Cocycle deformations of semisimple Hopf algebras

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By Chelsea Walton

joint w/ Adriana Mejía Castaño,
Susan Montgomery
Sonia Natale
Maria Vega

! GO WINART !

Why care about ~~cocycle deformations of semisimple~~ Hopf algebras?

... quantum symmetry

quantiz'n or deform'n ⋮

Classic Geometry

action:

$$G \times X \longrightarrow X$$

linear alg. group; affine variety

Commutative Algebra

Coaction:

$$\mathcal{O}(X) \longrightarrow \mathcal{O}(X) \otimes \mathcal{O}(G)$$

com. alg; com. Hopf alg

Noncommutative Geom.

action:

$$G_q \times X_q \longrightarrow X_q$$

quantum symmetry on quantum geom. object

{ Impossible to picture }

Noncom. Algebra

{ Study q. symmetries using Hopf co/actions }

Coaction:

$$\mathcal{O}(X_q) \longrightarrow \mathcal{O}(X_q) \otimes \mathcal{O}(G_q)$$

typically { noncom alg; noncom Hopf alg

Why care about ~~cocycle~~ deformations of ~~semisimple~~ Hopf algebras?

... many quantum symmetries arise from deforming classical symmetries

Classic Symmetries (geom.)

$G \curvearrowright X$ group action by automorphisms

$\mathfrak{g} \curvearrowright X$ Lie alg. action by derivations

– can't deform $G, \mathfrak{g}$ –

Classic Symmetries (alg)

$\Bbbk G \curvearrowright \mathcal{O}(X)$ action of group alg.

$U(\mathfrak{g}) \curvearrowright \mathcal{O}(X)$ action of univ. env. alg.

– can deform $\Bbbk G$, $U(\mathfrak{g})$ – as Hopf algebras

Quantum Symmetries:

$(\Bbbk G)_{def} \curvearrowright \mathcal{O}(X)_{def}$

$(U(\mathfrak{g}))_{def} \curvearrowright \mathcal{O}(X)_{def}$

How does one deform a Hopf algebra?

formal deformation

deformation-quantization

twisting:

- Drinfeld twist (leave alg. str. alone, twist coalg. structure & antipode)

- cocycle twist (leave coalg. str. alone, twist alg. structure & antipode)

Take a Hopf algebra $H = (H, m, u, \Delta, \varepsilon, S)$

- A linear form $\sigma : H \otimes H \longrightarrow \Bbbk$ is a 2-cocycle if

  * invertible under convolution product

  * $\sum \sigma(x_1, y_1)\, \sigma(x_2 y_2, z) = \sum \sigma(y_1, z_1)\, \sigma(x, y_2 z_2)$

  * $\sigma(x, 1) = \sigma(1, x) = \varepsilon(x)$.

using Sweedler notation $\Delta(x) = \sum x_1 \otimes x_2$

How does one deform a Hopf algebra?

Take a Hopf algebra $H = (H, m, u, \Delta, \varepsilon, S)$

Define H^σ, a Hopf algebra

$= H$ as $\mathbb{k}$-coalgebras with

new multiplication: $x \underset{\sigma}{*} y = \sum \sigma(x_1, y_1)\, x_2 y_2\, \sigma^{-1}(x_3, y_3)$

new antipode: $S^\sigma(x) = \sum \sigma(x_1, S(x_2))\, S(x_3)\, \sigma^{-1}(S(x_4), x_5)$

$\forall x, y \in H$.

$\left\{ \text{cocycle twist / deformation of } H \right\}$

Why care about cocycle deformations of ~~semisimple~~ Hopf algebras?

... get beautiful correspondences

$\boxed{L = H^{\sigma},\ 2\text{-cocycle def of } H}$

if H, L are fin-dim'l [Schauen..] ✓

can construct ~explicitly

use "Hopf crossed products"

$\boxed{\text{H-comod} \overset{\otimes}{\simeq} \text{L-comod as monoidal categs}}$

"Morita-Takeuchi equiv."

Can construct ~explicitly & uniquely

$\boxed{\exists \text{ (nonzero) } (L, H)\text{-bi Galois object } R}$

like a progenerator for Morita equivalence

dual to "gauge equiv."

H-mod $\underset{\otimes}{\simeq}$ L-mod

right H-comod alg & left L-com.alg

∋. Galois maps are isomorphisms

& H, L coact compatibly.

Why care about (cocycle deformations of) semisimple Hopf algebras?

The study &

Classification of Hopf algebras is an active program

finite-dimensional

as a $\Bbbk$-v.space

Semisimple

(semisimple as a $\Bbbk$-algebra)

pointed

(all simple comods are 1-dim'l)

Cocycle deformations of Semisimple Hopf algebras are still semisimple.

On finite-dim'l semisimple Hopf algebras ... Assume $\Bbbk = \overline{\Bbbk}$, $\operatorname{ch} \Bbbk = 0$

Semisimple Hopf algebras of certain dimension types are classified:

For distinct primes p, q, r: p [Zhu] $\quad p^2 \quad pq \quad p^3$ [Masuoka] $\quad pq^2$ [Natale] $\quad pqr \quad 2^4$ [Kashina]

{Open} for: $\boxed{p^4}_{p \neq 2} \quad \boxed{p^3 q} \quad \boxed{p^2 q^2} \quad \boxed{p^2 q r}$ in general

Helpful to understand cocycle def. of ss Hopf algebras of dim d to get results on ss Hopf algebras of dim d(prime #).

(eg. Kashina used Masuoka's classification of cocy. def. of ss Hopf alg of dim 8.)

Goal: Study Galois objects & cocycle def. of ss Hopf algs of dim p^3, pq^2

- An (G,H) biGalois object R is trivial if $R = H = G$.
- A cocycle def. H^σ of H is trivial if $H^\sigma \simeq H$ as Hopf algebras.

(assuming noncom & noncocom)

- Semisimple Hopf algebras of dimension p^3 classified by Masuoka

$\exists$ 1 isom. class for $p=2$

$\exists$ $p+1$ isom class for $p>2$

representatives:

$\boxed{A_{\xi,1}}$, $\boxed{A_{\xi^t,1}}$

$\boxed{A_{\xi,g}}$, $\boxed{A_{\xi^2,g}}$, $\cdots$, $\boxed{A_{\xi^{p-1},g}}$

$\xi = \sqrt[p]{1}$, $t \in \mathbb{F}_p$ quad residue, $g \neq 1$ certain gp-like elt

- Semisimple Hopf algs of dimension pq^2 classified by Natale

Denoted by:

$\boxed{A_l}$ for $p \equiv 1 \bmod q$

$\boxed{B_\lambda}$, $\boxed{B_\lambda^*}$ for $q \equiv 1 \bmod p$

$l, \lambda \in \mathbb{Z}$, $0 \leq l \leq q-1$

λ certain integer btw 0 & $p-2$

Main Result

Theorem [MMNVW = WINART group] "Galois object" = right Galois obj

For dimension p^3

① If $p=3$, then up to $\simeq$:

- $\boxed{A_{*,1}}$ only has triv. Galois objs
- $\boxed{A_{*,g}}$ each have 2 Galois objs

② If $p>3$, then up to $\simeq$:

- $\boxed{A_{*,g}}$ only has triv. Gal. objs
- $\boxed{A_{*,1}}$ each have p Galois objs

③ $\boxed{A_{*,*}}$ do not admit non-trivial cocycle deformations

For dimension pq^2

① $\boxed{B_\lambda}$, $\boxed{A_{\ell \neq 0}}$ only have triv. Galobjs up to $\simeq$

& do not admit non-triv. cocycle defs

② $\boxed{A_0}$ has q Galois objects up to $\simeq$

& it is a cocycle def of a Com. Hopf alg. ($\simeq (\Bbbk G)^*$ some G)

③ $\boxed{B_\lambda^*}$ have $\frac{p+q-1}{p}$ Galois obj, up to $\simeq$

& it is a cocycle def of a Com. Hopf algebra.

Proof technique for Galois object result for H of dimension p^3 or pq^2

I. Realize H as an abelian extension $\Bbbk \to (\Bbbk\Gamma)^* \to H \to \Bbbk F \to \Bbbk$

Γ, F finite group

II. Ulbrich: $\exists$ bijective corresp btw $\{\cong$ classes of right Galois objs of $H^*\}$

& $\{\cong$ classes of $\underbrace{\text{fiber functors}}$ from categ. $\underbrace{\mathcal{C}(F \bowtie \Gamma, \omega, F, 1)}\}$ α

(★)

faithful exact functor
$\mathcal{C} \to \mathrm{Vec}$

fusion category that is "group-theoretical", data:
F, Γ above, $\omega \in H^3(F \bowtie G, \Bbbk^\times)$
$\alpha \in Z^2(F, \Bbbk^\times)$

III. Ostrik & Natale: Parameterization of (★) given by pairs (L, β)

Subgrp of G $\quad$ $\in Z^2(L, \Bbbk^\times)$

so that $\omega|_{L^{\times 3}}$ triv, $G = LF$, $\alpha\beta^{-1}|_{F \cap L}$ is nondeg

Then we use group theory to classify pairs (L, β) for H above.

Proof technique was adapted recently by

- R. Xiong & Z. Yu (1704.05354)
- Z. Yu (1705.01694)

to study cocycle defs & Galois objects of ss Hopf algs of dimension: $\left.\begin{matrix}16\\ pqr\end{matrix}\right.$

(their works are under review)

Further Directions

① Use our result to aid in the classification of semisimple Hopf algebras of dim p^4, p^3q, p^2q^2, p^2qr

② Study semisimple Hopf algebras of dimension p^n, $n \in \mathbb{N}_{\geq 4}$.

$\ast$ $\exists$ structure result by Masuoka (1996)

$\ast$ Know that these Hopf algebras are group-theoretical.