H)$ for H a semisimple and cosemisimple Hopf algebra is a nondegenerate fusion category. Then, by [ENO04,], $\mathcal{C}$ is unimodular. If $\mathcal{M}$ is a semisimple left $\mathcal{C}$-module category, then by [ENO05, Theorem 2.18], $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ is multifusion. Hence, by Lemma 4.2, $\mathcal{M}$ is a unimodular left $\mathcal{C}$-module category.

(iii) Let H be a finite dimensional Hopf algebra. Then the category $\mathcal{C} = \mathsf{Rep}(H)$ is a finite tensor category. The forgetful functor $F : \mathsf{Rep}(H) \to \mathsf{Vec}$ turns Vec into a left $\mathsf{Rep}(H)$-module category. By [EGNO16, Example 7.12.26], we have that $\mathsf{Rex}_{\mathcal{C}}(\mathsf{Vec}) \cong \mathsf{Rep}(H^*)$, where H^* is the dual Hopf algebra of H. Therefore, Vec is a unimodular $\mathsf{Rep}(H)$-module category if and only if H^* is a unimodular Hopf algebra. This happens if and only if the distinguished grouplike element g_H of H is equal to its unit 1_H.

In Chapter 6, we generalize Example 4.4(iii) and classify unimodular $\mathsf{Rep}(H)$-module categories over for any finite dimensional Hopf algebra H.

The next remark shows that tensor categories that are not unimodular can admit unimodular module categories.

Remark 4.5. Let $\mathcal{C}$ be a finite tensor category that is not unimodular ($D_{\mathcal{C}} \ncong \mathbb{1}$). For instance, one can take $\mathcal{C}$ to be the category of representation of the Taft algebra. Now, consider the category $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$. Then, we get that

$$D_{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}} = \overline{\mathbb{N}}_{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}(\mathbb{1}_{\mathcal{C}\boxtimes\mathcal{C}^{\mathrm{rev}}}) \overset{(\dagger)}{\cong} (\overline{\mathbb{N}}_{\mathcal{C}}\boxtimes\overline{\mathbb{N}}_{\mathcal{C}^{\mathrm{rev}}})(\mathbb{1}\boxtimes\mathbb{1}) \overset{(\ddagger)}{\cong} \overline{\mathbb{N}}_{\mathcal{C}}(\mathbb{1})\boxtimes\overline{\mathbb{N}}_{\mathcal{C}}(\mathbb{1}) = D_{\mathcal{C}}\boxtimes D_{\mathcal{C}} \ncong \mathbb{1}\boxtimes\mathbb{1}.$$

Here, the isomorphism ($\dagger$) follows from [FSS20, Proposition 3.20]. Since, $\mathcal{C} = \mathcal{C}^{\mathrm{rev}}$ as categories, we have that $\overline{\mathbb{N}}_{\mathcal{C}} = \overline{\mathbb{N}}_{\mathcal{C}^{\mathrm{rev}}}$. Thus, the isomorphism ($\ddagger$) holds. Consequently, we get that $\mathcal{D}$ is not unimodular. Now, consider the left $\mathcal{D}$-module category $\mathcal{M} := \mathcal{C}$ with actions defined as

$$(X \boxtimes Y) \triangleright M := X \otimes M \otimes Y.$$

Then, we claim that the $\mathcal{D}$-module category $\mathcal{M}$ is unimodular. Indeed, by [EGNO16, 7.13.8], we have that $\mathsf{Rex}_{\mathcal{D}}(\mathcal{M}) \cong \mathcal{Z}(\mathcal{C})$. Further, by [EGNO16, Proposition 8.6.3], $\mathcal{Z}(\mathcal{C})$ is factorizable and by [EGNO16, 8.10.10], factorizable finite tensor categories are unimodular. Thus, $\mathcal{Z}(\mathcal{C})$ is unimodular. Equivalently, $\mathcal{M}$ is a unimodular $\mathcal{D}$-module category.

Next, we collect a few basic results about unimodular module categories. The following result shows that unimodular module categories are closed under direct sums.

Lemma 4.6. *Suppose that $\mathcal{M} = \oplus_{i\in I}\mathcal{M}_i$ where the set I is finite and each $\mathcal{M}_i$ is an indecomposable, exact left $\mathcal{C}$-module category. Then $\mathcal{M}$ is unimodular if and only each $\mathcal{M}_i$ is unimodular.*

Proof. Since $\mathcal{M}$ is decomposable, $\mathcal{D} := \mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ is a multitensor category. Thus, we can write $\mathcal{D}$ as a direct sum of its component subcategories $\mathcal{D}_{ij} := \mathsf{Rex}_{\mathcal{C}}(\mathcal{M})_{ij}$. But $\mathcal{D}_{ij} = \mathsf{Rex}_{\mathcal{C}}(\mathcal{M}_i, \mathcal{M}_j)$ by Lemma [EGNO16, Lemma 7.12.6]. Now, by the following equivalences, the claim follows.

$$\begin{aligned} \mathcal{M} \text{ unimodular} \quad \Longleftrightarrow \quad D_{\mathcal{D}} \cong_{\mathcal{C}} \mathsf{Id}_{\mathcal{M}} \quad &\overset{(2.60)}{\Longleftrightarrow} \quad D_{\mathcal{D}_{ii}} \cong_{\mathcal{C}} \mathsf{Id}_{\mathcal{M}_i}\ \forall i \in I \\ &\Longleftrightarrow \quad \mathcal{M}_i \text{ unimodular } \forall i \in I. \end{aligned}$$ ☐

Lemma 4.7. *Let $\mathcal{M}$ be an indecomposable, exact, left $\mathcal{C}$-module category. Then, a unimodular structure on $\mathcal{M}$, if it exists, is unique up to a scalar multiple.*

Proof. Since $\mathcal{M}$ is indecomposable, $\mathsf{Id}_{\mathcal{M}}$ is a simple object in $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$. Now, given two unimodular structures $\mathfrak{u}$ and $\mathfrak{u}'$ on $\mathcal{M}$, $\mathfrak{u}' \circ \mathfrak{u}^{-1}$ is an endomorphism of $\mathsf{Id}_{\mathcal{M}}$. Hence, by Schur's Lemma,

$$\mathfrak{u}' \circ \mathfrak{u}^{-1} = k\, \mathsf{Id}_{\mathcal{M}} \text{ for some } k \in \Bbbk \quad \Rightarrow \quad \mathfrak{u}' = k\, \mathfrak{u}.$$ ☐

Proposition 4.8. *Let $\mathcal{M}$ be a unimodular left $\mathcal{C}$-module category satisfying $\mathbb{N} \cong \mathsf{Id}_{\mathcal{M}}$. Then, for any $\mathcal{M} \in \mathcal{M}$, the internal End object $\underline{\mathrm{Hom}}(M, M)$, is a Frobenius algebra in $\mathcal{C}$.*

Proof. Since $\mathcal{M}$ is unimodular, we have a natural isomorphism $\mathfrak{u} : \mathsf{Id}_{\mathcal{M}} \to \mathbb{S}\,\mathbb{N}$. We also have a natural isomorphism $\tau : \mathbb{N} \to \mathsf{Id}_{\mathcal{M}}$. By combining these two isomorphisms, we get a natural isomorphism $p : \mathsf{Id}_{\mathcal{M}} \xrightarrow{\mathfrak{u}} \mathbb{S}\,\mathbb{N} \xrightarrow{\mathbb{S}\,\tau} \mathbb{S}$, thereby providing an isomorphism $p_M : M \to \mathbb{S}(M)$ for all $M \in \mathcal{M}$. Now, by [Shi19, Theorem 3.14], it follows that $\underline{\mathrm{Hom}}(M, M)$ is a Frobenius algebra in $\mathcal{C}$. ☐

Remark 4.9. A finite linear category $\mathcal{M}$ is said to be *symmetric Frobenius* if $\mathcal{M}$ is equivalent to the category of modules over a symmetric Frobenius algebra A [FSS20, Definition 3.23]. This definition is justified because the property of being a symmetric Frobenius algebra is Morita invariant. By [FSS20, Proposition 3.24], $\mathcal{M}$ is symmetric Frobenius if and only if $\mathbb{N}_{\mathcal{M}} \cong \mathsf{Id}_{\mathcal{M}}$. Thus, symmetric Frobenius module categories provide natural candidates to which the above proposition can be applied.

4.2 Frobenius algebras from unimodular module categories

In this section, we use unimodular module categories to provide a construction of (separable, special) Frobenius algebras in the Drinfeld center $\mathcal{Z}(\mathcal{C})$. We accomplish this by constructing appropriate Frobenius monoidal functors with target $\mathcal{Z}(\mathcal{C})$. To construct such functors, we employ the strategy outlined in [Yad22]. Namely, we consider the right adjoint of the functor Ψ defined below.

Definition 4.10. [Shi20, Section 3.6] Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ a left $\mathcal{C}$-module category. Consider the functor below

$$\Psi : \mathcal{Z}(\mathcal{C}) \to \mathsf{Rex}_{\mathcal{C}}(\mathcal{M}), \qquad (X, \sigma) \mapsto (X \triangleright -, s^{\sigma}), \tag{4.3}$$

where the left $\mathcal{C}$-module structure of $X \triangleright -$ is

$$s^{\sigma}_{Y,M} : Y \triangleright (X \triangleright (M)) = (Y \otimes X) \triangleright M \xrightarrow{\sigma_Y \triangleright \mathsf{Id}_M} (X \otimes Y) \triangleright M = X \triangleright (Y \triangleright M).$$

In order to better understand Ψ, we define the following functors.

Definition 4.11. Set $\mathcal{D} := \mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$.

- For any monoidal category $\mathcal{C}$, consider the forgetful functor

$$U'_{\mathcal{C}} : \mathcal{Z}(\mathcal{C})^{\mathrm{mir}} \to \mathcal{C} \qquad (X, \sigma) \mapsto X. \tag{4.4}$$

  As a monoidal functor, $U_{\mathcal{C}}$ and $U'_{\mathcal{C}}$ are identical. Thus, many facts about $U_{\mathcal{C}}$ also hold true for $U'_{\mathcal{C}}$. In particular, $U'_{\mathcal{C}}$ is a strong monoidal functor. When $\mathcal{C}$ is a finite tensor category, $U'_{\mathcal{C}}$ admits a right adjoint $R_{\mathcal{C}}$.
- Schauenburg's [Sch01, Theorem 3.3] established the following braided equivalence between the Drinfeld centers of $\mathcal{C}$ and $\mathcal{D}^{\mathrm{rev}}$.

$$\Theta_{\mathcal{M}} : \mathcal{Z}(\mathcal{C}) \xrightarrow{\sim} \mathcal{Z}(\mathcal{D}^{\mathrm{rev}}), \;\; (X, \sigma) \mapsto ((X \triangleright -, s^{\sigma}), \Sigma). \tag{4.5}$$

  For the definition of Σ, see [Shi20, Section 3.7].
- For any monoidal category $\mathcal{C}$, by [EGNO16, Exercise 8.5.2], we have the following braided equivalence

$$\Omega_{\mathcal{C}} : \mathcal{Z}(\mathcal{C}^{\mathrm{rev}}) \cong \mathcal{Z}(\mathcal{C})^{\mathrm{mir}}, \qquad (X, \sigma) \mapsto (X, \sigma^{-1}). \tag{4.6}$$

Now consider the following result.

Lemma 4.12. [Shi20, Theorem 3.14] *The functor Ψ is equal to the composition $U'_{\mathcal{D}} \circ \Omega_{\mathcal{D}} \circ \Theta_{\mathcal{M}}$. Further, Ψ is an exact, strong monoidal functor.*

Proof. Observe that

$$\begin{aligned} U'_{\mathcal{D}} \circ \Omega_{\mathcal{D}} \circ \Theta_{\mathcal{M}}(X,\sigma) &\overset{(4.5)}{=} U'_{\mathcal{D}} \circ \Omega_{\mathcal{D}}((X \triangleright -, s^{\sigma}), \Sigma) &&\overset{(4.6)}{=} U'_{\mathcal{D}}((X \triangleright -, s^{\sigma}), \Sigma^{-1}) \\ &\overset{(4.4)}{=} (X \triangleright -, s^{\sigma}). &&\overset{(4.3)}{=} \Psi(X,\sigma). \end{aligned}$$

Since $U'_{\mathcal{D}}$, $\Omega_{\mathcal{D}}$ and $\Theta_{\mathcal{M}}$ are each strong monoidal, Ψ is a strong monoidal functor. Furthermore, by [Shi20, Theorem 3.11], Ψ admits a right adjoint Ψ^{ra}. Since the categories $\mathcal{Z}(\mathcal{C})$ and $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ are rigid, a left adjoint also exists. Since, $\mathcal{Z}(\mathcal{C})$ and $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ are finite categories, the existence of adjoints implies that Ψ is exact. $\square$

In the following discussion, the algebra object $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}}) \in \mathcal{Z}(\mathcal{C})$ will be very important. By Theorem 2.62(a), $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is a commutative algebra in $\mathcal{Z}(\mathcal{C})$. To start, consider the following result, which follows from the work in [Shi20].

Theorem 4.13. *Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ a finite left $\mathcal{C}$-module category. Then, $\mathcal{M}$ is unimodular if and only if $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$.*

Proof. Let Θ^{-1}, Ω^{-1}, respectively, denote the quasi-inverse of the equivalences Θ, Ω. Then,

$$\Psi^{\mathsf{ra}} \cong \Theta^{\mathsf{ra}} \circ \Omega^{\mathsf{ra}} \circ U'^{\mathsf{ra}} \cong \Theta^{-1} \circ \Omega^{-1} \circ R \tag{4.7}$$

Suppose that $\mathcal{M}$ is indecomposable. Then, we get the following equivalences.

$$\begin{aligned} \Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}}) \in \mathsf{Frob}(\mathcal{Z}(\mathcal{C})) &\overset{(\diamondsuit)}{\Longleftrightarrow} R(\mathsf{Id}_{\mathcal{M}}) \in \mathsf{Frob}(\mathcal{Z}(\mathsf{Rex}_{\mathcal{C}}(\mathcal{M}))) \\ &\overset{(\spadesuit)}{\Longleftrightarrow} \mathsf{Rex}_{\mathcal{C}}(\mathcal{M}) \text{ unimodular} \\ &\overset{(4.2)}{\Longleftrightarrow} \mathcal{M} \text{ unimodular.} \end{aligned}$$

The equivalence $(\diamondsuit)$ holds because Θ^{-1} and Ω^{-1} are monoidal equivalences, and thereby they preserve Frobenius algebras. The equivalence $(\spadesuit)$ follows from Theorem 2.62(b) because $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ is a finite tensor category.

Now suppose that $\mathcal{M}$ is decomposable and $\mathcal{M} = \oplus_{i \in I} \mathcal{M}_i$ where $\mathcal{M}_i$ are indecomposable $\mathcal{C}$-module categories and the set I is finite. First observe that

$$\Psi^{\mathsf{ra}}_{\mathcal{M}}(\mathsf{Id}_{\mathcal{M}}) = \Psi^{\mathsf{ra}}_{\mathcal{M}}(\oplus_{i \in I} \mathsf{Id}_{\mathcal{M}_i}) = \oplus_{i \in I} \Psi^{\mathsf{ra}}_{\mathcal{M}_i}(\mathsf{Id}_{\mathcal{M}_i})$$

Now by the following equivalences, the claim follows. Below, we write Frob to denote $\mathsf{Frob}(\mathcal{Z}(\mathcal{C}))$.

$$\begin{aligned} \Psi^{\mathsf{ra}}_{\mathcal{M}}(\mathsf{Id}_{\mathcal{M}}) \in \mathsf{Frob} \overset{(2.42)}{\Longleftrightarrow} \Psi^{\mathsf{ra}}_{\mathcal{M}_i}(\mathsf{Id}_{\mathcal{M}_i}) \in \mathsf{Frob}\ \forall i \in I &\Longleftrightarrow \mathcal{M}_i \text{ unimodular } \forall i \in I \\ &\overset{(4.6)}{\Longleftrightarrow} \mathcal{M} \text{ unimodular.} \end{aligned}$$

$\square$

Proposition 4.14. *The adjunction $\Psi \dashv \Psi^{\mathsf{ra}}$ satisfies Condition 3.1.*

Proof. We know that Ψ is a strong monoidal functor between abelian monoidal categories. Since it is an exact functor (by Lemma 4.12) between finite abelian categories, it admits a right adjoint Ψ^{ra}. By Theorem 2.50(a), $\Psi \dashv \Psi^{\mathsf{ra}}$ is a comonoidal adjunction.

(a) As $\mathcal{Z}(\mathcal{C})$, $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ are rigid, by Remark 3.2, $\Psi \dashv \Psi^{\mathsf{ra}}$ is a coHopf adjunction.

(b) The functor Ψ^{ra} admits a left adjoint, namely Ψ. Since Ψ^{ra} is a functor between rigid categories, it also admits a right adjoint. Thus, Ψ^{ra} is a functor between finite tensor categories, $\mathcal{Z}(\mathcal{C})$ and $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$, admitting both adjoints. This implies that Ψ^{ra} is exact.

(c) Since $\mathcal{M}$ is indecomposable, $\mathcal{D} = \mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ satisfies $\mathrm{End}_{\mathcal{D}}(\mathsf{Id}_{\mathcal{M}}) \cong \Bbbk$. Thus, by [Shi16, Corollary 5.9], we get that the functor $(U'_{\mathcal{D}})^{\mathsf{ra}}$ is faithful. Since $\Theta_{\mathcal{M}}^{\mathsf{ra}}$ and $\Omega_{\mathcal{D}}$ are category equivalences, we get that $\Psi^{\mathsf{ra}} = \Theta_{\mathcal{M}}^{\mathsf{ra}} \circ \Omega_{\mathcal{D}}^{\mathsf{ra}} \circ U'^{\mathsf{ra}}_{\mathcal{D}}$ is faithful. $\square$

In the following theorem, we collect many characterizations of unimodular module categories. This result also highlights the importance of the functor Ψ^{ra} to the problem of constructing commutative Frobenius algebras in the Drinfeld center.

Theorem 4.15. *Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ an exact, left $\mathcal{C}$-module category. Then, the following are equivalent.*

(a) *$\mathcal{M}$ is a unimodular module category.*

(b) *$\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ is a unimodular multitensor category.*

(c) *$\mathbb{S}\,\mathbb{N} \cong \mathsf{Id}_{\mathcal{M}}$ as left $\mathcal{C}$-module functors.*

(d) *$\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$.*

If furthermore, $\mathcal{M}$ is indecomposable, then above conditions are equivalent to the following.

(e) *Ψ^{ra} is a Frobenius monoidal functor.*

Proof. (a)$\Leftrightarrow$(c) and (a) $\Leftrightarrow$(b) are clear from Definition 4.1 and Lemma 4.2, respectively. Also, (a)$\Leftrightarrow$(d) is known by Theorem 4.13.

Lastly, we need to show that when $\mathcal{M}$ is indecomposable, (d)$\Leftrightarrow$(e). By Proposition 4.14, the adjunctions $\Psi \dashv \Psi^{\mathsf{ra}}$ satisfies Condition 3.1. Thus, by Theorem 3.5, the claim follows. $\square$

By applying Theorem 3.9 to the functor Ψ, we get the following result.

Theorem 4.16. *Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ be an indecomposable, unimodular left $\mathcal{C}$-module category. Then, Ψ^{ra} is a separable (resp. special) Frobenius monoidal functor if and only if the Frobenius algebra $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ in $\mathcal{Z}(\mathcal{C})$ is separable (resp. special).*

Proof. By Proposition 4.14, the adjunctions $\Psi \dashv \Psi^{\mathsf{ra}}$ satisfies Condition 3.1. Thus, by Theorem 3.9 the claim follows. $\square$

For future use, we also record the following result.

Lemma 4.17. *Let $\mathcal{M}$ be an indecomposable left $\mathcal{C}$-module category. Then, $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is connected.*

Proof. The proof follows from the following computation.

$$\begin{aligned} \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbb{1}_{\mathcal{Z}(\mathcal{C})}, \Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})) \cong \mathrm{Hom}_{\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})}(\Psi(\mathbb{1}_{\mathcal{Z}(\mathcal{C})}), \mathsf{Id}_{\mathcal{M}}) &= \mathrm{Hom}_{\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})}(\mathsf{Id}_{\mathcal{M}}, \mathsf{Id}_{\mathcal{M}}) \\ &\cong \Bbbk. \end{aligned}$$

$\square$

4.3 Pivotal case

Next, we consider the pivotal case when $\mathcal{C}$ is a pivotal finite tensor category. This assumption is needed in order to construct symmetric Frobenius algebras in $\mathcal{Z}(\mathcal{C})$.

Let $\mathcal{C}$ be a pivotal tensor category with pivotal structure $\mathfrak{p} : \mathsf{Id}_{\mathcal{C}} \xrightarrow{\cong} {}^{\vee\vee}(-)$. Recall that, by Theorem 2.55(a), if $\mathcal{M}$ is an exact left $\mathcal{C}$-module category, then the (right) relative Serre functor $\mathbb{S}$ of $\mathcal{M}$ exists. Then, $\mathbb{S}$ is a left $\mathcal{C}$-module functor with module constraint given by

$$\mathbb{S}(X \triangleright M) \xrightarrow{\mathfrak{s}_{X,M}} {}^{\vee\vee}X \triangleright \mathbb{S}(M) \xrightarrow{\mathfrak{p}_X^{-1} \triangleright \mathsf{Id}_M} X \triangleright \mathbb{S}(M).$$

This allows one to define a pivotal structure on an exact $\mathcal{C}$-module category.

Definition 4.18. [Shi19, Definition 3.11] Let $\mathcal{C}$ be a pivotal tensor category. An *pivotal structure* on an exact left $\mathcal{C}$-module category $\mathcal{M}$ is a left $\mathcal{C}$-module natural isomorphism $\widetilde{\mathfrak{p}} : \mathsf{Id}_{\mathcal{M}} \to \mathbb{S}$. A *pivotal left $\mathcal{C}$-module category* is an exact left $\mathcal{C}$-module category equipped with a pivotal structure.

Shimizu proved the following interesting property of pivotal module categories.

Theorem 4.19. [Shi19, Theorem 3.13] *If $\mathcal{C}$ is a pivotal finite tensor category and $\mathcal{M}$ is a pivotal left $\mathcal{C}$-module category, then $(\mathsf{Rex}_{\mathcal{C}}\mathcal{M})^{\mathrm{rev}}$ is a pivotal finite multitensor category.* □

We use the notation $F^{\mathsf{lla}} := (F^{\mathsf{la}})^{\mathsf{la}}$ and $F^{\mathsf{rra}} = (F^{\mathsf{ra}})^{\mathsf{ra}}$. Under the assumptions of Theorem 4.19, $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ is also a pivotal monoidal category. In particular, its pivotal structure is given by:

$$\mathfrak{p}_F^{\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})} := (F \xrightarrow{F \circ \widetilde{\mathfrak{p}}} F \circ \mathbb{S} \xrightarrow{\omega^{-1}_{F^{\mathsf{lla}}}} \mathbb{S} \circ F^{\mathsf{lla}} \xrightarrow{\widetilde{\mathfrak{p}}^{-1} \circ F^{\mathsf{lla}}} F^{\mathsf{lla}}),$$

where $\omega_F : \mathbb{S}_{\mathcal{N}} \circ F \to F^{\mathsf{rra}} \circ \mathbb{S}_{\mathcal{M}}$ for $F \in \mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ is the natural isomorphism of functors from [Shi19, Theorem 3.10].

Lemma 4.20. *If $\mathcal{C}$ is a pivotal finite tensor category and $\mathcal{M}$ is a pivotal left $\mathcal{C}$-module category, then Ψ is a pivotal functor.*

Proof. By Lemma 4.12, $\Psi = U'_{\mathcal{D}} \circ \Omega_{\mathcal{D}} \circ \Theta_{\mathcal{M}}$. By [FGJS22, Proposition 5.14], $\Theta_{\mathcal{M}}$ is a pivotal functor. It is straightforward to check that $\Omega_{\mathcal{D}}$ is a pivotal functor. Also, for any pivotal monoidal category $\mathcal{D}$, the forgetful functor $\mathcal{Z}(\mathcal{D}) \to \mathcal{D}$ is pivotal (see e.g. [TV17, Section 5.2.2]). Thus, $U'_{\mathcal{D}}$ is pivotal. By Lemma 2.26, the composition of pivotal functors is pivotal. Hence, we conclude that Ψ is pivotal. □

Next, we prove the two main result of this section.

Theorem 4.21. *Let $\mathcal{C}$ be a pivotal finite tensor category and $\mathcal{M}$ be a indecomposable, pivotal, unimodular left $\mathcal{C}$-module category. Then Ψ^{ra} is a pivotal Frobenius monoidal functor.*

Proof. By Proposition 4.14, the adjunctions $\Psi \dashv \Psi^{\mathsf{ra}}$ satisfies Condition 3.1. Thus, by applying Theorem 3.11 to the adjunction $\Psi \dashv \Psi^{\mathsf{ra}}$, we obtain that Ψ^{ra} is a pivotal functor if $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is a symmetric Frobenius algebra in $\mathcal{Z}(\mathcal{C})$. First, recall that $\Psi = U' \circ \Omega \circ \Theta$. Thus, by (4.7),

$$\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}}) \cong \Theta^{-1} \circ \Omega^{-1} \circ (U')^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}}).$$

As $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ is pivotal finite tensor category, by Theorem 2.62(c), $(U')^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is a symmetric Frobenius algebra. It is straightforward that Ω is a pivotal equivalence and by [FGJS22, Proposition 1], Θ is a pivotal equivalence as well. Hence, Ω^{-1}, Θ^{-1} are also pivotal functors, and, by Lemma 2.26, so is $\Omega^{-1} \circ \Theta^{-1}$. As pivotal functors preserve symmetric Frobenius algebras, we conclude that $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is a symmetric Frobenius algebra in $\mathcal{Z}(\mathcal{C})$. So, we are done by Theorem 3.11. □

Theorem 4.22. *Let $\mathcal{C}$ be a pivotal finite tensor category and $\mathcal{M}$ be a indecomposable, pivotal, unimodular left $\mathcal{C}$-module category. Then, Ψ^{ra} is a special, pivotal Frobenius monoidal functor if and only if* $\dim(\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})) \neq 0$.

Proof. By Theorem 4.21, we know that Ψ^{ra} is a pivotal Frobenius pivotal functor. As such functors preserve symmetric Frobenius algebras, and $\mathsf{Id}_{\mathcal{M}}$ is a symmetric Frobenius algebra in $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$, it follows that $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is a symmetric Frobenius algebra in $\mathcal{Z}(\mathcal{C})$.

($\Rightarrow$) Suppose that Ψ^{ra} is special with nonzero constants β_0 and β_2. Then, by Lemma 2.28,

$$\dim(\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})) = \beta_0\beta_2 \dim(\mathsf{Id}_{\mathcal{M}}) = \beta_0\beta_2 \neq 0.$$

($\Leftarrow$) We know that $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is a connected (Lemma 4.17) Frobenius algebra of nonzero dimension. Thus, by Lemma 2.38, $\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is special Frobenius. Hence, by Theorem 4.16, Ψ^{ra} is a special, pivotal Frobenius monoidal functor. □

Remark 4.23. The starting point for our work [Yad23] was the work [FS21b] in which a recipe for constructing (commutative) symmetric Frobenius algebras in the Drinfeld center was provided. Concretely, for $\mathcal{C}$ a finite tensor category and $\mathcal{M}$, $\mathcal{N}$ exact left $\mathcal{C}$-module categories, the authors used right exact $\mathcal{C}$-module functors $F, G : \mathcal{M} \to \mathcal{N}$ to construct certain objects $\underline{\mathrm{Nat}}(F, G)$ in the Drinfeld center $\mathcal{Z}(\mathcal{C})$.

We observed that the objects $\underline{\mathrm{Nat}}(F, G)$ can be obtained functorially as follows. Consider the following functor.

$$\Psi_{\mathcal{M}} : \mathcal{Z}(\mathcal{C}) \to \mathsf{Rex}_{\mathcal{C}}(\mathcal{M}), \qquad (c, \sigma) \mapsto (c \triangleright -, s^{\sigma}).$$

Then we get observed that $\underline{\mathrm{Nat}}(F, G) = \Psi^{\mathsf{ra}}_{\mathcal{N}}(F \circ G^{\mathsf{la}})$. Hence, we can use the functor Ψ^{ra} for constructing Frobenius algebras in $\mathcal{Z}(\mathcal{C})$. This prompted us to investigate the functor $\Psi_{\mathcal{M}}$ and its right adjoint $\Psi^{\mathsf{ra}}_{\mathcal{M}}$.

Chapter 5

Tensor category of representations of a Hopf algebras

Let H be a finite dimensional Hopf algebra. In this chapter, we study the category $\mathcal{C} = \mathsf{Rep}(H)$ of finite dimensional left H-modules. In particular, we collect various results about $\mathcal{C}$ and also its module categories. These results will be used in Chapter 6 to classify unimodular $\mathcal{C}$-module categories.

The material in this chapter is based on [Yad23, Sections 4.1 and 4.2]. We start in Section 5.1 by providing background material on exact H-comodule algebras A. We explicitly describing the Nakayama functor of the $\mathsf{Rep}(H)$-module category $\mathsf{Rep}(A)$ and its twisted $\mathsf{Rep}(H)$-module structure. Using this, we describe the Radford isomorphism of $\mathsf{Rep}(H)$ and the relative Serre functor of its module categories $\mathsf{Rep}(A)$ in Section 5.2.

Definition 5.1. A *bialgebra* is a tuple $(H, m, u, \Delta, \varepsilon)$ such that

- (H, m, u) is an algebra over $\Bbbk$,
- (H, Δ, ε) is a coalgebra over $\Bbbk$, and
- Δ, ε are algebra maps.

In this section and the next one, we will frequently use the Sweedler notation for denoting coproducts and coations. For example, $\Delta(h)$ will be denoted as $h_1 \otimes h_2$.

Definition 5.2. A bialgebra $(H, m, u, \Delta, \varepsilon)$ is called a *Hopf algebra* is it admits a map $S : H \to H$ (called the *antipode*) which satisfies that $S(h_2)h_2 = \varepsilon(h)1_H = h_1 S(h_2)$.

Here are a few immediate properties that we will need in the following. See [Rad11] for details.

(a) H finite dimensional $\Rightarrow$ S is bijective.
(b) $H^* = \mathrm{Hom}(H, \Bbbk)$ is also a Hopf algebra over $\Bbbk$.

From here on, we fix $\mathcal{C} = \mathsf{Rep}(H)$ for H a finite dimensional Hopf algebra over $\Bbbk$. Then, we get the following result.

Lemma 5.3. *Let H be a finite-dimensional Hopf algebra. Then, $\mathcal{C} = \mathsf{Rep}(H)$ is a finite tensor category.*

Proof. Observe that

- $\mathcal{C}$ is $\Bbbk$-*linear*, *abelian*.
- $\mathcal{C}$ is *monoidal*: unit object is $\Bbbk \in \mathcal{C}$ with action $h \cdot 1_{\Bbbk} = \varepsilon(h)$; for $X, Y \in \mathcal{C}$, $X \otimes Y \in \mathcal{C}$ via action $h \cdot (x \otimes y) = h_1 \cdot x \otimes h_2 \cdot y$.
- $\mathcal{C}$ is *rigid*: for $X \in \mathcal{C}$, define ${}^{\vee}X, X^{\vee} = \mathrm{Hom}(X, \Bbbk) = X^*$ as vector spaces. Then ${}^{\vee}X \in \mathcal{C}$ via $(h \cdot f)(v) = f(S(h) \cdot v)$ and $X^{\vee} \in \mathcal{C}$ via $(h \cdot f)(v) = f(S^{-1}(h) \cdot v)$.
- The tensor product bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is biexact and $\mathrm{End}(\mathbb{1}) \cong \Bbbk$.

Thus, $\mathcal{C}$ is a *finite tensor category.* $\square$

5.1 Exact comodule algebras and the Nakayama functor

A *left H-comodule algebra* is a left H-comodule (A, ρ) with an algebra structure such that the multiplication and unit maps are H-comodule maps, that is,

$$\rho(aa') = a_{(-1)}a'_{(-1)} \otimes_{\Bbbk} a_{(0)}a'_{(0)}, \qquad \rho(1_A) = 1_H \otimes_{\Bbbk} 1_A \qquad (\forall\, a, a' \in A) \tag{5.1}$$

where $\rho(a)$ is denoted as $a_{(-1)} \otimes a_{(0)} \in H \otimes A$. Then the category $\mathsf{Rep}(A)$ is a left $\mathsf{Rep}(H)$-module category via the action $\triangleright : \mathsf{Rep}(H) \times \mathsf{Rep}(A) \to \mathsf{Rep}(A)$ where $X \triangleright M = X \otimes_{\Bbbk} M$ as vector space, and the A-action on $X \triangleright M$ is defined as

$$a \cdot (x \otimes m) = a_{(-1)}x \otimes a_{(0)}m \qquad (a \in A,\ x \in X,\ m \in M).$$

In fact, by [AM07], every left $\mathcal{C}$-module category $\mathcal{M}$ is of the form $\mathcal{M} = \mathsf{Rep}(A)$ for A a left H-comodule algebra. In this setting, A is called *exact* (resp., *indecomposable*) if the $\mathcal{C}$-module category $\mathcal{M}$ is exact (resp., indecomposable).

By [Shi19, Lemma 4.5] (see also [Skr07]), every exact left H-comodule algebra A is a Frobenius algebra. Thus, we can endow A with a *Frobenius system*, that is, a triple $(\lambda_A, \{a^i\}, \{b_i\})$ with $1 \leq i \leq r = \dim(A)$. Here $\lambda_A : A \to \Bbbk$ is a linear map and $\{a^i\}$, $\{b_i\}$ are two bases of A such that $\langle \lambda_A, a^i b_j \rangle = \delta_{i,j}$ for all $i, j = 1, \ldots, r$. The *Nakayama automorphism* of A (with respect to λ_A) is the unique algebra automorphism $\nu := \nu_A : A \to A$ characterized by

$$\langle \lambda_A, ab \rangle = \langle \lambda_A, \nu_A(b)a \rangle \qquad (a, b \in A). \tag{5.2}$$

By [Shi19, Lemma 4.2] , the following equalities hold for all $c \in A$:

$$\langle \lambda_A, a^i \rangle b_i = 1_A = \langle \lambda_A, b_i \rangle a^i, \quad a^i c \otimes b_i = a^i \otimes c b_i, \quad \nu_A(c) a^i \otimes b_i = a^i \otimes b_i c. \tag{5.3}$$

Notation 5.4. We will use the following notations going forward.

(a) For a vector space M, we will denote the basis of M by m_i and the dual basis of M^* by m^i. These satisfy $\langle m^i, m\rangle m_i = m$ for all $m \in M$.

(b) If $(X, \cdot)$ is a left H-module and $f : H' \to H$ is an algebra map. We use the notation $({}_f X, \cdot_f)$ to denote the H'-module X with action given by $h' \cdot_f x := f(h') \cdot x$ for $h' \in H', x \in X$.

(c) $\phi_M^N : M^* \otimes_A N \xrightarrow{\sim} \mathrm{Hom}_A(M, N)$ is an isomorphism given by $m^* \otimes_A n \mapsto \langle m^*, ?\rangle n$ with inverse $f \mapsto m^i \otimes_A \langle f, m_i\rangle$.

Nakayama functor of $\mathsf{Rep}(A)$

Let A be a Frobenius algebra. We first provide a description of the Nakayama functor of $\mathsf{Rep}(A)$.

Theorem 5.5. *Let A be a Frobenius algebra with Nakayama automorphism ν. Then, we have that*

$$\mathbb{N}_{\mathsf{Rep}(A)}(M) = \int^{N \in \mathsf{Rep}(A)} \mathrm{Hom}_A(M, N)^* \blacktriangleright N = {}_\nu M. \tag{5.4}$$

The projection maps $\mathbb{i}_{M,N} : \mathrm{Hom}_A(M, N)^* \blacktriangleright N \to {}_\nu M$ of the coend are given by

$$\mathbb{i}_{M,N}(\xi \otimes_{\Bbbk} n) = \langle \xi, \phi_M^N(m^i \otimes_A a^j \cdot n)\rangle \nu(b_j) \cdot m_i \tag{5.5}$$

for all $\xi \in \mathrm{Hom}_A(M, N)^*$ and $n \in N$.

While this result is well-known to the experts, we could not find a direct proof of it in the literature. Hence, for the reader's convenience, we provide a direct proof in Appendix A.0.1.

Now suppose that A is an exact left H-comodule algebra. Then, by Theorem 2.56(b), $\mathbb{N}_{\mathsf{Rep}(A)}$ is a twisted left $\mathsf{Rep}(H)$-module functor. The following result explicitly describes this structure.

Theorem 5.6. *Let A be an exact left H-comodule algebra. The twisted left $\mathsf{Rep}(H)$-module structure $\mathfrak{n}^l_{X,M} : {}_\nu(X \triangleright M) \to X^{\vee\vee} \triangleright {}_\nu M$ of the Nakayama functor $\mathbb{N}$ of $\mathsf{Rep}(A)$ is given by*

$$\mathfrak{n}^l_{X,M}(x \otimes_{\Bbbk} m) = \langle \lambda_A, a^i_{(0)}\rangle \phi_X(S^{-1}(a^i_{-1}) \cdot x) \otimes_{\Bbbk} \nu(b_i) \cdot m. \tag{5.6}$$

The inverse of $\mathfrak{n}^l_{X,M}$ is given by

$$\overline{\mathfrak{n}}^l_{X,M}(\phi_X(x) \otimes_{\Bbbk} m) = \langle \lambda_A, a^i_{(0)}\rangle \nu(b_i) \cdot \left(S^{-2}(a^i_{(-1)}) \cdot x \otimes_{\Bbbk} m\right). \tag{5.7}$$

Now, let B be a right H-comodule algebra. Then $\mathsf{Rep}(B)$ is a right $\mathsf{Rep}(H)$-module category. In this case, the Nakayama functor $\mathbb{N}_{\mathsf{Rep}(B)}$ is a twisted right $\mathsf{Rep}(H)$-module functor and the following result describes this structure.

Theorem 5.7. *Let B be an exact right H-comodule algebra. The twisted right $\mathcal{C}$-module structure $\mathfrak{n}^r_{X,M} : {}_\nu(M \triangleleft X) \to {}_\nu M \triangleleft {}^{\vee\vee}X$ of the Nakayama functor $\mathbb{N}$ of $\mathsf{Rep}(B)$ is given by*

$$\mathfrak{n}^r_{X,M}(m \otimes_{\Bbbk} x) = \langle \lambda_B, a^i_{(0)} \rangle \nu(b_i) \cdot m \otimes_{\Bbbk} \phi_X(S(a^i_{(1)}) \cdot x). \tag{5.8}$$

The inverse of $\mathfrak{n}^r_{X,M}$ is given by

$$\overline{\mathfrak{n}}^r_{X,M}(m \otimes_{\Bbbk} \phi_X(x)) = \langle \lambda_B, a^i_{(0)} \rangle \nu(b_i) \cdot \Big(m \otimes_{\Bbbk} S^2(a^i_{(1)}) \cdot x\Big). \tag{5.9}$$

The proof of Theorem 5.7, which can be inferred from [SS21, Theorem 7.3], is provided in Appendix A.0.2. Then, Theorem 5.6 is proved by applying Theorem 5.7 to the exact right H-comodule algebra A^{op}.

5.2 Radford's isomorphism and the Serre functor

In this section, we calculate the Radford isomorphism (2.15) for the finite tensor category $\mathcal{C} = \mathsf{Rep}(H)$. This will allow us to explicitly describe the Serre functor of the module category $\mathsf{Rep}(A)$. We emphasize the Radford isomorphism for $\mathsf{Rep}(H)$ and Serre functor of $\mathsf{Rep}(A)$ are known by prior work of Shimizu [Shi19].

5.2.1 (Co)integrals of a Hopf algebra

Definition 5.8. Let H be a finite-dimensional Hopf algebras. Going forward, we will need the following notions:

(a) An element $g \in H$ is called a *grouplike elements* of H if it satisfies that $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$.

(b) An element $\Lambda \in H$ satisfying $h\Lambda = \varepsilon(h)\Lambda$ for all $h \in H$ is called a *left integral* of H.

(c) A *right cointegral* of H is any element $\lambda_H \in H^*$ satisfying

$$\langle \lambda_H, h_{(1)} \rangle h_{(2)} = \langle \lambda_H, h \rangle 1_H \text{ for all } h \in H. \tag{5.10}$$

(d) For any left integral Λ, $\Lambda h = \Lambda \alpha_H(h)$ for some $\alpha_H \in G(H^*)$. Here α_H is called the *distinguished character* of H^*.

(e) The *distinguished grouplike* element of H is an element $g_H \in H$ satisfying

$$h_{(1)}\langle \lambda_H, h_{(2)}\rangle = \langle \lambda_H, h\rangle g_H \text{ for all } h \in H. \tag{5.11}$$

Next, we recall from [Rad11] certain results about (co)integrals that will be needed in later sections.

Theorem 5.9. (a) *Every finite dimensional Hopf algebra admits a non-zero left integral Λ and a non-zero right integral λ_H. We can choose Λ and λ_H such that $\langle \lambda_H, \Lambda\rangle = 1$.*

(b) *Invertible elements are in bijection with $G(H^*)$ Given $\beta \in G(H^*)$, the corresponding invertible object is $\Bbbk_\beta$ which is $\Bbbk$ as a vector space and H-action is given by $h \cdot c := \beta(h)c$.*

(c) *Every finite dimensional Hopf algebra is a Frobenius algebra with any right cointegral λ_H as the Frobenius form.*

(d) *The Nakayama automorphism* ν of the Frobenius algebra (H, λ) is $\nu(h) = \alpha(y_1)S^2(y_2)$.*

Notation 5.10. In light of Theorem 5.9(a), for every f.d. Hopf algebra H, we fix a left integral Λ and right cointegral λ_H satisfying $\langle \lambda_H, \Lambda\rangle = 1$. Also, set $\overline{\alpha}_H := \alpha_H \circ S$ and $\overline{g_H} := g_H^{-1}$.

5.2.2 Radford isomorphism

It is well known that finite dimensional Hopf algebras are Frobenius with the Frobenius form given by any right cointegral λ. By [Rad94, Theorem 3(a,b)], we get that the Nakayama automorphism of H is given by $\nu_H(h) = \langle \alpha_H, h_1\rangle S^2(h_2)$ and its inverse $\overline{\nu}$ is

$$\overline{\nu}_H(h) = S^2(\overline{g_H} h_1 g_H)\langle \alpha_H, g_H S(h_2)\overline{g_H}\rangle \overset{(\dagger)}{=} S^2(\overline{g_H} h_1 g_H)\langle \alpha_H, S(h_2)\rangle.$$

Here the equality (†) holds because α is grouplike. Thus,

$$\langle \alpha_H, g_H S(h_2)\overline{g_H}\rangle = \langle \alpha_H, g_H\rangle\langle \alpha_H, S(h_2)\rangle\langle \alpha_H, \overline{g_H}\rangle = \langle \alpha_H, S(h_2)\rangle.$$

Lemma 5.11. *We have the following results for $\mathbb{1}_{\mathsf{Rep}(H)} = \Bbbk$.*

(a) $\mathbb{N}_{\mathsf{Rep}(H)}(\Bbbk) = \Bbbk$ *as a vector space with H-action given by $h \star c := \langle \alpha_H, h\rangle c$.*

(b) $\overline{\mathbb{N}}_{\mathsf{Rep}(H)}(\Bbbk) = \Bbbk$ *as vector space with H-action given by $h \overline{\star} c := \langle \alpha_H, S(h)\rangle c$.*

*ν is defined as the unique automorphism of H such that $\lambda(xy) = \lambda(\nu(y)x)$. By, [Rad11, Theorem 10.5.4(e)] the claim follows.

Proof. By Theorem 5.5, $\mathbb{N}(\Bbbk) = {}_{\nu}\Bbbk$. Since, the H-action on $\Bbbk$ is given by $h \cdot c = \langle \varepsilon_H, h\rangle c$, we get that that the H-action $\star$ on ${}_{\nu_H}\Bbbk$ is given by

$$h \star c = \langle \varepsilon_H, \nu_H(h)\rangle c = \langle \alpha_H, h_1\rangle\langle \varepsilon_H, S^2(h_2)\rangle c = \langle \alpha_H, h_1\rangle\langle \varepsilon_H, h_2\rangle = \langle \alpha_H, h\rangle c. \quad (5.12)$$

As the functor $\overline{\mathbb{N}}$ is the quasi-inverse of $\mathbb{N}$, it clear that $\overline{\mathbb{N}}(M) = {}_{\overline{\nu}}M$. Now using the same argument as above, the claim follows. □

The following material is based on the discussion in [SS21, §6.3]. The vector space H is a left and right H-comodule algebras. Thus, the category $\mathsf{Rep}(H)$ is $\mathsf{Rep}(H)$-bimodule category. By using Theorems 5.6 and 5.7 with $A = B = H$, we get the following map.

$$g_X : \mathbb{N}(\mathbb{1}) \triangleleft {}^{\vee\vee}X \xrightarrow{\overline{\mathfrak{n}}^r_{X,\mathbb{1}}} \mathbb{N}(\mathbb{1} \triangleleft X) \xrightarrow{\text{flip}} \mathbb{N}(X \triangleright \mathbb{1}) \xrightarrow{\mathfrak{n}^l_{X,\mathbb{1}}} X^{\vee\vee} \triangleright \mathbb{N}(\mathbb{1}) \qquad (5.13)$$

By [FSS20, Lemma 4.11], $D = \overline{\mathbb{N}}(\mathbb{1})$ is the right dual of $\mathbb{N}(\mathbb{1})$. The evaluation and coevaluation maps are trivial identity maps. As explained in [Shi17, Remark 4.11], the map $\mathfrak{R}_X$ from [ENO04] is equal to the following composition.

$${}^{\vee\vee}X \otimes D \xrightarrow{\mathrm{coev}_D \otimes \mathsf{Id}} D \otimes \mathbb{N}(\mathbb{1}) \otimes {}^{\vee\vee}X \otimes D \xrightarrow{\mathsf{Id} \otimes g_X \otimes \mathsf{Id}} D \otimes X^{\vee\vee} \otimes \mathbb{N}(\mathbb{1}) \otimes D \xrightarrow{\mathsf{Id} \otimes \mathrm{ev}_D} D \otimes X^{\vee\vee} \qquad (5.14)$$

We follow this strategy to calculate the Radford isomorphism. To start, we calculate the map g_X.

Lemma 5.12. *The map g_X in (5.13) is given by $c \otimes_{\Bbbk} \phi_X(x) \mapsto \phi_X(\overline{g_H} \cdot x) \otimes_{\Bbbk} c$.*

Proof. Fix any Frobenius system $(\lambda_H, \{a^i\}, \{b_i\})$ of H where λ_H is a right integral of H.

$$\begin{aligned}
c \otimes_{\Bbbk} \phi_X(x) \quad &\mapsto \quad \mathfrak{n}^l_{X,\mathbb{1}} \circ \text{flip} \circ \overline{\mathfrak{n}}^r_{X,\mathbb{1}}(c \otimes_{\Bbbk} \phi_X(x)) \\
&\overset{(5.9)}{=} \mathfrak{n}^l_{X,\mathbb{1}} \circ \text{flip}\Big[\langle \lambda_H, a^i_{(1)}\rangle \nu(b_i) \cdot \Big(c \otimes_{\Bbbk} S^2(a^i_1) \cdot x\Big)\Big] \\
&= \mathfrak{n}^l_{X,\mathbb{1}}\Big[\langle \lambda_H, a^i_{(1)}\rangle \Big(\nu(b_i)_{(2)} S^2(a^i_2) \cdot x \otimes_{\Bbbk} \nu(b_i)_{(1)} \cdot c\Big)\Big] \\
&\overset{(5.6)}{=} \langle \lambda_H, a^j_{(2)}\rangle\langle \lambda_H, a^i_{(1)}\rangle \phi_X(S^{-1}(a^j_1)\nu(b_i)_{(2)} S^2(a^i_2) \cdot x) \otimes_{\Bbbk} \nu(b_j)\nu(b_i)_{(1)} \cdot c \\
&\overset{(5.10,5.11)}{=} \langle \lambda_H, a^j\rangle\langle \lambda_H, a^i\rangle \phi_X(S^{-1}(g_H)\nu(b_i)_{(2)} S^2(1_H) \cdot x) \otimes_{\Bbbk} \nu(b_j)\nu(b_i)_{(1)} \cdot c \\
&\overset{(5.3)}{=} \phi_X(\overline{g_H} \cdot x) \otimes_{\Bbbk} c
\end{aligned}$$

□

Remark 5.13. For every finite dimensional Hopf algebra H, Radford [Rad76] proved that

$$S^4(h) = g_H[\alpha_H(h_1)h_2\overline{\alpha}_H(h_3)]\overline{g_H}.$$

The fact that g_X is a morphism of left H-modules is equivalent to above mentioned formula for the fourth power of antipode after plugging $h = \alpha_H \rightharpoonup S^{-2}(h)$.

Proposition 5.14. *The Radford isomorphism (2.15) for $\mathcal{C} = \mathsf{Rep}(H)$ is given by*

$$\mathfrak{R}_X : {}^{\vee\vee}X \otimes D \to D \otimes X^{\vee\vee}, \quad \phi_X(x) \otimes c \mapsto c \otimes \phi_X(\overline{g_H} \cdot x) \qquad \text{for } c \in D, x \in X. \tag{5.15}$$

Proof. Plugging the formula for g_X into (5.14), we that

$$\phi_X(x) \otimes_{\Bbbk} c \xmapsto{\mathrm{coev}_D \otimes \mathsf{Id}} 1 \otimes_{\Bbbk} 1 \otimes_{\Bbbk} \phi_X(c) \otimes_{\Bbbk} c \xmapsto{\mathsf{Id} \otimes g_X \otimes \mathsf{Id}} 1 \otimes_{\Bbbk} \phi_X(\overline{g_H} \cdot x) \otimes_{\Bbbk} 1 \otimes_{\Bbbk} c \xmapsto{\mathsf{Id} \otimes \mathrm{ev}_D} c \otimes_{\Bbbk} \phi_X(\overline{g_H} \cdot x).$$

Hence, the claim follows. ☐

5.2.3 Serre functor

By [FSS20, Theorem 4.26], the relative Serre functor satisfies that $\mathbb{S} \cong D \triangleright \mathbb{N}$ as a left $\mathcal{C}$-module functor. Since, the Serre functor is unique up to isomorphism, we take the above as the definition of it. Then we get the following result.

Theorem 5.15. *The Serre functor of $\mathsf{Rep}(A)$ is given by $\mathbb{S}(M) = {}_{\nu'}(M)$ where*

$$\nu'(a) = \langle \alpha_H, S(a_{(-1)}) \rangle \nu(a_{(0)}). \tag{5.16}$$

The twisted left $\mathsf{Rep}(H)$-module structure $\mathfrak{s}^l_{X,M} : {}_{\nu'}(X \triangleright M) \xrightarrow{\sim} {}^{\vee\vee}X \triangleright {}_{\nu'}M$ is given by

$$\mathfrak{s}(x \otimes_{\Bbbk} m) = \langle \lambda, a^i_{(0)} \rangle \phi_X(g_H S^{-1}(a^i_{(-1)}) \cdot x) \otimes_{\Bbbk} \nu(b_i) \cdot m \tag{5.17}$$

Proof. The formula (5.16) from the formula of the Nakayama functor (5.4) and the description of D in Lemma 5.11(b). The formula (5.17) follows by composing the twisted left $\mathsf{Rep}(H)$-module structure of $\mathbb{N}_{\mathsf{Rep}(A)}$ given in (5.6) the inverse $\mathfrak{R}_X^{-1}$ of the Radford isomorphism (5.15). ☐

Remark 5.16. A different formula for the relative Serre functor was also provided in [Shi19].

Chapter 6

Unimodular structures on $\mathsf{Rep}(A)$

Let H be a finite-dimensional Hopf algebra. In this chapter, we classify unimodular $\mathsf{Rep}(H)$-module categories. This chapter is based on [Yad23, Sections 4.3-4.5].

We use the description of the Nakayama functor and the relative Serre functor of $\mathsf{Rep}(A)$ provided in Sections 5.1 and 5.2.3, respectively, to describe the $\mathcal{C}$-module functor $\mathbb{S}_{\mathsf{Rep}(A)}\mathbb{N}_{\mathsf{Rep}(A)}$ in Section 6.1. Then we introduce unimodular elements of an exact H-comodule algebra A. Using them, in Section 6.2, we characterize unimodular structures on $\mathsf{Rep}(A)$. As the definition of unimodular elements is involved, in Section 6.3 we discuss the simpler case when A admits a grouplike cointegral. In this case, the definition of unimodular elements is much simpler. To illustrate our results, we consider the example of Taft Hopf algebras in Section 6.4. Finally, we end with some remarks and question in Section 6.5.

6.1 Description of the functor $\mathbb{S}_{\mathsf{Rep}(A)}\mathbb{N}_{\mathsf{Rep}(A)}$

Theorem 6.1. *For $M \in \mathsf{Rep}(A)$, we have that $\mathbb{S}\,\mathbb{N}(M) = {}_{\widetilde{\nu}}M \in \mathsf{Rep}(A)$ where*

$$\widetilde{\nu}(a) = \langle \alpha_H, S(a_{-1})\rangle \nu^2(a_0). \tag{6.1}$$

The left $\mathcal{C}$-module structure of $\mathbb{S}\,\mathbb{N}$ is given by

$$\mathfrak{d}_{X,M}(x \triangleright m) = (\Im_H \cdot x) \triangleright (\Im_L \cdot m), \ \textit{where} \tag{6.2}$$

$$\Im = \Im_H \otimes \Im_L := \langle \lambda_A, a_0^i\rangle\langle \lambda_A, a_0^j\rangle\ g_H S^{-3}(a_{-1}^j)S^{-1}(a_{-1}^i) \otimes_{\Bbbk} \nu(b_j b_i) \in H \otimes_{\Bbbk} L. \tag{6.3}$$

Proof. By the descriptions of the Nakayama functor (5.4) and the relative Serre functor (5.16), it is clear that as a functor, $\mathbb{S}\,\mathbb{N}$ is given by (6.1). Further, using the the twisted left $\mathcal{C}$-module structures of the relative Serre functor (5.17) and the Nakayama functor (5.6) , we get that, $\mathbb{S}\,\mathbb{N} : \mathcal{M} \to \mathcal{M}$ is a left $\mathcal{C}$-module functor with the left $\mathcal{C}$-module structure $\mathfrak{d}_{X,M}$ given by the following composition.

$$\mathfrak{d}_{X,M}\ :\ \mathbb{S}\mathbb{N}(X \triangleright M) \xrightarrow{\mathbb{S}(\mathfrak{n}_{X,M})} \mathbb{S}(X^{\vee\vee} \triangleright \mathbb{N}(M)) \xrightarrow{\mathfrak{s}_{X^{\vee\vee},\mathbb{N}(M)}} {}^{\vee\vee}X^{\vee\vee} \triangleright \mathbb{S}\mathbb{N}(M) = X \triangleright \mathbb{S}\mathbb{N}(M).$$

The following calculation yields an explicit description of the left $\mathcal{C}$-module structure of the functor $\mathbb{S}\,\mathbb{N}$.

$$
\begin{aligned}
& x \otimes_{\Bbbk} m \\
& \xmapsto{\mathbb{S}(\mathfrak{n}_{X,M})} \langle \lambda, a^i_0 \rangle \phi_X(S^{-1}(a^i_{-1}) \cdot x) \otimes_{\Bbbk} \nu(b_i) \cdot m \\
& \xmapsto{\mathfrak{s}_{X^{\vee\vee},\mathbb{N}(M)}} \langle \lambda_A, a^j_0 \rangle \langle \lambda_A, a^i_0 \rangle \phi_{X^{\vee\vee}}(g_H S^{-1}(a^j_{-1}) \cdot \phi_X(S^{-1}(a^i_{-1}) \cdot x)) \otimes_{\Bbbk} \nu(b_j)\nu(b_i) \cdot m \\
& \overset{(\spadesuit)}{=} \langle \lambda_A, a^j_0 \rangle \langle \lambda_A, a^i_0 \rangle \phi_{X^{\vee\vee}} \circ \phi_X(S^{-2}(g_H S^{-1}(a^j_{-1}))S^{-1}(a^i_{-1}) \cdot x) \otimes_{\Bbbk} \nu(b_j)\nu(b_i) \cdot m \\
& \overset{(\diamondsuit)}{=} \langle \lambda_A, a^j_0 \rangle \langle \lambda_A, a^i_0 \rangle S^{-2}(g_H S^{-1}(a^j_{-1}))S^{-1}(a^i_{-1}) \cdot x \otimes_{\Bbbk} \nu(b_j b_i) \cdot m \\
& \overset{(\heartsuit)}{=} \langle \lambda_A, a^j_0 \rangle \langle \lambda_A, a^i_0 \rangle g_H S^{-3}(a^j_{-1})S^{-1}(a^i_{-1}) \cdot x \otimes_{\Bbbk} \nu(b_j b_i) \cdot m
\end{aligned}
$$

Here $(\lambda_A, \{a^i\}, \{b_i\})$ and $(\lambda_A, \{a^j\}, \{b_j\})$ are Frobenius systems of A. The equality $(\spadesuit)$ holds because the left action of H on $X^{\vee\vee}$ is given $h \cdot \phi_X(x) = \phi_X(S^{-2}(h) \cdot x)$. The equality $(\diamondsuit)$ holds because we identify X and ${}^{\vee\vee}X^{\vee\vee}$ via the map $\phi_{X^{\vee\vee}} \circ \phi_X$ and ν is an algebra map. Lastly, $(\heartsuit)$ holds because g_H is grouplike. From the above computation, it follows that the left $\mathcal{C}$-module structure of $\mathbb{S}\,\mathbb{N}$ is as described by equations (6.2) and (6.3). □

6.2 Unimodular elements in exact H-comodule algebras

From Definition 4.1, recall that $\mathsf{Rep}(A)$ is a unimodular $\mathsf{Rep}(H)$-module category if and only if there exists a $\mathcal{C}$-module natural isomorphism $\mathfrak{u} : \mathsf{Id}_{\mathcal{M}} \to \mathbb{S}\,\mathbb{N}$. Next, we will use Theorem 6.1, to characterize such natural isomorphisms using certain invertible elements of the algebra A. To this end, consider the following notion.

Definition 6.2. Let A be an exact, left H-comodule algebra. A *unimodular element* of A is an invertible element $\widetilde{g} \in A$ satisfying the following two relations:

$$\widetilde{g} a \widetilde{g}^{-1} = \widetilde{\nu}(a) \qquad (\forall\, a \in A) \tag{6.4}$$

$$1_H \otimes \widetilde{g} = \Im \cdot \delta(\widetilde{g}) \tag{6.5}$$

Here δ is comodule structure of A. For the definition of $\widetilde{\nu}$ and $\Im$, see (6.1) and (6.2), respectively.

Using this, we obtain the following result.

Theorem 6.3. *Let A be an exact, left H-comodule algebra. Then, unimodular structures on the* $\mathsf{Rep}(H)$*-module category* $\mathsf{Rep}(A)$ *are in bijection with unimodular elements of A.*

Proof. Suppose that we have a unimodular structure $\mathfrak{u} : \mathsf{Id}_{\mathcal{M}} \to \mathbb{S}\,\mathbb{N} \overset{(6.1)}{=} {}_{\widetilde{\nu}}(-)$. Then, we get the element $\widetilde{g} = \mathfrak{u}_A(1_A) \in A$. By naturality of $\mathfrak{u}$, it follows that $\mathfrak{u}_X(x) = \widetilde{g} \cdot x$ for all $x \in X$ and $X \in \mathsf{Rep}(A)$. As $\mathfrak{u}_A$ is an isomorphism, $\widetilde{g}$ is an invertible element in A.

- As $\mathfrak{u}_A = \widetilde{g} \cdot (-) : A \to {}_{\widetilde{\nu}}A$ is a map of left A-modules, condition (6.4) is satisfied.
- As $\mathfrak{u}$ is a $\mathcal{C}$-module natural isomorphism, the diagram (4.2) commutes. Using $\mathfrak{u}_X(x) = \widetilde{g} \cdot x$ and that the $\mathcal{C}$-module structure of $\mathbb{S}\,\mathbb{N}$ is given by $\Im$ (6.2), we get that (6.5) is satisfied.

Thus, $\widetilde{g}$ is a unimodular element of A.

Conversely, given a unimodular element $\widetilde{g}$ of A, we define the natural isomorphism

$$\mathfrak{u} = \{\mathfrak{u}_M : M \to \mathbb{S}\,\mathbb{N}(M), \quad m \mapsto \widetilde{g} \cdot m\}_{M \in \mathsf{Rep}(A)}.$$

Then, repeating the above arguments backwards, we get that $\mathfrak{u} = \{\mathfrak{u}_M\}$ is a unimodular structure on $\mathsf{Rep}(A)$. $\square$

6.3 Grouplike cointegrals on comodule algebras

Consider the following notion.

Definition 6.4. [Kas18] Let H be a Hopf algebra and A a left H-comodule algebra. A *grouplike cointegral* on A is a pair (g, λ) consisting of a grouplike element $g \in H$ and a linear form $\lambda : A \to \Bbbk$ such that the equation

$$a_{(-1)}\langle \lambda, a_{(0)} \rangle = \langle \lambda, a \rangle g \tag{6.6}$$

holds for all elements $a \in A$. In this situation, λ is called a *g-cointegral* on A. If λ is a Frobenius form on A, the g-cointegral λ is called *non-degenerate*.

Next, we see the the $\mathcal{C}$-module structure of the functor $\mathbb{S}\,\mathbb{N}$ simplifies when we have a grouplike cointegral on A.

Theorem 6.5. *Recall the element $\Im$ from (6.3) and the left $\mathcal{C}$-module structure of $D_{\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})} = \mathbb{S}\,\mathbb{N}$ (6.2). If the Frobenius form λ_A of A is a g_A-grouplike integral for some grouplike element $g_A \in H$, then we have that*

$$\Im = g_A^{-2} g_H \otimes 1_A. \tag{6.7}$$

Thus, for $X \in \mathcal{C}$ and $M \in \mathcal{M}$, the left $\mathcal{C}$-module structure of $\mathbb{S}\,\mathbb{N}$ given by:

$$\mathbb{S}\,\mathbb{N}(X \triangleright M) \to X \triangleright \mathbb{S}\,\mathbb{N}(M), \qquad x \triangleright m \mapsto \phi_X(g_A^{-2} g_H \cdot x) \triangleright m.$$

Proof. Observe that

$$\begin{aligned}
\Im &\overset{(6.3)}{=} \langle \lambda_A, a_0^i \rangle \langle \lambda_A, a_0^j \rangle \, g_H S^{-3}(a_{-1}^j) S^{-1}(a_{-1}^i) \otimes_{\Bbbk} \nu(b_j b_i) \\
&\overset{(6.6)}{=} \langle \lambda_A, a^i \rangle \langle \lambda_A, a^j \rangle \, g_H S^{-3}(g_A) S^{-1}(g_A) \otimes_{\Bbbk} \nu(b_j b_i) \\
&\overset{(5.3)}{=} g_H S^{-3}(g_A) S^{-1}(g_A) \otimes_{\Bbbk} \nu(1_A) \\
&\overset{(\spadesuit)}{=} g_H g_A^{-1} g_A^{-1} \otimes 1_A \\
&\overset{(\diamondsuit)}{=} g_A^{-2} g_H \otimes 1_A.
\end{aligned}$$

Here, the equality $(\spadesuit)$ holds because g_A is grouplike element of H and ν is an algebra map. The equality $(\diamondsuit)$ holds because g_H commutes with all grouplike elements of H. □

When the Frobenius form on the H-comodule algebra under consideration is a grouplike cointegral, Theorem 6.3 simplifies a lot and we recover [Shi22, Corollary 7.10].

Corollary 6.6. *If the Frobenius form λ_A of A is a g_A-grouplike cointegral for some $g_A \in H$, then, the unimodular structures on $\mathsf{Rep}(A)$ are in bijection with invertible elements $\widetilde{g} \in A$ satisfying:*

$$\widetilde{g} a \widetilde{g}^{-1} = \widetilde{\nu}(a) = \langle \alpha_H, S(a_{-1}) \rangle \nu^2(a_0), \qquad g_H^{-1} g_A^2 \otimes \widetilde{g} = \delta(\widetilde{g}) \qquad \forall\, a \in A. \tag{6.8}$$

□

The category $\mathsf{Vec} = \mathsf{Rep}(\Bbbk)$ is an exact left H-module category as $\Bbbk$ is an exact left H-comodule algebra with the H-coaction given by $1 \mapsto 1_H \otimes 1$. Then, by applying Corollary 6.6 to the H-comodule algebra $\Bbbk$, we obtain $\Bbbk$ admits a unimodular structure if and only if $g_H = 1_H$, that is, H^* is unimodular. This is consistent with our observation in Example 4.4(iii).

6.4 Taft algebras example

In this section, we study the case of $\mathcal{C} = \mathsf{Rep}(H)$ for H being the Taft algebra $T(\omega)$. Let $\Bbbk$ be an algebraically closed field of characteristic 0. To define Taft algebras, fix an integer $N > 1$ and a primitive N-th root of unity $\omega \in \Bbbk$. Then $T(\omega)$ is defined as the $\Bbbk$-algebra generated by g and x subject to the relations

$$x^N = 0, \quad g^N = 1 \quad \text{and} \quad gx = \omega x g. \tag{6.9}$$

Equipped with the following comultiplication and antipode maps, $T(\omega)$ becomes a Hopf algebra.

$$\Delta(g) = g \otimes g, \ \ \Delta(x) = x \otimes 1 + g \otimes x \quad \text{and} \quad S(g) = g^{-1}, \ \ S(x) = -g^{-1}x. \tag{6.10}$$

From this, we get the following information.

- The element $\Lambda^l = \sum_{i=0}^{N-1} g^i x^{N-1}$ is a non-zero left integral of $T(\omega)$. The distinguished character of $T(\omega)$ is given by $\alpha_{T(\omega)}(g) = \omega$ and $\alpha_{T(\omega)}(x) = 0$.
- The functional $\lambda_{T(\omega)} : T(\omega) \to \Bbbk$ given by $\lambda_{T(\omega)}(x^r g^s) = \delta_{r,N-1}\delta_{s,0}$ for $r, s = 0, \dots, N-1$ is a right cointegral of $T(\omega)$. The distinguished grouplike element of $T(\omega)$ is given by $g_{T(\omega)} = g^{-1}$.

As $g_{T(\omega)} \neq 1_H$, we have that $T(\omega)^*$ is not unimodular. Thus, by Example 4.4, we have that Vec is not a unimodular $\mathsf{Rep}(T(\omega))$-module category. In fact, as we will see below, $\mathsf{Rep}(T(\omega))$ does not admit a unimodular module category.

Indecomposable, left, exact $T(\omega)$-comodule algebras (or equivalently, indecomposable, exact $\mathsf{Rep}(T(\omega))$-module categories) were classified by Mombelli [Mom10]. Shimizu [Shi19, §5.1] showed that these comodule algebras admit grouplike cointegrals and described them explicitly. Using these results and Corollary 6.6, we obtain the following result on non-existence of unimodular module categories.

Theorem 6.7. *The Taft algebra $T(\omega)$ does not admit a unimodular comodule algebra. In other words, the category $\mathsf{Rep}(T(\omega))$ does not admit a unimodular module category.*

Proof. Choose a divisor $d|N$ and an element $\xi \in \Bbbk$. Set $m = N/d$ and consider the following algebras:

(a) $A_0(d) = \Bbbk\langle G | G^d = 1\rangle$.
(b) $A_1(d, \xi) = \Bbbk\langle G, X | G^d = 1, X^N = \xi, GX = \omega^m XG\rangle$.

They are $T(\omega)$-comodule algebras with the coaction determined by

$$\delta(G) = g^m \otimes G \ \ , \delta(X) = x \otimes 1 + g \otimes X. \tag{6.11}$$

By [Mom10, Proposition 8.3], every indecomposable exact left module category over $\mathsf{Rep}(T(\omega))$ is equivalent to $\mathsf{Rep}(A)$ where A is one of the comodule algebras listed above. Next, we recall the grouplike cointegrals for these comodule algebras and use them to show that the module categories $\mathsf{Rep}(A)$ are not unimodular.

The comodule algebras $A_0(d)$: Define the linear map $\lambda_A : A_0(d) \to \Bbbk$ by $\lambda_A(G^r) = \delta_{0,r}$ (for $r \in \mathbb{Z}/d\mathbb{Z}$). Then, by [Shi19, Section 5.1.1], λ_A is a g_A-grouplike cointegral with $g_A = 1$ on $A_0(d)$. Further, λ_A is a Frobenius form on $A_0(d)$. The Nakayama automorphism with respect to λ_A is $\nu = \mathsf{Id}_{A_0(d)}$. Next, we calculate the automorphism $\widetilde{\nu}$ defined in (6.1).

$$\widetilde{\nu}(G) = \langle \alpha_{T(\omega)}, S(g^m)\rangle \nu^2(G) = \langle \alpha_{T(\omega)}, g^{-m}\rangle G = \omega^{-m} G$$

Plugging this, $g_{T(\omega)} = g^{-1}$ and $g_A = 1$ into Corollary 6.6, we get that A admits a unimodular structure if and only if there exists an element $\widetilde{G} \in A_0(d)$ such that

$$\widetilde{G}G^r = \widetilde{\nu}(G^r)\widetilde{G} = \omega^{-mr}G^r\widetilde{G} = \omega^{-mr}\widetilde{G}G^r \quad \text{and} \quad g \otimes_{\Bbbk} \widetilde{G} = \delta(\widetilde{G}) \quad \text{hold } \forall\, r \in \mathbb{Z}/d\mathbb{Z}. \tag{6.12}$$

The first condition of (6.12) is satisfied if and only if $\omega^{-mr} = 1$ for all r. This is satisfied only if $m = N$ and $d = 1$. With this choice d, $A_0(d) \cong \Bbbk$. Then, the second condition of (6.12) is not satisfied for any $\widetilde{G}$. Hence, the module categories $\mathsf{Rep}(A_0(d))$ are not unimodular for any d.

The comodule algebras $A_1(d, \xi)$: Observe that the set $\{X^rG^s | r = 0, \ldots, N-1; s = 0, \ldots, d-1\}$ is a basis of $A_1(d, \xi)$. By [Shi19, Section 5.1.2], the linear map $\lambda_A : A_1(d, \xi) \to \Bbbk$ given by

$$\lambda_A(X^rG^s) = \delta_{r,N-1}\delta_{s,0} \qquad (\text{for } r \in \{0, \ldots, N-1\} \text{ and } s \in \mathbb{Z}/d\mathbb{Z})$$

is a g^{-1}-cointegral and a Frobenius form on $A_1(d, \xi)$. The Nakayama permutation ν with respect to λ_A is given by $\nu(X) = X$, $\nu(G) = \omega^m G$. Using the formula (6.1), we get that

$$\begin{aligned}
\widetilde{\nu}(G) &= \langle \alpha_{T(\omega)}, S(g^m)\rangle \nu^2(G) = \omega^m G, \\
\widetilde{\nu}(X) &= \langle \alpha_{T(\omega)}, S(x)\rangle \nu^2(1) + \langle \alpha_{T(\omega)}, S(g)\rangle \nu^2(X) = \omega^{-1} X.
\end{aligned}$$

Hence, $\widetilde{\nu}(X^rG^s) = \omega^{ms-r}X^rG^s$. For $A_1(d, \xi)$ to be unimodular, by the first condition of (6.8), we want $\widetilde{\nu}$ to be an inner automorphism. As in [Shi19, Section 5.1.2], by a case-by-case analysis, we show that $\widetilde{\nu}$ is not an inner automorphism.

- $\underline{\xi = 0,\ d > 1}$: Consider the non-zero algebra map $\epsilon : A_1(d, \xi) \to \Bbbk$ given by $\epsilon(X) = 0$ and $\epsilon(G) = 1$. Then, $\epsilon \circ \widetilde{\nu}(G) = \omega^m \neq 1$ as $m < N$. Thus, $\epsilon \circ \widetilde{\nu}(G) \neq \epsilon(G)$, and so $\widetilde{\nu}$ is can not be an inner automorphism.
- $\underline{\xi = 0,\ d = 1}$: In this case, the only invertible element in $A_1(d, \xi)$ is 1. But, $\widetilde{\nu}(X) = \omega^{-1}X \neq X$. Hence, $\widetilde{\nu}$ is not inner.

- $\underline{\xi \neq 0,\ d < N}$: Fix a N-th root ζ of ξ. Define the left $A_1(d,\xi)$-module V as follows. A basis of V is given by $\{v_i\}_{i\in\mathbb{Z}/d\mathbb{Z}}$ and the action is given by
$$X \cdot v_i = \zeta v_{i+1}, \qquad G \cdot v_i = \omega^{mi} v_i \qquad\qquad (i \in \mathbb{Z}/d\mathbb{Z}).$$
Also, consider the $\widetilde{\nu}$-twisted module ${}_{\widetilde{\nu}}V$. Then X^d acts on V and ${}_{\widetilde{\nu}}V$ as scalars ζ^d and $\omega^{-d}\zeta^d$, respectively. As $d < N$, we have that $\omega^{-d} \neq 1$. Thus, V and ${}_{\widetilde{\nu}}V$ are not isomorphic as left $A_1(d,\xi)$-modules. Hence, $\widetilde{\nu}$ can not be an inner automorphism.
- $\underline{\xi \neq 0,\ d = N}$: Consider the non-zero algebra map $\epsilon : A_1(N,\xi) \to \Bbbk$ given by $\epsilon(G) = 1$ and $\epsilon(X) = \zeta$ for ζ a N-th root of ξ. Then, $\epsilon \circ \widetilde{\nu}(X) = \omega^{-1}\zeta \neq \zeta$. Thus, $\epsilon \circ \widetilde{\nu}(X) \neq \epsilon(X)$, and so $\widetilde{\nu}$ is not an inner automorphism.

Hence, the module categories $\mathsf{Rep}(A_1(d,\xi))$ are not unimodular for any d and ξ. $\square$

6.5 Further remarks and questions

We end this section by listing some remarks and directions for further investigation.

First, we show that Theorem 6.3 answers a question of Shimizu [Shi22, Question 7.15]. Take H to be a finite dimensional Hopf algebra over a field $\Bbbk$. Let $\mathcal{D} = \mathsf{Corep}(H)$ denote the category of left H-comodules. Consider a left H-comodule algebra A. Then, A is nothing but an algebra object in the category $\mathcal{D}$. Furthermore, the category of A-bimodules in the $\mathcal{D}$, denoted ${}_A\mathcal{D}_A$, monoidally equivalent to $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ for $\mathcal{C} = \mathsf{Rep}(H)$ and $\mathcal{M} = \mathsf{Rep}(A)$. Moreover, when A is exact, both these are multitensor categories. So, one can ask when they are unimodular. In [Shi22], the unimodularity of ${}_A\mathcal{D}_A$ was studied under the assumption that the algebra A admits a grouplike cointegral, see Definition 6.4. Further, in [Shi22, §7.5], an example of an exact left H-comodule algebra that does not admit a grouplike cointegral was provided and it was asked if there is an easy criterion for determining the unimodularity of ${}_A\mathcal{D}_A$ in the general case.

By Lemma 4.2, the multitensor category $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ (or ${}_A\mathcal{D}_A$) is unimodular if and only if $\mathcal{M}$ is a unimodular $\mathcal{C}$-module category. Thus, the following Corollary of Theorem 6.3 provides an answer to Shimizu's question.

Corollary 6.8. *For $\mathcal{C} = \mathsf{Rep}(H)$ and $\mathcal{M} = \mathsf{Rep}(A)$, the category $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ is unimodular if and only if A admits a unimodular element.* $\square$

While Corollary 6.8 answers Shimizu's question, it is not an easy criterion in general. For instance, see Section 6.4 for an example of a computation. This inspires the following discussion.

A finite dimensional Hopf algebras is unimodular if and only if it admits a two sided integral. To define the integrals, the counit ε, which is an algebra map from H to $\Bbbk$, is crucially used. It would be interesting to find a similar characterization of unimodularity of exact H-comodule algebras. However, we do not know whether such algebras A admits an algebra map to $\Bbbk$. This raises the following question.

Question 1. *Let A be an exact left H-comodule algebra. Is there a way to define left and right integrals for A. If so, can the integrals be used to characterize the unimodularity of the exact left $\mathsf{Rep}(H)$-module category $\mathsf{Rep}(A)$?*

Next, two finite tensor categories $\mathcal{C}, \mathcal{D}$ are called *categorically Morita equivalent* is there exists an indecomposable exact left $\mathcal{C}$-module category $\mathcal{M}$ such that $\mathcal{D}^{\mathrm{rev}} \cong \mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ as finite tensor categories. It is clear that a tensor category admits a unimodular module category if and only if it is categorically Morita equivalent to a unimodular tensor category. Thus, Theorem 6.7 established that the category $\mathsf{Rep}(T(\omega))$ is not categorically Morita equivalent to a unimodular tensor category. This naturally leads to the following question.

Question 2. *Find a characterization of finite tensor categories that do not admit a unimodular module category.*

Remark 6.9. Let $(\mathcal{C}, \mathfrak{p})$ be a pivotal finite tensor category. Then, we call $\mathcal{C}$ *trace-spherical* if $\dim(X) = \dim({}^{\vee}X)$ holds for all objects $X \in \mathcal{C}$. On the other hand, a pivotal finite tensor category $(\mathcal{C}, \mathfrak{p})$ is called *(DSPS-)spherical* [DSPS18] if $\mathcal{C}$ is unimodular and it satisfies that

$$\mathfrak{p}_X \circ \mathfrak{p}_{X^{\vee\vee}} = (\mathfrak{R}_X)^{-1} : X^{\vee\vee} \to {}^{\vee\vee}X \text{ for all } X \in \mathcal{C}.$$

It is known that these two notion of sphericality are not the same. For instance, it was shown in [DSPS18] that $\mathsf{Rep}(T(\omega))$ is trace-spherical but not DSPS-spherical. By Theorem 6.7, we obtain that $\mathsf{Rep}(T(\omega))$ is not categorically Morita equivalent to a unimodular tensor category. This, in particular, implies that $\mathsf{Rep}(T(\omega))$ can not be categorically Morita equivalent to a DSPS-spherical tensor category. This establishes that the two notions of sphericality are not equivalent even when one considers the weaker notion of categorical Morita equivalence.

Appendix A

Nakayama functor of $\mathsf{Rep}(A)$ and its twisted $\mathsf{Rep}(H)$-module structure

Let H be a finite-dimensional Hopf algebra and A be a left H-comodule algebra. In this appendix we provide proofs of Theorems 5.5, 5.6 and 5.7 which pertain to the (right) Nakayama functor of the $\mathsf{Rep}(H)$-module category $\mathsf{Rep}(A)$ and its twisted module structure.

Notation A.1. Consider three $\Bbbk$-vector spaces M, N, N' and a $\Bbbk$-linear map $f : N \to N'$. We will denote by $f^\natural : \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N')$, the map $g \mapsto f^\natural(g) = f \circ g$. Also, for any algebra A, we have that A^* is a A-bimodule via the actions

$$\langle a' \rightharpoonup f \leftharpoonup a'', a\rangle := \langle f, a''aa'\rangle \qquad (a, a', a'' \in A,\ f \in A^*). \tag{A.1}$$

For this section, we fix A to be an exact left H-comodule algebra. Let λ_A and ν denote the Frobenius form and the Nakayama automorphism of A, respectively. We will need the following result.

Lemma A.2. *Let A be an exact left H-comodule algebra. Then, the following results hold.*

(a) *For $V \in \mathsf{Vec}$ and $M \in \mathsf{Rep}(A)$, the canonical Vec action is given by $V \blacktriangleright M = V \otimes_\Bbbk M$ as a vector space and $a \cdot (v \otimes_\Bbbk m) = v \otimes_\Bbbk a \cdot m$.*

(b) *If A is a left H-comodule algebra, A^* also becomes a left H-comodule with the coaction given by $\rho_{A^*}(f) := f_{(-1)} \otimes f_{(0)} \in H \otimes A^*$ where,*

$$f_{(-1)}\langle f_{(0)}, a\rangle = \langle f, a_{(0)}\rangle S^{-1}(a_{(-1)}) \qquad (a \in A,\ f \in A^*). \tag{A.2}$$

(c) *Let M, N be left A-modules. Then, the map $\psi : N^* \otimes_A M \to \mathrm{Hom}_A(M, N)^*$ given by $n^* \otimes_A m \mapsto \langle n^*, ?(m)\rangle$ is an isomorphism of vector spaces .*

(d) *The endofunctors ${}_{(\nu)}(-)$ and $A^* \otimes_A -$ of the category $\mathsf{Rep}(A)$ are isomorphic via the following natural isomorphisms*

$$\alpha_M : A^* \otimes_A M \to {}_\nu M, \qquad f \otimes_A m \mapsto \langle f, a^i\rangle \nu_A(b_i) \cdot m. \tag{A.3}$$

$$\beta_M : {}_\nu M \to A^* \otimes_A M, \qquad m \mapsto \lambda_A \otimes_A m, \tag{A.4}$$

Proof. Parts (a) and (b) are straightforward to check. Part (c) is [SS21, Lemma 4.1]. Thus, we only prove part (d) below.

It is straightforward to check that α_M and β_M are natural in M and are maps of left A-modules. Below, we check they are isomorphisms.

$$\begin{aligned}
\beta_M[\alpha_M(f \otimes_L m)] &\overset{(A.3)}{=} \beta_M[\langle f, a^i\rangle \nu(b_i) \cdot m] &&\overset{(A.4)}{=} \lambda_A \otimes_A \langle f, a^i\rangle \nu(b_i) \cdot m \\
&\overset{(\diamondsuit)}{=} \langle f, a^i\rangle\ (\lambda_L \leftharpoonup \nu(b_i)) \otimes_A m &&\overset{(A.1)}{=} \langle f, a^i\rangle\ \langle \lambda_A, \nu(b_i)\ ?\rangle \otimes_A m \\
&\overset{(5.2)}{=} \langle f, a^i\rangle\ \langle \lambda_A, ?\ b_i\rangle \otimes_A m &&\overset{(5.3)}{=} \langle f, a^i\ ?\rangle\ \langle \lambda_A, b_i\rangle \otimes_A m \\
&= \langle f, a^i\langle \lambda_A, b_i\rangle\ ?\rangle \otimes_A m &&\overset{(5.3)}{=} f \otimes_A m, \\
\alpha_M[\beta_M(m)] &\overset{(A.4)}{=} \alpha_M[\lambda_A \otimes_A m] &&\overset{(A.3)}{=} \langle \lambda_M, a^i\rangle \nu(b_i) \cdot m \\
&= \nu(\langle \lambda_M, a^i\rangle b_i) \cdot m &&\overset{(5.3)}{=} \nu(1_A) \cdot m \quad = \quad m.
\end{aligned}$$

Here the equality $(\diamondsuit)$ holds because $\nu(b_i) \in A$, hence we can move it across the tensor product over A. Thus α is natural isomorphism with inverse β. $\square$

A.0.1 Proof of Theorem 5.5

Recall from (5.5) that the maps $\mathbb{i}_{M,N} : \mathrm{Hom}_A(M,N)^* \blacktriangleright M \to {}_\nu M$ are given by

$$\mathbb{i}_{M,N}(\xi \otimes_{\Bbbk} n) = \langle \xi, \phi_M^N(m^i \otimes_A a^j \cdot n)\rangle \nu(b_j) \cdot m_i \qquad (\xi \in \mathrm{Hom}_A(M,N)^*,\ n \in N)$$

We first show that the component maps $\mathbb{i}_{M,N}$ admit a right inverse.

Lemma A.3. *The map $\omega : {}_\nu M \to \mathrm{Hom}_A(M,A)^* \blacktriangleright A$ given by $m \mapsto \langle \lambda_A, ?(m)\rangle \otimes_{\Bbbk} 1_A$ satisfies that $\mathbb{i}_{M,A} \circ \omega = \mathsf{Id}_{{}_\nu M}$.*

Proof. Observe that

$$\begin{aligned}
\mathbb{i}_{M,A} \circ \omega(m) &= \mathbb{i}_{M,A}\big(\langle \lambda_A, ?(m)\rangle \otimes_{\Bbbk} 1_A\big) \\
&= \langle \langle \lambda_A, ?(m)\rangle, \phi_M^A(m^i \otimes_A a^j 1_A)\rangle\ \nu(b_j) \cdot m_i \\
&= \langle \lambda_A, \langle m^i, m\rangle a^j\rangle\ \nu(b_j) \cdot m_i = \langle \lambda_A, a^j\rangle \nu(b_j) \cdot m = m.
\end{aligned}$$

Thus, $\mathbb{i}_{M,A}$ admits a right inverse. $\square$

To establish that ${}_\nu M$ is equal to the coend $\mathbb{N}(M)$, we show that the maps $\mathbb{i}_{M,N}$ are dinatural.

Lemma A.4. *The maps $\mathbb{i}_{M,N}$ are morphism in $\mathsf{Rep}(A)$, and they are dinatural.*

Proof. Observe that

$$\begin{aligned}
\mathbb{i}_{M,N}(a \cdot (\xi \otimes_{\Bbbk} n)) &= \mathbb{i}_{M,N}(\xi \otimes_{\Bbbk} a \cdot n) \\
&= \langle \xi, \phi_M^N(m^i \otimes_A a^j a \cdot n)\rangle \nu(b_j) \cdot m_i \\
&= \langle \xi, \phi_M^N(m^i \leftharpoonup a^j a \otimes_A n)\rangle \nu(b_j) \cdot m_i \\
&\overset{(5.3)}{=} \langle \xi, \phi_M^N(m^i \otimes_A n)\rangle (\nu(b_j) a^j a) \cdot m_i \\
&\overset{(5.3)}{=} \langle \xi, \phi_M^N(m^i \otimes_A n)\rangle (\nu(a b_j) a^j) \cdot m_i \\
&= \nu(a) \cdot \mathbb{i}_{M,N}(\xi \otimes_{\Bbbk} n) = a \cdot_\nu \mathbb{i}_{M,N}(\xi \otimes_{\Bbbk} n)
\end{aligned}$$

Thus, $\mathbb{i}_{M,N}$ is a map of left A-modules. Further, for $f : N \to N' \in \mathsf{Rep}(A)$ and $\xi \in \mathrm{Hom}_A(M, N')^*$, we get that

$$\begin{aligned}
\mathbb{i}_{M,N'}(\xi \otimes_{\Bbbk} f(n)) &= \langle \xi, \phi_M^{N'}(m^i \otimes_A a^j \cdot f(n))\rangle \nu(b_j) \cdot m_i \\
&= \langle \xi, f \circ \phi_M^N(m^i \otimes_A n)\rangle \nu(b_j) a^j \cdot m_i \\
&= \mathbb{i}_{M,N}(\xi \circ f^\natural \otimes_{\Bbbk} n)
\end{aligned}$$

This proves dinaturality. □

Next, we show that the pair $({}_\nu M, \mathbb{i}_M)$ satisfies the universal property of coends.

Lemma A.5. *The pair $({}_\nu M, \mathbb{i}_M)$ satisfies the universal property of coends.*

Proof. Suppose that there exists an object $X \in \mathsf{Rep}(A)$ along with dinatural maps

$$\mathbb{j}_{M,N} : \mathrm{Hom}_A(M, N)^* \blacktriangleright N \to X.$$

We will show that there exists a unique map $\kappa_X : {}_\nu M \to X$ such that $\kappa_X \circ \mathbb{i}_{M,N} = \mathbb{j}_{M,N}$ for all $M \in \mathcal{M}$. Since $\mathbb{j}$ is dinatural in N, we get that for all $\xi \in \mathrm{Hom}_{\mathcal{M}}(M, N)^*$, $n \in N$ and $f : N \to N'$

$$\mathbb{j}_{M,N'}(\xi \otimes_{\Bbbk} f(n)) = \mathbb{j}_{M,N}(\xi \circ f^\natural \otimes_{\Bbbk} n) \tag{A.5}$$

Plugging $f_n : A \to N$ where $a \mapsto a \cdot n$ in the above equation yields us

$$\mathbb{j}_{M,N}(\xi \otimes_{\Bbbk} f_n(a)) = \mathbb{j}_{M,A}(\xi \circ f_n^\natural \otimes_{\Bbbk} a) \tag{A.6}$$

With $a = 1_A$, LHS is equal to $\mathbb{j}_{M,N}(\xi \otimes_{\Bbbk} n)$. On the other hand,

$$\begin{aligned}
\omega \circ \mathbb{i}_{M,N}(\xi \otimes_{\Bbbk} n) &= \omega\Big(\langle \xi, \phi_M^N(m^i \otimes_A a^j \cdot n)\rangle \nu(b_j) \cdot m_i\Big) \\
&= \langle \xi, \phi_M^N(m^i \otimes_A a^j \cdot n)\rangle \langle \lambda_A, ?(\nu(b_j) \cdot m_i)\rangle \otimes_{\Bbbk} 1_A \\
&= \langle \xi, \phi_M^N(m^i \otimes_A a^j \cdot n)\rangle \langle \lambda_A, \nu(b_j) ?(m_i)\rangle \otimes_{\Bbbk} 1_A
\end{aligned}$$

$$
\begin{aligned}
&= \langle \xi, \phi_M^N(m^i \otimes_A a^j \cdot n)\rangle \langle \lambda_A, ?(m_i) b_j\rangle \otimes_{\Bbbk} 1_A \\
&= \langle \xi, \phi_M^N(m^i \otimes_A a^j ?(m_i) \cdot n)\rangle \langle \lambda_A, b_j\rangle \otimes_{\Bbbk} 1_A \\
&= \langle \xi, \phi_M^N(m^i \otimes_A ?(m_i) \cdot n)\rangle \otimes_{\Bbbk} 1_A \\
&= \langle \xi, \phi_M^N(m^i \otimes_A ?(m_i) \cdot f_n(1_A))\rangle \otimes_{\Bbbk} 1_A \\
&= \langle \xi, \phi_M^N(m^i \otimes_A f_n(?(m_i)))\rangle \otimes_{\Bbbk} 1_A \\
&= \langle \xi, f_n(\phi_M^N(m^i \otimes_A ?(m_i)))\rangle \otimes_{\Bbbk} 1_A \\
&= \langle \xi \circ f_n^{\natural}, \phi_M^N(m^i \otimes_A ?(m_i))\rangle \otimes_{\Bbbk} 1_A \\
&= \langle \xi \circ f_n^{\natural}, ?\rangle \otimes_{\Bbbk} 1_A \;=\; \xi \circ f_n^{\natural} \otimes_{\Bbbk} 1_A
\end{aligned}
$$

Plugging the above formula in (A.6) at $a = 1$, we get that for all $N \in \mathsf{Rep}(A)$

$$\mathbb{j}_{M,N}(\xi \otimes_{\Bbbk} n) = \mathbb{j}_{M,A}(\xi \circ f_n^{\natural} \otimes_{\Bbbk} 1_L) = \mathbb{j}_{M,A} \circ \omega \circ \mathbb{i}_{M,N}(\xi \otimes_{\Bbbk} n) = \kappa \circ \mathbb{i}_{M,N}(\xi \otimes_{\Bbbk} n), \quad \text{(A.7)}$$

where $\kappa_X = \mathbb{j}_{M,A} \circ \omega$. In order to check that κ_X is the unique map so that (A.7) holds, we plug $N = A$ in (A.7) to get $\mathbb{j}_{M,A} = \kappa_X \circ \mathbb{i}_{M,A}$. By Lemma A.3, $\mathbb{i}_{M,A}$ admits a right inverse ω. Therefore, κ_X is the unique map such that $\mathbb{j}_{M,N} = \kappa_X \circ \mathbb{i}_{M,N}$ holds. □

Proof of Theorem 5.5. Together Lemma A.4 and Lemma A.5 imply the claim. □

A.0.2 Proof of Theorem 5.7

Let A be an exact right H-comodule algebra. By Lemma A.2(d), we know that $A^* \otimes_A M \xrightarrow{\sim} {}_\nu M$. Thus, we will employ [SS21, Theorem 7.3], which provided the twisted right $\mathcal{C}$-module structure of the functor $A^* \otimes_A - : \mathsf{Rep}(A) \to \mathsf{Rep}(A)$, to get the twisted right $\mathcal{C}$-module structure of $\mathbb{N}(M) = {}_\nu M$.

Using [SS21, Theorem 7.3], the twisted right $\mathcal{C}$-module structure

$$\Psi_{M,X} : (A^* \otimes_A M) \triangleleft {}^{\vee\vee}X \to A^* \otimes_A (M \triangleleft X) \quad \text{(A.8)}$$

of the functor $A^* \otimes_A - : \mathsf{Rep}(A) \to \mathsf{Rep}(A)$ is given by

$$\Psi_{M,X}[(a^* \otimes_A m) \otimes_{\Bbbk} \phi_X(x)] = a^*_{(0)} \otimes_A (m \otimes_{\Bbbk} S(a^*_{(1)}) \cdot x) \quad \text{(A.9)}$$

Thus, the that twisted right $\mathcal{C}$-module structure of $\mathbb{N}(M) = {}_\nu M$ is given by

$$\overline{\mathfrak{n}}^r_{X,M} = (\alpha_{M \triangleleft X} \circ \Psi_{M,X} \circ (\beta_M \triangleleft \mathsf{Id}_{{}^{\vee\vee}X})) : \ {}_\nu M \triangleleft {}^{\vee\vee}X \ \to \ {}_\nu(M \triangleleft X).$$

The following computation provides an explicit formula for $\overline{\mathfrak{n}}^r_{X,M}$.

$$
\begin{aligned}
\overline{\mathfrak{n}}^r_{X,M}(m \otimes_{\Bbbk} \phi_X(x)) &= \alpha_{M \triangleleft X} \circ \Psi_{M,X} \circ (\beta_M \triangleleft \mathsf{Id}_{{}^{\vee\vee}X})\big(m \otimes_{\Bbbk} \phi_X(x)\big) \\
&= \alpha_{M \triangleleft X} \circ \Psi_{M,X}\big((\lambda \otimes_A m) \otimes_{\Bbbk} \phi_X(x)\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_{M \triangleleft X}\Big(\lambda_{(0)} \otimes_A (m \otimes_{\Bbbk} S(\lambda_{(1)}) \cdot x)\Big) \\
&= \langle \lambda_{(0)}, a^i \rangle \nu(b_i) \cdot [m \otimes_{\Bbbk} S(\lambda_{(1)}) \cdot x] \\
&= \langle \lambda, a^i_{(0)} \rangle \nu(b_i) \cdot [m \otimes_{\Bbbk} S(S(a^i_1)) \cdot x]
\end{aligned}
$$

One can check that the inverse of the map (A.8) is given by

$$
\Psi^{-1}_{M,X}[a^* \otimes_A (m \otimes_{\Bbbk} x)] = (a^*_{(0)} \otimes_A m) \otimes_{\Bbbk} \phi_X(a^*_{(1)} \cdot x)
$$

Now again using the isomorphism $A^* \otimes_A M \xrightarrow{\cong} {}_{\nu}M$ from Lemma A.2(d), we get that the maps $\mathfrak{n}^r_{X,M} : {}_{\nu}(M \triangleleft X) \to {}_{\nu}M \triangleleft {}^{\vee\vee}X$ is given by the following map.

$$
\begin{aligned}
\mathfrak{n}^r_{X,M}(m \otimes_{\Bbbk} x) &= (\alpha_M \triangleleft \mathsf{Id}_{{}^{\vee\vee}X}) \circ \Psi^{-1}_{M,X} \circ \beta_{M \triangleleft X}\Big(m \otimes_{\Bbbk} x\Big) \\
&= (\alpha_M \triangleleft \mathsf{Id}_{{}^{\vee\vee}X}) \circ \Psi^{-1}_{M,X}\Big(\lambda \otimes_A (m \otimes_{\Bbbk} x)\Big) \\
&= (\alpha_M \triangleleft \mathsf{Id}_{{}^{\vee\vee}X})\Big((\lambda_{(0)} \otimes_A m) \otimes_{\Bbbk} \phi_X(\lambda_{(1)} \cdot x)\Big) \\
&= \langle \lambda_{(0)}, a^i \rangle \nu(b_i) \cdot m \otimes_{\Bbbk} \phi_X(\lambda_{(1)} \cdot x) \\
&= \langle \lambda, a^i_{(0)} \rangle \nu(b_i) \cdot m \otimes_{\Bbbk} \phi_X(S(a^i_{(1)}) \cdot x)
\end{aligned}
$$

Hence, the proof is finished. □

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