Division algebras and extended Frobenius structures in monoidal categories

Thesis Defense
Rice University Department of Mathematics

Jacob Kesten
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Overview

1. Introduction to categorical algebra
2. Division algebras in monoidal categories
3. Extended Frobenius structures in monoidal categories

Introduction to Categorical Algebra

Categories and Functors

Ref: *Symmetries of Algebras* by Walton, 2024

Categories and Functors

- A *category* is a collection of objects and the maps between them.

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"Morphisms"

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“Morphisms” “Arrows”

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"Morphisms"

"Arrows"

$$f : X \to Y$$

or

$$X \xrightarrow{f} Y$$

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  - Axioms: Composition and Existence of Identities

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Given $f: X \to Y$ and $g: Y \to Z$,
we can compose to get
$g \circ f: X \to Z$.

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Given $f : X \to Y$ and $g : Y \to Z$,
we can compose to get
$g \circ f : X \to Z$.

For every object X,
there is a map $\mathrm{id}_X : X \to X$
satisfying $\mathrm{id}_X \circ f = f = f \circ \mathrm{id}_X$.

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Examples

Category	Objects	Morphisms

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Category	Objects	Morphisms
Set	Sets	Functions

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Monoid	Monoids	Monoid Homs

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Org	Cells	Effector signaling

Ref: Burgos and Salcedo, “A qualitative mathematical model of immunocompetence with applications to SARS-CoV-2 immunity.” 2021

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Category	Objects	Morphisms
Set	Sets	Functions
Monoid	Monoids	Monoid Homs
Vec	Vector Spaces	Linear Maps
Org	Cells	Effector signaling
Neur	Neural activity over time and space	Identities Only

Ref: Northoff, Tsuchiya, and Saigo, "Mathematics and the Brain: A category theoretical approach to go beyond the neural correlates of consciousness." 2019

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  (Both objects and morphisms)

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$$\mathsf{Forg} : \mathsf{Monoid} \to \mathsf{Set}$$

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$$\mathsf{Free} : \mathsf{Set} \to \mathsf{Vec}$$

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$$- \otimes_{\Bbbk} V : \mathsf{Vec} \to \mathsf{Vec}$$

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$$- \otimes_{\mathbb{k}} V : \mathsf{Vec} \to \mathsf{Vec}$$

$$\mathsf{Stimulus} : \mathsf{Neur\text{-}Pre} \to \mathsf{Neur\text{-}Post}$$

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Categories and Functors

- A *category* is a collection of objects and the maps between them.
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- A *natural transformation* is a nice map between functors.

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- A *natural transformation* is a nice map between functors.
  - For $F, G : \mathscr{C} \to \mathscr{D}$, a natural transformation $\phi : F \Rightarrow G$ is a collection of morphisms $\phi_X : F(X) \to G(X)$ in $\mathscr{D}$ that is *natural*.

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"Plays well with morphisms"

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Categorifying Algebra

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  A set M with a multiplication $\cdot$
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  $1_A \cdot x = x = x \cdot 1_A$
  $x \cdot (y + z) = x \cdot y + x \cdot z$
  $(x + y) \cdot z = x \cdot z + y \cdot z$
  $(ax) \cdot (by) = ab(x \cdot y)$

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$$\cdot \text{ is a linear map} \left\{ \begin{array}{l} x \cdot (y + z) = x \cdot y + x \cdot z \\ (x + y) \cdot z = x \cdot z + y \cdot z \\ (ax) \cdot (by) = ab(x \cdot y) \end{array} \right.$$

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$\cdot$ is a morphism in Vec:

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The same structure!!

Categorifying Algebra

Monoidal Categories

Ref: *Tensor Categories* by EGNO, 2015

Categorifying Algebra

Monoidal Categories

Definition.

Ref: *Tensor Categories* by EGNO, 2015

Categorifying Algebra

Monoidal Categories

Definition. A monoidal category is a category $\mathscr{C}$

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Categorifying Algebra

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Definition. A monoidal category is a category $\mathscr{C}$ equipped with a monoidal product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, a unit object $\mathbf{1} \in \mathscr{C}$,

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$$a := \{a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)\}_{X,Y,Z \in \mathscr{C}},$$

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Example

Vec is a monoidal category with monoidal product $\otimes_{\Bbbk}$ and unit $\Bbbk$, since

$$(V \otimes_{\Bbbk} W) \otimes_{\Bbbk} U \cong V \otimes_{\Bbbk} (W \otimes_{\Bbbk} U),$$

$$V \otimes_{\Bbbk} \Bbbk \cong V, \text{ and } \Bbbk \otimes_{\Bbbk} V \cong V.$$

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Categorifying Algebra

Algebra in Monoidal Categories

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Algebra in $(\mathscr{C}, \otimes, \mathbf{1})$:

An object $A \in \mathscr{C}$ with a multiplication $m : A \otimes A \to A$ in $\mathscr{C}$ and a unit $u : \mathbf{1} \to A$ in $\mathscr{C}$ satisfying

$m(\mathrm{id}_A \otimes m) = m(m \otimes \mathrm{id}_A)$

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$\Bbbk \to A$ in Vec

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Algebra objects in Vec are algebras!

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Algebra objects in Vec are algebras!

Algebra objects in Set are monoids!

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Examples

Algebra objects in Vec are algebras!

Algebra objects in Set are monoids!

$\mathbf{1}$ is always an algebra objects.

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Categorifying Algebra
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Examples

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Algebra objects in Set are monoids!

$\mathbf{1}$ is always an algebra objects.

$\mathbf{0}$ is an algebra object (when it exists).

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Categorifying Algebra

Modules

Ref: *Tensor Categories* by EGNO, 2015

Categorifying Algebra
Modules

(Left) Module over A in Vec:

A vector space M with a linear action map $\triangleright : A \otimes_{\Bbbk} M \to M$ satisfying

$a \triangleright (b \triangleright x) = (ab) \triangleright x$

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Categorifying Algebra
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Example

Vectors with action map given by scaling:

$$2 \triangleright (3,4) = (6,8)$$

Ref: *Tensor Categories* by EGNO, 2015

Categorifying Algebra
Modules

(Left) Module over A in Vec:

A vector space M with a linear action map $\triangleright : A \otimes_{\Bbbk} M \to M$ satisfying

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$1_A \triangleright x = x$

$\rightsquigarrow$

(Left) Module over A in $(\mathscr{C}, \otimes, \mathbf{1})$:

An object $M \in \mathscr{C}$ with an action map $\triangleright : A \otimes M \to M$ in $\mathscr{C}$ satisfying

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This is called the *regular (left) A-module.*

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Division Algebras in Monoidal Categories

Division Algebra over a Field

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Definition. Let A be a non-zero, associative, unital $\mathbb{k}$-algebra. We say A is a *division algebra over* $\mathbb{k}$ if every non-zero element of A is left invertible.

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$a \in A$
$a \neq 0$

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$$\begin{matrix} a \in A \\ a \neq 0 \end{matrix} \quad \Rightarrow \quad \exists \;\; a^{-1} \in A \;\text{ such that }\; a^{-1} \cdot a = 1_A$$

Division Algebras over a Field - Motivation

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The only real division algebras are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.

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Theorem. (Cartan-Artin-Wedderburn-Noether-Hopkins, 1898-1939)

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- Every semisimple algebra over a field is a product of matrix algebras over division algebras.

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The only submodules of A are 0 and A

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Definition. (K-Walton, 2025) A non-zero algebra A in $\mathscr{C}$ (abelian monoidal) is a simplistic division algebra in $\mathscr{C}$

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Ref: Grossman-Snyder, 2016; Grossman, 2019; Kong-Zheng, 2019.

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Simplistic?

Since $\mathbf{1}\text{-}\operatorname{mod}(\mathscr{C}) \cong \mathscr{C}$,
$\mathbf{1}$ is simplistic if and only if it is
a simple object in $\mathscr{C}$.

Essential?

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But they are still not equivalent!

(We produce simplistic, non-essential division algebra in the Fibonacci fusion category and in $\mathsf{fdRep}(G)$.)

On Essential Division Algebras

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For $\mathscr{C}$ an *abelian* monoidal category:

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A has more than one isoclass of modules.

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(Unit)

All modules over the monad $A \otimes - : \mathscr{A} \to \mathscr{A}$ are free.

Monadic Division Algebras

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For $\mathscr{A}$ any monoidal category:

Definition. (K-Walton, 2025)

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EM → Modules over the monad

Kleisli → Free modules over the monad

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Modules over the monad (EM) — Free modules over the monad (Kleisli)

Proposition. (K-Walton, 2025)

$$\text{Monadic} \iff \text{Essential}.$$

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$\Searrow$ T needs to have equivalent EM and Kleisli categories.

Division Algebras from Monads

Proposition. (K-Walton, 2025) Let $\mathscr{A}$ be a strict monoidal category, and $T : \mathscr{A} \to \mathscr{A}$ a monad.

If T satisfies $T(X) \otimes Y \cong T(X \otimes Y)$ and has equivalent EM and Kleisli categories, then $T(\mathbf{1})$ is an essential division algebra in $\mathscr{A}$.

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- The monad $T(-) := (- \sqcup \{\star\})$ satisfies the two required conditions.
- Hence $T(\varnothing) = \varnothing \sqcup \{\star\} = \{\star\}$ is a right essential division algebra in Set.

Further Directions

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- Essential vs. Simplistic?

- Left vs. Right?
  -**Theorem** (Nakamura-Shibata-Shimizu, 2025). *The left/right distinction of simplistic division algebras is not necessary in finite tensor categories.*

Extended Frobenius Structures in Monoidal Categories

Topological Quantum Field Theories

Ref: Kock's 2-TQFT Book

Topological Quantum Field Theories

$$(2\text{-Cob}, \sqcup, \varnothing, \text{flip}) \xrightarrow{Z} (\mathscr{C}, \otimes, \mathbf{1}, c)$$

Symmetric monoidal functor

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For $2-\text{Cob}$:
Objects: Oriented, closed, 1-manifolds
Morphisms: Orientation preserving cobordisms
(2-manifolds having the objects of 2-Cob as boundary)

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Topological Quantum Field Theories

$$(2\text{-Cob}, \sqcup, \varnothing, \text{flip}) \xrightarrow[\text{Symmetric monoidal functor}]{Z} (\mathscr{C}, \otimes, \mathbf{1}, c)$$

For $2-\text{Cob}$:
Objects: Oriented, closed, 1-manifolds
Morphisms: Orientation preserving cobordisms
(2-manifolds having the objects of 2-Cob as boundary)

Example:

$(\mathscr{C}, \otimes, \mathbf{1})$ is a monoidal category;

$$c := \{c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X\}$$
$$c^2 = \text{Id}$$

+ Axioms

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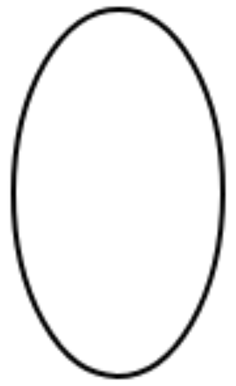

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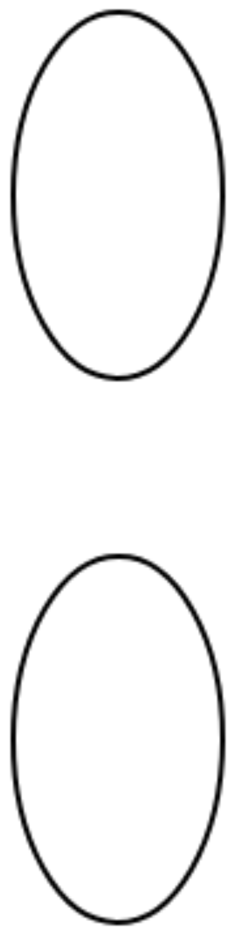

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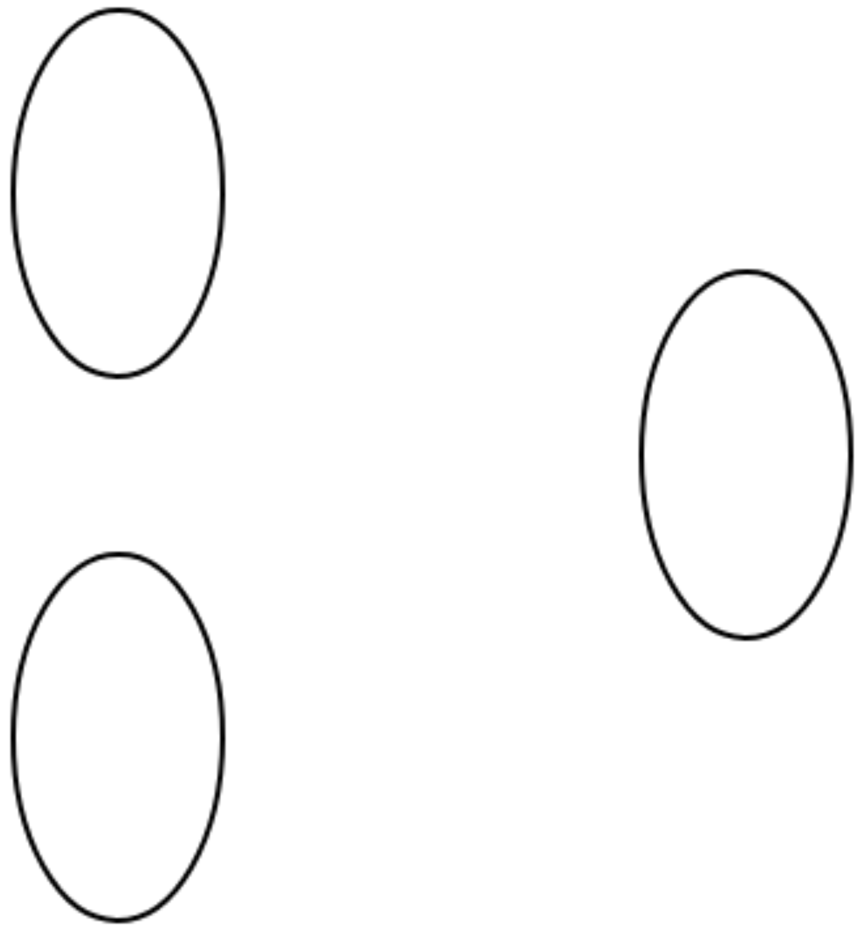

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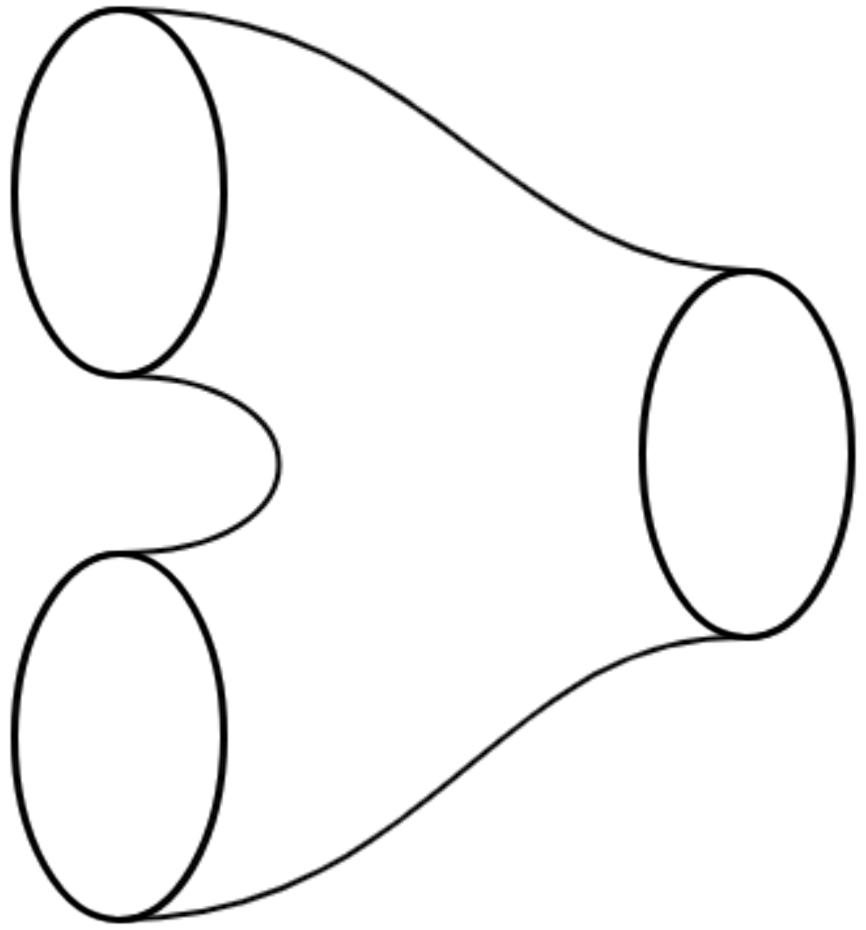

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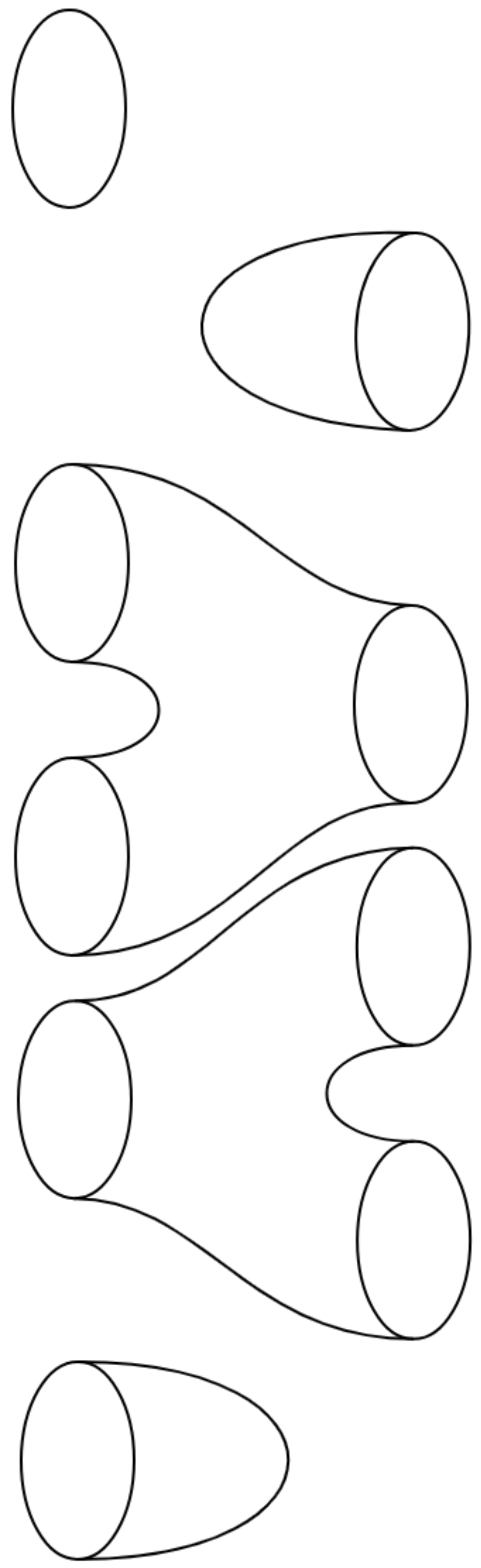

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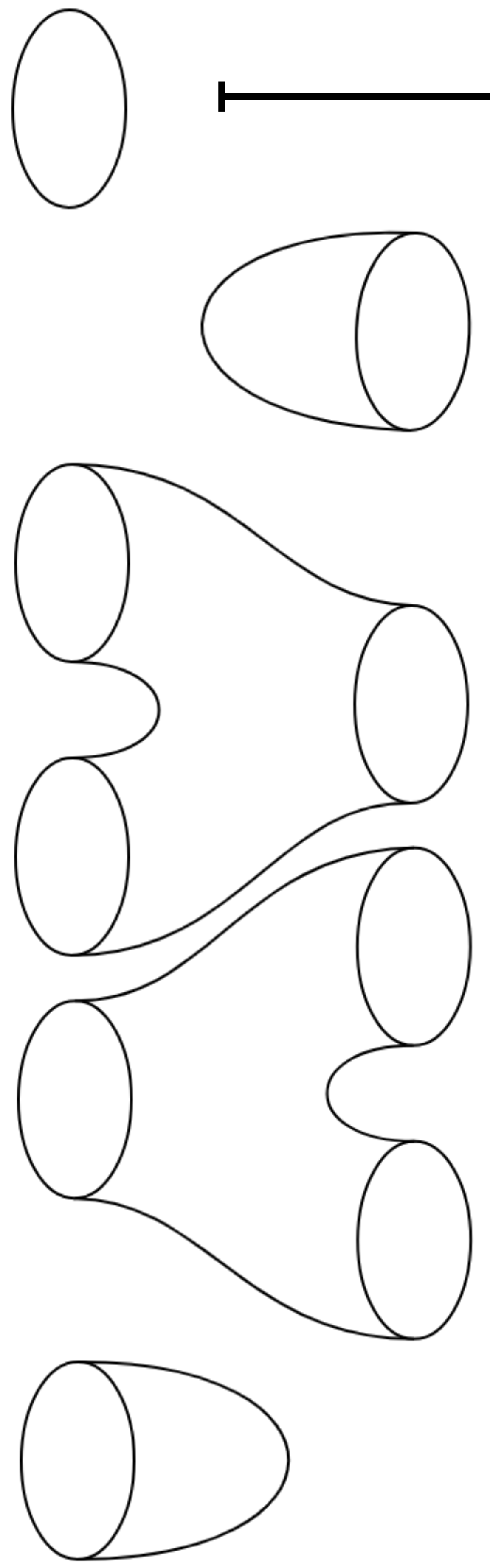

$$\longmapsto A$$

Ref: Kock's 2-TQFT Book

Topological Quantum Field Theories

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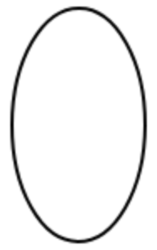
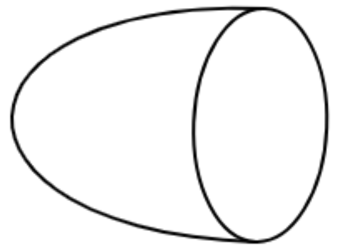
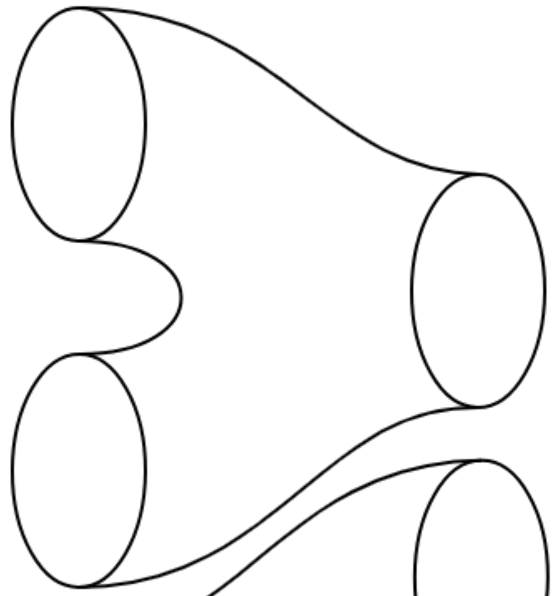
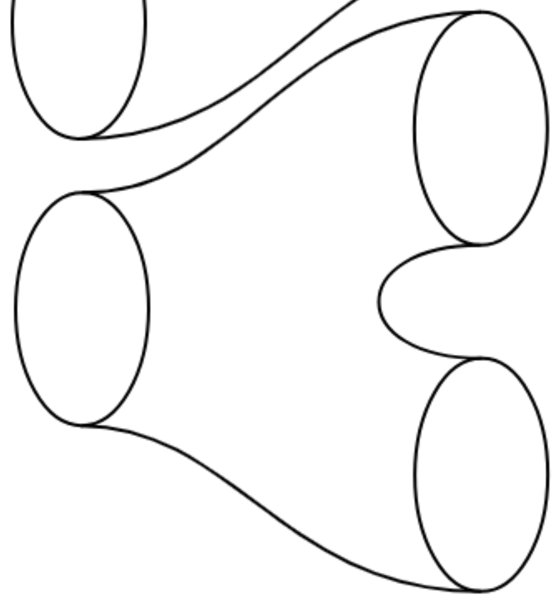
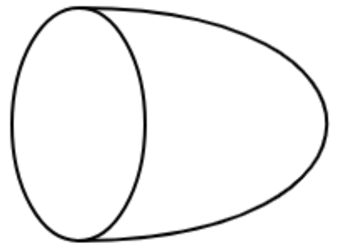
 $\longmapsto A$

 $\longmapsto u : \mathbf{1} \to A$







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Topological Quantum Field Theories

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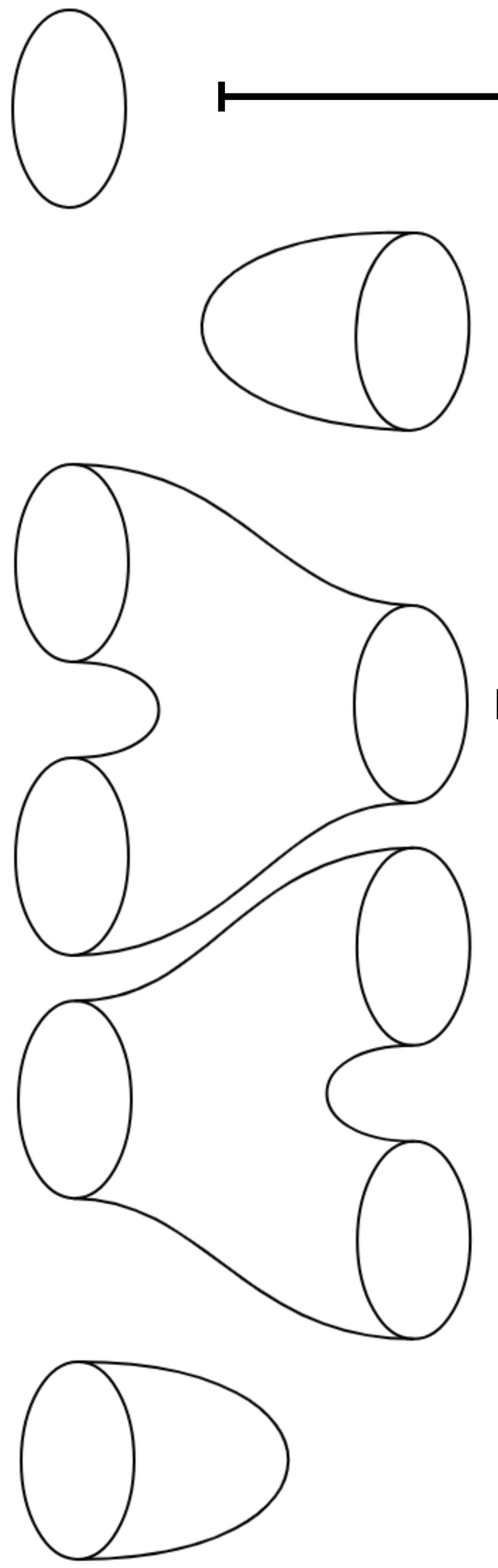

$\mapsto A$

$\mapsto u : \mathbf{1} \to A$

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Ref: Kock's 2-TQFT Book

Topological Quantum Field Theories

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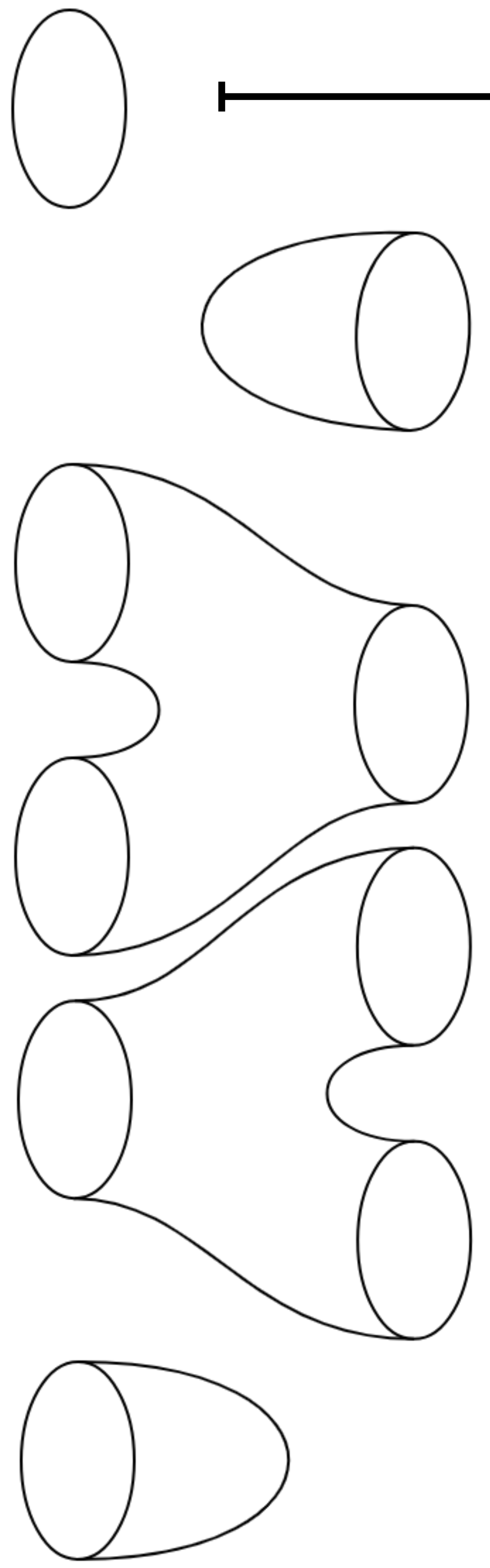

$\mapsto A$

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$\mapsto m : A \otimes A \to A$

$\mapsto \Delta : A \to A \otimes A$

Ref: Kock's 2-TQFT Book

Topological Quantum Field Theories

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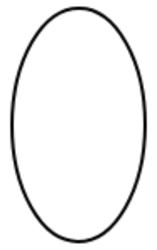
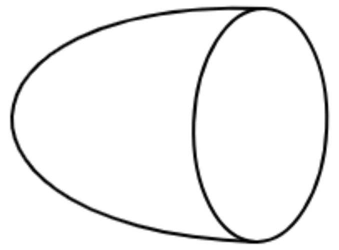
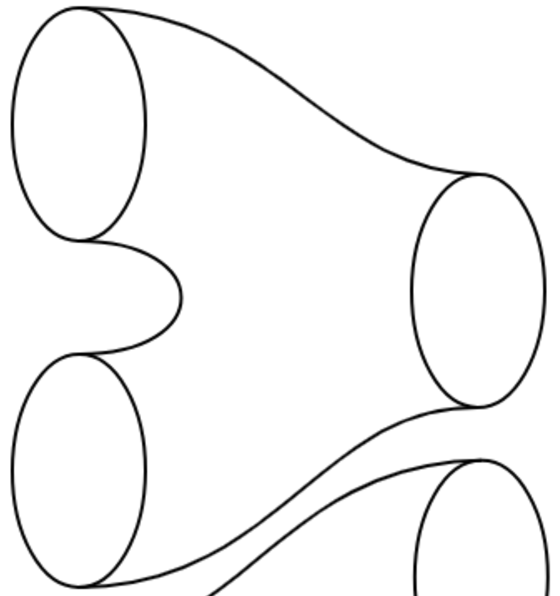
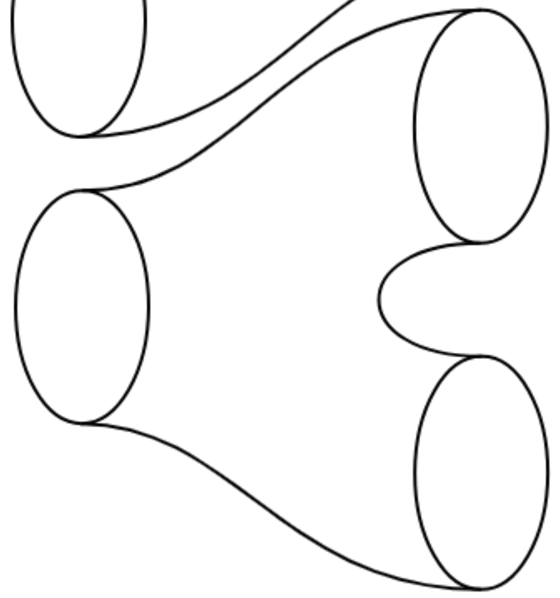
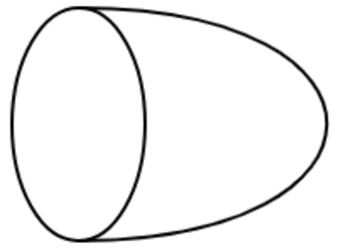
 $\longmapsto A$

 $\longmapsto u : \mathbf{1} \to A$

 $\longmapsto m : A \otimes A \to A$

 $\longmapsto \Delta : A \to A \otimes A$

 $\longmapsto \varepsilon : A \to \mathbf{1}$

Ref: Kock's 2-TQFT Book

Topological Quantum Field Theories

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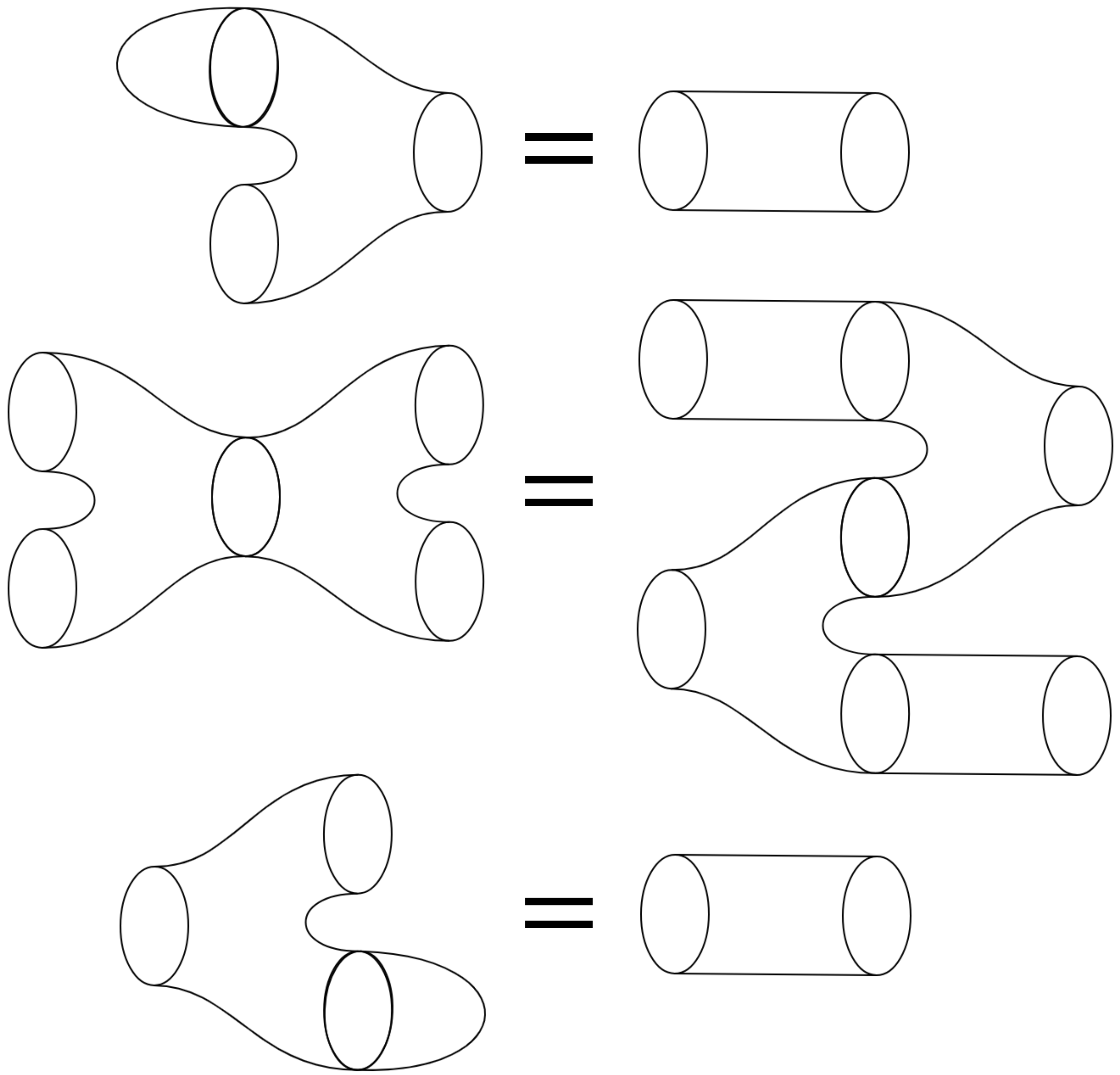

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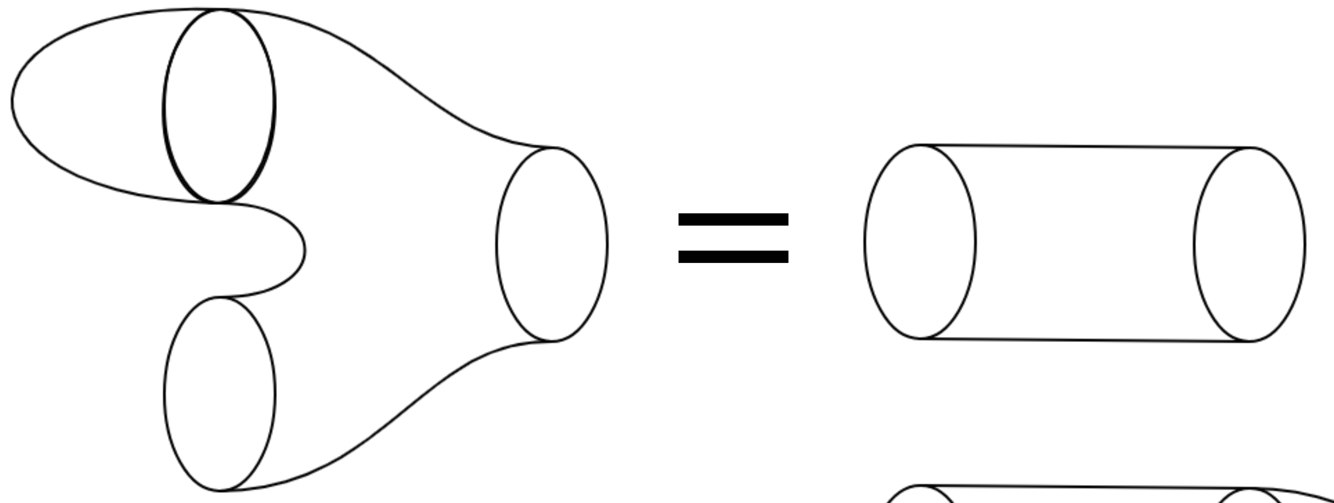
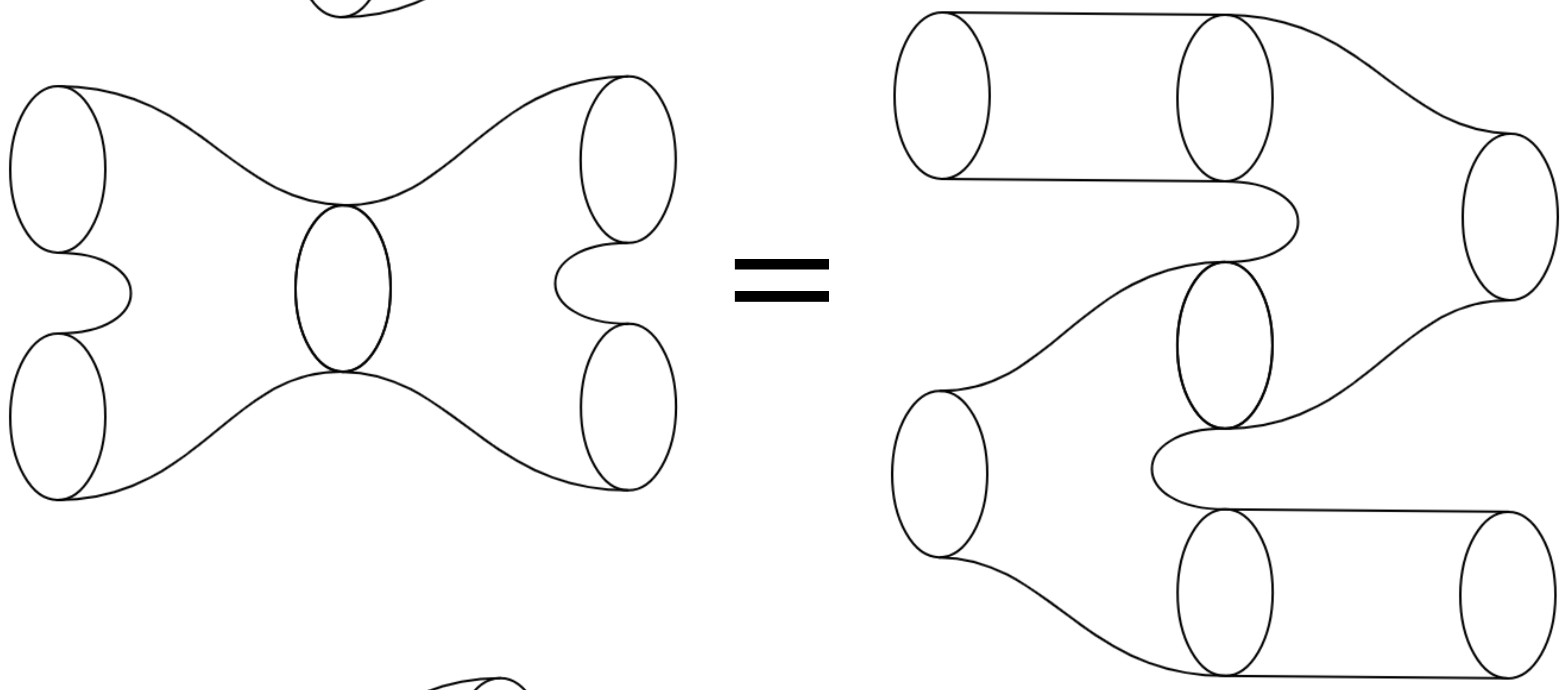
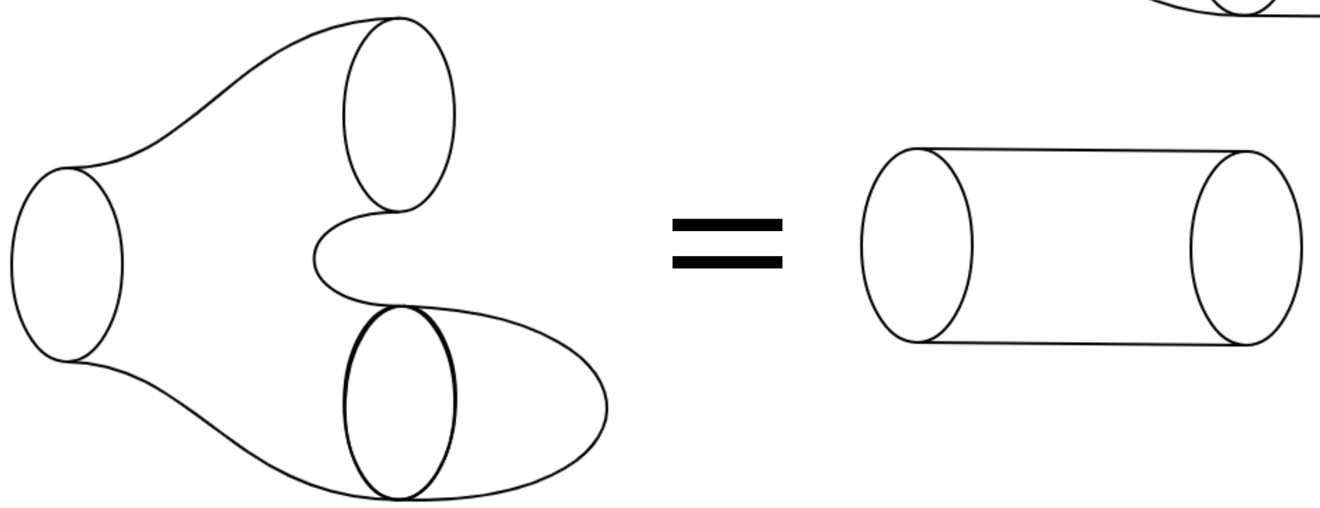


$\longmapsto$ $m(u \otimes \mathrm{id}_A) = \mathrm{id}_A$





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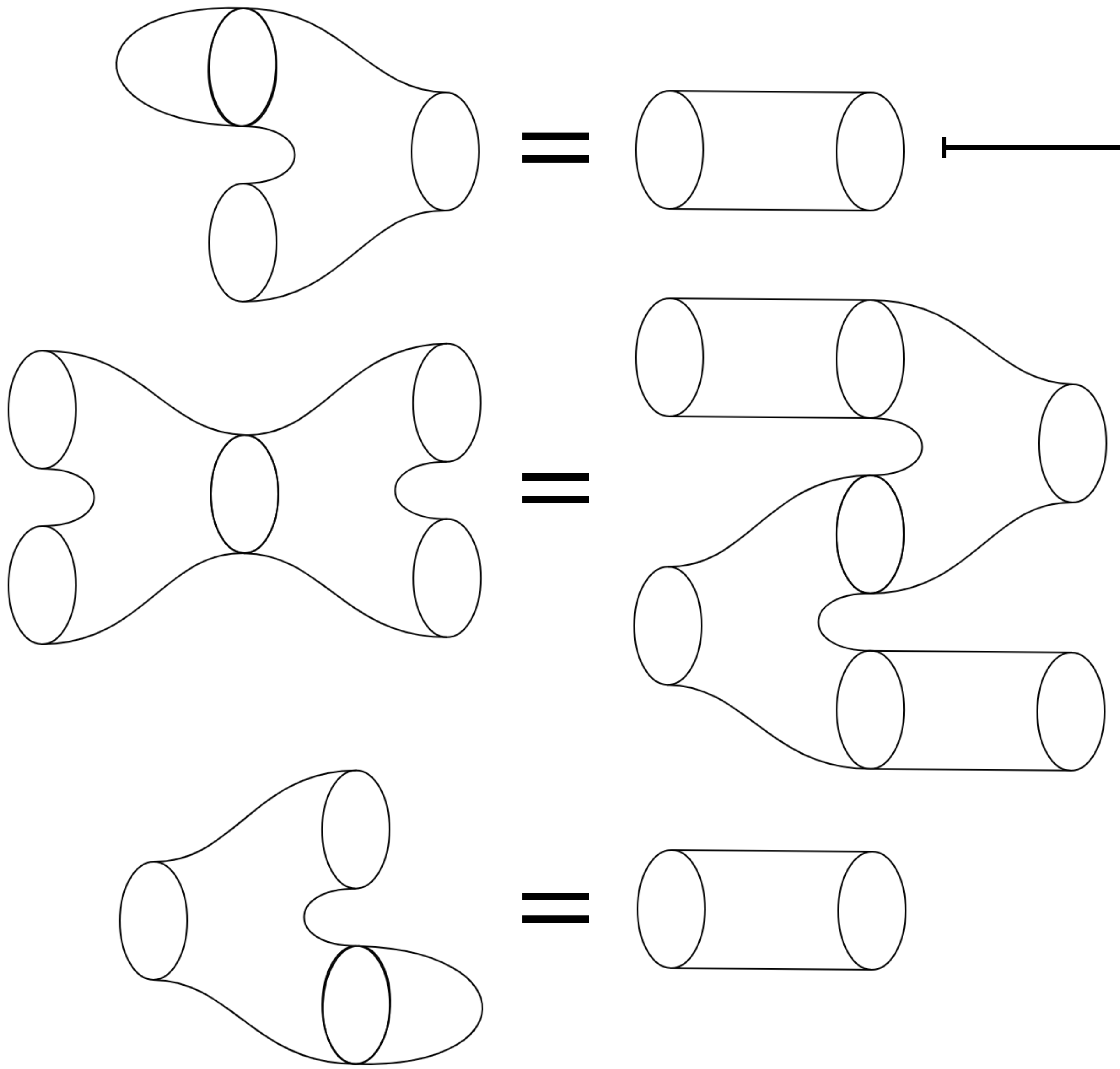

$\longmapsto$ $m(u \otimes \mathrm{id}_A) = \mathrm{id}_A$

$\longmapsto$ $\Delta m = (m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta)$

Ref: Kock's 2-TQFT Book

Topological Quantum Field Theories

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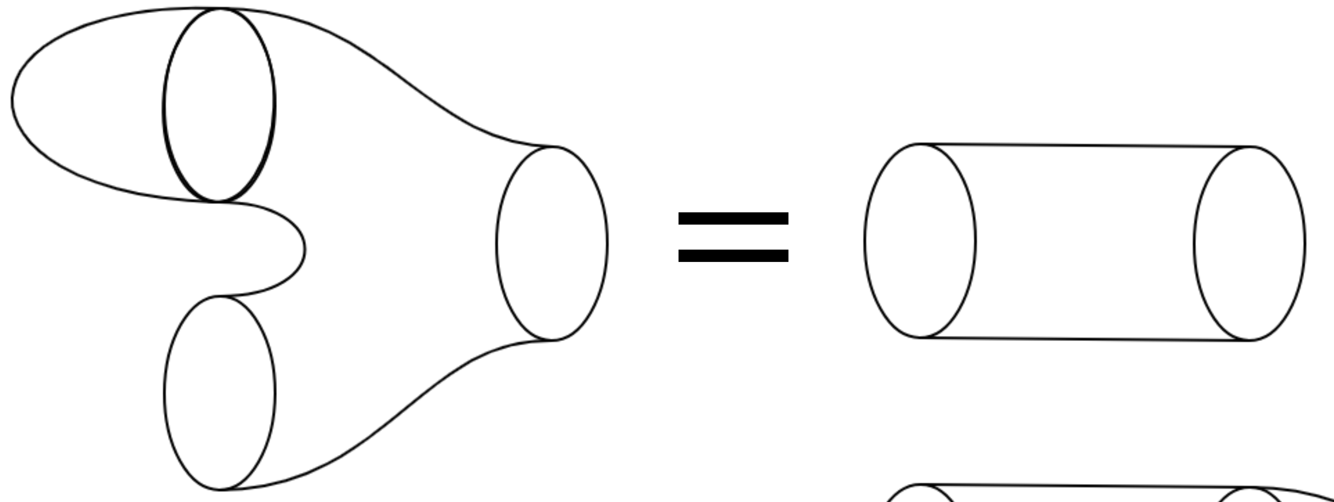
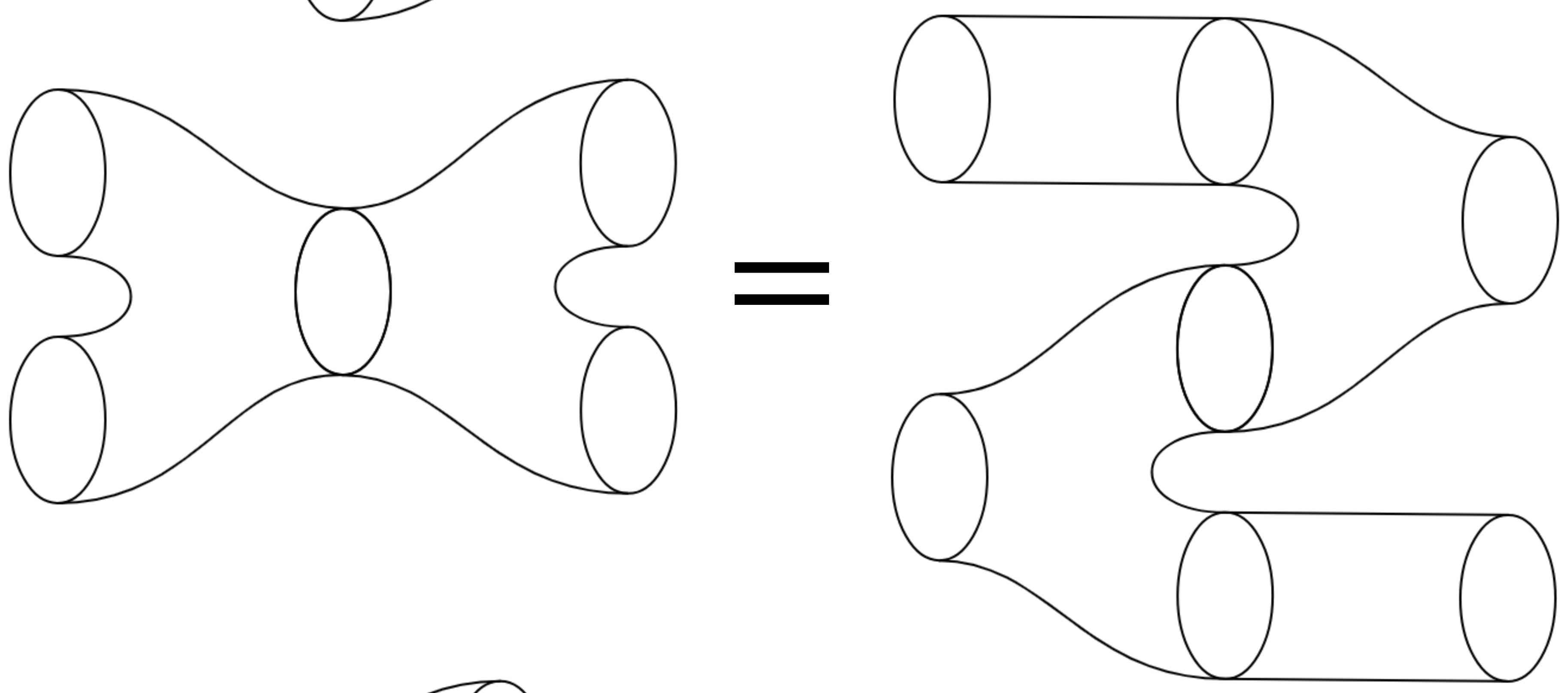
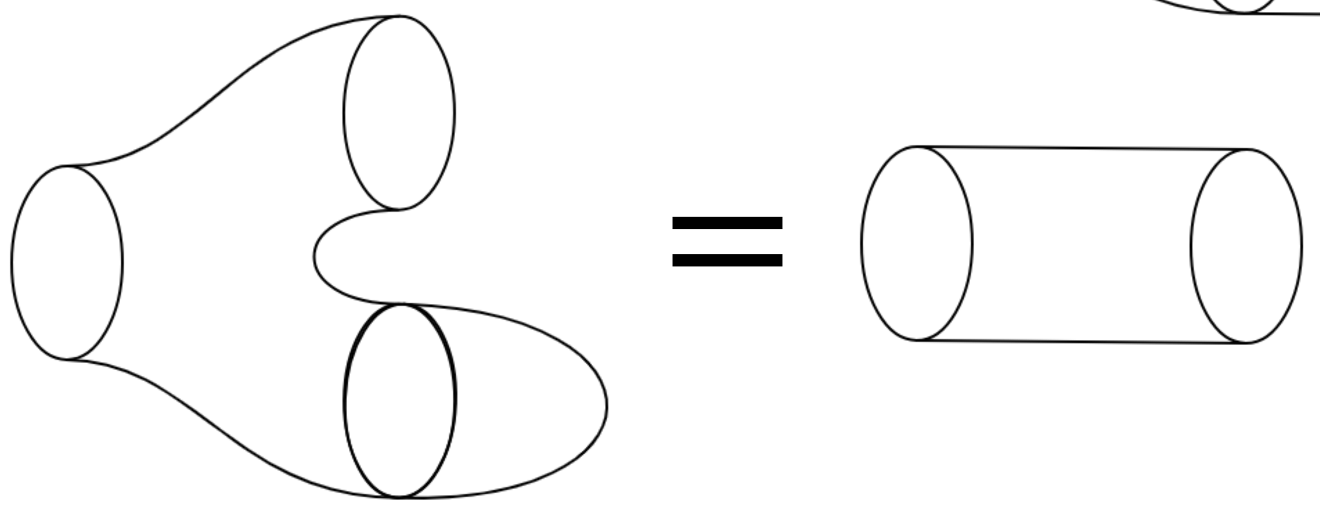


$$\longmapsto \quad m(u \otimes \text{id}_A) = \text{id}_A$$



$$\longmapsto \quad \Delta m = (m \otimes \text{id}_A)(\text{id}_A \otimes \Delta)$$



$$\longmapsto \quad (\text{id}_A \otimes \varepsilon)\Delta = \text{id}_A$$

Ref: Kock's 2-TQFT Book

Topological Quantum Field Theories

$$(2\text{-Cob}, \sqcup, \varnothing, \text{flip}) \xrightarrow[\text{Symmetric monoidal functor}]{Z} (\mathscr{C}, \otimes, \mathbf{1}, c)$$

- A structure in $\mathscr{C}$ satisfying the same relations as $Z(S^1)$ is called a commutative Frobenius algebra in $\mathscr{C}$.

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- Specifically, in any monoidal category, we can define these in a purely categorical way:

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  $A \in \mathscr{C}$ $\qquad$ $\varepsilon : A \to \mathbf{1}$

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Ref: Kock's 2-TQFT Book

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    - Coassociativity: $(\Delta \otimes \mathrm{id}_A)\Delta = (\mathrm{id}_A \otimes \Delta)\Delta$
    - Counitality: $(\varepsilon \otimes \mathrm{id}_A)\Delta = \mathrm{id}_A = (\mathrm{id}_A \otimes \varepsilon)\Delta$

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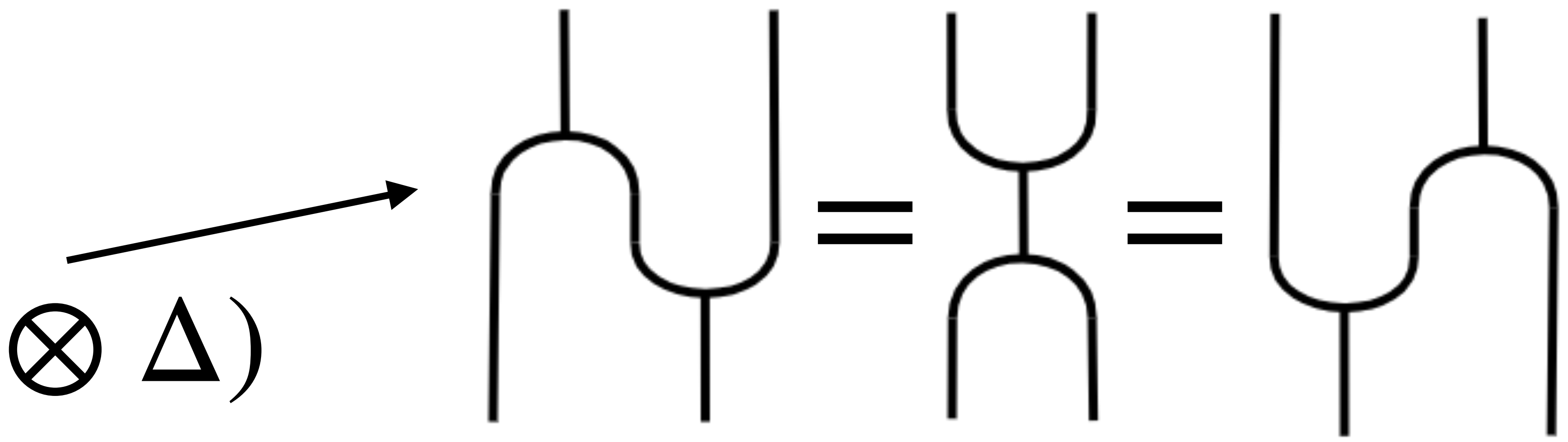
  - $(\mathrm{id}_A \otimes m)(\Delta \otimes \mathrm{id}_A) = \Delta m = (m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta)$



Ref: Kock's 2-TQFT Book

Topological Quantum Field Theories

- In fact, commutative Frobenius algebras completely describe 2-TQFTs via the following result:

$$\text{2-TQFT}(\mathscr{C}) \overset{\otimes}{\simeq} \text{ComFrobAlg}(\mathscr{C})$$

Ref: Kock's 2-TQFT Book

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- This is then extended in a number of ways.

Ref: Kock's 2-TQFT Book

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  - n-TQFTs are symmetric monoidal functors from n-Cob to a symmetric monoidal category $\mathscr{C}$.

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- This is then extended in a number of ways.
  - n-TQFTs are symmetric monoidal functors from n-Cob to a symmetric monoidal category $\mathscr{C}$.
  - Un-orient our cobordism categories and consider symmetric monoidal functors from n-UCob to a symmetric monoidal category $\mathscr{C}$.

Ref: Kock's 2-TQFT Book

Unoriented 2-TQFTs

$$(2\text{-UCob}, \sqcup, \varnothing, \text{flip}) \xrightarrow[\text{Symmetric monoidal functor}]{Z} (\mathscr{C}, \otimes, \mathbf{1}, c)$$

Ref: You Qi's TQFT Course Notes

Unoriented 2-TQFTs

$$(2\text{-UCob}, \sqcup, \varnothing, \text{flip}) \xrightarrow[\text{Symmetric monoidal functor}]{Z} (\mathscr{C}, \otimes, \mathbf{1}, c)$$

2-UCob is generated by all the generators of 2-Cob along with two new cobordisms:

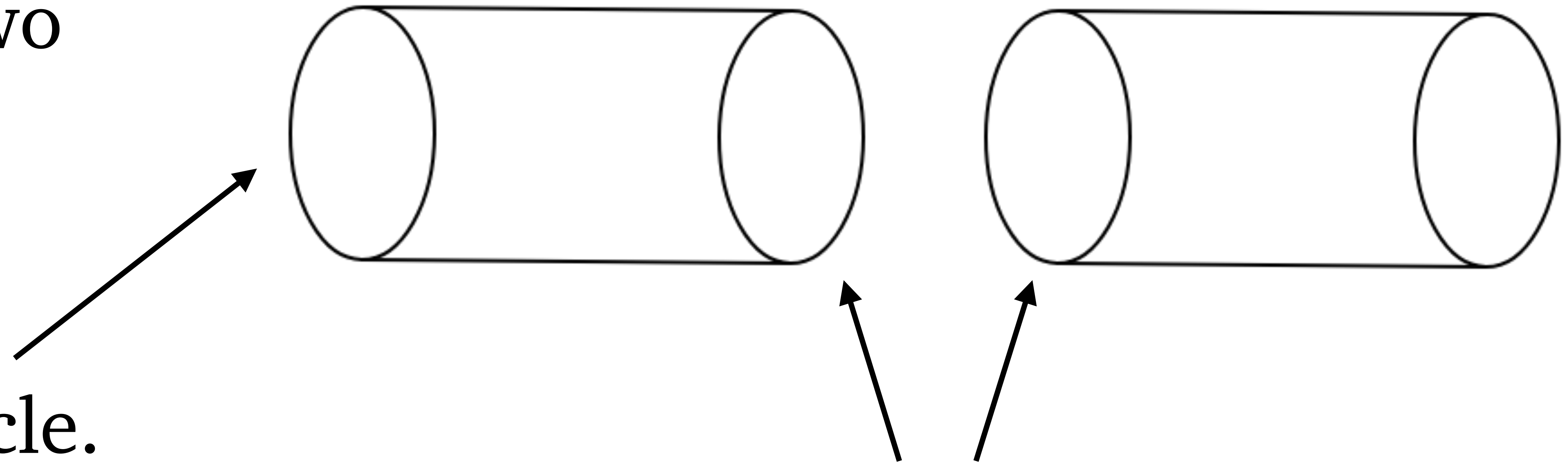
Ref: You Qi's TQFT Course Notes

Unoriented 2-TQFTs

$$(2\text{-UCob}, \sqcup, \emptyset, \text{flip}) \xrightarrow[\text{Symmetric monoidal functor}]{Z} (\mathscr{C}, \otimes, \mathbf{1}, c)$$

2-UCob is generated by all the generators of 2-Cob along with two new cobordisms:

1. A cobordism from S^1 to S^1 that switches the orientation of the circle.



Glue these two circles by a reflection

Ref: You Qi's TQFT Course Notes

Unoriented 2-TQFTs

$$(2\text{-UCob}, \sqcup, \varnothing, \text{flip}) \xrightarrow[\text{Symmetric monoidal functor}]{Z} (\mathscr{C}, \otimes, \mathbf{1}, c)$$

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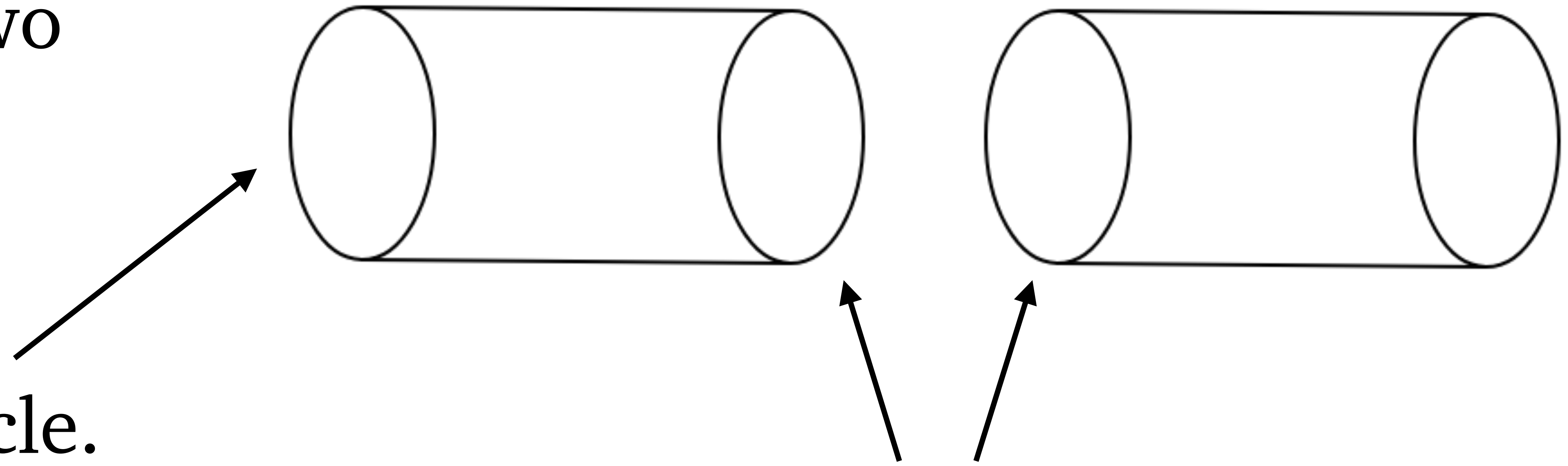

Glue these two circles by a reflection

2. The mobius band, which can be considered a cobordism from $\varnothing$ to S^1.

Ref: You Qi's TQFT Course Notes

Unoriented 2-TQFTs

$$(2\text{-UCob}, \sqcup, \varnothing, \text{flip}) \xrightarrow[\text{Symmetric monoidal functor}]{Z} (\mathscr{C}, \otimes, \mathbf{1}, c)$$

2-UCob is generated by all the generators of 2-Cob along with two new cobordisms:

1. A cobordism from S^1 to S^1 that switches the orientation of the circle. $\longmapsto$ $\phi : A \to A$ satisfying $\phi^2 = \mathrm{id}_A$

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2. The mobius band, which can be considered a cobordism from $\emptyset$ to S^1. $\longmapsto$ A special element of A: $\theta : \mathbf{1} \to A$

These two cobordisms also satisfy some other relations, giving us relations that ϕ and θ satisfy.

Ref: You Qi's TQFT Course Notes

Unoriented 2-TQFTs

An **extended Frobenius algebra in $\mathscr{C}$** is a Frobenius algebra $(A, m, u, \Delta, \varepsilon)$ equipped with two additional morphisms $\phi : A \to A$ and $\theta : \mathbf{1} \to A$ (called the **extended structure**) satisfying

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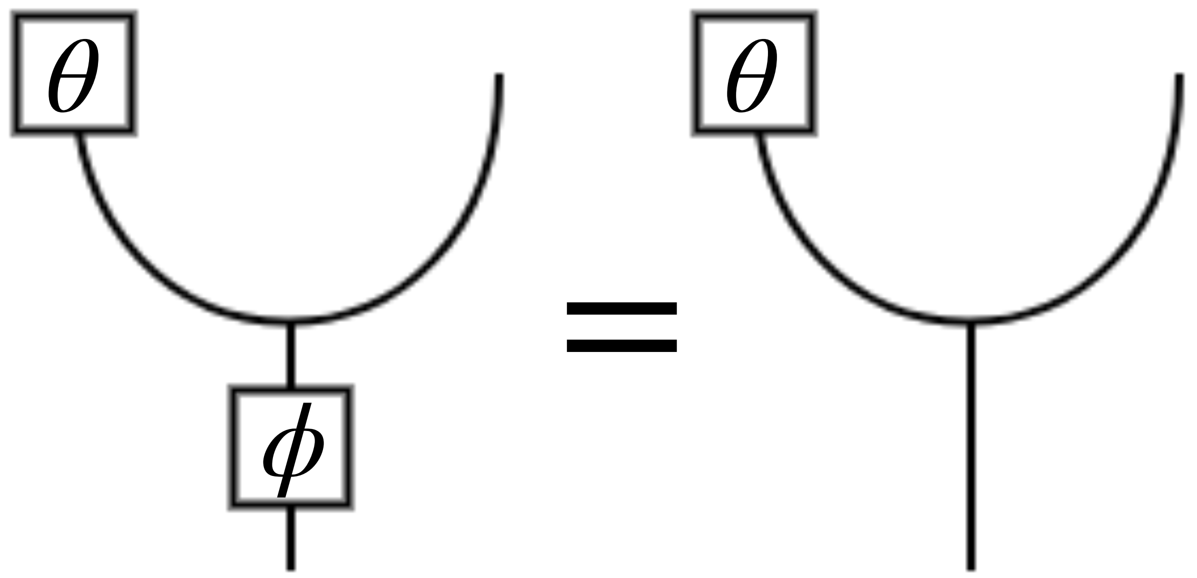

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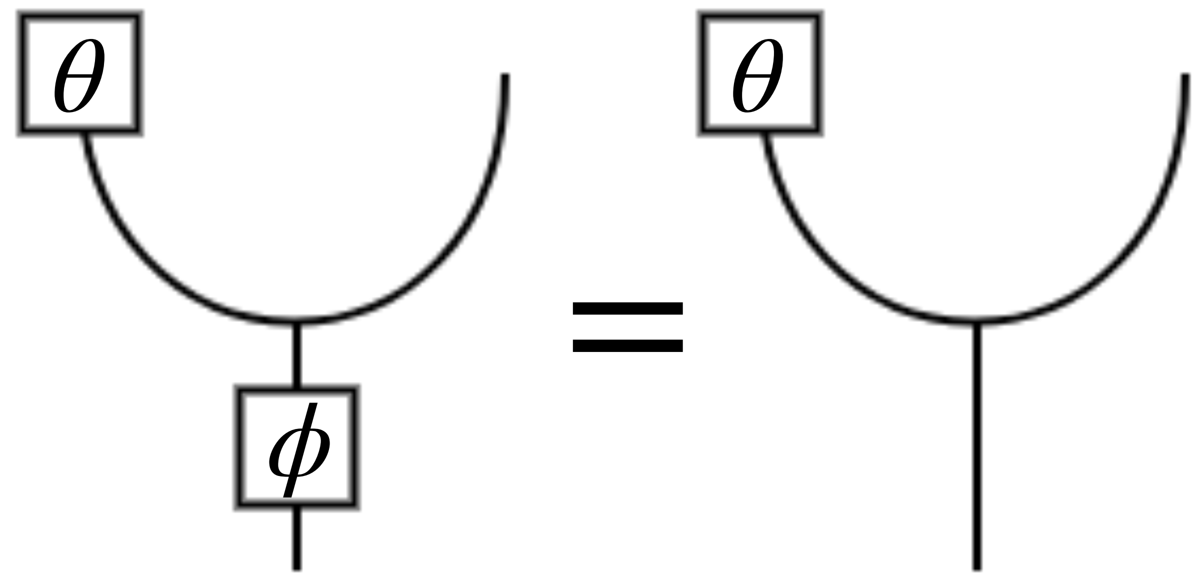

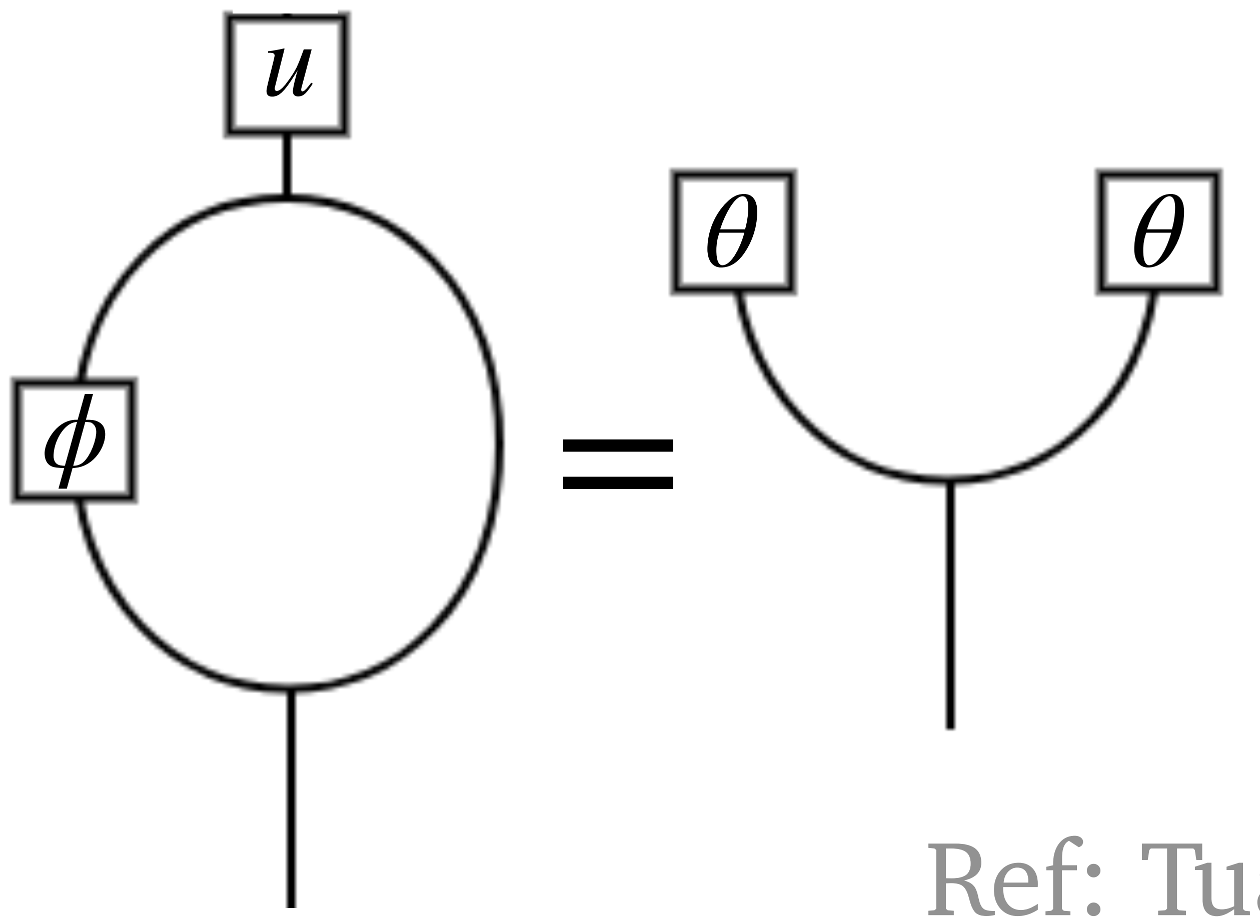

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(i.e. is the zero map
$0_{\mathbf{1},A} : \mathbf{1} \to A$
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Unoriented 2-TQFTs

- Again, extended Frobenius algebras completely describe 2-uTQFTs via the following result:

$$2\text{-uTQFT}(\mathscr{C}) \overset{\otimes}{\simeq} \mathsf{ExtFrobAlg}(\mathscr{C})$$
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Extended Frobenius Algebras over $\mathbb{k}$

Recall: Extended Frobenius algebra = Frobenius algebra + Extended structure

$(A, m, u, \Delta, \varepsilon)$ (ϕ, θ)

Extended Frobenius Algebras over $\Bbbk$

Theorem. (Czenky-K-Quinonez-Walton, 2024)

The extended structures of the following well known $\Bbbk$-Frobenius algebras are classified.

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Proposition. (Czenky-K-Quinonez-Walton, 2024)
Every f.d. Hopf algebra over $\Bbbk$ admits a ϕ-trivial extended Frobenius structure.

Recall: Extended Frobenius algebra = Frobenius algebra $(A, m, u, \Delta, \varepsilon)$ + Extended structure (ϕ, θ)

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(d) The group algebra $\Bbbk C_4$: extensions are either ϕ-trivial, θ-trivial, or ϕ maps a generator g of C_4 to $\omega_4 g^3$.

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Categorical Constructions

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Proposition. (Czenky-K-Quinonez-Walton, 2024)
Every separable Frobenius algebra admits an extended Frobenius algebra structure.

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extended → ϕ-trivial

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Functorial Constructions

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Algebra in $\mathscr{C}$:

Functorial Constructions

Algebra in $\mathscr{C}$:

- Object $A \in \mathscr{C}$;
- $m : A \otimes A \to A$;
- $u : \mathbf{1} \to A$;
- Axioms.

Functorial Constructions

Algebra in $\mathscr{C}$: $\rightsquigarrow$ Monoidal Functor $(\mathscr{C}, \otimes, \mathbf{1}) \to (\mathscr{C}', \otimes', \mathbf{1}')$:

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- Functor $F : \mathscr{C} \to \mathscr{C}'$;
- $F^{(2)} : F(-) \otimes' F(-) \Rightarrow F(- \otimes -)$;
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Ref: *Monoidal functors, species and Hopf algebras* by Aguiar and Mahajan, 2010

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Monoidal Functors Preserve Algebras!

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Coalgebra in $\mathscr{C}$:

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(Czenky-K-Quinonez-Walton, 2024)

Functorial Constructions

Extended Frobenius Algebra in $\mathscr{C}$:

(Czenky-K-Quinonez-Walton, 2024)

Functorial Constructions

Extended Frobenius Algebra in $\mathscr{C}$:

- $(A, m, u, \Delta, \varepsilon) \in \mathsf{FrobAlg}(\mathscr{C})$;
- Extended structure (ϕ, θ);
- $\phi m(\theta \otimes \mathrm{id}_A) = m(\theta \otimes \mathrm{id}_A)$
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Functorial Constructions

Extended Frobenius Algebra in $\mathscr{C}$: $\rightsquigarrow$ Extended Frobenius Monoidal Functor $\mathscr{C} \to \mathscr{C}'$:

- $(A, m, u, \Delta, \varepsilon) \in \mathsf{FrobAlg}(\mathscr{C})$;
- Extended structure (ϕ, θ); where $\phi: A \to A$, $\phi^2 = \mathrm{id}_A$ and $\theta: \mathbf{1} \to A$
- $\phi m(\theta \otimes \mathrm{id}_A) = m(\theta \otimes \mathrm{id}_A)$
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- Frobenius monoidal functor $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)})$;
- Extended structure $(\hat{F}, \check{F})$;
- $\hat{F}_{\mathbf{1}\otimes X} \circ F^{(2)}_{\mathbf{1},X} \circ (\check{F} \otimes' \mathrm{id}_{F(X)}) = F^{(2)}_{\mathbf{1},X} \circ (\check{F} \otimes' \mathrm{id}_{F(X)})$;
- $F^{(2)}_{\mathbf{1},\mathbf{1}} \circ (\hat{F}_{\mathbf{1}} \otimes' \mathrm{id}_{F(\mathbf{1})}) \circ F^{\mathbf{1},\mathbf{1}}_{(2)} \circ F^{(0)} = F^{(2)}_{\mathbf{1},\mathbf{1}} \circ (\check{F} \otimes' \check{F})$
- $F^{(2)}_{X,Y} \circ (\hat{F}_X \otimes' \mathrm{id}_{F(Y)}) \circ F^{X,Y}_{(2)} = F^{(2)}_{X\otimes Y,\mathbf{1}} \circ (\hat{F}_{X\otimes Y} \otimes' \mathrm{id}_{F(\mathbf{1})}) \circ F^{X\otimes Y,\mathbf{1}}_{(2)}$
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$\rightsquigarrow$

Extended Frobenius Monoidal Functor $\mathscr{C} \to \mathscr{C}'$:

- Frobenius monoidal functor $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)})$;
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  $\hat{F}: F \Rightarrow F$, $\hat{F}^2 = \mathrm{Id}$

- $\hat{F}_{\mathbf{1}\otimes X} \circ F^{(2)}_{\mathbf{1},X} \circ (\check{F} \otimes' \mathrm{id}_{F(X)}) = F^{(2)}_{\mathbf{1},X} \circ (\check{F} \otimes' \mathrm{id}_{F(X)})$;
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- Extended structure (ϕ, θ); $\phi: A \to A$, $\phi^2 = \mathrm{id}_A$; $\theta: \mathbf{1} \to A$
- $\phi m(\theta \otimes \mathrm{id}_A) = m(\theta \otimes \mathrm{id}_A)$
- $m(\phi \otimes \mathrm{id}_A)\Delta u = m(\theta \otimes \theta)$
- Axioms.

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Extended Frobenius Monoidal Functor $\mathscr{C} \to \mathscr{C}'$:

- Frobenius monoidal functor $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)})$;
- Extended structure $(\hat{F}, \check{F})$; $\hat{F}: F \Rightarrow F$, $\hat{F}^2 = \mathrm{Id}$; $\check{F}: \mathbf{1}' \to F(\mathbf{1})$
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(Czenky-K-Quinonez-Walton, 2024)

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(These functors are separable Frobenius $\iff$ the algebra A is separable Frobenius)

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Thank you!