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Division algebras and extended Frobenius structures in monoidal categories

by

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Abstract

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Due to the wide range of applications in logic, programming, and quantum physics, adapting algebraic objects to the monoidal setting has become an active area of current inquiry. This thesis adds to this field of categorical algebra by exploring generalizations of division algebras and extended Frobenius algebras in monoidal categories.

Division algebras were first introduced to the categorical setting in attempts to generalize structure results from classical algebra. Extended Frobenius algebras were introduced by Turaev and Turner in 2006 as a way to extend the correspondence between oriented 2-dimensional topological quantum field theories and commutative Frobenius algebras to the unoriented case. In this thesis, we explore the monoidal analogues of these objects. Concerning division algebras, we are especially interested in determining how analogues of the equivalent definitions of division algebras over a field relate in a variety of monoidal settings. We also find categorical and functorial constructions that interact well with division algebras and extended Frobenius algebras, and we use these constructions to produce examples.

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Contents

Chapter 1

Introduction

Category theory, first introduced by Eilenberg and MacLane [EM45], is a branch of mathematics that focuses on the relationships between abstract objects. The basic structures of categories and functors emphasize the importance of maps, or morphisms, between objects. Categories are defined as collections of objects with the morphisms between them, and functors are nice maps between categories that move both objects and morphisms. Eilenberg and MacLane used this setting to formalize the concept of naturality. They defined natural transformations as maps between functors that respect the full structure of the categories involved, both objects and morphsims. This allowed the naturality of isomorphisms from many disparate areas of mathematics to be coherently understood; for examples, see [EM45, Section 2.10].

Category theory has since been used in many areas of mathematics, notably in algebraic topology (e.g. [Ler45, Car51, ES52]) and algebraic geometry (e.g. [Ser55, Ser56, Gro57a, Gro57b, DG71]). An important development from this is the concept of adjoint functors, which stemmed from a desire to simplify homological computations. Adjunctions, first introduced by Kan [Kan58], are functors satisfying a natural morphism set correspondence mimicking that of the tensor and Hom functors for modules over rings. Adjunctions were also shown to be equivalent to the concept of monads [Hub61, EM65, Kle65]. Understanding the equivalence of adjunctions and monads led to the natural constructions of Eilenberg-Moore and Kleisli categories associated to every monad. Monads have since found a place in many applications, especially functional programming and logic; see, for example, [Mog89, PJW93, Wad95, Mul98, WZ99, FLF21].

Adjunctions are also used to define a number of categorical constructions. For example, free objects are constructed by defining the free functor as the left adjoint of the forgetful functor [Mac50]. Because adjunctions are a correspondence between morphisms, free objects could be defined explicitly by specifying the morphism property that they satisfy. Many categorical objects can also be defined by such a universal property, including (co)images, (co)kernels, and (co)products. Categories where all such objects exist are called abelian, coined by Bushbaum and Grothendieck during their work generalizing the algebraic concept of exact sequences [Buc55, Gro57b].

Generalizing the objects and results of classical algebra became a major goal of category theorists. Bénabou and MacLane adapted the structure of monoids from classical algebra by defining monoidal categories as categories equipped with a product functor and unit object [Bén63, ML63]. The ability to "multiply" objects of a category was necessary in defining algebraic objects internal to categories, thus birthing the field of categorical, or universal, algebra [Bén64, Wal70]. Algebras, coalgebras, bialgebras, Hopf algebras, and Frobenius algebras could all be defined as objects within a monoidal category equipped with morphisms mimicking the structure of the corresponding objects over a field [AM10]. Monoidal categories and algebraic objects therein have many applications, including logic, condensed matter physics, and quantum field theories; see, for example, [Lam68, Lam69, Min81, Koc04, ABK21, BB22].

Due to the wide range of applications, understanding algebraic objects in monoidal categories is an active area of current research. This thesis, based on the pair of papers [KW25, CKQW24], adds to the field of categorical algebra by exploring division algebras and extended Frobenius algebras in the monoidal setting. Both of these have been previously introduced to the categorical setting (see [GS16, Gro19, KZ19] and [TT06], respectively), but lack a substantial body of results and examples. We aim to remedy this, especially by investigating constructions that can lead to new examples of these algebras. Below, we give a more detailed introduction to these objects and the results included in this thesis.

In all that follows, $\Bbbk$ denotes an algebraically closed field of characteristic zero, and categories $\mathcal{C}$ are assumed to be locally small, unless otherwise stated.

1.1 Division algebras

Division algebras are fundamental objects in classical algebra. They were originally introduced in conjunction with the study of quaternions and octonions by Hamilton [Ham53], Graves [Gra45], and Cayley [Cay45]. This led to the famous classification theorems of Frobenius [Fro78], Hurwitz [Hur98, Hur22], and Zorn [Zor30], as they sought to list all examples of finite dimensional real division algebras. Division algebras also played a role in the classification of simple and semisimple algebras over a field. The work of Cartan [Car98] and the theorem of Artin and Wedderburn [Wed08] showed that a simple algebra is precisely a matrix algebra over a division algebra, and hence that semisimple algebras are finite products of matrix algebras over division algebras.

With the goal of extending the Artin-Wedderburn Theorem and similar results to the categorical setting, Kong and Zheng defined division algebras in multifusion categories [KZ19]. Grossman and Snyder also introduced a similar definition of division algebras in fusion categories as tools for studying Morita equivalence classes and autoequivalences [Gro19, GS16]. However, over a field there are a number of equivalent definitions of division algebras which had yet to be adapted and explored categorically. We recall the definitions over a field below.

Definition 1.1.1 (Definition 2.1.1)**.** Let A be a non-zero, associative, unital $\Bbbk$-algebra. We say that A is a *division algebra over* $\Bbbk$ if it satisfies any, and hence all, of the following equivalent conditions.

(i) Every non-zero element of A is left (or right) invertible.

(ii) Every left (or right) A-module is free.

(iii) The regular left (or right) A-module is a simple module.

Kong, Zheng, Grossman, and Snyder all adapted Definition 1.1.1(iii). In this thesis, as in the paper [KW25], we generalize Definition 1.1.1(ii) for the monoidal setting, create also a monad-theoretic description of division algebras, and then explore how these different definitions of division algebras relate to one another. The definitions of categorical division algebras used throughout the thesis are briefly presented below.

Definition 1.1.2 (Definitions 3.1.1, 3.1.2, 3.2.2)**.** Take $\mathcal{C}$ to be an abelian monoidal category.

(i) A non-zero algebra in $\mathcal{C}$ is called a *right (left) monadic division algebra* if its associated "tensor on the right (left) monad" has equivalent Kleisli and Eilenberg-Moore categories.

(ii) A non-zero algebra in $\mathcal{C}$ is called a *right (left) essential division algebra* if the right (left) free-module functor is essentially surjective.

(iii) A non-zero algebra is called a *right (left) simplistic division algebra* if the right (left) regular module is simple.

The following theorem and Figure 1.1 summarize the relationships found between the different division algebra structures introduced by Definition 1.1.2.

Theorem 1.1.3 (Propositions 3.1.4, 3.1.16, 3.2.3)**.** *Let A be a non-zero algebra in an abelian monoidal category $\mathcal{C}$.*

(i) *A is a monadic division algebra in $\mathcal{C}$ precisely when it is an essential division algebra in $\mathcal{C}$.*

(ii) *Suppose that $\mathcal{C}$ is rigid with simple unit, and A has a simple module in $\mathcal{C}$. If A is an essential division algebra, then A is a simplistic division algebra.*

(iii) *When $\mathcal{C}$ is a pivotal multifusion category, then each left version of a division algebra in $\mathcal{C}$ is equivalent to its right version in $\mathcal{C}$.*

The hypothesis on A in part (ii) holds in many settings, including in semisimple categories [Remark 3.1.5]. Also, $\mathcal{C}$ need be abelian only when working with simplistic division algebras. Results on essential and monadic division algebras can be given in not-necessarily-abelian monoidal categories by replacing "non-zero" with the condition that the algebra admits more than one isoclass of modules in $\mathcal{C}$; see Lemma 2.3.4.

Throughout the thesis, we also supply several examples for the division algebras of Definition 1.1.2, especially to show how the types differ in various monoidal categories.

- We provide a sufficient condition for a monad T on a monoidal category $\mathcal{C}$ to ensure that $T(\mathbb{1})$ is a monadic division algebra in $\mathcal{C}$ [Proposition 3.2.7]. We use this to produce a monadic division algebra in the non-abelian monoidal category Set [Example 3.2.9].

- For certain semisimple, rigid, abelian monoidal categories with simple unit, we show that simplistic $\not\Rightarrow$ essential [Examples 3.1.8, 3.1.9]; cf. Theorem 1.1.3(ii).

- For a monoidal category $\mathcal{C}$ with non-simple unit, we show that the unit is an essential division algebra in $\mathcal{C}$ that is not a simplistic division algebra in $\mathcal{C}$ [Example 3.1.3]; cf. Theorem 1.1.3(ii).

- In rigid categories $\mathcal{C}$, take the algebras $X \otimes X^*$ and ${}^*X \otimes X$, for $X \in \mathcal{C}$. These are simplistic division algebras in $\mathcal{C}$ precisely when X is a simple object in $\mathcal{C}$, and are essential division algebras in $\mathcal{C}$ precisely when X is a one-sided invertible object in $\mathcal{C}$ [Proposition 3.1.6].

Regarding the last item, one-sided invertibility implies simplicity under certain conditions on $\mathcal{C}$ [Lemma 2.2.2], so this item illustrates Theorem 1.1.3(ii) (see Remark 3.1.7). The algebras in the last item are also examples of internal End algebras, and the result there holds in settings where Ostrik's Theorem [Theorem 2.4.4] is valid (e.g., in multifusion categories); see Lemma 2.2.1, Propositions 3.1.10, 3.1.12. This yields more examples of simplistic and essential division algebras in monoidal categories.

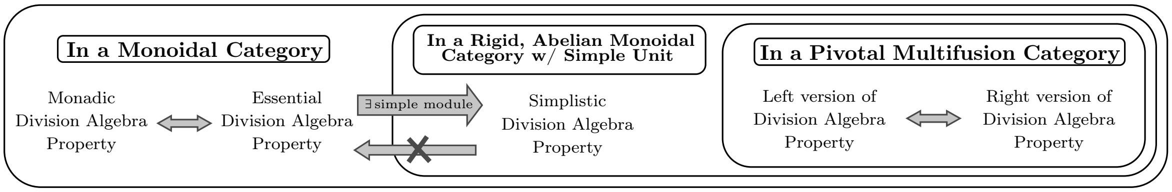

Figure 1.1 : Summary of connections between division algebra properties.

1.2 Extended Frobenius algebras

Topological quantum field theories (TQFTs) are certain categorical constructions that yield topological invariants. Loosely speaking, a TQFT is a functor from a category of topological data to a target category with extra structure. In the 2-dimensional case, 2-TQFTs are symmetric monoidal functors from the symmetric monoidal category of 1-manifolds and 2-cobordisms to a chosen symmetric monoidal category $\mathcal{C}$. Often, $\mathcal{C}$ is taken to be the symmetric monoidal category Vec of $\Bbbk$-vector spaces. A classical result is that a 2-TQFT with values in $\mathcal{C}$ is classified by where it sends the circle, which in the oriented setting, is a commutative Frobenius algebra in $\mathcal{C}$; see, e.g., [Koc04]. Turaev and Turner expanded this correspondence to the unoriented setting by introducing the concept of extended Frobenius algebras [TT06, Section 2].

Turaev and Turner's 2-TQFT Result ($\star$): *Isomorphism classes of unoriented 2-dimensional TQFTs in* Vec *are in 1-1 correspondence with isomorphism classes of commutative extended Frobenius algebras over* $\Bbbk$.

Since then, extended Frobenius algebras have appeared in many works, such as in an adaptation of ($\star$) to compute virtual link homologies [Tub14], in an analogue of ($\star$) for homotopy quantum field theories [Tag12], in a modification of ($\star$) to examine linearized TQFTs [Cze24], in a categorical expansion of ($\star$) [Oca24], and in a study of topological invariants of ribbon graphs [CL24].

We expect that extended Frobenius algebras will continue to play a crucial role in the TQFT program. Thus, we focus on the algebraic side and study extended Frobenius algebras in detail– producing numerous examples, classification results, and general constructions. We first work in Vec. Consider the terminology below.

Definition 1.2.1 (Definitions 2.1.5, 4.1.1)**.** (a) A *Frobenius algebra* over $\Bbbk$ is a vector space equipped both with the structure of an associative, unital $\Bbbk$-algebra and the structure of a coassociative, counital $\Bbbk$-coalgebra, which are compatible via the Frobenius law.

(b) [TT06, Definition 2.5] A Frobenius algebra A over $\Bbbk$ is an *extended Frobenius algebra* over $\Bbbk$ if it is equipped with a Frobenius algebra involution $\phi : A \to A$ and a special element $\theta \in A$ that satisfy certain axioms. We call the pair (ϕ, θ) the *extended structure* of the Frobenius algebra.

Our first main result is the classification of extended structures for various well-known examples of Frobenius algebras over $\Bbbk$.

Theorem 1.2.2 (Propositions 4.1.10–4.1.12, 4.1.14–4.1.16, 4.1.18–4.1.19)**.** *The extended structures for the following Frobenius algebras are classified:* $\Bbbk$ *over itself;* $\mathbb{C}$ *over* $\mathbb{R}$*; the polynomial algebra* $\Bbbk[x]/x^n$ *for* $n \geqslant 2$*; the group algebras* $\Bbbk C_2$*,* $\Bbbk C_3$*,* $\Bbbk C_4$*, and* $\Bbbk(C_2 \times C_2)$*; and the Sweedler algebra* $T_2(-1)$*.*

Next, we move to the general monoidal setting. Following [TT06, Section 2.2], we adapt Definition 1.2.1 to the categorical setting [Definition 4.2.1] and explore some preliminary results. Particularly interesting is the connection between separable Frobenius algebras [Definition 4.2.4] and extended Frobenius algebras. Separability (or *specialness*) is a widely used condition in quantum theory (see, e.g., [MÖ3, RFFS07, HV19]). In particular, it is used to construct *state sum 2-TQFTs* [NR15]. We produce the following result.

Proposition 1.2.3 (Proposition 4.2.5)**.** *A separable Frobenius algebra in a monoidal category is always extendable.*

Then, letting $\mathsf{ExtFrobAlg}(\mathcal{C})$ denote the category of extended Frobenius algebras in $\mathcal{C}$, we establish monoidal structures on $\mathsf{ExtFrobAlg}(\mathcal{C})$. Namely, if $\mathcal{C}$ is also symmetric, then $\mathsf{ExtFrobAlg}(\mathcal{C})$ is monoidal with $\otimes = \otimes^{\mathcal{C}}$ and $\mathbb{1} = \mathbb{1}^{\mathcal{C}}$ [Proposition 4.2.7]. Moreover, if $\mathcal{C}$ is additive monoidal, then $\mathsf{ExtFrobAlg}(\mathcal{C})$ is monoidal with $\otimes$ being the biproduct of $\mathcal{C}$ and $\mathbb{1}$ being the zero object of $\mathcal{C}$ [Proposition 4.2.8].

Next, we explore functors that preserve extended Frobenius algebras in monoidal categories. To start, take monoidal categories $\mathcal{C}$ and $\mathcal{C}'$, and note that a *Frobenius monoidal functor* $\mathcal{C} \to \mathcal{C}'$ [Definition 2.5.1] sends Frobenius algebras in $\mathcal{C}$ to those in $\mathcal{C}'$. It is also known that the separability condition is preserved when such a functor is separable, see [DP08] and [BÏ8, Chapter 6] for more details. We extend this theory of Frobenius monoidal functors by introducing the notion of an *extended Frobenius monoidal functor* [Definition 4.3.1]. We establish that this construction satisfies many desirable conditions as discussed below.

Theorem 1.2.4 (Propositions 4.3.2 and 4.3.4, Theorem 4.3.5)**.** *The following statements hold.*

(a) *A separable Frobenius monoidal functor is extended Frobenius monoidal.*

(b) *Extended Frobenius monoidal functors preserve extended Frobenius algebras.*

(c) *The composition of two extended Frobenius monoidal functors is extended Frobenius monoidal.*

Various separable Frobenius monoidal functors appear in the literature; see, e.g., [Szl05, MS10, Mor12, BT15, HLRC23, FHL23, Yad24]. So, parts (a,b) above imply that each of these constructions produce extended Frobenius algebras in monoidal categories. There are also extended Frobenius monoidal functors that are not necessarily separable [Examples 4.3.7, 4.3.8].

Finally, we turn our attention to Hopf algebras, which also play a role in quantum theory and TQFTs (see, e.g., [KL01, BBG21, CCC22]). It is well-known that finite-dimensional Hopf algebras over $\Bbbk$ admit a Frobenius structure. A lesser known result is that in a *symmetric* monoidal category $\mathcal{C}$, *integral Hopf algebras* in $\mathcal{C}$ [Definition 2.3.3] are Frobenius [Proposition 4.4.3]. A graphical proof of this result is in Appendix B. Building on this, we introduce *extended Hopf algebras* in symmetric monoidal categories [Definition 4.4.7], and obtain the result below.

Proposition 1.2.5 (Proposition 4.4.8)**.** *If an integral Hopf algebra in a symmetric monoidal category is extendable, then so is its corresponding Frobenius structure (via Proposition 4.4.3).*

We also explore functorial constructions that preserve extended Hopf algebras, as we did with extended Frobenius algebras. This leads to the construction of integral Hopf monoidal functors [Definition 4.4.11] and extended Hopf monoidal functors [Definition 4.4.15], which preserve integral Hopf algebras and extended Hopf algebras, respectively [Propositions 4.4.12 and 4.4.16].

1.3 Organization

We give a brief overview of the structure of this thesis. Chapter 2 is dedicated to providing all necessary background on monoidal categories, the algebraic structures of interest both over a field and in a monoidal category, and the major results that will be used throughout. Chapter 3 is based on the paper [KW25], presenting results on division algebras in monoidal categories. Chapter 4 gives the results on extended Frobenius algebras, including the functorial constructions that preserve such structures. Many of the longer commutative diagram arguments from Chapter 4 are presented in Appendix A. Finally, Appendix B is dedicated to the proof that integral Hopf algebras admit Frobenius algebra stuctures, a result that is used in Section 4.4. Chapter 4 and Appendices A and B all follow [CKQW24].

Chapter 2

Background material

We begin in Section 2.1 by recalling the definitions and major results pertaining to different types of algebras over a field that will be considered throughout the thesis. We give background on monoidal categories and functors in Section 2.2, and define basic algebraic objects in this setting in Section 2.3. The tools necessary for, and the statements of, the major theorems of Morita and of Ostrik are presented in Section 2.4. Finally, we discuss functorial analogues of algebraic objects in Section 2.5.

Sometimes, we impose that categories are abelian. In this case, the zero object is denoted by $\mathtt{0}$, the biproduct is denoted by $\sqcup$, and an object is called *simple* if its only subobjects are itself and $\mathtt{0}$.

2.1 Preliminary definitions and results over a field

We use this section to present the classical definitions and results related to the structures considered in this thesis. Division algebras are explored in §2.1.1; Frobenius and Hopf algebras are in §2.1.2.

The information on division algebras below can be found in any noncommutative algebra textbook. See, for example, [Coh02, Coh04, GW04, Bre14]. For Frobenius and Hopf algebras over a field, we refer the reader to [Koc04] and [Rad12], respectively.

Recall that in all that follows, $\Bbbk$ denotes an algebraically closed field of characteristic zero.

2.1.1 Division algebras over a field

We begin with the definition.

Definition 2.1.1. Let A be a non-zero, associative, unital $\Bbbk$-algebra. We say that A is a *division algebra over* $\Bbbk$ if it satisfies any, and hence all, of the following equivalent conditions.

(i) Every non-zero element of A is left (or right) invertible.

(ii) Every left (or right) A-module is free.

(iii) The regular left (or right) A-module is a simple module.

These are instrumental in a number important classical results. Below is one such result, due to Schur [Sch05].

Lemma 2.1.2 (Schur's Lemma)**.** *Let A be an algebra, and let M and N be simple left A-modules. Then, any element in* $\mathrm{Hom}_{A\text{-}\mathsf{Mod}}(M, N)$ *is either the zero map or an isomorphism. In particular,* $\mathrm{End}_{A\text{-}\mathsf{Mod}}(M)$ *is a division algebra.* $\square$

Also, we have the following structure theorem, originally stated in the finite dimensional case by Wedderburn in 1908 [Wed08]. Improvements were then done by Artin [Art27], Noether [Noe29], and Hopkins [Hop39], resulting in the statement below.

Theorem 2.1.3 (Artin-Wedderburn Theorem)**.** *An algebra A over $\Bbbk$ is a semisimple algebra if and only if A is a product of matrix algebras over division algebras.* $\square$

Finally, we have a classification theorem of Frobenius [Fro78].

Theorem 2.1.4 (Frobenius Theorem)**.** *The only finite dimensional division algebras over $\mathbb{R}$ are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$.* $\square$

2.1.2 Frobenius and Hopf algebras over a field

Again, we start with the relevant definitions.

Definition 2.1.5. A *Frobenius algebra over* $\Bbbk$ is a tuple $(A, m, 1_A, \Delta, \varepsilon)$, where $(A, m, 1_A)$ is an associative, unital $\Bbbk$-algebra, where (A, Δ, ε) is a coassociative, counital $\Bbbk$-coalgebra, and which satisfies the Frobenius law:

$$(a \otimes 1_A)\Delta(b) = \Delta(ab) = \Delta(a)(1_A \otimes b) \text{ for all } a, b \in A.$$

We note that every Frobenius algebra must be finite dimensional over $\Bbbk$. Moreover, Frobenius algebras should not be confused with bialgebras over $\Bbbk$, defined below.

Definition 2.1.6. A *bialgebra over* $\Bbbk$ is a tuple $(A, m, 1_A, \Delta, \varepsilon)$, where $(A, m, 1_A)$ is an associative, unital $\Bbbk$-algebra, where (A, Δ, ε) is a coassociative, counital $\Bbbk$-coalgebra, and which satisfies the condition that its comultiplication Δ and counit ε are morphisms of algebras. In particular, for any $a, b \in A$, we have

$$\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}; \qquad \varepsilon(ab) = \varepsilon(a)\varepsilon(b)$$

$$\Delta(1_A) = 1_A \otimes 1_A; \qquad \varepsilon(1_A) = 1_{\Bbbk}$$

where the comultiplication Δ is defined by $\Delta(x) := x_{(1)} \otimes x_{(2)}$ for any $x \in A$.

By equipping a bialgebra over a field with an extra property, we obtain the concept of a Hopf algebra over a field.

Definition 2.1.7. A *Hopf algebra over* $\Bbbk$ is a bialgebra H over $\Bbbk$, together with a $\Bbbk$-linear *antipode* map $S : H \to H$ which, for any $a \in H$, satisfies the antipode axiom

$$S(a_{(1)})a_{(2)} = \varepsilon(a)1_H = a_{(1)}S(a_{(2)}),$$

where the comultiplication Δ is again defined by $\Delta(a) := a_{(1)} \otimes a_{(2)}$, and the counit is denoted by $\varepsilon : H \to \Bbbk$.

Frobenius and Hopf algebras over a field are connected via the following theorem.

Theorem 2.1.8. [LS69], [Par71, Theorem 1] *Every finite dimensional Hopf algebra over* $\Bbbk$ *admits the structure of a Frobenius algebra.* $\square$

2.2 Monoidal categories and monoidal functors

In this section we recall the basics of monoidal categories. In §§2.2.1, 2.2.2, the definitions of monoidal categories, monoidal functors, and monoidal natural transformations are presented. We discuss rigidity in §2.2.3 and other properties of monoidal categories in §2.2.4. Finally, invertible objects are introduced in §2.2.5.

The standard reference for background on monoidal categories is [EGNO15]. We also refer the reader to [Wal24, Chapter 3] for a pedagogical introduction.

2.2.1 Monoidal categories

A *monoidal category* consists of a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an associator natural isomorphism $a := \{a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)\}_{X,Y,Z \in \mathcal{C}}$, a unit object $\mathbb{1} \in \mathcal{C}$, and unitality natural isomorphisms $\ell := \{\ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X\}_{X \in \mathcal{C}}$ and $r := \{r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X\}_{X \in \mathcal{C}}$, such that the diagrams in Figure 2.1 commute.

$$\begin{tikzcd}
& ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes \mathrm{id}_Z"'] \arrow[dr, "a_{W \otimes X,Y,Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W,X \otimes Y,Z}"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_W \otimes a_{X,Y,Z}"'] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

Figure 2.1 : Pentagon axiom for a monoidal category.

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes \ell_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

Figure 2.2 : Triangle axiom for a monoidal category.

Unless stated otherwise, by MacLane's strictness theorem [ML98, §VII.2], we assume that all monoidal categories are *strict* in the sense that

$$X \otimes Y \otimes Z := (X \otimes Y) \otimes Z = X \otimes (Y \otimes Z), \qquad X := \mathbb{1} \otimes X = X \otimes \mathbb{1},$$

for all $X, Y, Z \in \mathcal{C}$; that is, a, ℓ, and r are identity natural isomorphisms.

2.2.2 Monoidal functors and monoidal natural transformations

To move between monoidal categories, we want to consider functors that respect the monoidal structures of the two categories. Specifically, given monoidal categories $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{C}', \otimes', \mathbb{1}')$, a *monoidal functor* between them is a tuple $(F, F^{(2)}, F^{(0)})$, where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, $F^{(2)} := \{F^{(2)}_{X,Y} : F(X) \otimes' F(Y) \to F(X \otimes Y)\}$ is a natural transformation, and $F^{(0)} : \mathbb{1}' \to F(\mathbb{1})$ is a morphism in $\mathcal{C}'$, which satisfy associativity and unitality axioms.

A monoidal functor $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \to \mathcal{C}'$ is said to be *strict* if $F^{(2)}$ is an identity natural transformation and $F^{(0)}$ is an identity morphism. The monoidal functor $(F, F^{(2)}, F^{(0)})$ is called *strong* if $F^{(2)}$ is a natural isomorphism and $F^{(0)}$ is an iso in $\mathcal{C}'$.

We say that two monoidal categories $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{C}', \otimes', \mathbb{1}')$ are *equivalent as monoidal categories*, or *monoidally equivalent*, if there exists a monoidal functor $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \to \mathcal{C}'$ whose underlying functor F is an equivalence of categories.

Now, take two monoidal functors $(F, F^{(2)}, F^{(0)})$ and $(G, G^{(2)}, G^{(0)})$ from $\mathcal{C}$ to $\mathcal{C}'$. A *monoidal natural transformation* (resp., *monoidal natural isomorphism*) from $(F, F^{(2)}, F^{(0)})$ to $(G, G^{(2)}, G^{(0)})$ is a natural transformation $\phi : F \Rightarrow G$ (resp., natural isomorphism $\phi : F \overset{\sim}{\Rightarrow} G$) such that the diagrams in Figure 2.3 commute for all objects $X, Y \in \mathcal{C}$.

$$\begin{tikzcd} F(X) \otimes' F(Y) \arrow[r, "F^{(2)}_{X,Y}"] \arrow[d, "\phi_X \otimes \phi_Y"'] & F(X \otimes Y) \arrow[d, "\phi_{X \otimes Y}"] \\ G(X) \otimes' G(Y) \arrow[r, "G^{(2)}_{X \otimes Y}"] & G(X \otimes Y) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1}' \arrow[rr, "F^{(0)}"] \arrow[dr, "G^{(0)}"'] & & F(\mathbb{1}) \arrow[dl, "\phi_{\mathbb{1}}"] \\ & G(\mathbb{1}) & \end{tikzcd}$$

Figure 2.3 : Compatibility requirements for monoidal natural transformations.

2.2.3 Rigidity and pivotality

We say that $\mathcal{C}$ is *rigid* if each $X \in \mathcal{C}$ has a *left dual* $X^* \in \mathcal{C}$ with (co)evaluation maps $\mathrm{ev}_X^L : X^* \otimes X \to \mathbb{1}$ and $\mathrm{coev}_X^L : \mathbb{1} \to X \otimes X^*$, and a *right dual* ${}^*X \in \mathcal{C}$ with (co)evaluation maps $\mathrm{ev}_X^R : X \otimes {}^*X \to \mathbb{1}$, $\mathrm{coev}_X^R : \mathbb{1} \to {}^*X \otimes X$, satisfying coherence conditions. Here, ${}^*(X^*) \cong X \cong ({}^*X)^*$ in $\mathcal{C}$.

For a morphism $f : X \to Y$ in a rigid category $\mathcal{C}$, one can define the left dual morphism $f^* : Y^* \to X^*$ and right dual morphism ${}^*f : {}^*Y \to {}^*X$. This gives that $(-)^*$ and ${}^*(-)$ are contravariant (strong monoidal) autoequivalences of $\mathcal{C}$, called the *left* and *right duality functors*, respectively.

A rigid category is called *pivotal* if every object is naturally isomorphic to its double duals. More specifically, this means there exists a monoidal natural isomorphism $j : \mathrm{Id}_{\mathcal{C}} \overset{\sim}{\Rightarrow} (-)^{**}$. We note that this is equivalent to there existing a monoidal natural isomorphism $\hat{j} : (-)^* \overset{\sim}{\Rightarrow} {}^*(-)$.

2.2.4 Types of monoidal categories

A monoidal category $\mathcal{C}$ is *symmetric* if it is equipped with a natural isomorphism $c := \{c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X\}_{X,Y \in \mathcal{C}}$ which satisfies $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all $X, Y \in \mathcal{C}$, and which obeys the hexagon axioms. The component $c_{X,Y}$ of c, the $c^2 = \mathrm{id}$ property, the naturality of c at a morphism $f \in \mathcal{C}$, and unit coherence of c are all depicted in Figure 2.4 with string diagrams.

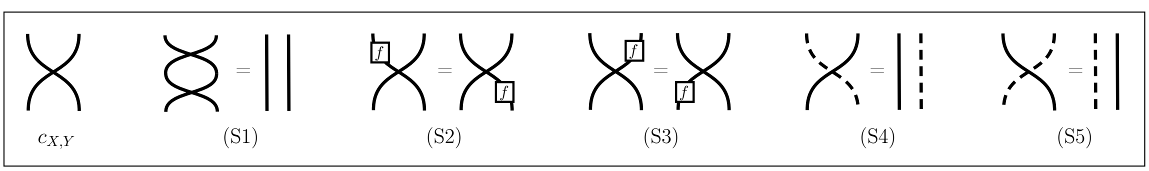

Figure 2.4 : Some axioms for a symmetric monoidal category.

We say that $\mathcal{C}$ is *additive monoidal* if its underlying category is additive and the endofunctors $(X \otimes -)$ and $(- \otimes X)$ of $\mathcal{C}$ are additive, for each $X \in \mathcal{C}$. It is *abelian monoidal* if is additive monoidal, and moreover, its underlying category is abelian.

Let $\Bbbk$ denote an algebraically closed field of characteristic 0. A category is $\Bbbk$-*linear* if all Hom-sets are $\Bbbk$-vector spaces, where composition distributes over addition and scalar multiplication. We say that $\mathcal{C}$ is ($\Bbbk$-)*linear monoidal* if the underlying category is $\Bbbk$-linear, and the endofunctors $(X \otimes -)$ and $(- \otimes X)$ of $\mathcal{C}$ are linear, for each $X \in \mathcal{C}$.

A $\Bbbk$-linear abelian category $\mathcal{C}$ is *locally finite* if every object has finite length and each Hom-space is finite dimensional. It is *finite* if it is locally finite, has enough projectives, and has only finitely many isoclasses of simple objects. We say that $\mathcal{C}$ is *fusion* if it is abelian, $\Bbbk$-linear monoidal, finite, rigid, semisimple, and satisfies $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk$. When the last condition is omitted, $\mathcal{C}$ is *multifusion*.

2.2.5 Invertible objects in monoidal categories

There are a few notions of invertible objects in monoidal categories. Here, $X \in \mathcal{C}$ is *left invertible* if there exists $X^L \in \mathcal{C}$ such that $X^L \otimes X \cong \mathbb{1}$, and is *right invertible* if there exists $X^R \in \mathcal{C}$ such that $X \otimes X^R \cong \mathbb{1}$. We also say that X is *invertible* if it is both left and right invertible; here, $X^L \cong X^R$.

The result below is straightforward to verify.

Lemma 2.2.1. *Consider an object $X \in \mathcal{C}$.*

(i) *X is left invertible if and only if $(- \otimes X) : \mathcal{C} \to \mathcal{C}$ is essentially surjective.*

(ii) *X is right invertible if and only if $(X \otimes -) : \mathcal{C} \to \mathcal{C}$ is essentially surjective.*

(iii) *If $X \in \mathcal{C}$ is invertible, then both functors $(- \otimes X) : \mathcal{C} \to \mathcal{C}$ and $(X \otimes -) : \mathcal{C} \to \mathcal{C}$ are equivalences.* □

In a rigid category $\mathcal{C}$, a stronger notion of invertibility for $X \in \mathcal{C}$ is to require that it is invertible via (co)evaluation morphisms [EGNO15, §2.11]. Next, we connect invertibility here with simplicity.

Lemma 2.2.2. *Take an abelian monoidal category $\mathcal{C}$ with simple $\mathbb{1}$, and where all objects have finite length. If $X \in \mathcal{C}$ is left or right invertible, then X is simple.*

Proof. This follows since $\mathrm{length}(X \otimes Y) \geqslant \mathrm{length}(X)\,\mathrm{length}(Y)$ when $X, Y \in \mathcal{C}$ have finite length [EGNO15, Exercise 4.3.11(1,2)], and objects of length 1 are precisely the simple objects of $\mathcal{C}$. □

2.3 Algebraic structures in monoidal categories

Here, we adapt algebraic objects from §2.1 to the monoidal setting. We start with algebras and coalgebras in monoidal categories is §2.3.1. Modules are introduced in §2.3.2, and their properties and operations are discussed in §§2.3.3, 2.3.4.

For details on algebras in monoidal categories, see, for example, [Koc04, Chapter 3], [TV17, Parts I and II], or [Wal24, Chapter 4]. The first of these also includes an introduction to Frobenius algebras in monoidal categories, and the last also includes background on modules in monoidal categories. For background on coalgebras, bialgebras, and Hopf algebras in monoidal categories, see [AM10, Section 1.2], noting that the term monoid is used instead of the term algebra for all of these structures.

2.3.1 Algebraic objects in monoidal categories

Let $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category. An *algebra in* $\mathcal{C}$ is an object $A \in \mathcal{C}$, equipped with morphisms $m : A \otimes A \to A$ and $u : \mathbb{1} \to A$ in $\mathcal{C}$, subject to the associativity and unitality axioms given below.

$$m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m), \qquad m(u \otimes \mathrm{id}_A) = \mathrm{id}_A = m(\mathrm{id}_A \otimes u).$$

Algebras in $\mathcal{C}$ form a category, $\mathsf{Alg}(\mathcal{C})$, where a morphism $(A, m_A, u_A) \to (B, m_B, u_B)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $f\, m_A = m_B(f \otimes f)$ and $f\, u_A = u_B$.

For example, $\mathbb{1} \in \mathsf{Alg}(\mathcal{C})$, with $m_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ being the unitor isomorphism, and $u_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$. In an abelian monoidal category, $\mathsf{0} \in \mathsf{Alg}(\mathcal{C})$ with $m_{\mathsf{0}} : \mathsf{0} \otimes \mathsf{0} \to \mathsf{0}$ and $u_{\mathsf{0}} : \mathbb{1} \to \mathsf{0}$ coming from $\mathsf{0}$ being terminal.

A *coalgebra in* $\mathcal{C}$ is an object $A \in \mathcal{C}$, equipped with morphisms $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbb{1}$ in $\mathcal{C}$, subject to coassociativity and counitality axioms:

$$(\Delta \otimes \mathrm{id}_A)\Delta = (\mathrm{id}_A \otimes \Delta)\Delta, \qquad (\varepsilon \otimes \mathrm{id}_A)\Delta = \mathrm{id}_A = \varepsilon(\mathrm{id}_A \otimes u)\Delta.$$

Coalgebras in $\mathcal{C}$ form a category, where a morphism $(A, \Delta_A, \varepsilon_A) \to (B, \Delta_B, \varepsilon_B)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $\Delta_B f = (f \otimes f)\Delta_A$ and $\varepsilon_B f = \varepsilon_A$. This category is denoted $\mathsf{Coalg}(\mathcal{C})$.

We continue our examples of $\mathbb{1}$ and $\mathsf{0}$. Specifically, for any monoidal category $\mathcal{C}$, we have that $\mathbb{1}$ is a coalgebra in $\mathcal{C}$ with $\Delta_{\mathbb{1}} : \mathbb{1} \to \mathbb{1} \otimes \mathbb{1}$ being the inverse of the unitor isomorphism, and $\varepsilon_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$. If $\mathcal{C}$ is abelian, then $\mathsf{0} \in \mathsf{Coalg}(\mathcal{C})$ with $\Delta_{\mathsf{0}} : \mathsf{0} \to \mathsf{0} \otimes \mathsf{0}$ and $\varepsilon_{\mathsf{0}} : \mathsf{0} \to \mathbb{1}$ both coming from $\mathsf{0}$ being initial.

We now explore ways in which objects in $\mathcal{C}$ can be simultaneously an algebra and a coalgebra. The first, which requires no additional structure on the monoidal category $\mathcal{C}$ is given in the following definition.

Definition 2.3.1. A *Frobenius algebra in* $\mathcal{C}$ is a tuple $(A, m, u, \Delta, \varepsilon)$ where (A, m, u) is an algebra in $\mathcal{C}$, (A, Δ, ε) is a coalgebra in $\mathcal{C}$, and which satisfies the Frobenius law:

$$(m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta) = \Delta m = (\mathrm{id}_A \otimes m)(\Delta \otimes \mathrm{id}_A).$$

A *morphism of Frobenius algebras in* $\mathcal{C}$ is a morphism of the underlying algebras and coalgebras in $\mathcal{C}$. The above objects and morphisms then form a category, $\mathsf{FrobAlg}(\mathcal{C})$.

When the monoidal category $\mathcal{C}$ is symmetric via natural isomorphism c, as discussed in §2.2.4, we can also define bialgebras and Hopf algebras.

Definition 2.3.2. A *bialgebra in a symmetric monoidal category* $\mathcal{C}$ is a tuple $(A, m, u, \Delta, \varepsilon)$ where $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$, where $(A, \Delta, \varepsilon) \in \mathsf{Coalg}(\mathcal{C})$, and which satisfies the condition that Δ and ε are morphisms of algebras, or equivalently that m and u are morphisms of coalgebras. Again, together with morphisms that are simultaneously algebra and coalgebra morphisms, these objects form a category, $\mathsf{Bialg}(\mathcal{C})$.

Imposing further structure on bialgebras produces Hopf algebras, which are defined below.

Definition 2.3.3. A *Hopf algebra in a symmetric monoidal category* $\mathcal{C}$ is a bialgebra $(H, m, u, \Delta, \varepsilon)$, equipped with a morphism $S : H \to H$ in $\mathcal{C}$, called the *antipode*, satisfying the antipode axiom:

$$m(S \otimes \mathrm{id}_H)\Delta = u\varepsilon = m(\mathrm{id}_H \otimes S)\Delta.$$

Again, we obtain a category, $\mathsf{HopfAlg}(\mathcal{C})$, whose objects are Hopf algebras and whose morphisms are those morphisms of $\mathcal{C}$ which are simultaneously algebra and coalgebra morphisms.

As elementary examples, we note that it is easy to verify that $\mathbb{1}$ is a Frobenius algebra in any monoidal category, and is a bialgebra and Hopf algebra in any symmetric monoidal category. In the Hopf case, the antipode is given by $S = \mathrm{id}_{\mathbb{1}}$. Similarly, in the case that the (symmetric) monoidal category is abelian, one can check that $\mathsf{0}$ is a Frobenius algebra, bialgebra, and Hopf algebra with antipode $S = \mathrm{id}_{\mathsf{0}}$.

Note also that all of these above definitions coincide with the corresponding definitions over a field $\Bbbk$ from Section 2.1 when we take $\mathcal{C} = \mathsf{Vec}_{\Bbbk}$.

2.3.2 Modules in monoidal categories

Fix an algebra (A, m_A, u_A) in $\mathcal{C}$. A *right A-module in $\mathcal{C}$* is a pair $(M, \triangleleft)$, where $M \in \mathcal{C}$ and $\triangleleft : M \otimes A \to M$ in $\mathcal{C}$ satisfying associativity and unitality axioms. These structures form a category $\mathsf{Mod}\text{-}A(\mathcal{C})$, where morphisms are those morphisms in $\mathcal{C}$ that respect the right module structures. *Left A-modules $(N, \triangleright)$ in $\mathcal{C}$* and the category $A\text{-}\mathsf{Mod}(\mathcal{C})$ are defined likewise.

If (B, m_B, u_B) is another algebra in $\mathcal{C}$, an *(A, B)-bimodule in $\mathcal{C}$* is a tuple $(Q, \triangleright, \triangleleft)$ such that $(Q, \triangleright) \in A\text{-}\mathsf{Mod}(\mathcal{C})$, $(Q, \triangleleft) \in \mathsf{Mod}\text{-}B(\mathcal{C})$, satisfying a middle associativity axiom. With morphisms that are simultaneously left and right module morphisms, we obtain the category $(A, B)\text{-}\mathsf{Bimod}(\mathcal{C})$.

For example, for any algebra A in an abelian monoidal category $\mathcal{C}$, we have that $\mathsf{0} \in A\text{-}\mathsf{Mod}(\mathcal{C})$, with $\rhd : A \otimes \mathsf{0} \to \mathsf{0}$ coming from $\mathsf{0}$ being terminal.

A *left ideal* of an algebra A in $\mathcal{C}$ is a subobject of $A_{\text{reg}} \in A\text{-}\mathsf{Mod}(\mathcal{C})$. In other words, it is an object (I, λ) with a mono $\iota_I^A : (I, \lambda) \hookrightarrow (A, m_A)$ in $A\text{-}\mathsf{Mod}(\mathcal{C})$. Similarly, a *right ideal of* A is a subobject of $A_{\text{reg}} \in \mathsf{Mod}\text{-}A(\mathcal{C})$, and a (*two-sided*) *ideal of* A is a subobject of $A_{\text{reg}} \in (A, A)\text{-}\mathsf{Bimod}(\mathcal{C})$.

In an abelian monoidal category $\mathcal{C}$, a non-zero algebra A is *simple* if its only ideals are itself and zero (i.e., A_{reg} is a simple object in $(A, A)\text{-}\mathsf{Bimod}(\mathcal{C})$). Next, consider the preliminary result below.

Lemma 2.3.4. *In an abelian monoidal category $\mathcal{C}$, an algebra A in $\mathcal{C}$ is the zero algebra $\mathsf{0}$ if and only if $A\text{-}\mathsf{Mod}(\mathcal{C})$ and $\mathsf{Mod}\text{-}A(\mathcal{C})$ each have one object, namely the zero module, up to isomorphism.*

Proof. If $A = \mathsf{0}$ and $(M, \rhd) \in \mathsf{0}\text{-}\mathsf{Mod}(\mathcal{C})$, then $\mathrm{id}_M = \rhd \circ (u_{\mathsf{0}} \otimes \mathrm{id}_M)$ is a zero morphism. Hence, M is both initial and terminal, and $M \cong \mathsf{0}$. Similarly, all right modules over $A = \mathsf{0}$ are also zero. On the other hand, if A is non-zero, then $A\text{-}\mathsf{Mod}(\mathcal{C})$ (or $\mathsf{Mod}\text{-}A(\mathcal{C})$) contains the zero module and the regular module, and these are not isomorphic. □

2.3.3 Properties of modules

A non-zero right module $M \in \mathsf{Mod}\text{-}A(\mathcal{C})$ is called *simple* if it is a simple object in the category $\mathsf{Mod}\text{-}A(\mathcal{C})$; a similar notion holds for right modules and bimodules in $\mathcal{C}$.

The *regular right* (resp., *left*) *A-module in $\mathcal{C}$* is (A, m_A) in $\mathsf{Mod}\text{-}A(\mathcal{C})$ (resp., in $A\text{-}\mathsf{Mod}(\mathcal{C})$), and *the regular (A, A)-bimodule in $\mathcal{C}$* is (A, m_A, m_A) in $(A, A)\text{-}\mathsf{Bimod}(\mathcal{C})$, which is denoted by A_{reg} or A.

Moreover, a right A-module $(M, \lhd)$ in $\mathcal{C}$ is said to be *free* if there is an object $X \in \mathcal{C}$ so that $(M, \lhd) \cong (X \otimes A, \mathrm{id}_X \otimes m_A)$ in $\mathsf{Mod}\text{-}A(\mathcal{C})$. Similarly, a left A-module $(N, \rhd)$ in $\mathcal{C}$ is said to be *free* if there is an object $Y \in \mathcal{C}$ so that $(N, \rhd) \cong (A \otimes Y, m_A \otimes \mathrm{id}_Y)$

in A-$\mathsf{Mod}(\mathcal{C})$. For instance, the regular left and right A-modules are the free modules over A on the object $\mathbb{1} \in \mathcal{C}$.

2.3.4 Operations on modules

Now let $(M, \triangleleft) \in \mathsf{Mod}$-$A(\mathcal{C})$. If ${}^*M$ exists in $\mathcal{C}$, then ${}^*M$ is a left A-module in $\mathcal{C}$. Similarly, given $(N, \triangleright) \in A$-$\mathsf{Mod}(\mathcal{C})$, if N^* exists in $\mathcal{C}$, then N^* is a right A-module in $\mathcal{C}$. When $\mathcal{C}$ is rigid, restricting the duality functors from §2.2.3 to categories of modules, we obtain the equivalences of categories $(-)^* : A\text{-}\mathsf{Mod}(\mathcal{C}) \xrightarrow{\sim} \mathsf{Mod}\text{-}A(\mathcal{C})$ and ${}^*(-) : \mathsf{Mod}\text{-}A(\mathcal{C}) \xrightarrow{\sim} A\text{-}\mathsf{Mod}(\mathcal{C})$, for any $A \in \mathsf{Alg}(\mathcal{C})$.

Given $(M, \triangleleft) \in \mathsf{Mod}$-$A(\mathcal{C})$ and $(N, \triangleright) \in A$-$\mathsf{Mod}(\mathcal{C})$, the *tensor product of M and N over A* is the coequalizer of the morphisms $\mathrm{id}_M \otimes \triangleright$ and $\triangleleft \otimes \mathrm{id}_N$, denoted by $M \otimes_A N$, if it exists in $\mathcal{C}$. In any case, for any $Q \in (A, B)$-$\mathsf{Bimod}(\mathcal{C})$ and $P \in (B, C)$-$\mathsf{Bimod}(\mathcal{C})$, we get that $Q \otimes_B P$ is in (A, C)-$\mathsf{Bimod}(\mathcal{C})$. Here, (A, A)-$\mathsf{Bimod}(\mathcal{C})$ is monoidal with $\otimes := \otimes_A$ and $\mathbb{1} := A_{\mathrm{reg}}$.

2.4 Morita's and Ostrik's theorems

Here, instead of doing algebra in monoidal categories, we do algebra on monoidal categories, thinking of monoidal categories as the analogue of algebras. We introduce module categories and functors in §2.4.1. Internal Homs and Ends are in §2.4.2. Finally, Morita's Theorem and Ostrik's Theorem are presented in §§2.4.3, 2.4.4.

For further details see [EGNO15, Chapter 7] or [Wal24, Chapters 3 and 4].

2.4.1 Module categories and functors

A *left $\mathcal{C}$-module category* is a category $\mathcal{M}$ with a left action bifunctor $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and associativity and unitality natural isomorphisms which satisfy the pentagon and triangle axioms. *Right $\mathcal{C}$-module categories* $(\mathcal{N}, \triangleleft)$ are defined likewise.

The *regular left (resp. right) $\mathcal{C}$-module category* is given by $\mathcal{C}$, with action bifunctor $\triangleright := \otimes$ (resp. $\triangleleft := \otimes$). We also have that for any algebra $A \in \mathsf{Alg}(\mathcal{C})$, the category

$\mathsf{Mod}\text{-}A(\mathcal{C})$ is a left $\mathcal{C}$-module category and $A\text{-}\mathsf{Mod}(\mathcal{C})$ is a right $\mathcal{C}$-module category, again with action bifunctors given by $\otimes$.

A *left $\mathcal{C}$-module functor* is a tuple (F, s) where F is a functor between left $\mathcal{C}$-module categories $(\mathcal{M}, \triangleright)$ and $(\mathcal{M}', \triangleright')$ and $s := \{s_{X,M} : F(X \triangleright M) \xrightarrow{\sim} X \triangleright' F(M)\}_{X \in \mathcal{C}, M \in \mathcal{M}}$ is a natural isomorphism satisfying a pentagon and triangle axiom. One can analogously define a *right $\mathcal{C}$-module functor*.

We say two left $\mathcal{C}$-module categories are *equivalent as left $\mathcal{C}$-module categories* if there is a left $\mathcal{C}$-module functor between them such that the underlying functor is an equivalence. *Right $\mathcal{C}$-module equivalence of categories* is defined analogously.

2.4.2 Internal Homs

A left $\mathcal{C}$-module category $\mathcal{M} := (\mathcal{M}, \triangleright)$ is *closed* if, for each $M \in \mathcal{M}$ (resp,. $N \in \mathcal{N}$), the functor $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ has a right adjoint: $\underline{\mathrm{Hom}}_{\mathcal{M}}(M, -) : \mathcal{M} \to \mathcal{C}$. We call $\underline{\mathrm{Hom}}_{\mathcal{M}}(M, N)$ the *internal Hom of M and N*. Also, $\underline{\mathrm{End}}_{\mathcal{M}}(M) := \underline{\mathrm{Hom}}_{\mathcal{M}}(M, M)$ is called the *internal End of M*. Similar notions hold for right $\mathcal{C}$-module categories.

For any $M \in (\mathcal{M}, \triangleright)$ and any $N \in (\mathcal{N}, \triangleleft)$, the objects $\underline{\mathrm{End}}_{\mathcal{M}}(M)$ and $\underline{\mathrm{End}}_{\mathcal{N}}(N)$ are algebras in $\mathcal{C}$. Given $M' \in \mathcal{M}$ and $N' \in \mathcal{N}$, we obtain that $\underline{\mathrm{Hom}}_{\mathcal{M}}(M, M')$ is a right $\underline{\mathrm{End}}_{\mathcal{M}}(M)$-module in $\mathcal{C}$. Similarly, $\underline{\mathrm{Hom}}_{\mathcal{N}}(N, N')$ is a left $\underline{\mathrm{End}}_{\mathcal{N}}(N')$-module in $\mathcal{C}$. From this, we obtain the functors $\underline{\mathrm{Hom}}_{\mathcal{M}}(M, -) : \mathcal{M} \to \mathsf{Mod}\text{-}\underline{\mathrm{End}}_{\mathcal{M}}(M)(\mathcal{C})$ and $\underline{\mathrm{Hom}}_{\mathcal{N}}(-, N') : \mathcal{N} \to \underline{\mathrm{End}}_{\mathcal{N}}(N')\text{-}\mathsf{Mod}(\mathcal{C})$.

As an example, if the category $\mathcal{C}$ is rigid, the regular left $\mathcal{C}$-module category is closed, with $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \cong Y \otimes X^*$. The algebra and module structures on these internal Homs and Ends are then given by appropriate (co)evaluation maps.

Given an algebra $A \in \mathcal{C}$, we see that the left $\mathcal{C}$-module category $\mathsf{Mod}\text{-}A(\mathcal{C})$ and the right $\mathcal{C}$-module category $A\text{-}\mathsf{Mod}(\mathcal{C})$ are both closed, with internal Homs given by $\underline{\mathrm{Hom}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M, M') \cong (M \otimes_A {}^*(M'))^*$ and $\underline{\mathrm{Hom}}_{A\text{-}\mathsf{Mod}(\mathcal{C})}(N, N') \cong {}^*((N')^* \otimes_A N)$ in $\mathcal{C}$. The next result is also useful.

Lemma 2.4.1. *For $\mathcal{C}$ rigid with $A \in \mathsf{Alg}(\mathcal{C})$, we have that $A \cong \underline{\mathrm{End}}_{A\text{-}\mathsf{Mod}(\mathcal{C})}(A)$ and $A \cong \underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(A)$ as algebras in $\mathcal{C}$.*

Proof. Take the projection $\pi : A^* \otimes A \to A^* \otimes_A A$ from the coequalizer property. Next, note that $m_{\underline{\mathrm{End}}_{A\text{-}\mathsf{Mod}(\mathcal{C})}(A)} = {}^*\mu$ where $\mu := \mathrm{id}_{A^*} \otimes_A \mathrm{coev}_A^L \otimes_A \mathrm{id}_A$, and $u_{\underline{\mathrm{End}}_{A\text{-}\mathsf{Mod}(\mathcal{C})}(A)} = {}^*\eta$ such that $\mathrm{ev}_A^L = \eta\,\pi$. Moreover, the isomorphism $A^* \otimes_A A \cong A^*$ in $\mathcal{C}$ is given by mutually inverse morphisms $\phi : A^* \otimes_A A \to A^*$ and $\psi : A^* \to A^* \otimes_A A$ in $\mathcal{C}$, where $\phi\,\pi = \triangleleft_{A^*} = (\mathrm{ev}_A^L \otimes \mathrm{id}_{A^*})(\mathrm{id}_{A^*} \otimes m_A \otimes \mathrm{id}_{A^*})(\mathrm{id}_{A^*} \otimes \mathrm{id}_A \otimes \mathrm{coev}_A^L)$ and $\psi = \pi\,(\mathrm{id}_{A^*} \otimes u_A)$.

Now one can check that $(m_A)^* = (\phi \otimes \phi)\,\mu\,\psi$ and $(u_A)^* = \eta\,\psi$. Thus, ${}^*\psi$ yields the first algebra isomorphism in $\mathcal{C}$. Similarly, $A \cong \underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(A)$ as algebras in $\mathcal{C}$. □

2.4.3 Morita's Theorem

We say that algebras A and B in $\mathcal{C}$ are *Morita equivalent* in $\mathcal{C}$ if their categories of modules are equivalent as $\mathcal{C}$-module categories. The following gives the formal definition, as well as a useful characterization of this notion in terms of bimodules.

Theorem 2.4.2 (Generalized Morita's Theorem)**.** *Let $\mathcal{C}$ be an abelian monoidal category such that the functors $- \otimes X$ and $X \otimes -$ are right exact for each $X \in \mathcal{C}$. Take algebras $A, B \in \mathsf{Alg}(\mathcal{C})$. Then the following statements are equivalent.*

(a) *A-$\mathsf{Mod}(\mathcal{C})$ and B-$\mathsf{Mod}(\mathcal{C})$ are equivalent as right $\mathcal{C}$-module categories.*

(b) *Mod-$A(\mathcal{C})$ and Mod-$B(\mathcal{C})$ are equivalent as left $\mathcal{C}$-module categories.*

(c) *There exists bimodules $P \in (A,B)$-$\mathsf{Bimod}(\mathcal{C})$ and $Q \in (B,A)$-$\mathsf{Bimod}(\mathcal{C})$ such that $P \otimes_B Q \cong A_{reg}$ as A-bimodules and $Q \otimes_A P \cong B_{reg}$ as B-bimodules.* □

In particular, the proof of this theorem shows that if we have the bimodules P and Q, then the equivalences of module categories are given by

$$Q \otimes_A - : A\text{-}\mathsf{Mod}(\mathcal{C}) \to B\text{-}\mathsf{Mod}(\mathcal{C}); \qquad P \otimes_B - : B\text{-}\mathsf{Mod}(\mathcal{C}) \to A\text{-}\mathsf{Mod}(\mathcal{C})$$

$$- \otimes_A P : \mathsf{Mod}\text{-}A(\mathcal{C}) \to \mathsf{Mod}\text{-}B(\mathcal{C}); \qquad - \otimes_B Q : \mathsf{Mod}\text{-}B(\mathcal{C}) \to \mathsf{Mod}\text{-}A(\mathcal{C}).$$

For instance, let $\mathcal{C}$ be abelian rigid monoidal with simple unit. Then, for any non-zero $X \in \mathcal{C}$, the internal End of X, given by $X \otimes X^*$, is Morita equivalent to $\mathbb{1}$ in $\mathcal{C}$ via one of the functors below:

$$(- \otimes X^*) : \mathcal{C} \xrightarrow{\sim} \mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C}); \qquad (X \otimes -) : \mathcal{C} \xrightarrow{\sim} (X \otimes X^*)\text{-}\mathsf{Mod}(\mathcal{C}). \quad (2.4.3)$$

Their respective quasi-inverses are given by:

$$(- \otimes_{X \otimes X^*} X) : \mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C}) \to \mathcal{C}; \qquad (X^* \otimes_{X \otimes X^*} -) : (X \otimes X^*)\text{-}\mathsf{Mod}(\mathcal{C}) \to \mathcal{C}.$$

See, e.g., [Wal24, Example 4.58]. Similarly, the algebra ${}^*X \otimes X$ is also Morita equivalent to $\mathbb{1}$ in $\mathcal{C}$.

2.4.4 Ostrik's Theorem

The categories A-$\mathsf{Mod}(\mathcal{C})$ and Mod-$A(\mathcal{C})$ are the prototypical examples of $\mathcal{C}$-module categories, and the following theorem from [Ost03] addresses when any given $\mathcal{C}$-module category is of this form.

Theorem 2.4.4 (Ostrik's Theorem)**.** *Let $\mathcal{C}$ be a multifusion category, with $\mathcal{M}$ and $\mathcal{N}$ non-zero, indecomposable left and right $\mathcal{C}$-module categories, respectively. Then, for any non-zero $M \in \mathcal{M}$ and any non-zero $N' \in \mathcal{N}$, we have that*

$$\mathcal{M} \simeq \mathsf{Mod}\text{-}\underline{\mathrm{End}}_{\mathcal{M}}(M)(\mathcal{C}) \text{ and } \mathcal{N} \simeq \underline{\mathrm{End}}_{\mathcal{N}}(N')\text{-}\mathsf{Mod}(\mathcal{C}),$$

as left and right $\mathcal{C}$-module categories, respectively, via the functors

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(M, -) : \mathcal{M} \xrightarrow{\sim} \mathsf{Mod}\text{-}\underline{\mathrm{End}}_{\mathcal{M}}(M)(\mathcal{C}) \quad \text{and}$$
$$\underline{\mathrm{Hom}}_{\mathcal{N}}(-, N') : \mathcal{N} \xrightarrow{\sim} \underline{\mathrm{End}}_{\mathcal{N}}(N')\text{-}\mathsf{Mod}(\mathcal{C}).$$

□

2.5 Functorial constructions

In this section, we begin by expanding the analogy between algebras in monoidal categories and monoidal functors to include all the different algebraic structures introduced in §2.3.1. This is done in §2.5.1. Then we introduce Monads and the Eilenberg-Moore and Kleisli categories corresponding to them in §§2.5.2, 2.5.3.

A good overview of the material concerning functorial preservation of algebra objects can be found in [AM10, Chapter 3]. Further information can be found in [Wal24, DP08, BÏ8], with specific locations provided at the beginning of the theorems below. References on monads include [Rie17, Chapter 5] and [Wal24, §§4.3.2, 4.4.3].

2.5.1 Functors preserving algebraic objects

Algebras in monoidal categories and monoidal functors are linked by the fact that monoidal functors preserve algebras (see Proposition 2.5.4 below). Similarly, there are specific types of functors between (symmetric) monoidal categories that preserve coalgebras, Frobenius algebras, bialgebras, and Hopf algebras.

First, A functor $F : \mathcal{C} \to \mathcal{C}'$ is a *comonoidal functor* if it is equipped with a natural transformation $F_{(2)} := \{F_{(2)}^{X,Y} : F(X \otimes Y) \to F(X) \otimes' F(Y)\}_{X,Y \in \mathcal{C}}$ and a morphism $F_{(0)} : F(\mathbb{1}) \to \mathbb{1}'$ in $\mathcal{C}'$ which satisfy coassociativity and counitality constraints.

To ease notation when working with a monoidal functor $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \to \mathcal{C}'$, we will write $F^{(2)}_{X,Y,Z} : F(X) \otimes' F(Y) \otimes' F(Z) \to F(X \otimes Y \otimes Z)$ for the composition $F^{(2)} \circ F^{(2)}$, which is unambiguous by associativity. Similarly, for comonoidal functors, we will write $F_{(2)}^{X,Y,Z} : F(X \otimes Y \otimes Z) \to F(X) \otimes' F(Y) \otimes' F(Z)$ by coassociativity.

Definition 2.5.1. A *Frobenius monoidal functor* between monoidal categories $\mathcal{C}$ and $\mathcal{C}'$ is a tuple $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)})$ such that $(F, F^{(2)}, F^{(0)})$ is a monoidal functor between $\mathcal{C}$ and $\mathcal{C}'$, and $(F, F_{(2)}, F_{(0)})$ is a comonoidal functor between $\mathcal{C}$ and $\mathcal{C}'$, and which satisfies the Frobenius conditions given below, for all $X, Y, Z \in \mathcal{C}$.

$$\left(F^{(2)}_{X,Y} \otimes' \mathrm{id}_{F(Z)}\right)\left(\mathrm{id}_{F(X)} \otimes' F_{(2)}^{Y,Z}\right) = F_{(2)}^{X\otimes Y,Z} \circ F^{(2)}_{X,Y\otimes Z},$$

$$\left(\mathrm{id}_{F(X)} \otimes' F^{(2)}_{Y,Z}\right)\left(F_{(2)}^{Y,Z} \otimes' \mathrm{id}_{F(Z)}\right) = F_{(2)}^{X,Y\otimes Z} \circ F^{(2)}_{X\otimes Y,Z}.$$

In the case that $F^{(2)} \circ F_{(2)} = \mathrm{id}$, we call F a *separable Frobenius monoidal functor*.

As with bialgebras and Hopf algebras, to define corresponding functors we must require that we are working in symmetric monoidal categories.

Definition 2.5.2. A *bi-monoidal functor* between the symmetric monoidal categories $(\mathcal{C}, c)$ and $(\mathcal{C}', c')$ is a tuple $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)})$ such that $(F, F^{(2)}, F^{(0)})$ is a monoidal functor between $\mathcal{C}$ and $\mathcal{C}'$, and $(F, F_{(2)}, F_{(0)})$ is a comonoidal functor between $\mathcal{C}$ and $\mathcal{C}'$, and which satisfies the three unitality axioms given below and the braiding axiom shown in Figure 2.5 for any $X, Y, Z, W \in \mathcal{C}$:

$$F_{(2)}^{\mathbb{1},\mathbb{1}} \circ F^{(0)} = F^{(0)} \otimes' F^{(0)}; \qquad F_{(0)} \circ F^{(2)}_{\mathbb{1},\mathbb{1}} = F_{(0)} \otimes' F_{(0)}; \qquad F_{(0)} \circ F^{(0)} = \mathrm{id}_{\mathbb{1}'}.$$

$$\begin{tikzcd}
 & F(X \otimes Y) \otimes' F(Z \otimes W) \arrow[dl, "F^{(2)}_{X\otimes Y, Z\otimes W}"'] \arrow[dr, "F_{(2)}^{X,Y} \otimes' F_{(2)}^{Z,W}"] & \\
F(X \otimes Y \otimes Z \otimes W) \arrow[d, "F(\mathrm{id}_X \otimes c_{Y,Z} \otimes \mathrm{id}_W)"'] & & F(X) \otimes' F(Y) \otimes' F(Z) \otimes' F(W) \arrow[d, "\mathrm{id}_{F(X)} \otimes' c_{F(Y),F(Z)} \otimes' \mathrm{id}_{F(W)}"] \\
F(X \otimes Z \otimes Y \otimes W) \arrow[dr, "F_{(2)}^{X\otimes Z, Y\otimes W}"'] & & F(X) \otimes' F(Z) \otimes' F(Y) \otimes' F(W) \arrow[dl, "F^{(2)}_{X,Z} \otimes' F^{(2)}_{Y,W}"] \\
 & F(X \otimes Z) \otimes' F(Y \otimes W) &
\end{tikzcd}$$

Figure 2.5 : Braiding axiom for bi-monoidal functors.

Lastly, we define Hopf monoidal functors.

Definition 2.5.3. A *Hopf monoidal functor* between symmetric monoidal categories $(\mathcal{C}, c)$ and $(\mathcal{C}', c')$ is a tuple $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}, \Upsilon)$ where $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)})$ is a bi-monoidal functor, and $\Upsilon : F \Rightarrow F$ is a natural transformation, again called the *antipode*, such that the following conditions are satisfied for any $X, Y, Z \in \mathcal{C}$.

$$F^{(2)}_{X,Y,Z} \left(\mathrm{id}_F(X) \otimes' \Upsilon_Y \otimes' \mathrm{id}_{F(Z)}\right) F_{(2)}^{X,Y,Z} = \mathrm{id}_{F(X\otimes Y\otimes Z)},$$

$$F^{(2)}_{X,Y,Z} \left(\Upsilon_X \otimes' \mathrm{id}_{F(Y)} \otimes' \Upsilon_Z\right) F_{(2)}^{X,Y,Z} = \Upsilon_{X\otimes Y\otimes Z},$$

$$F^{(2)}_{\mathbb{1},\mathbb{1}} \left(\mathrm{id}_{F(\mathbb{1})} \otimes' \Upsilon_{\mathbb{1}}\right) F_{(2)}^{\mathbb{1},\mathbb{1}} = F^{(0)} F_{(0)} = F^{(2)}_{\mathbb{1},\mathbb{1}} \left(\Upsilon_{\mathbb{1}} \otimes' \mathrm{id}_{F(\mathbb{1})}\right) F_{(2)}^{\mathbb{1},\mathbb{1}}.$$

Now, with all these definitions we obtain the following results.

Proposition 2.5.4. [Wal24, Proposition 4.3] [DP08, Corollary 5] [BÏ8, Lemma 6.10] [AM10, Proposition 3.31 and Theorem 3.70] *Take monoidal categories $\mathcal{C}$ and $\mathcal{C}'$.*

(a) *Given a monoidal functor $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \to \mathcal{C}'$ and an algebra (A, m_A, u_A) in $\mathcal{C}$, we obtain that*

$$\big(F(A),\ m_{F(A)} := F(m_A)F^{(2)}_{A,A},\ u_{F(A)} := F(u_A)F^{(0)}\big) \in \mathsf{Alg}(\mathcal{C}').$$

In particular, F induces a functor $\mathsf{Alg}(\mathcal{C}) \to \mathsf{Alg}(\mathcal{C}')$.

(b) *Given A comonoidal functor $(F, F_{(2)}, F_{(0)}) : \mathcal{C} \to \mathcal{C}'$ and a coalgebra $(A, \Delta_A, \varepsilon_A)$ in $\mathcal{C}$, we obtain that*

$$\big(F(A),\ \ \Delta_{F(A)} := F_{(2)}^{A,A}\, F(\Delta_A),\ \ \varepsilon_{F(A)} := F_{(0)}\, F(\varepsilon_A)\big) \in \mathsf{Coalg}(\mathcal{C}').$$

In particular, F induces a functor $\mathsf{Colg}(\mathcal{C}) \to \mathsf{Coalg}(\mathcal{C}')$.

(c) *Moreover, given a Frobenius monoidal functor $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}) : \mathcal{C} \to \mathcal{C}'$ and Frobenius algebra $(A, m_A, u_A, \Delta_A, \varepsilon_A)$ in $\mathcal{C}$, we have that $F(A)$ is a Frobenius algebra in $\mathcal{C}'$ by using the formulas from parts (a) and (b). In particular, F induces a functor $\mathsf{FrobAlg}(\mathcal{C}) \to \mathsf{FrobAlg}(\mathcal{C}')$.*

(d) *Now, letting $\mathcal{C}$ and $\mathcal{C}'$ be symmetric monoidal, given a bi-monoidal functor $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}) : \mathcal{C} \to \mathcal{C}'$ and bialgebra $(A, m_A, u_A, \Delta_A, \varepsilon_A)$ in $\mathcal{C}$, we have that $F(A)$ is a bialgebra in $\mathcal{C}'$ via the formulas from parts (a) and (b). In particular, F induces a functor $\mathsf{Bialg}(\mathcal{C}) \to \mathsf{Bialg}(\mathcal{C}')$.*

(e) *Likewise, given a Hopf monoidal functor $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}, \Upsilon) : \mathcal{C} \to \mathcal{C}'$ between symmetric monoidal categories, and a Hopf algebra $(A, m_A, u_A, \Delta_A, \varepsilon_A, S)$ in $\mathcal{C}$, it follows that $F(A)$ is a Hopf algebra in $\mathcal{C}'$ using the formulas from (a) and (b), and with antipode given by $S_{F(A)} := \Upsilon_H F(S) = F(S)\Upsilon_H$. In particular, F induces a functor $\mathsf{HopfAlg}(\mathcal{C}) \to \mathsf{HopfAlg}(\mathcal{C}')$.* □

Another nice feature of these functors is that they are closed under composition.

Proposition 2.5.5. [Wal24, Exercise 3.4] [DP08, Proposition 4] [BÏ8, Exercise 3.10 and 6.4] [AM10, Proposition 3.10 and Theorem 3.72] *Take monoidal categories $\mathcal{C}$, $\mathcal{C}'$, and $\mathcal{C}''$.*

(a) *Let $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \to \mathcal{C}'$ and $(G, G^{(2)}, G^{(0)}) : \mathcal{C}' \to \mathcal{C}''$ be monoidal functors. Then, the composition $GF : \mathcal{C} \to \mathcal{C}''$ is monoidal, with $(GF)^{(2)}$ and $(GF)^{(0)}$ defined by:*

$$(GF)^{(2)}_{X,Y} := G(F^{(2)}_{X,Y}) \circ G^{(2)}_{F(X),F(Y)} \ \ \forall X, Y \in \mathcal{C}, \qquad (GF)^{(0)} := G(F^{(0)}) \circ G^{(0)}.$$

(b) *Let $(F, F_{(2)}, F_{(0)}) : \mathcal{C} \to \mathcal{C}'$ and $(G, G_{(2)}, G_{(0)}) : \mathcal{C}' \to \mathcal{C}''$ be comonoidal functors. Then, the composition $GF : \mathcal{C} \to \mathcal{C}''$ is comonoidal, with $(GF)_{(2)}$ and $(GF)_{(0)}$ defined by:*

$$(GF)^{X,Y}_{(2)} := G^{F(X),F(Y)}_{(2)} \circ G(F^{X,Y}_{(2)}) \ \ \forall X, Y \in \mathcal{C}, \qquad (GF)_{(0)} := G_{(0)} \circ G(F_{(0)}).$$

(c) *Let $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}) : \mathcal{C} \to \mathcal{C}'$ and $(G, G^{(2)}, G^{(0)}, G_{(2)}, G_{(0)}) : \mathcal{C}' \to \mathcal{C}''$ be Frobenius monoidal functors. Then, the composition $GF : \mathcal{C} \to \mathcal{C}''$ is Frobenius monoidal by using the formulas from parts (a) and (b).*

(d) *Now, letting $\mathcal{C}$, $\mathcal{C}'$, and $\mathcal{C}''$ all be symmetric, the composition of two bi-monoidal functors is also bi-monoidal, using the formulas from parts (a) and (b).*

(e) *Similarly, the composition, GF, of two Hopf monoidal functors F and G, with antipodes Υ^F and Υ^G respectively, is also a Hopf monoidal functor, with antipode $\Upsilon^{GF}_X := \Upsilon^G_{F(X)} \circ G(\Upsilon^F_X) = G(\Upsilon^F_X) \circ \Upsilon^G_{F(X)}$ for any $X \in \mathcal{C}$.* □

2.5.2 Monads

Let $\mathcal{A}$ be any category. A *monad on* $\mathcal{A}$ is an algebra in the monoidal category $(\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$. More explicitly, a monad is a tuple (T, μ, η) where $T : \mathcal{A} \to \mathcal{A}$ is an endofunctor, and $\mu : T \circ T \Rightarrow T$ and $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$ are natural transformations satisfying associativity and unitality axioms.

For example, given an adjunction, $(F : \mathcal{A} \to \mathcal{B}) \dashv (G : \mathcal{B} \to \mathcal{A})$ with unit η and counit ε, we get that $(GF, G\varepsilon F, \eta)$ is a monad on $\mathcal{A}$.

2.5.3 Eilenberg-Moore and Kleisli categories

Next, for a monad (T, μ, η) on $\mathcal{A}$, the *Eilenberg-Moore category* $\mathcal{A}^T$ *of* T is the category with objects (Y, ξ_Y) where $Y \in \mathcal{A}$ and $\xi_Y : T(Y) \to Y \in \mathcal{A}$. In $\mathcal{A}^T$, a morphism $f : (Y, \xi_Y) \to (Z, \xi_Z)$ is a morphism $f : Y \to Z \in \mathcal{A}$, satisfying $f \circ \xi_Y = \xi_Z \circ T(f)$. This construction produces an adjunction

$$(\mathrm{Free}^T : \mathcal{A} \to \mathcal{A}^T) \;\dashv\; (\mathrm{Forg}^T : \mathcal{A}^T \to \mathcal{A}),$$

where the functor Free^T is defined by $\mathrm{Free}^T(Y) := (T(Y), \mu_Y)$, $\mathrm{Free}^T(f) := T(f)$, and Forg^T is defined by $\mathrm{Forg}^T(Y, \xi_Y) := Y$, $\mathrm{Forg}^T(f) := f$. The monad associated to this adjunction coincides with the original monad T.

Alternatively, the *Kleisli category* $\mathcal{A}_T$ *of* T is the category whose objects are the objects of $\mathcal{A}$, where $\mathrm{Hom}_{\mathcal{A}_T}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, T(Y))$. The composition of morphisms $f \in \mathrm{Hom}_{\mathcal{A}_T}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{A}_T}(Y, Z)$ in $\mathcal{A}_T$ is given by $g \circ_T f := \mu_Z \circ T(g) \circ f$, which is in $\mathrm{Hom}_{\mathcal{A}}(X, T(Z))$. Again, this produces an adjunction:

$$(F_T : \mathcal{A} \to \mathcal{A}_T) \;\dashv\; (U_T : \mathcal{A}_T \to \mathcal{A}),$$

where the functor F_T is defined by $F_T(Y) := Y$, $F_T(f : Y \to Z) := \eta_Z \circ f$, and the functor U_T is defined by $U_T(Y) := T(Y)$, $U_T(f : Y \to Z) := \mu_Z \circ T(f)$. Again, the monad associated to this adjunction coincides with the original monad T.

The category $\mathcal{A}_T$ can be identified with the essential image of Free^T in $\mathcal{A}^T$, via the embedding $K : \mathcal{A}_T \to \mathcal{A}^T$ given by $K(Y) = T(Y)$ and $K(f : Y \to Z) = \mu_Z \circ T(f)$. Hence, the objects of the Kleisli category of T are considered as the free objects of the Eilenberg-Moore category of T.

The Eilenberg-Moore and Kleilsi categories are, respectively, the terminal and initial solutions to the problem of finding an adjunction which gives rise to a certain monad. Namely, given a monad T on $\mathcal{A}$, consider the category Adj_T, whose objects are adjunctions $(F : \mathcal{A} \to \mathcal{B}) \dashv (G : \mathcal{B} \to \mathcal{A})$ which induce the monad T. Morphisms in Adj_T are defined as follows. Given adjunctions $\mathbb{A}_1 := (F_1 : \mathcal{A} \to \mathcal{B}_1) \dashv (G_1 : \mathcal{B}_1 \to \mathcal{A})$ and $\mathbb{A}_2 := (F_2 : \mathcal{A} \to \mathcal{B}_2) \dashv (G_2 : \mathcal{B}_2 \to \mathcal{A})$ in Adj_T, a morphism $K : \mathbb{A}_1 \to \mathbb{A}_2$ is a functor $K : \mathcal{B}_1 \to \mathcal{B}_2$ which satisfies $K \circ F_1 = F_2$ and $G_2 \circ K = G_1$.

In this category Adj_T, the Eilenberg-Moore adjunction is terminal and the Kleisli adjunction is initial. This produces a unique functor $K_T : \mathcal{A}_T \to \mathcal{A}^T$ which satisfies $K_T \circ F_T = \mathrm{Free}^T$ and $\mathrm{Forg}^T \circ K_T = U_T$. This functor is called the *comparison functor*, and it coincides with the embedding K of $\mathcal{A}_T$ into $\mathcal{A}^T$ mentioned above.

Chapter 3

Division algebras in monoidal categories

This chapter of the thesis is based on the paper [KW25]. Our goal is to adapt the equivalent definitions of division algebras over a field, given in Defintion 2.1.1, to the monoidal setting. In Section 3.1 we use module-theoretic techniques to accomplish this, and explore how these definitions relate in a variety of monoidal categories. In Section 3.2, we instead use monads to define division algebras and examine their connection to the previously proposed definitions. Finally, we discuss possible future directions that may be of interest in Section 3.3.

3.1 Module-theoretic division algebras

In this part, we adapt Definition 2.1.1(ii,iii) to the abelian monoidal setting via module theoretic techniques. In §3.1.1, we introduce module-theoretic definitions of division algebras in abelian monoidal categories. We then explore these structures in rigid, multifusion, and pivotal multifusion categories in §§3.1.2, 3.1.3, 3.1.4, respectively.

3.1.1 In abelian monoidal categories

Let $\mathcal{C}$ denote an abelian monoidal category.

Definition 3.1.1. A non-zero algebra $A \in \mathsf{Alg}(\mathcal{C})$ is a *left* (resp., *right*) *simplistic division algebra* in $\mathcal{C}$ if the regular module A_{reg} in A-$\mathsf{Mod}(\mathcal{C})$ (resp., in Mod-$A(\mathcal{C})$) is simple, and we say that A is a *simplistic division algebra* in $\mathcal{C}$ if both conditions hold.

The full subcategories of $\mathsf{Alg}(\mathcal{C})$ on these objects are denoted by $\ell.\mathsf{SimpDivAlg}(\mathcal{C})$, by $r.\mathsf{SimpDivAlg}(\mathcal{C})$, and by $\mathsf{SimpDivAlg}(\mathcal{C})$, respectively.

Definition 3.1.2. A non-zero algebra $A \in \mathsf{Alg}(\mathcal{C})$ is a *left* (resp., *right*) *essential division algebra in* $\mathcal{C}$ if the functor $(A \otimes -) : \mathcal{C} \to A\text{-}\mathsf{Mod}(\mathcal{C})$ (resp., the functor $(- \otimes A) : \mathcal{C} \to \mathsf{Mod}\text{-}A(\mathcal{C}))$ is essentially surjective. We say that A is an *essential division algebra in* $\mathcal{C}$ if both conditions hold.

The full subcategories of $\mathsf{Alg}(\mathcal{C})$ on these objects are denoted by $\ell.\mathsf{EssDivAlg}(\mathcal{C})$, by $r.\mathsf{EssDivAlg}(\mathcal{C})$, and by $\mathsf{EssDivAlg}(\mathcal{C})$, respectively.

Note that Definition 3.1.1 was used in previous works involving division algebras in abelian monoidal categories [GS16, Gro19, KZ19], as this recovers Definition 2.1.1(iii) when $\mathcal{C}$ is the monoidal category of $\Bbbk$-vector spaces, $(\mathsf{Vec}, \otimes_{\Bbbk}, \Bbbk)$. On the other hand, Definition 3.1.2 recovers Definition 2.1.1(ii) when $\mathcal{C}$ is Vec since if the functor $(-\otimes A)$ is essentially surjective, then every right A-module in $\mathcal{C}$ is isomorphic to one in the image of $(- \otimes A)$, hence free. Similarly, if $(A \otimes -)$ is essentially surjective, then every left A-module in $\mathcal{C}$ is free. Moreover, the hypothesis that A is non-zero in the terminology above is needed; else, by Lemma 2.3.4, the conditions in Definitions 3.1.1 and 3.1.2 hold vacuously.

Example 3.1.3. If $\mathbb{1} \neq 0$, then $\mathbb{1}$ is a simplistic division algebra precisely when $\mathbb{1}$ is a simple object in $\mathcal{C}$. But, $\mathbb{1}$ is always an essential division algebra since, by unitality, the functors $(\mathbb{1} \otimes -) : \mathcal{C} \to \mathcal{C}$ and $(- \otimes \mathbb{1}) : \mathcal{C} \to \mathcal{C}$ are essentially surjective.

With the quick example above, we see that these two types of division algebras differ in the general abelian monoidal setting beyond Vec.

3.1.2 In rigid, abelian monoidal categories with simple unit

In this part, assume that $\mathcal{C}$ is a rigid, abelian monoidal category with simple $\mathbb{1}$. We will show that essential division algebras in $\mathcal{C}$ are simplistic division algebras in $\mathcal{C}$. We will also present examples of simplistic, non-essential division algebras in $\mathcal{C}$, showing that these definitions remain distinct in this setting.

Proposition 3.1.4. *Take $A \in \mathsf{Alg}(\mathcal{C})$ where A admits a simple left (resp., right) module in $\mathcal{C}$. If A is in $\ell./r.\mathsf{EssDivAlg}(\mathcal{C})$, then A is in $\ell./r.\mathsf{SimpDivAlg}(\mathcal{C})$.*

Proof. We start with $A \in \mathsf{Alg}(\mathcal{C})$ that is a left essential division algebra admitting a simple left A-module S in $\mathcal{C}$. We aim to show that $A_{\text{reg}} \in A\text{-}\mathsf{Mod}(\mathcal{C})$ is simple. Now take a non-zero left ideal $\iota : I \to A$ in $\mathcal{C}$; it suffices to show that the mono ι is an isomorphism in $\mathcal{C}$.

Since A is a left essential division algebra, the module S is free. So, there is an object $X \in \mathcal{C}$ such that $S \cong A \otimes X \in A\text{-}\mathsf{Mod}(\mathcal{C})$. By exactness of the functor $(- \otimes X)$, a consequence of rigidity, we have that monos are preserved. Thus, we obtain a submodule $\iota \otimes \mathrm{id}_X : I \otimes X \to A \otimes X$. By simplicity of $S \cong A \otimes X$, either $I \otimes X = \mathsf{0}$ or $\iota \otimes \mathrm{id}_X$ is an isomorphism.

Next, note that $(- \otimes X) : \mathcal{C} \to \mathsf{Mod}\text{-}({}^*X \otimes X)(\mathcal{C})$ is an equivalence of categories via the right dual version of (2.4.3). Since $I \neq \mathsf{0}$, we conclude that $I \otimes X$, the image of I under the equivalence $(- \otimes X)$, is also non-zero. Hence, $\iota \otimes \mathrm{id}_X$ is an isomorphism in $\mathcal{C}$, and hence in $({}^*X \otimes X)\text{-}\mathsf{Mod}(\mathcal{C})$. But, equivalences of categories reflect isomorphisms, so ι must be an isomorphism, as desired.

The right version argument is similar, using instead the equivalence of categories $(X \otimes -) : \mathcal{C} \xrightarrow{\sim} (X \otimes X^*)\text{-}\mathsf{Mod}(\mathcal{C})$. □

Remark 3.1.5. The hypothesis on A in Proposition 3.1.4 holds when the regular module in $A\text{-}\mathsf{Mod}(\mathcal{C})$ (resp., $\mathsf{Mod}\text{-}A(\mathcal{C})$) is left (resp., right) Artinian, is left (resp., right) Noetherian, or is semisimple.

Next, we construct simplistic, non-essential division algebras in $\mathcal{C}$. To do this, we study when the internal End algebras of the regular left and right $\mathcal{C}$-module categories are division algebras in $\mathcal{C}$.

Proposition 3.1.6. *Take $\mathcal{C}$ as above, and take an object X in $\mathcal{C}$.*

(i) *$X \otimes X^*$ is a simplistic division algebra in $\mathcal{C}$ if and only if X is simple.*

(ii) *${}^*X \otimes X$ is a simplistic division algebra in $\mathcal{C}$ if and only if X is simple.*

(iii) *$X \otimes X^*$ is an essential division algebra in $\mathcal{C}$ if and only if X is left invertible.*

(iv) *${}^*X \otimes X$ is an essential division algebra in $\mathcal{C}$ if and only if X is right invertible.*

Proof. For (i), recall the equivalence $(- \otimes X^*) : \mathcal{C} \xrightarrow{\sim} \mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C})$ from (2.4.3). Applying this to X, we obtain that X is simple in $\mathcal{C}$ if and only if $X \otimes X^*$ is simple in $\mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C})$, if and only if $X \otimes X^*$ is a right simplistic division algebra. Again, using (2.4.3), we get an equivalence $X \otimes (-)^* : \mathcal{C} \xrightarrow{\sim} (X \otimes X^*)\text{-}\mathsf{Mod}(\mathcal{C})$, and applying this to X, we obtain that X is simple in $\mathcal{C}$ if and only if $X \otimes X^*$ is a left simplistic division algebra. The proof of part (ii) follows likewise.

For (iii), by way of Lemma 2.2.1(i), we first show that $X \otimes X^*$ is a right essential division algebra if and only if $(- \otimes X) : \mathcal{C} \to \mathcal{C}$ is essentially surjective. For the forward direction, assume that $X \otimes X^*$ is a right essential division algebra, and let $Z \in \mathcal{C}$ be any object. Take the module $Z \otimes X^*$ in $\mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C})$, and by the assumption, there is an object $\tilde{Z} \in \mathcal{C}$ such that $\tilde{Z} \otimes X \otimes X^* \cong Z \otimes X^*$ in $\mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C})$. Again, $(- \otimes X^*) : \mathcal{C} \xrightarrow{\sim} \mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C})$ is an equivalence, so apply its quasi-inverse to get that $\tilde{Z} \otimes X \cong Z$ in $\mathcal{C}$. Thus, Z is in the essential image of $(- \otimes X)$.

Conversely, assume that $(- \otimes X) : \mathcal{C} \to \mathcal{C}$ is essentially surjective, and take any right module $M \in \mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C})$. Since $(- \otimes X^*)$ is essentially surjective onto $\mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C})$, there exists an object $\tilde{M} \in \mathcal{C}$ such that $\tilde{M} \otimes X^* \cong M$ in $\mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C})$. Moreover, since $(- \otimes X)$ is essentially surjective onto $\mathcal{C}$, there exists an object $\tilde{X} \in \mathcal{C}$ such that $\tilde{X} \otimes X \cong \mathbb{1}$ in $\mathcal{C}$. Then, $(\tilde{M} \otimes \tilde{X}) \otimes (X \otimes X^*) \cong \tilde{M} \otimes X^* \cong M$ in $\mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C})$, so that M is in the essential image of the functor $(- \otimes (X \otimes X^*)) : \mathcal{C} \to \mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C})$, completing the direction.

Likewise, $X \otimes X^*$ is a left essential division algebra if and only if $(- \otimes X) : \mathcal{C} \to \mathcal{C}$ is essentially surjective, by the equivalence $(X \otimes -) : \mathcal{C} \xrightarrow{\sim} \mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C})$ from

(2.4.3), and by duality functors. Now apply Lemma 2.2.1(i) to conclude part (iii). The proof of part (iv) follows similarly. $\square$

Remark 3.1.7. We can recover Proposition 3.1.4 for the algebras $X \otimes X^*$ and ${}^*X \otimes X$ in $\mathcal{C}$, in the finite length case. First, applying (2.4.3) to the simple object $\mathbb{1} \in \mathcal{C}$, we get simple modules $X^* \in \mathsf{Mod}\text{-}(X \otimes X^*)(\mathcal{C})$ and $X \in (X \otimes X^*)\text{-}\mathsf{Mod}(\mathcal{C})$. So, the hypotheses of Proposition 3.1.4 hold for the algebra $X \otimes X^*$ in $\mathcal{C}$. Now assume that $X \otimes X^*$ is an essential division algebra in $\mathcal{C}$. Then, X is left invertible [Proposition 3.1.6(iii)], so X is simple [Lemma 2.2.2], and hence $X \otimes X^*$ is a simplistic division algebra [Proposition 3.1.6(i)]. Similar arguments work for the algebra ${}^*X \otimes X$ in $\mathcal{C}$.

Next, we provide examples of simplistic, non-essential division algebras in certain fusion categories. Indeed, fusion categories satisfy the hypotheses on $\mathcal{C}$ here, including those in Remark 3.1.7.

Example 3.1.8. Take the Fibonacci fusion category, Fib, which has simple objects $\mathbb{1}$ and τ satisfying the fusion rules: $\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$, and $\mathbb{1} \otimes \tau \cong \tau \cong \tau \otimes \mathbb{1}$, and $\tau \otimes \tau \cong \mathbb{1} \sqcup \tau$. See e.g., [Wal24, §3.9] or [BD12]. We have that $\tau \otimes \tau^* \cong \mathbb{1} \sqcup \tau$. Since τ is simple in $\mathcal{C}$, Proposition 3.1.6(i) implies that $1 \sqcup \tau$ is a simplistic division algebra in Fib.

But, $(- \otimes \tau) : \mathsf{Fib} \to \mathsf{Fib}$ is not essentially surjective. Indeed, since Fib is semisimple, each object in Fib is isomorphic to $\mathbb{1}^n \sqcup \tau^m$, for some $m, n \geqslant 0$. Then, the essential image of $(- \otimes \tau)$ has objects $(\mathbb{1}^n \sqcup \tau^m) \otimes \tau \cong \mathbb{1}^m \sqcup \tau^{m+n}$, for $m, n \geqslant 0$. So, $\mathbb{1}$ is not in the essential image of $(- \otimes \tau)$, and by Lemma 2.2.1(i) with Proposition 3.1.6(iii), $\mathbb{1} \sqcup \tau$ is not an essential division algebra in Fib.

Example 3.1.9. Take a finite non-abelian group G, and take its (fusion) category $\mathsf{FdRep}(G)$ of finite-dimensional representations over $\Bbbk$. Here, $\otimes := \otimes_{\Bbbk}$ and $\mathbb{1} := \Bbbk$. Now $\mathsf{FdRep}(G)$ has a simple object Z with $\dim_{\Bbbk}(Z) > 1$. So, $Z \otimes Z^*$ is a simplistic division algebra in $\mathsf{FdRep}(G)$ [Proposition 3.1.6(i)].

But, $\dim_{\Bbbk}(\mathbb{1}) = 1$, and $\dim_{\Bbbk}(X \otimes Y) = \dim_{\Bbbk}(X) \dim_{\Bbbk}(Y)$ for $X, Y \in \mathsf{FdRep}(G)$. So, Z above is not left invertible, and $Z \otimes Z^*$ is not an essential division algebra in $\mathsf{FdRep}(G)$ [Proposition 3.1.6(iii)].

3.1.3 In multifusion categories

Proposition 3.1.6 used the Morita equivalence of $\mathbb{1}$ and $X \otimes X^*$ for any non-zero $X \in \mathcal{C}$. More generally, the results below use a Morita equivalence from Ostrik's Theorem [Theorem 2.4.4].

Proposition 3.1.10. *Let $\mathcal{C}$ be multifusion with $A \in \mathsf{Alg}(\mathcal{C})$ whose categories of modules in $\mathcal{C}$ satisfy the hypothesis of Ostrik's Theorem.*

(i) $\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M)$ *is a right simplistic division algebra if and only if M is simple in* $\mathsf{Mod}\text{-}A(\mathcal{C})$.

(ii) $\underline{\mathrm{End}}_{A\text{-}\mathsf{Mod}(\mathcal{C})}(N)$ *is a left simplistic division algebra if and only if N is simple in* $A\text{-}\mathsf{Mod}(\mathcal{C})$.

Proof. Applying Ostrik's Theorem first to $\mathsf{Mod}\text{-}A(\mathcal{C})$, we obtain an equivalence of categories $\underline{\mathrm{Hom}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M, -) : \mathsf{Mod}\text{-}A(\mathcal{C}) \to \mathsf{Mod}\text{-}\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M)(\mathcal{C})$ for any non-zero $M \in \mathsf{Mod}\text{-}A(\mathcal{C})$. Applying this equivalence to $M \in \mathsf{Mod}\text{-}A(\mathcal{C})$, it follows that $\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M)$ is simple in $\mathsf{Mod}\text{-}\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M)(\mathcal{C})$ if and only if M is simple in $\mathsf{Mod}\text{-}A(\mathcal{C})$, proving (i). Similarly, applying Ostrik's Theorem to the right $\mathcal{C}$-module category $A\text{-}\mathsf{Mod}(\mathcal{C})$ gives (ii). □

Remark 3.1.11. Deriving simplistic division algebras from internal End algebras of simple objects was considered in [Gro19, Theorem 2.5], [GS16, Theorem 2.8], and [KZ19, Lemma 3.6(4)], without considering the converse statement.

Proposition 3.1.12. *Let $\mathcal{C}$ be multifusion with $A \in \mathsf{Alg}(\mathcal{C})$ whose categories of modules in $\mathcal{C}$ satisfy the hypothesis of Ostrik's Theorem.*

(i) *For any $M \in \mathsf{Mod}\text{-}A(\mathcal{C})$, we have that $\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M)$ is a right essential division algebra if and only if $(- \otimes M) : \mathcal{C} \to \mathsf{Mod}\text{-}A(\mathcal{C})$ is essentially surjective.*

(ii) *For any $N \in A\text{-}\mathsf{Mod}(\mathcal{C})$, we have that $\underline{\mathrm{End}}_{A\text{-}\mathsf{Mod}(\mathcal{C})}(N)$ is a left essential division algebra if and only if $(N \otimes -) : \mathcal{C} \to A\text{-}\mathsf{Mod}(\mathcal{C})$ is essentially surjective.*

Proof. To prove (i), note that $X \otimes \underline{\mathrm{Hom}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M, M') \cong \underline{\mathrm{Hom}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M, X \otimes M')$, for any $X \in \mathcal{C}$ and $M, M' \in \mathsf{Mod}\text{-}A(\mathcal{C})$; see [EGNO15, Lemma 7.9.4]. Therefore, we get that

$$(- \otimes \underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M)) \;\cong\; \underline{\mathrm{Hom}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M, - \otimes M)$$

as functors from $\mathcal{C}$ to $\mathsf{Mod}\text{-}\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M)(\mathcal{C})$. Moreover, $\underline{\mathrm{Hom}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M, - \otimes M)$ is the composition of the functor $(- \otimes M) : \mathcal{C} \to \mathsf{Mod}\text{-}A(\mathcal{C})$ with the functor $\underline{\mathrm{Hom}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M, -) : \mathsf{Mod}\text{-}A(\mathcal{C}) \to \mathsf{Mod}\text{-}\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M)(\mathcal{C})$, where the second is an equivalence of categories by Ostrik's Theorem. Hence

$$(- \otimes \underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M)) \;\cong\; \underline{\mathrm{Hom}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M, -) \circ (- \otimes M)$$

is essentially surjective if and only if $(- \otimes M)$ is essentially surjective, and we are done. The proof of (ii) is analogous. $\square$

Remark 3.1.13. Propositions 3.1.10 and 3.1.12 are analogues of Schur's Lemma [Lemma 2.1.2].

Example 3.1.14. Let $A \in \mathsf{Alg}(\mathcal{C})$ such that $\mathsf{Mod}\text{-}A(\mathcal{C})$ satisfies the hypothesis of Ostrik's Theorem. Also, let $M \in \mathsf{Mod}\text{-}A(\mathcal{C})$ be a left invertible module, i.e., there is some $N \in A\text{-}\mathsf{Mod}(\mathcal{C})$ satisfying $N \otimes M \cong A_{\mathrm{reg}} \in (A, A)\text{-}\mathsf{Bimod}(\mathcal{C})$. Then we have that $M' \cong M' \otimes_A A_{\mathrm{reg}} \cong M' \otimes_A N \otimes M$ in $\mathsf{Mod}\text{-}A(\mathcal{C})$, for any $M' \in \mathsf{Mod}\text{-}A(\mathcal{C})$. Hence, $(- \otimes M) : \mathcal{C} \to \mathsf{Mod}\text{-}A(\mathcal{C})$ is essentially surjective, and Proposition 3.1.12 gives that $\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M) \cong (M \otimes_A {}^*M)^*$ is a right essential division algebra in $\mathcal{C}$.

When $A = \mathbb{1}$, we recover Proposition 3.1.6(iii): namely, if an object $X \in \mathcal{C}$ is left invertible, then $(X \otimes {}^*X)^* \cong X \otimes X^*$ is a right essential division algebra in $\mathcal{C}$.

3.1.4 In pivotal multifusion categories

We now address whether the distinction between left and right division algebras is necessary, and we find that in a pivotal multifusion category, the distinction is not needed. Note that such categories are abundant, as it is conjectured that every fusion category must be pivotal [ENO05, Conjecture 2.8].

Lemma 3.1.15. *Let $\mathcal{C}$ be a pivotal abelian monoidal category with $A \in \mathsf{Alg}(\mathcal{C})$. Then, for any $M \in \mathsf{Mod}\text{-}A(\mathcal{C})$, the algebras $\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M)$ and $\underline{\mathrm{End}}_{A\text{-}\mathsf{Mod}(\mathcal{C})}({}^*M)$ are isomorphic as algebras in $\mathcal{C}$.*

Proof. We have the following isomorphism:

$$\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(M) \cong (M \otimes_A {}^*M)^* \xrightarrow{\hat{\jmath}_{M\otimes_A {}^*M}} {}^*(M \otimes_A {}^*M) \cong \underline{\mathrm{End}}_{A\text{-}\mathsf{Mod}(\mathcal{C})}({}^*M).$$

Thus, it is suffices to show that $\hat{\jmath}_{M\otimes_A {}^*M}$ is an algebra map. To do this, recall that the algebra structure maps of $(M \otimes_A {}^*M)^*$ are given by $m = (\mu)^*$ and $u = (\eta)^*$, while the algebra structure maps of ${}^*(M \otimes_A {}^*M)$ are given by $m' = {}^*\mu$ and $u' = {}^*\eta$, where $\mu := \mathrm{id}_M \otimes_A \mathrm{coev}^R_M \otimes_A \mathrm{id}_{{}^*M}$. Then, η is defined as the map from $M \otimes_A {}^*M$ to $\mathbb{1}$ satisfying $\mathrm{ev}^R_M = \eta\,\pi$, where π is the coequalizer projection morphism associated to $M \otimes_A {}^*M = \mathrm{coeq}(\triangleleft \otimes \mathrm{id}_{{}^*M}, \mathrm{id}_M \otimes \triangleright)$. Using this structure on the internal Ends, and the fact that $\hat{\jmath}$ is a monoidal natural transformation, it is straightforward to verify that $\hat{\jmath}$ is an algebra isomorphism. □

Proposition 3.1.16. *Let $\mathcal{C}$ be a pivotal multifusion category with $A \in \mathsf{Alg}(\mathcal{C})$.*

(i) *$A \in \ell.\mathsf{SimpDivAlg}(\mathcal{C})$ if and only if $A \in r.\mathsf{SimpDivAlg}(\mathcal{C})$.*

(ii) *$A \in \ell.\mathsf{EssDivAlg}(\mathcal{C})$ if and only if $A \in r.\mathsf{EssDivAlg}(\mathcal{C})$.*

Proof. Start with algebras A and B in $\mathcal{C}$ that are a left simplistic division algebra and left essential division algebra, respectively. Using the equivalence of categories $(-)^*$ from left modules to right modules we obtain that A^* is simple in $\mathsf{Mod}\text{-}A(\mathcal{C})$, and that $(- \otimes B^*) : \mathcal{C} \to \mathsf{Mod}\text{-}B(\mathcal{C})$ is essentially surjective. Proposition 3.1.10(i) then gives that $\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(A^*)$ is a right simplistic division algebra, and Proposition 3.1.12(i) gives that $\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}B(\mathcal{C})}(B^*)$ is a right essential division algebra. By Lemmas 2.4.1 and 3.1.15, we get that as algebras,

$$\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}A(\mathcal{C})}(A^*) \;\cong\; \underline{\mathrm{End}}_{A\text{-}\mathsf{Mod}(\mathcal{C})}({}^*(A^*)) \;\cong\; \underline{\mathrm{End}}_{A\text{-}\mathsf{Mod}(\mathcal{C})}(A) \;\cong\; A.$$

Similarly, $\underline{\mathrm{End}}_{\mathsf{Mod}\text{-}B(\mathcal{C})}(B^*) \cong B$. Thus, A is a right simplistic division algebra, and B is a right essential division algebra. The backwards direction is analogous. $\square$

3.2 Monad-theoretic division algebras

Previously, we were restricted to working in abelian monoidal and (multi)fusion categories to study simplistic division algebras. But essential division algebras can be defined in any monoidal category; we will see here that they can be examined via monads. Monadic division algebras are introduced in §3.2.1, and connections to essential division algebras are discussed there. We provide examples of monadic division algebras in §3.2.2.

3.2.1 Monadic division algebras

We first direct the reader to §2.5.3 to recall the Eilenberg-Moore and Kleisli categories of a monad, along with the comparison functor between these categories.

Definition 3.2.1. Let $T : \mathcal{A} \to \mathcal{A}$ be a monad on any monoidal category $\mathcal{A}$. We say that T is *adjunction-trivial* if the comparison functor $K_T : \mathcal{A}_T \to \mathcal{A}^T$ is an equivalence of categories.

The term adjunction-trivial was chosen to describe such a monad because when this condition is satisfied, the category Adj_T has only one object, up to isomorphism. Thinking of the Kleisli category as the free modules over the monad T, this condition is an analogue of the division algebra property that all modules over a division algebra are free. We use this analogue to define monad-theoretic division algebras in Definition 3.2.2 below.

Now, let $\mathcal{C}$ be a strict monoidal category. For any $A \in \mathsf{Alg}(\mathcal{C})$, we obtain two monads on $\mathcal{C}$: $\big((A \otimes -),\, m_A \otimes \mathrm{id}_{(-)},\, u_A \otimes \mathrm{id}_{(-)}\big)$ and $\big((- \otimes A),\, \mathrm{id}_{(-)} \otimes m_A,\, \mathrm{id}_{(-)} \otimes u_A\big)$.

To be consistent with the exclusion of the zero algebra as a division algebra in abelian monoidal categories, via Lemma 2.3.4, we will consider the condition below:

$$\begin{gathered} A \in \mathsf{Alg}(\mathcal{C}) \text{ satisfies that } A\text{-}\mathsf{Mod}(\mathcal{C}) \text{ and } \mathsf{Mod}\text{-}A(\mathcal{C}) \\ \text{both have more than one isoclass of objects.} \end{gathered} \tag{$\star$}$$

Definition 3.2.2. An algebra A in $\mathcal{C}$ subject to $(\star)$ is called a *left (resp., right) monadic division algebra* if the monad $(A \otimes -)$ (resp., $(- \otimes A)$) on $\mathcal{C}$ is adjunction-trivial.

The full subcategory of $\mathsf{Alg}(\mathcal{C})$ on these algebras is denoted by $\ell.\mathsf{MonDivAlg}(\mathcal{C})$ (resp., $r.\mathsf{MonDivAlg}(\mathcal{C})$).

Note that essential division algebras can also be defined in $\mathcal{C}$ by replacing the non-zero condition on A with $(\star)$. The connection to monadic division algebras in $\mathcal{C}$ is given below.

Proposition 3.2.3. *Take $A \in \mathsf{Alg}(\mathcal{C})$ subject to $(\star)$. Then, $A \in \ell./r.\mathsf{MonDivAlg}(\mathcal{C})$ if and only if $A \in \ell./r.\mathsf{EssDivAlg}(\mathcal{C})$.*

Proof. This follows as $\mathcal{C}^{(A\otimes -)} \simeq A\text{-}\mathsf{Mod}(\mathcal{C})$ and $\mathcal{C}^{(-\otimes A)} \simeq \mathsf{Mod}\text{-}A(\mathcal{C})$, and because under these equivalences, $\mathcal{C}_{(A\otimes -)}$ and $\mathcal{C}_{(-\otimes A)}$ are the left and right free A-modules in $\mathcal{C}$, respectively. □

Example 3.2.4. For the monoidal category of $\Bbbk$-vector spaces, $(\mathsf{Vec}, \otimes_{\Bbbk}, \Bbbk)$, with $A \in \mathsf{Alg}(\mathsf{Vec})$, consider the monad $(- \otimes_{\Bbbk} A)$ on Vec. Then, $\mathsf{Vec}^{(-\otimes_{\Bbbk} A)} \simeq \mathsf{Vec}_{(-\otimes_{\Bbbk} A)}$ if and only if every right A-module over $\Bbbk$ is free, which happens precisely when A is a division algebra over $\Bbbk$. So, monadic division algebras in Vec again recover division algebras over $\Bbbk$.

3.2.2 Examples of monadic division algebras

To construct more examples of monadic division algebras, we use monads that satisfy the following property. For further information, see [Mog91] or [MU22].

Definition 3.2.5. A monad (T, μ, η) on $\mathcal{C}$ is *left strong* if it is equipped with a natural transformation $\theta := \{\theta_{X,Y} : X \otimes T(Y) \to T(X \otimes Y)\}_{X,Y \in \mathcal{C}}$ (*left strength*) such that for all $X, Y, Z \in \mathcal{C}$:

(i) $\theta_{X,Y\otimes Z}(\mathrm{id}_X \otimes \theta_{Y,Z}) = \theta_{X\otimes Y,Z}$; (iii) $\theta_{X,Y}(\mathrm{id}_X \otimes \mu_Y) = \mu_{X\otimes Y} T(\theta_{X,Y})\theta_{X,T(Y)}$;

(ii) $\theta_{\mathbb{1},X} = \mathrm{id}_{T(X)}$; (iv) $\theta_{X,Y}(\mathrm{id}_X \otimes \eta_Y) = \eta_{X\otimes Y}$.

A left strong monad (T, μ, η, θ) is said to be *left very strong* if θ is a natural isomorphism. *Right strong* and *right very strong* monads on $\mathcal{C}$ are defined analogously.

The next result is straightforward to verify.

Lemma 3.2.6. *If (T, μ, η, θ) is left (resp., right) strong, then $T(\mathbb{1}) \in \mathsf{Alg}(\mathcal{C})$. Here, $m_{T(\mathbb{1})} := \mu_{\mathbb{1}}\, \theta_{T(\mathbb{1}),\mathbb{1}}$ (resp., $\mu_{\mathbb{1}}\, \theta_{\mathbb{1},T(\mathbb{1})}$), and $u_{T(\mathbb{1})} := \eta_{\mathbb{1}}$.* □

Proposition 3.2.7. *Let T be a monad on a strict monoidal category $\mathcal{C}$.*

(i) *If T is left very strong, then T is adjunction-trivial if and only if $T(\mathbb{1})$ is a right monadic division algebra.*

(ii) *If T is right very strong, then T is adjunction-trivial if and only if $T(\mathbb{1})$ is a left monadic division algebra.*

Proof. In both cases, $T(\mathbb{1})$ is an algebra in $\mathcal{C}$ by Lemma 3.2.6. Next, let T be left very strong, with left strength θ. Then, $T \cong (- \otimes T(\mathbb{1}))$ via the natural isomorphism $\theta^{-1}_{-,\mathbb{1}}$. It follows that T is adjunction-trivial if and only if $(- \otimes T(\mathbb{1}))$ is adjunction-trivial, if and only if $T(\mathbb{1}) \in r.\mathsf{MonDivAlg}(\mathcal{C})$ by definition. Similarly, when T is right very strong, $T \cong (T(\mathbb{1}) \otimes -)$, and part (ii) holds. □

Example 3.2.8. Continuing Example 3.2.4 for $\mathcal{C} = \mathsf{Vec}$ and $A \in \mathsf{Alg}(\mathsf{Vec})$, we have that $(- \otimes_{\Bbbk} A)$ is a left very strong monad on Vec with strength being the associativity of $\otimes_{\Bbbk}$. Now, $\mathsf{Vec}^{(-\otimes_{\Bbbk} A)} \simeq \mathsf{Vec}_{(-\otimes_{\Bbbk} A)}$ if and only if $\Bbbk \otimes_{\Bbbk} A \cong A$ is a right monadic division algebra in Vec (which happens precisely when A is a division algebra over $\Bbbk$).

Example 3.2.9. Here, we consider the maybe monad T on $(\mathsf{Set}, \sqcup, \varnothing)$, given by $T(-) := (- \sqcup \{*\})$, where $\sqcup$ is disjoint union. See [Rie17, Examples 5.1.4(i) and 5.3.2] for details; in particular, it is adjunction-trivial. Also, T is left very strong by the associativity of $\sqcup$. Proposition 3.2.7 implies that $T(\varnothing) \cong \{*\}$ is a right monadic division algebra in $(\mathsf{Set}, \sqcup, \varnothing)$. Using that $X \sqcup Y \cong Y \sqcup X$ for $X, Y \in \mathsf{Set}$, we see that T is right very strong. Hence, $\{*\}$ is also a left monadic division algebra in $(\mathsf{Set}, \sqcup, \varnothing)$.

Example 3.2.10. Consider the free vector space monad T on $(\mathsf{Set}, \times, \{*\})$, given by $T(X) := \Bbbk^X$, consisting of finitely supported functions $f : X \to \Bbbk$. See [Rie17, Example 5.1.4(iii)]. We obtain that $\mathsf{Set}^T \simeq \mathsf{Vec} \simeq \mathsf{Set}_T$. However, T is not left very strong as, in general, $X \times \Bbbk^Y \ncong \Bbbk^{X \times Y}$ as sets. So, we cannot use Proposition 3.2.7 to get a left monadic division algebra in Set. Still, see §3.3.1 below.

3.3 Discussion

We briefly discuss here potential research directions that may be of interest to the reader.

3.3.1 On division monads

One may want to refer to a monad T on $\mathcal{A}$ as a "division monad" when $\mathcal{A}_T \simeq \mathcal{A}^T$, instead of calling such monads adjunction-trivial. This would include monads that are not necessarily very strong, such as in Example 3.2.10. We inquire whether the scarcity of these types of monads mirrors the scarcity of division $\Bbbk$-algebras among the collection of $\Bbbk$-algebras.

3.3.2 On structural results for algebras in monoidal categories

There are several classical results using division $\Bbbk$-algebras that could be expanded to general monoidal settings, e.g., Artin-Wedderburn Theorem. Moreover, the classification of division algebras in various monoidal settings is open. For example, we expect an analogue of Frobenius's Theorem (i.e., the only finite-dimensional division algebra over an algebraically closed field is the field itself) to hold in finite monoidal settings.

3.3.3 On the essential condition versus the simplistic condition

If one uses simplistic division algebras as done in previous works (e.g., in [GS16, Gro19, KZ19]), then the supply of division algebras may be too abundant to make satisfactory progress. For instance, any simple module over a finite group G yields a simplistic division algebra in the monoidal category of G-modules [Example 3.1.9]. We propose it is that better to use the more restrictive class of essential/monadic division algebras to examine pertinent results for algebras in monoidal categories.

3.3.4 On the left versus right division algebra conditions

In Proposition 3.1.16, we proved that a left simplistic (resp., essential) division algebra in a pivotal multifusion category $\mathcal{C}$ is a right simplistic (resp., essential) division algebra in $\mathcal{C}$, and vice versa. It is shown in recent work of Nakamura, Shibata, and Shimizu that the result in the simplistic case holds when $\mathcal{C}$ is a finite tensor category [NSS25, Lemma 2.12]. There, the more common terminology, *left/right simple algebra in* $\mathcal{C}$, is used instead of our terminology here. We all expect that such "Left $\Leftrightarrow$ Right" results hold in more general monoidal settings [Shi25].

Chapter 4

Extended Frobenius algebras in monoidal categories

This chapter follows the paper [CKQW24]. We begin by exploring extended Frobenius algebras over a field in Section 4.1 and in a monoidal category in Section 4.2. We define functors that preserve extended Frobenius algebras in Section 4.3. Section 4.4 is dedicated to using the connection between Frobenius algebras and Hopf algebras to create more extended Frobenius algebras by extending Hopf algebras and defining functors that preserve these structures. We end with a discussion on some possible future research directions in Section 4.5.

4.1 Extended Frobenius algebras over a field

In this section, we study extended Frobenius algebras over a field $\Bbbk$, originally defined in [TT06] as follows.

Definition 4.1.1. (a) [TT06, Definition 2.5] A Frobenius algebra $(A, m, u, \Delta, \varepsilon)$ (see Definition 2.1.5) is an *extended Frobenius algebra* over $\Bbbk$ if it is equipped with a morphism $\phi : A \to A$ and an element $\theta \in A$ such that:

(i) $\phi : A \to A$ is an involution of Frobenius algebras,

(ii) $\theta \in A$ satisfies $\phi(\theta a) = \theta a$, for all $a \in A$,

(iii) $m(\phi \otimes \mathrm{id}_A)\Delta(1_A) = \theta^2$.

A *morphism* $f : (A, \phi_A, \theta_A) \to (B, \phi_B, \theta_B)$ *of extended Frobenius algebras over* $\Bbbk$ is a morphism $f : A \to B$ of $\Bbbk$-Frobenius algebras such that $f\phi_A = \phi_B f$ and $f(\theta_A) = \theta_B$.

(b) We refer to (ϕ, θ) in part (b) as the *extended structure* of the underlying Frobenius algebra A, and say that A is *extendable* when ϕ and θ exist. We also call an extended structure (ϕ, θ) on A *ϕ-trivial* when $\phi = \mathrm{id}_A$, and call it *θ-trivial* when $\theta = 0$.

The roman numerals (i), (ii), (iii) *in this section will refer to the conditions in Definition 4.1.1(a).*

We provide many examples of, and preliminary results for, such structures in §4.1.1. Then, §4.1.2 is dedicated to establishing the following theorem classifying extended structures for several Frobenius algebras over $\Bbbk$.

Theorem 4.1.2 (Propositions 4.1.10–4.1.12, 4.1.14–4.1.16, 4.1.18–4.1.19)**.** *Take an integer $n \geqslant 2$, and $\omega_n \in \Bbbk$ an n-th root of unity. The extended structures for the Frobenius algebras below are classified, recapped as follows.*

(a) $\Bbbk$: *all extensions are ϕ-trivial.*

(b) $\mathbb{C}$ *over* $\mathbb{R}$*: all extensions are ϕ-trivial or θ-trivial.*

(c) $\Bbbk[x]/(x^n)$*: all extensions are ϕ trivial when n is odd, and is not extendable when n is even.*

(d) $\Bbbk C_2$*: all extensions are ϕ-trivial or θ-trivial.*

(e) $\Bbbk C_3$*: all extensions are ϕ-trivial or ϕ maps a generator g of C_3 to $\omega_3 g^2$.*

(f) $\Bbbk C_4$*: all extensions are ϕ-trivial, or θ-trivial, or ϕ takes a generator g of C_4 to $\omega_4 g^3$.*

(g) $\Bbbk(C_2 \times C_2)$*: here, ϕ maps g to $\omega_2 g'$, where g, g' are generators of $C_2 \times C_2$.*

(h) $T_2(-1) := \Bbbk\langle g, x\rangle/(g^2 - 1, x^2, gx + xg)$: *all extensions are ϕ-trivial.*

4.1.1 Preliminary results and examples

We begin with some useful preliminary results on (extended) Frobenius algebras A over $\Bbbk$. First, the Frobenius law from Definition 2.1.5 implies that

$$\Delta(a) = a(1_A)^1 \otimes (1_A)^2, \quad \text{where } \Delta(1_A) := (1_A)^1 \otimes (1_A)^2, \tag{4.1.3}$$

for any $a \in A$. So, $\Delta(1_A)$ determines the Frobenius structure of A.

Lemma 4.1.4. *If A is a Frobenius algebra that is also a domain, then an extended structure of A (if it exists) must be either ϕ-trivial or θ-trivial.*

Proof. Suppose that an extended structure (A, ϕ, θ) exists. Then, for all $a \in A$ we have that $\theta\phi(a) = \phi(\theta)\phi(a) = \phi(\theta a) = \theta a$, by condition (i). Hence, $\theta(\phi(a) - a) = 0$ for all $a \in A$, and the result follows from A being a domain. $\square$

Lemma 4.1.5. *Let A be a Frobenius algebra over $\Bbbk$, and let (A, ϕ, θ) and (A, ϕ', θ') be two extended structures of A. If $\theta \in \Bbbk 1_A$ and $\theta \neq \theta'$, then an extended Frobenius algebra morphism from (A, ϕ, θ) to (A, ϕ', θ') does not exist.*

Proof. Suppose by way of contrapositive that $\theta = \lambda 1_A$ for some $\lambda \in \Bbbk$ and there is a morphism $f : (A, \phi, \theta) \to (A, \phi', \theta')$ of extended Frobenius algebras. Since f is unital and preserves the extended structure, $\theta = \lambda 1_A = \lambda f(1_A) = f(\lambda 1_A) = f(\theta) = \theta'$, as desired. $\square$

We will see in Proposition 4.1.14 that Lemma 4.1.5 fails when $\theta \notin \Bbbk 1_A$. We now include some examples of extended structures for well-known Frobenius algebras.

Example 4.1.6. Let G be a finite group. Its group algebra $\Bbbk G$ has a Frobenius algebra structure determined by $\Delta(e_G) = \sum_{h \in G} h \otimes h^{-1}$. Then,

$$\phi = \mathrm{id}_{\Bbbk G}, \qquad \theta = \pm\sqrt{|G|} \cdot e_G$$

yield extended structures of $\Bbbk G$. Now, conditions (i) and (ii) are trivially satisfied. Condition (iii) holds as $m(\phi \otimes \mathrm{id}_{\Bbbk G})\Delta(e_G) = m\left(\sum_{h \in G} h \otimes h^{-1}\right) = |G| \cdot e_G = \theta^2$.

Example 4.1.7. Let C_n denote the cyclic group of order $n \geqslant 2$, and let g denote a generator of C_n. Consider the Frobenius structure on $\Bbbk C_n$ from Example 4.1.6. Then,

$$\phi(g) = \omega_n g^{-1}, \qquad \theta = \pm \tfrac{1}{\sqrt{n}} \textstyle\sum_{j=0}^{n-1} \omega_n^j g^{-2j}$$

are extended structures of $\Bbbk C_n$ for any n-th root of unity $\omega_n \in \Bbbk$. It is a quick check that condition (i) holds. Towards condition (ii), let $a := \sum_{i=0}^{n-1} a_i g^i$ be an element in $\Bbbk C_n$. Then,

$$\begin{aligned} \phi(a\theta) &= \pm \tfrac{1}{\sqrt{n}} \textstyle\sum_{i,j=0}^{n-1} a_i \omega_n^j \phi(g)^{i-2j} = \pm \tfrac{1}{\sqrt{n}} \textstyle\sum_{i,j=0}^{n-1} a_i \omega_n^{i-j} g^{-i+2j} \\ &= \pm \tfrac{1}{\sqrt{n}} \textstyle\sum_{i,k=0}^{n-1} a_i \omega_n^k g^{i-2k} = a\theta. \end{aligned}$$

For condition (iii), we compute:

$$\begin{aligned} m(\phi \otimes \mathrm{id}_{\Bbbk G})\Delta(e_{C_n}) &= m(\phi \otimes \mathrm{id}_{\Bbbk C_n}) \left(\textstyle\sum_{j=0}^{n-1} g^j \otimes g^{-j} \right) \\ &= \textstyle\sum_{j=0}^{n-1} \omega_n^j g^{-2j} = \tfrac{1}{n} \textstyle\sum_{i=0}^{n-1} \sum_{k=0}^{n-1} \omega_n^k g^{-2k} \\ &= \tfrac{1}{n} \textstyle\sum_{i,j=0}^{n-1} \omega_n^{i+j} g^{-2(i+j)} = \tfrac{1}{n} \left(\textstyle\sum_{j=0}^{n-1} \omega_n^j g^{-2j} \right)^2 = \theta^2. \end{aligned}$$

Example 4.1.8. Let $\omega := \omega_n$ be a primitive n-th root of unity, for $n \geqslant 2$. Consider the Taft algebra, $T_n(\omega) := \Bbbk\langle g, x \rangle / (g^n - 1, x^n, gx - \omega xg)$, with Frobenius structure defined by $\Delta(1_{T_n(\omega)}) = \sum_{j=0}^{n-1} \left(-\omega^j g^{j+1} \otimes g^{-(j+1)} x + g^j x \otimes g^{-j} \right)$. This Frobenius structure on $T_n(\omega)$ can be extended via

$$\phi = \mathrm{id}_{T_n(\omega)}, \qquad \theta \in \textstyle\bigoplus_{j=0,k=1}^{n-1} \Bbbk g^j x^k.$$

To show this, we compute: $m(\phi \otimes \mathrm{id}_{T_n(\omega)})\Delta(1) = 0 = \theta^2$, so condition (iii) holds. Conditions (i) and (ii) are trivially satisfied.

Example 4.1.9. Let $\mathrm{Mat}_n(\Bbbk)$ be the algebra of $n \times n$ matrices over $\Bbbk$, with basis $\{E_{i,j}\}_{i,j=1}^n$ of elementary matrices. Consider the Frobenius structure determined by $\Delta(E_{i,j}) = \sum_{\ell=1}^n E_{i,\ell} \otimes E_{\ell,j}$, for all $1 \leqslant i, j \leqslant n$. Then,

$$\phi = \mathrm{id}_{\mathrm{Mat}_n(\Bbbk)}, \qquad \theta = \pm\sqrt{n} \cdot I_n$$

give extended structures of $\mathrm{Mat}_n(\Bbbk)$.

Indeed, $m(\phi \otimes \mathrm{id}_{\mathrm{Mat}_n(\Bbbk)})\Delta(I_n) = \sum_{i,\ell=1}^{n} E_{i,\ell}E_{\ell,i} = n \cdot I_n = \theta^2$, so condition (iii) holds. Moreover, conditions (i) and (ii) are trivially satisfied.

4.1.2 Classification results

Now we proceed to establish Theorem 4.1.2, starting with the results for the Frobenius algebras: $\Bbbk$ over $\Bbbk$, $\mathbb{C}$ over $\mathbb{R}$, and the nilpotent algebra $\Bbbk[x]/(x^n)$ over $\Bbbk$.

Proposition 4.1.10. *The only extended structures of the Frobenius algebra $\Bbbk$ where $\Delta_{\Bbbk}$ is given by the isomorphism $\Bbbk \xrightarrow{\sim} \Bbbk \otimes \Bbbk$ are ϕ-trivial, with $\theta = \pm 1_{\Bbbk}$. Moreover, these two extended Frobenius algebra structures are non-isomorphic.*

Proof. Suppose ϕ and θ give an extended structure of $\Bbbk$. Since $\phi : \Bbbk \to \Bbbk$ is a morphism of algebras, the only possible choice is $\phi = \mathrm{id}_{\Bbbk}$, which satisfies conditions (i) and (ii) trivially. Condition (iii) implies that $\theta = \pm 1_{\Bbbk}$. Lastly, the structures are non-isomorphic by Lemma 4.1.5. □

Proposition 4.1.11. *Consider the Frobenius algebra $\mathbb{C}$ over $\mathbb{R}$ with Δ defined by $\Delta(1) = 1 \otimes 1 - i \otimes i$. Then,*

(a) *$\phi = \mathrm{id}_{\mathbb{C}}$ and $\theta = \pm\sqrt{2}$, and*

(b) *$\phi(z) = \overline{z}$ for all $z \in \mathbb{C}$, and $\theta = 0$,*

are all of the extended structures of $\mathbb{C}$, and these extended Frobenius algebras are all non-isomorphic.

Proof. By Lemma 4.1.4, an extended structure of $\mathbb{C}$ should be ϕ-trivial or θ-trivial. If $\phi = \mathrm{id}_{\mathbb{C}}$, then $\theta^2 = m(\phi \otimes \mathrm{id}_{\mathbb{C}})\Delta(1) = m(1 \otimes 1 - i \otimes i) = 2$, and so $\theta = \pm\sqrt{2}$. On the other hand, if $\theta = 0$, then $0 = m(\phi \otimes \mathrm{id}_{\mathbb{C}})\Delta(1) = 1 - \phi(i)i$. Hence, $\phi(i) = -i$ and it follows that ϕ must be complex conjugation. Now condition (iii) holds, and it is a quick check that conditions (i) and (ii) are satisfied for these choices. Lastly, it follows from Lemma 4.1.5 that these structures are all non-isomorphic. □

Proposition 4.1.12. *Consider the algebra* $\Bbbk[x]/(x^n)$*, for* $n \geqslant 2$*, with Frobenius structure determined by* $\Delta(1) = \sum_{i=0}^{n-1} x^i \otimes x^{n-i-1}$*. Then, the following statements hold.*

(a) *For* n *even, the Frobenius algebra* $\Bbbk[x]/(x^n)$ *is not extendable.*

(b) *For* n *odd, all extended structures of the Frobenius algebra* $\Bbbk[x]/(x^n)$ *are* ϕ*-trivial, with* $\theta = \pm\sqrt{n}x^{\frac{n-1}{2}} + \sum_{j=\frac{n+1}{2}}^{n-1} \theta_j x^j$ *for some* $\theta_{\frac{n+1}{2}}, \ldots, \theta_{n-1} \in \Bbbk$*.*

Proof. Suppose that ϕ and θ give an extended structure of $\Bbbk[x]/(x^n)$. Then, a routine calculation with ϕ being multiplicative and $\phi^2 = \mathrm{id}$ (from condition (i)) implies that $\phi(x) = \pm x$. So, in the rest of the proof, we look at the cases $\phi = \mathrm{id}$ and $\phi(x) = -x$, and conclude the latter is never possible, while the former is only possible for n odd.

Suppose first that $\phi = \mathrm{id}$. Then, conditions (i) and (ii) are satisfied trivially. Let $\theta_0, \ldots, \theta_{n-1} \in \Bbbk$ such that $\theta = \sum_{i=0}^{n-1} \theta_i x^i$. Then, condition (iii) implies that

$$nx^{n-1} = \sum_{i=0}^{n-1} \theta_i^2 x^{2i} + \sum_{i \neq j} \theta_i \theta_i x^{i+j}. \tag{4.1.13}$$

From the coefficient of 1, it follows that $\theta_0 = 0$. We can argue by induction that $\theta_i = 0$ for all $0 \leqslant i \leqslant \frac{n-1}{2} - 1$ if n is odd, and for all $0 \leqslant i \leqslant \frac{n}{2} - 1$ if n is even. It follows that if n is even, then the coefficient of x^{n-1} in (4.1.13) leads to the contradiction: $n = 2\sum_{i=0}^{\frac{n}{2}-1} \theta_i \theta_{n-1-i} = 0$. Thus, $\phi = \mathrm{id}$ is not possible when n is even. On the other hand, if n is odd, then the coefficient of x^{n-1} in (4.1.13) yields $n = (\theta_{\frac{n-1}{2}})^2 + 2\sum_{i=0}^{\frac{n-1}{2}-1} \theta_i \theta_{n-1-i}$, which implies that $\theta_{\frac{n-1}{2}} = \pm\sqrt{n} \cdot 1_{\Bbbk}$. So, $\phi = \mathrm{id}$ and $\theta = \pm\sqrt{n}x^{\frac{n-1}{2}} + \sum_{j=\frac{n+1}{2}}^{n-1} \theta_j x^j$ precisely satisfy conditions (i), (ii), and (iii) yielding an extended structure on the Frobenius algebra $\Bbbk[x]/(x^n)$ when n is odd.

It remains to look at the case $\phi(x) = -x$. It follows from ϕ being a morphism of coalgebras that this is not possible when n is even, since we get the following contradiction:

$$\begin{aligned} \sum_{i=0}^{n-1} x^i \otimes x^{n-i-1} &= \Delta(\phi(1)) = (\phi \otimes \phi)\Delta(1) \\ &= \sum_{i=0}^{n-1} (-1)^{n-1} x^i \otimes x^{n-i-1} = -\sum_{i=0}^{n-1} x^i \otimes x^{n-i-1}. \end{aligned}$$

When n is odd, the equalities $\phi(\theta) = \theta$ and $\phi(x\theta) = x\theta$ from condition (ii) yield the equations

$$\textstyle\sum_{i=0}^{n-1} \theta_i x^i = \sum_{i=0}^{n-1} (-1)^i \theta_i x^i \qquad \text{and} \qquad \sum_{i=0}^{n-2} \theta_i x^{i+1} = \sum_{i=0}^{n-2} (-1)^{i+1} \theta_i x^{i+1},$$

respectively. Hence $\theta_i = 0$ for $1 \leqslant i \leqslant n-2$, and we have that $\theta = \theta_{n-1} x^{n-1}$. But then this would imply $0 = \theta^2 = m(\phi \otimes \mathrm{id})\Delta(1) = x^{n-1}$. Hence, $\phi(x) = -x$ is also not possible when n is odd. □

For a group G, consider the Frobenius algebra $\Bbbk G$ of Example 4.1.6. We provide classification results for extended structures of $\Bbbk G$ when $G = C_2, C_3, C_4$, and $C_2 \times C_2$.

Proposition 4.1.14. *Let g be a generator of C_2. The extended structures of $\Bbbk C_2$ are:*

(a) $\phi = \mathrm{id}_{\Bbbk C_2}$ *and* $\theta \in \{\pm\sqrt{2} e_{C_2},\ \pm\sqrt{2} g\}$, *and*

(b) $\phi(g) = -g$ *and* $\theta = 0$.

Moreover, $(\Bbbk C_2, \mathrm{id}_{\Bbbk C_2}, \sqrt{2} g) \cong (\Bbbk C_2, \mathrm{id}_{\Bbbk C_2}, -\sqrt{2} g)$ *as extended Frobenius algebras, and all other structures are non-isomorphic. That is, there are four isomorphism classes of extended Frobenius structures on* $\Bbbk C_2$.

Proof. Suppose that ϕ and θ define an extended Frobenius structure on $\Bbbk C_2$, with $\phi(g) = \phi_0 e_{C_2} + \phi_1 g$ and $\theta = \theta_0 e_{C_2} + \theta_1 g$ for $\phi_0, \phi_1, \theta_0, \theta_1 \in \Bbbk$. By the counitality of ϕ, we have that $\phi_0 = \varepsilon(\phi(g)) = \varepsilon(g) = 0$, and $\phi_1^2 = \varepsilon(\phi_1^2 g^2) = \varepsilon(\phi(g^2)) = \varepsilon(g^2) = 1$. So, $\phi_1 = \pm 1$. Both choices are involutions and it is a quick check that they satisfy condition (i). We look now at the conditions (ii) and (iii).

When $\phi = \mathrm{id}$, we have that $\theta_0^2 + \theta_1^2 = 2e_{C_2}$ and $2\theta_0\theta_1 = 0$, and so either $\theta = \pm\sqrt{2} e_{C_2}$ or $\theta = \pm\sqrt{2} g$. Both of these satisfy conditions (ii) and (iii). When $\phi(g) = -g$, condition (iii) yields $\theta_0^2 + \theta_1^2 = 0$ and $2\theta_0\theta_1 = 0$. Hence, $\theta = 0$, and condition (ii) is satisfied in this case.

Lastly, it follows from Lemma 4.1.5 that an isomorphism can only exist between $(\Bbbk C_2, \mathrm{id}_{\Bbbk C_2}, \sqrt{2} g)$ and $(\Bbbk C_2, \mathrm{id}_{\Bbbk C_2}, -\sqrt{2} g)$, which are in fact isomorphic via the morphism of extended Frobenius algebras $f : \Bbbk C_2 \to \Bbbk C_2$ defined by $g \mapsto -g$. □

Proposition 4.1.15. *Let g be a generator of C_3. The extended structures of $\Bbbk C_3$ are:*

(a) $\phi = \mathrm{id}_{\Bbbk C_3}$ *and* $\theta \in \{\pm\sqrt{3}e_{C_3},\ \pm\frac{1}{\sqrt{3}}(e_{C_3} - 2\omega_3 g - 2\omega_3^2 g^2)\}$,

(b) $\phi(g) = \omega_3 g^2$ *and* $\theta = \pm\frac{1}{\sqrt{3}}(e_{C_3} + \omega_3 g + \omega_3^2 g^2)$,

where $\omega_3 \in \Bbbk$ is a 3-rd root of unity. Moreover, these structures are all non-isomorphic.

Proof. Suppose that ϕ and θ define an extended Frobenius structure of $\Bbbk C_3$, where $\phi(g) = \phi_0 e_{C_3} + \phi_1 g + \phi_2 g^2$ and $\theta = \theta_0 e_{C_3} + \theta_1 g + \theta_2 g^2$, for $\phi_i, \theta_i \in \Bbbk$. By condition (i), we get that $\phi = \mathrm{id}$ or $\phi(g) = \omega_3 g^2$. We now examine the conditions (ii) and (iii): $m(\phi \otimes \mathrm{id}_{\Bbbk C_3})\Delta(e_{C_3}) = \theta^2$, and $\phi(\theta a) = \theta a$ for $a \in \Bbbk C_3$.

When $\phi = \mathrm{id}$, this gives the equation $\theta^2 = 3e_{C_3}$. Hence, $\theta_0 \neq 0$, and if $\theta_1 = 0$ or $\theta_2 = 0$, these imply $\theta = \pm\sqrt{3}e_{C_3}$. Otherwise, we have that both $\theta_1, \theta_2 \neq 0$, and it follows that $\theta = \pm\frac{1}{\sqrt{3}}(e_{C_3} - 2\omega_3 g - 2\omega_3^2 g^2)$ for some 3-rd root of unity ω_3. Condition (ii) is trivially satisfied for these cases. When $\phi(g) = \omega_3 g^2$, then condition (iii) implies that $\theta^2 = e_{C_3} + \omega_3 g + \omega_3^2 g^2$. We also require $\theta = \phi(\theta) = \theta_0 e_{C_3} + \theta_1 \omega_3 g^2 + \theta_2 \omega_3^2 g$, and thus $\theta_2 = \omega_3 \theta_1$. Therefore, we get that $\theta = \pm\frac{1}{\sqrt{3}}(e_{C_3} + \omega_3 g + \omega_3^2 g^2)$. One can check that these choices satisfy condition (ii); see Example 4.1.7.

Lastly, any morphism f of extended Frobenius algebras between these possible structures is counital, so $f(g) = cg$ or $f(g) = cg^2$ for some $c \in \Bbbk$ such that $c^3 = 1$. From this and Lemma 4.1.5, we conclude there are no such morphisms between the different extended structures. □

Proposition 4.1.16. *Let g be a generator of C_4. The extended structures of $\Bbbk C_4$ are given by*

(a) $\phi = \mathrm{id}_{\Bbbk C_4}$ *and* $\theta \in \{\pm 2e_{C_4},\ \pm 2g^2,\ \pm(1-i)(g + ig^3),\ \pm(1+i)(g - ig^3)\}$;

(b) $\phi(g) = -g$ *and* $\theta = 0$;

(c) $\phi(g) = \omega_4 g^3$ *and* $\theta \in \left\{\pm\frac{1+\omega_4^2}{2}(e_{C_4} - g^2),\ \pm i\frac{1+\omega_4^2}{2}(g - g^3)\right\}$,

for any 4-th root of unity $\omega_4 \in \Bbbk$. These form eight isomorphism classes of extended structures.

Proof. Suppose that ϕ and θ define an extended structure on $\Bbbk C_4$, where for $\phi_i, \theta_i \in \Bbbk$, we have $\phi(g) = \phi_0 e_{C_3} + \phi_1 g + \phi_2 g^2 + \phi_3 g^3$ and $\theta = \theta_0 e_{C_3} + \theta_1 g + \theta_2 g^2 + \theta_3 g^3$. By condition (i), we get that $\phi_2 = 0$ with $\phi(g) = \phi_1 g$ or $\phi(g) = \phi_3 g^3$; else, $\phi_2 \neq 0$ with $\phi_1^2 + \phi_3^2 = 0$. But a routine computation using $\phi^2(g) = g$ and condition (iii) shows that the $\phi_2 \neq 0$ case is not possible. So, either $\phi(g) = \phi_1 g$ or $\phi(g) = \phi_3 g^3$. Since $\phi^2(g) = g$, we obtain $\phi(g) = \pm g$ or $\phi(g) = \omega_4 g^3$ for some $\omega_4 \in \Bbbk$.

Suppose that $\phi = \mathrm{id}_{\Bbbk C_4}$. Then, condition (ii) is trivially satisfied. Condition (iii) implies that $4e_{C_4} = \theta^2$, and we get the choices for θ in part (a). Condition (ii) implies that when $\phi(g) = -g$, we must have that $\theta_1 = \theta_3 = 0$. So, by condition (iii), we obtain that $\theta_0^2 + 2\theta_0\theta_2 g^2 + \theta_2^2 = 0$, and it follows that $\theta = 0$. This yields the choice in part (b). Lastly, if $\phi(g) = \omega_4 g^3$, then from condition (ii), we know that $\theta_1 = \omega_4^3 \theta_3$. Also from condition (iii), we get that $\theta^2 = (1+\omega_4^2)e_{C_4} + (\omega_4 + \omega_4^3)g^2$. Solving for θ^2 in $\Bbbk C_4$, we get the two choices for θ in part (c). The former coincides with the choice of structure given in Example 4.1.7. For the latter, it is easy to check that condition (ii) still holds.

We prove now that there are exactly eight isomorphism classes of extended structures. It follows from Lemma 4.1.5 that three such classes are given by

$$\{(\Bbbk C_4, \mathrm{id}_{\Bbbk C_4}, 2e_{C_4})\}, \;\; \{(\Bbbk C_4, \mathrm{id}_{\Bbbk C_4}, -2e_{C_4})\}, \;\; \{(\Bbbk C_4, \phi(g) = -g, 0)\}.$$

Next, there can be no isomorphisms $f : (\Bbbk C_4, \mathrm{id}_{\Bbbk C_4}, \theta) \to (\Bbbk C_4, \phi(g) = \omega_4 g^3, \theta')$, as this would imply $f(g) = f(\omega_4 g^3)$. Also, the algebra isomorphisms $f, f' : \Bbbk C_4 \to \Bbbk C_4$ defined by $f(g) = -g$ and $f'(g) = ig$ imply that

$$\{(\Bbbk C_4, \mathrm{id}_{\Bbbk C_4}, \pm(1-i)(g+ig^3)), \;\; \{(\Bbbk C_4, \mathrm{id}_{\Bbbk C_4}, \pm(1+i)(g-ig^3))\}, \;\; \{(\Bbbk C_4, \mathrm{id}_{\Bbbk C_4}, \pm 2g^2)\}$$

are isomorphism classes of extended structures. The remaining isomorphism classes are then

$$\{(\Bbbk C_4,\ \phi(g) = \omega_4 g^3,\ \pm\tfrac{1+\omega_4^2}{2}(e_{C_4} - g^2))\}, \;\; \{(\Bbbk C_4,\ \phi(g) = \omega_4 g^3,\ \pm i\tfrac{1+\omega_4^2}{2}(g - g^3))\}$$

by a routine calculation. $\square$

Given the results in Proposition 4.1.14, 4.1.15, 4.1.16, we propose the following.

Conjecture 4.1.17. Let g be a generator of C_n. The following are the only possibilities for the Frobenius automorphism ϕ for an extended structure on $\Bbbk C_n$:

(a) $\phi(g) = \pm g$ or $\phi(g) = \omega_n g^{-1}$ when n is even,

(b) $\phi(g) = g$ or $\phi(g) = \omega_n g^{-1}$ when n is odd,

where $\omega_n \in \Bbbk$ is any n-th root of unity.

The remainder of Theorem 4.1.2 is established in the next two results.

Proposition 4.1.18. *The extended structures of* $\Bbbk(C_2 \times C_2)$ *are:*

(a) $\phi = \mathrm{id}_{\Bbbk(C_2\times C_2)}$ *and* $\theta \in \{\pm 2e,\ \pm 2g_i,\ \pm(e+g_\ell) \pm (g_i - g_j),\ \pm(e-g_\ell) \pm (g_i+g_j)\}$*;*

(b) $\phi(g_i) = -g_i,\ \ \phi(g_j) = -g_j,\ \ \phi(g_\ell) = g_\ell,\quad$ *and* $\theta = 0$*;*

(c) $\phi(g_i) = g_j,\ \ \phi(g_j) = g_i,\ \ \phi(g_\ell) = g_\ell,\ $ *and* $\theta \in \{\pm(e+g_\ell),\ \pm(g_i+g_j)\}$*;*

(d) $\phi(g_i) = -g_j,\ \ \phi(g_j) = -g_i, \phi(g_\ell) = g_\ell,\quad$ *and* $\theta \in \{\pm(e-g_\ell),\ \pm(g_i-g_j)\}$*;*

where $C_2 \times C_2 = \{e, g_1, g_2, g_3\}$ *and* $\{i,j,\ell\} = \{1,2,3\}$.

Proof. It follows from ϕ being counital that $\phi(g_i) = a_{i,1}g_1 + a_{i,2}g_2 + a_{i,3}g_3$ for $a_{i,p} \in \Bbbk$, for all $1 \leqslant i, p \leqslant 3$. Since ϕ is multiplicative, we then get that

$$e = \phi(g_i^2) = \phi(g_i)^2 = (a_{i,1}^2 + a_{i,2}^2 + a_{i,3}^2)e + 2a_{i,1}a_{i,2}g_3 + 2a_{i,1}a_{i,3}g_2 + 2a_{i,2}a_{i,3}g_1.$$

Hence, $\phi(g_i) = \pm g_j$ for some $1 \leqslant j \leqslant 3$. But $\phi^2 = \mathrm{id}_{\Bbbk(C_2\times C_2)}$, and thus the remaining possibilities for ϕ are the ones listed in the statement. It remains to find suitable θ for each possible ϕ. Let $\theta_0, \theta_1, \theta_2, \theta_3 \in \Bbbk$ such that $\theta = \theta_0 e + \theta_1 g_2 + \theta_2 g_2 + \theta_3 g_3$.

We compute $\theta^2 = \phi(e)e + \sum_{i=1}^3 \phi(g_i)g_i$. When $\phi = \mathrm{id}_{\Bbbk(C_2\times C_2)}$, one can check that we get the choices of θ in part (a) by condition (iii). When $\phi(g_i) = -g_i$, $\phi(g_j) = -g_j$ and $\phi(g_\ell) = g_\ell$ for $\{i,j,\ell\} = \{1,2,3\}$, condition (iii) implies $\theta^2 = 0$, so $\theta = 0$; this implies part (b). When $\phi(g_i) = g_j$, $\phi(g_j) = g_i$ and $\phi(g_\ell) = g_\ell$ for $\{i,j,\ell\} = \{1,2,3\}$, conditions (ii) and (iii) yield the choices of θ in part (c). The case $\phi(g_i) = -g_j$, $\phi(g_j) = -g_i$ and $\phi(g_\ell) = g_\ell$ for $\{i, j, \ell\} = \{1, 2, 3\}$ is analogous. $\square$

Proposition 4.1.19. *Consider the Taft algebra $T_2(-1) := \Bbbk\langle g, x\rangle/(g^2-1, x^2, gx+xg)$ as defined in Example 4.1.8. All extensions of $T_2(-1)$ are ϕ-trivial, with $\theta \in \Bbbk x \oplus \Bbbk gx$.*

Proof. First, note that $\Delta(1) = -g \otimes gx + x \otimes 1 + 1 \otimes x + gx \otimes g$. So, by (4.1.3), we get that $\Delta(g) = -1 \otimes gx + gx \otimes 1 + g \otimes x + x \otimes g$, $\Delta(x) = gx \otimes gx + x \otimes x$, and $\Delta(gx) = x \otimes gx + gx \otimes x$. Hence, $\varepsilon(1) = \varepsilon(g) = \varepsilon(gx) = 0$ and $\varepsilon(x) = 1$. Now suppose that $\phi : T_2(-1) \to T_2(-1)$ and $\theta \in T_2(-1)$ define an extended structure on $T_2(-1)$. Let $a_i, b_i \in \Bbbk$ such that $\phi(g) = a_1 + a_2 g + a_3 x + a_4 gx$ and $\phi(x) = b_1 + b_2 g + b_3 x + b_4 gx$. Since ϕ is an algebra morphism, we have that

$$1 = \phi(g)^2 = a_1^2 + a_2^2 + 2a_1a_2g + 2a_1a_3x + 2a_1a_4gx,$$
$$0 = \phi(x)^2 = b_1^2 + b_2^2 + 2b_1b_2g + 2b_1b_3x + 2b_1b_4gx.$$

It follows that $\phi(g) = \pm g + a_3x + a_4gx$ and $\phi(x) = b_3x + b_4gx$. On the other hand, since ϕ is counital, we get $0 = \varepsilon(\phi(g)) = a_3$ and $1 = \varepsilon(\phi(x)) = b_3$. So, $\phi(g) = \pm g + a_4gx$ and $\phi(x) = x + b_4gx$. Also, because ϕ is an involution, we have that $g = \phi(\pm g + a_4gx) = \pm(g + a_4gx) \pm a_4(gx + b_4x)$. It follows that $\phi = \mathrm{id}_{T_2(-1)}$. Lastly, $\theta^2 = m(\phi \otimes \mathrm{id}_{T_2(-1)})\Delta(1) = 0$, and thus $\theta \in \Bbbk x \oplus \Bbbk gx$. $\square$

Conjecture 4.1.20. Recall the Taft algebras $T_n(\omega) := \Bbbk\langle g, x\rangle/(g^n - 1, x^n, gx - \omega xg)$ from Example 4.1.8. All extended Frobenius structures of $T_n(\omega)$ are ϕ-trivial, with $\theta \in \Bbbk x \oplus \Bbbk gx \oplus \cdots \oplus \Bbbk g^{n-1}x$.

4.2 Extended Frobenius algebras in a monoidal category

In this section, we first generalize Definition 4.1.1 to the monoidal setting, following [TT06, Section 2.2], and give some preliminary results in the monoidal setting in §4.2.1. Then, in §4.2.2 we put monoidal structures on the category of extended Frobenius algebras.

4.2.1 Monoidal definition and preliminary results

Recall the basics of monoidal categories and functors from Section 2.2, as well as Definition 2.3.1 of Frobenius algebras in a monoidal category. We build on this definition below to produce our main algebraic structures of interest in this chapter.

Definition 4.2.1. Let $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category.

(a) An *extended Frobenius algebra in* $\mathcal{C}$ is a tuple $(A, m, u, \Delta, \varepsilon, \phi, \theta)$, where the tuple $(A, m, u, \Delta, \varepsilon)$ is a Frobenius algebra in $\mathcal{C}$, with $\phi : A \to A$ and $\theta : \mathbb{1} \to A$ being morphisms in $\mathcal{C}$ such that

  (i) ϕ is a morphism of Frobenius algebras in $\mathcal{C}$, with $\phi^2 = \mathrm{id}_A$;

  (ii) $\phi m(\theta \otimes \mathrm{id}_A) = m(\theta \otimes \mathrm{id}_A)$;

  (iii) $m(\phi \otimes \mathrm{id}_A)\Delta u = m(\theta \otimes \theta)$.

A *morphism* $f : (A, \phi_A, \theta_A) \to (B, \phi_B, \theta_B)$ *of extended Frobenius algebras in* $\mathcal{C}$ is a morphism $f : A \to B$ of Frobenius algebras in $\mathcal{C}$, such that $f\,\phi_A = \phi_B\, f$ and $f\,\theta_A = \theta_B$. The above objects and morphisms form a category, $\mathsf{ExtFrobAlg}(\mathcal{C})$.

(b) The morphisms ϕ and θ in part (a) are the *extended structure* of the underlying Frobenius algebra. When ϕ and θ exist, we say that the underlying Frobenius algebra is *extendable*.

(c) An extended structure (ϕ, θ) on a Frobenius algebra A is said to be ϕ*-trivial* if ϕ is the identity morphism, and is θ*-trivial* if θ is the zero morphism (when these exist in $\mathcal{C}$).

The structure morphisms for an extended Frobenius algebra in $\mathcal{C}$ are depicted in Figure 4.1, and the axioms that they satisfy are depicted in Figure 4.2. Here, we read diagrams from top down.

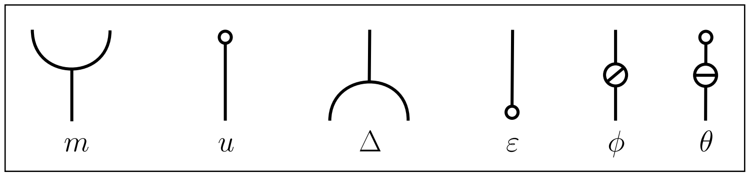

Figure 4.1 : Structure morphisms for an extended Frobenius algebra in $\mathcal{C}$.

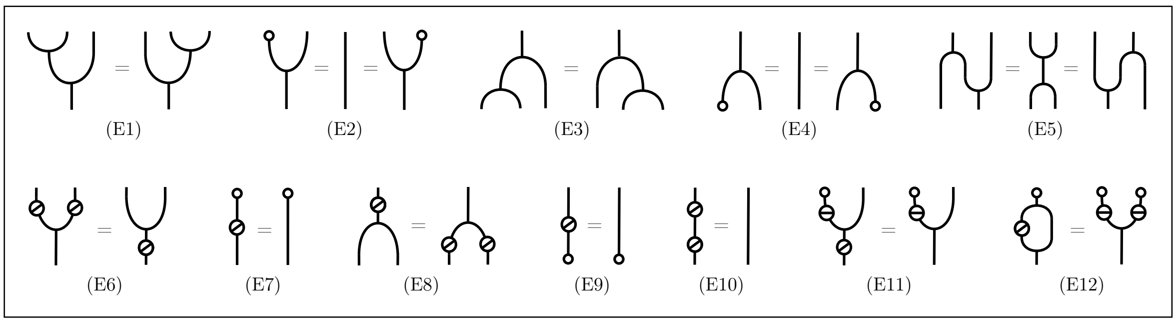

Figure 4.2 : Axioms for an extended Frobenius algebra in $\mathcal{C}$.

One useful lemma is the following, adapted from [TT06, Lemma 2.8] for the monoidal setting.

Lemma 4.2.2. *If $(A, m, u, \Delta, \varepsilon, \phi, \theta)$ is an extended Frobenius algebra in $\mathcal{C}$, then*

$$m(\phi \otimes \mathrm{id}_A)\Delta \;=\; m\big(m(\theta \otimes \theta) \otimes \mathrm{id}_A\big).$$

Proof. This is proved in Figure 4.3 with references to Figures 4.1 and 4.2. $\square$

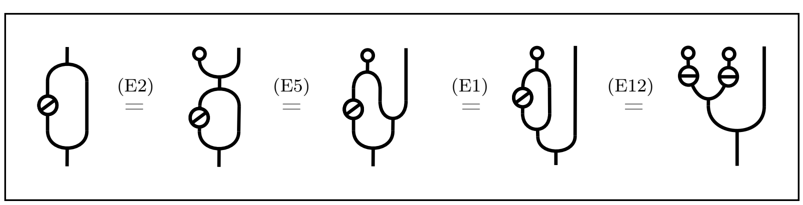

Figure 4.3 : Proof of Lemma 4.2.2.

Proposition 4.2.3. *A morphism of extended Frobenius algebras in $\mathcal{C}$ must be an isomorphism.*

Proof. This follows from the well-known fact that a morphism of Frobenius algebras in $\mathcal{C}$ must be an isomorphism. We repeat the proof here for the reader's convenience. Take a morphism of (extended) Frobenius algebras $f : A \to B$ in $\mathcal{C}$, that is, f is a morphism of the underlying algebras and coalgebras in $\mathcal{C}$. In graphical calculus, we will denote the (extended) Frobenius structure morphisms on A by those given in Figure 4.1, and the (extended) Frobenius structure morphisms on B will be denoted according to Figure 4.4. We then define a morphism $g : B \to A$ in Figure 4.5, and show that $gf = \mathrm{id}_A$ and $fg = \mathrm{id}_B$ using graphical calculus in Figure 4.6. $\square$

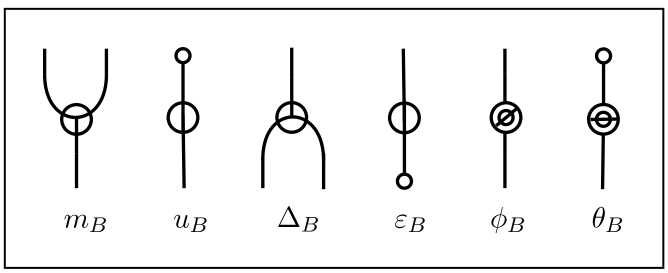

Figure 4.4 : Extended Frobenius structure on B.

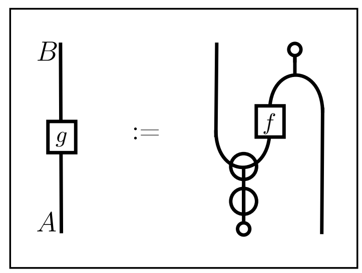

Figure 4.5 : Defining g.

We now recall the definition of separable Frobenius algebras, and show that they are all extendable.

Definition 4.2.4. (a) We say that an algebra $A := (A, m, u)$ in $\mathcal{C}$ is *separable* if there exists a morphism $t : A \to A \otimes A$ such that $mt = \mathrm{id}_A$, and

$$(m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes t) \;=\; tm \;=\; (\mathrm{id}_A \otimes m)(t \otimes \mathrm{id}_A).$$

(b) A Frobenius algebra $A := (A, m, u, \Delta, \varepsilon)$ is *separable Frobenius* if $m\Delta = \mathrm{id}_A$.

These structures form full subcategories as indicated below:

$$\mathsf{SepAlg}(\mathcal{C}) \subset \mathsf{Alg}(\mathcal{C}), \qquad \mathsf{SepFrobAlg}(\mathcal{C}) \subset \mathsf{FrobAlg}(\mathcal{C}).$$

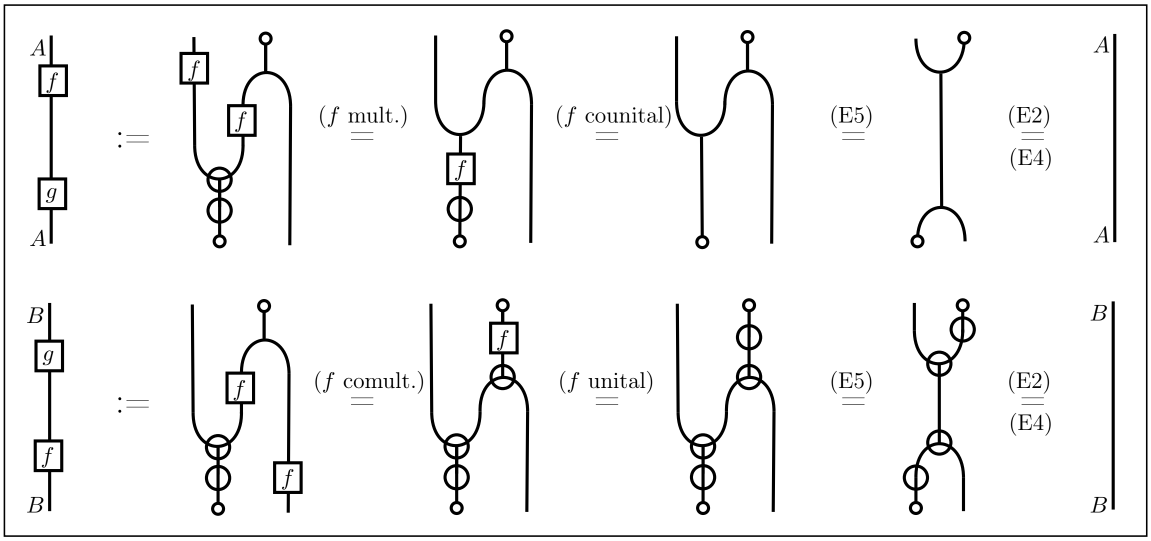

Figure 4.6 : Proof that $gf = \mathrm{id}_A$ and $fg = \mathrm{id}_B$.

Proposition 4.2.5. *If A is a separable Frobenius algebra in $\mathcal{C}$, then A is extendable.*

Proof. Suppose that $A := (A, m, u, \Delta, \varepsilon)$ is a separable Frobenius algebra, and take $\phi := \mathrm{id}_A$ and $\theta := u$. Then, conditions (i) and (ii) of Definition 4.2.1(b) are trivially satisfied. Condition (iii) of Definition 4.2.1(b) holds by the computation below:

$$m(\phi \otimes \mathrm{id}_A)\Delta u \ = \ m\Delta u \ = \ u \ = \ m(u \otimes u) \ = \ m(\theta \otimes \theta),$$

where the third equality follows from a unitality axiom of A. □

Example 4.2.6. The monoidal unit $\mathbb{1} \in \mathcal{C}$ is a separable Frobenius algebra, with m and Δ identified as $\mathrm{id}_{\mathbb{1}}$, and with $u = \varepsilon = \mathrm{id}_{\mathbb{1}}$. The Frobenius structure is then extended with $\phi = \theta = \mathrm{id}_{\mathbb{1}}$.

4.2.2 Structural results

Recall the category $\mathsf{ExtFrobAlg}(\mathcal{C})$ defined in Definition 4.2.1. We put monoidal structures on this category, using two distinct monoidal products, in the following results.

Proposition 4.2.7. *Let $(\mathcal{C}, \otimes, \mathbb{1}, c)$ be a symmetric monoidal category. Then, the category $\mathsf{ExtFrobAlg}(\mathcal{C})$ is monoidal with $\otimes := \otimes^{\mathcal{C}}$ and $\mathbb{1} := \mathbb{1}^{\mathcal{C}}$.*

Proof. We first note that $\mathbb{1}^{\mathcal{C}} = (\mathbb{1}^{\mathcal{C}}, \ell_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}}, \ell_{\mathbb{1}}^{-1}, \mathrm{id}_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}})$ is an extended Frobenius algebra in $\mathcal{C}$.

Next, we show that the monoidal product of two extended Frobenius algebras is extended Frobenius. Namely, we verify that given extended Frobenius algebras $(A, m_A, u_A, \Delta_A, \varepsilon_A, \phi_A, \theta_A)$ and $(B, m_B, u_B, \Delta_B, \varepsilon_B, \phi_B, \theta_B)$, then the tensor product $(A \otimes B, \tilde{m}, \tilde{u}, \tilde{\Delta}, \tilde{\varepsilon}, \tilde{\phi}, \tilde{\theta})$ is an extended Frobenius algebra, where

$$\tilde{m} := (m_A \otimes m_B)(\mathrm{id}_A \otimes c_{B,A} \otimes \mathrm{id}_B), \qquad \tilde{\Delta} := (\mathrm{id}_A \otimes c_{A,B} \otimes \mathrm{id}_B)(\Delta_A \otimes \Delta_B)$$

$$\tilde{u} := u_A \otimes u_B, \qquad \tilde{\varepsilon} := \varepsilon_A \otimes \varepsilon_B, \qquad \tilde{\phi} := \phi_A \otimes \phi_B, \qquad \tilde{\theta} := \theta_A \otimes \theta_B.$$

Figure 4.7 shows what these morphisms look like in graphical calculus, using the symbols from Figure 4.1 for A and the symbols from Figure 4.4 for B, as in Proposition 4.2.3. Recall also the axioms for a symmetric monoidal category from Figure 2.4.

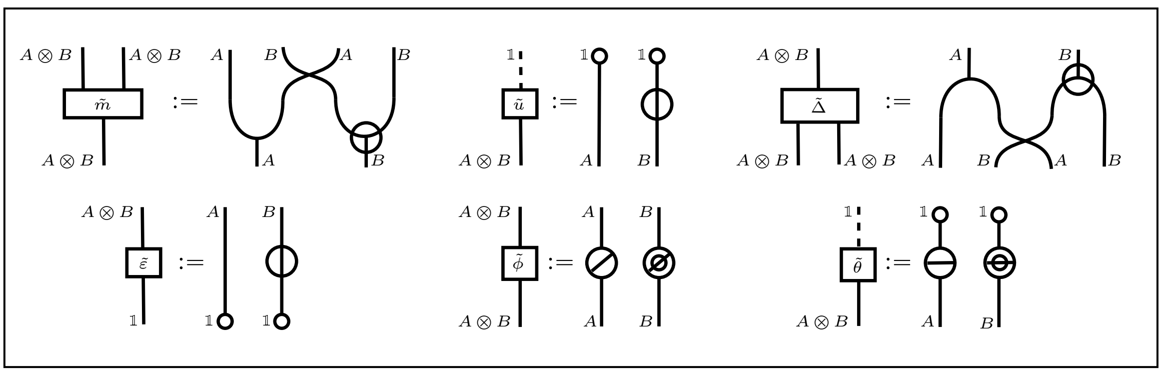

Figure 4.7 : Extended Frobenius structure morphisms for $A \otimes B$.

We then have that $(A \otimes B, \tilde{m}, \tilde{u}, \tilde{\Delta}, \tilde{\varepsilon}) \in \mathsf{FrobAlg}(\mathcal{C})$ by [Koc04, Section 2.4]. To see that this Frobenius algebra is extended via $\tilde{\phi}$ and $\tilde{\theta}$, we verify the three required conditions in Definition 4.2.1(b).

(i) We see that $\tilde{\phi}$ is an involution since both ϕ_A and ϕ_B are involutions. Moreover, since both ϕ_A, ϕ_B are Frobenius morphisms, so is their monoidal product in $\mathcal{C}$.

(ii) Figure 4.8 gives that $\tilde{\phi}\,\tilde{m}(\tilde{\theta} \otimes \mathrm{id}_{A\otimes B}) = \tilde{m}(\tilde{\theta} \otimes \mathrm{id}_{A\otimes B})$.

(iii) Finally, Figure 4.9 gives that $\tilde{m}(\tilde{\phi} \otimes \mathrm{id}_{A\otimes B})\tilde{\Delta}\tilde{u} = \tilde{m}(\tilde{\theta} \otimes \tilde{\theta})$.

Lastly, we note that by taking $\mathbb{1}^{\mathcal{C}}$ as the unit and $\otimes^{\mathcal{C}}$ as the monoidal product in $\mathsf{ExtFrobAlg}(\mathcal{C})$, with extended structures behaving as described above, we obtain that the required pentagon and triangle axioms (refer to Figures 2.1 and 2.2) in $(\mathsf{ExtFrobAlg}(\mathcal{C}), \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$ are both inherited from the same axioms in $(\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$. Thus, we conclude that $(\mathsf{ExtFrobAlg}(\mathcal{C}), \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$ is a monoidal category. $\square$

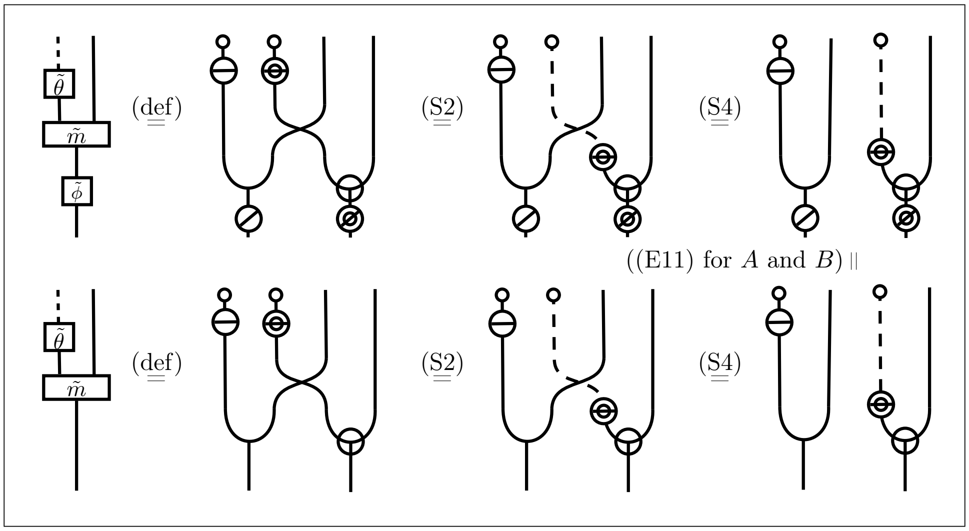

Figure 4.8 : Proof that $A \otimes B$ satisfies Definition 4.2.1(b)(ii).

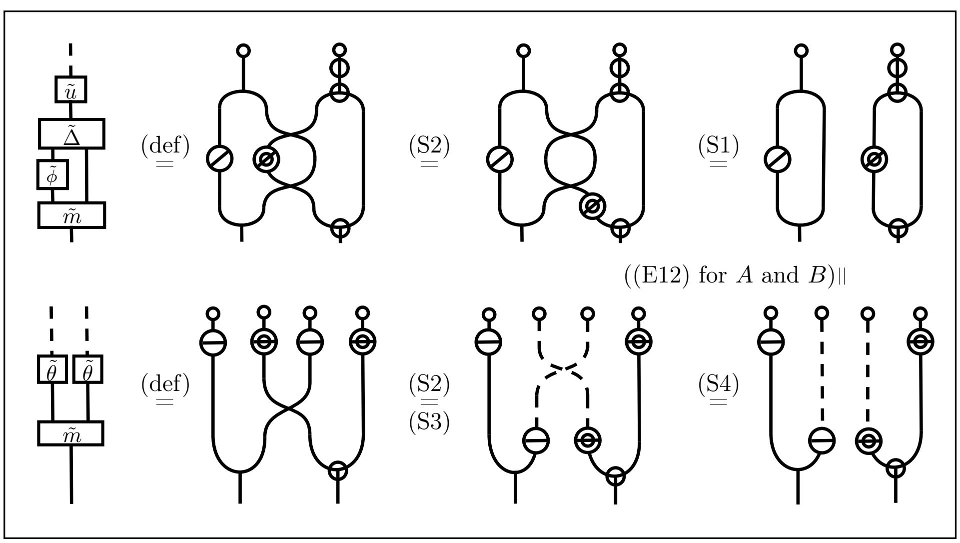

Figure 4.9 : Proof that $A \otimes B$ satisfies Definition 4.2.1(b)(iii).

Now we turn our attention to extended Frobenius algebras in additive monoidal categories. See §2.2.4 for a brief introduction to such categories, and [Wal24, Section 3.1.3] for further background material.

Proposition 4.2.8. *Let* $(\mathcal{C}, \otimes, \mathbb{1})$ *be an additive monoidal category. Then, the category* $\mathsf{ExtFrobAlg}(\mathcal{C})$ *is monoidal with* $\otimes$ *being the biproduct* $\sqcup$*, and* $\mathbb{1}$ *being the zero object* $\mathsf{0}$*.*

Proof. We first note that $\mathsf{0}$ is an extended Frobenius algebra in $\mathcal{C}$, with structure morphisms $m, u, \Delta, \varepsilon$, and θ all being zero morphisms, and $\phi = \mathrm{id}_{\mathsf{0}}$. We next note that similar to the previous proposition, the pentagon and triangle axioms in $(\mathsf{ExtFrobAlg}(\mathcal{C}), \sqcup, \mathsf{0})$ will be inherited from these same axioms on the strict monoidal category $(\mathcal{C}, \sqcup, \mathsf{0})$. Hence, to finish the proof, it suffices to show that the biproduct of two extended Frobenius algebras is again extended Frobenius. To do so, let $(A, m_A, u_A, \Delta_A, \varepsilon_A, \phi_A, \theta_A)$ and $(B, m_B, u_B, \Delta_B, \varepsilon_B, \phi_B, \theta_B)$ be two extended Frobenius algebras in $\mathcal{C}$. We will show that $(A \sqcup B, \tilde{m}, \tilde{u}, \tilde{\Delta}, \tilde{\varepsilon}, \tilde{\phi}, \tilde{\theta})$ is an extended Frobenius algebra, where $\tilde{m}$, $\tilde{u}$, $\tilde{\Delta}$, $\tilde{\varepsilon}$, $\tilde{\phi}$, and $\tilde{\theta}$ are defined by universal property diagrams in Figure 4.10.

It is well known that with the above constructions, $(A \sqcup B, \tilde{m}, \tilde{u}, \tilde{\Delta}, \tilde{\varepsilon})$ is a Frobenius algebra. See [Koc04, Exercises 2.2.7 and 2.2.8] for the case where $\mathcal{C} = \mathsf{Vec}$. Thus, we only need to verify that $\tilde{\phi}$ and $\tilde{\theta}$ extend this Frobenius algebra. The three required properties from Definition 4.2.1(b) can be verified by respectively considering each of the universal property diagrams in Figure 4.11. Using uniqueness of the completing map in each of these diagrams, it follows that

(i) $(\tilde{\phi})^2 = \mathrm{id}_{A \sqcup B}$,

(ii) $\tilde{m}(\tilde{\theta} \otimes \mathrm{id}_{A \sqcup B}) = \tilde{\phi}(\tilde{m}(\tilde{\theta} \otimes \mathrm{id}_{A \sqcup B}))$,

(iii) $\tilde{m}(\tilde{\phi} \otimes \mathrm{id}_{A \sqcup B})(\tilde{\Delta}(\tilde{u})) = \tilde{m}(\tilde{\theta} \otimes \tilde{\theta})$,

which completes the proof that $(A \sqcup B, \tilde{\phi}, \tilde{\theta})$ is an extended Frobenius algebras in $\mathcal{C}$, hence giving that $(\mathsf{ExtFrobAlg}(\mathcal{C}), \sqcup, 0)$ is a monoidal category. □

$$
\begin{tikzcd}
& (A \sqcup B) \otimes (A \sqcup B) \arrow[d, dashed, "\exists!\,\tilde{m}"] \arrow[ddl, bend right, "m_A \circ \pi_{A\otimes A}"'] \arrow[ddr, bend left, "m_B \circ \pi_{B\otimes B}"] & \\
& A \sqcup B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\
A & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& (A \sqcup B) \otimes (A \sqcup B) & \\
& A \sqcup B \arrow[u, dashed, "\exists!\,\tilde{\Delta}"'] & \\
A \arrow[ur, "\iota_A"] \arrow[uur, bend left, "\iota_{A\otimes A} \circ \Delta_A"] & & B \arrow[ul, "\iota_B"'] \arrow[uul, bend right, "\iota_{B\otimes B} \circ \Delta_B"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathbb{1} \arrow[d, dashed, "\exists!\,\tilde{u}"] \arrow[ddl, bend right, "u_A"'] \arrow[ddr, bend left, "u_B"] & \\
& A \sqcup B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\
A & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{1} & \\
& A \sqcup B \arrow[u, dashed, "\exists!\,\tilde{\varepsilon}"'] & \\
A \arrow[ur, "\iota_A"] \arrow[uur, bend left, "\varepsilon_A"] & & B \arrow[ul, "\iota_B"'] \arrow[uul, bend right, "\varepsilon_B"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
& A \sqcup B \arrow[d, dashed, "\exists!\,\tilde{\phi}"] \arrow[ddl, bend right, "\phi_A \circ \pi_A"'] \arrow[ddr, bend left, "\phi_B \circ \pi_B"] & \\
& A \sqcup B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\
A & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{1} \arrow[d, dashed, "\exists!\,\tilde{\theta}"] \arrow[ddl, bend right, "\theta_A"'] \arrow[ddr, bend left, "\theta_B"] & \\
& A \sqcup B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\
A & & B
\end{tikzcd}
$$

Figure 4.10 : Defining the extended Frobenius algebra structure on $A \sqcup B$.

$$
\begin{tikzcd}
& A \sqcup B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] \arrow[d, dashed, "\exists!"] & \\
A \arrow[d, "\phi_A^2 = \mathrm{id}_A"'] & A \sqcup B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & B \arrow[d, "\phi_B^2 = \mathrm{id}_B"] \\
A & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \sqcup B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] \arrow[d, dashed, "\exists!"] & \\
A \arrow[d, "\substack{m_A(\theta_A \otimes \mathrm{id}_A) \\ \parallel \\ \phi_A(m_A(\theta_A \otimes \mathrm{id}_A))}"'] & A \sqcup B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & B \arrow[d, "\substack{m_B(\theta_B \otimes \mathrm{id}_B) \\ \parallel \\ \phi_B(m_B(\theta_B \otimes \mathrm{id}_B))}"] \\
A & & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathbb{1} \arrow[d, dashed, "\exists!"] \arrow[ddl, bend right, "\substack{m_A(\phi_A \otimes \mathrm{id}_A)(\Delta_A(u_A)) \\ \parallel \\ m_A(\theta_A \otimes \theta_A)}"'] \arrow[ddr, bend left, "\substack{m_B(\phi_B \otimes \mathrm{id}_B)(\Delta_B(u_B)) \\ \parallel \\ m_B(\theta_B \otimes \theta_B)}"] & \\
& A \sqcup B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\
A & & B
\end{tikzcd}
$$

Figure 4.11 : Proof that $A \sqcup B$ is an extended Frobenius algebra.

4.3 Extended Frobenius monoidal functors

In this section, we introduce the construction of an extended Frobenius monoidal functor, which preserves extended Frobenius algebras [Proposition 4.3.4]. The main construction is covered in §4.3.1 and examples are presented in §4.3.2.

4.3.1 Main construction and results

Here, we extend the results in Proposition 2.5.4 and Proposition 2.5.5 to the category $\mathsf{ExtFrobAlg}(\mathcal{C})$. In particular, we will define a type of functor that preserves extended Frobenius algebras and show that this type of functor is closed under composition.

Definition 4.3.1. A Frobenius monoidal functor $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)})$ between the monoidal categories $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{C}', \otimes', \mathbb{1}')$ is called an *extended Frobenius monoidal functor* (or is *extendable*) if there exist a natural transformation $\widehat{F} : F \Rightarrow F$ and a morphism $\check{F} : \mathbb{1}' \to F(\mathbb{1}) \in \mathcal{C}'$ such that the conditions below hold.

(a) $\widehat{F}$ is both a monoidal and comonoidal natural transformation.

(b) $F^{(2)}_{\mathbb{1},\mathbb{1}} \circ (\widehat{F}_{\mathbb{1}} \otimes' \mathrm{id}_{F(\mathbb{1})}) \circ F^{\mathbb{1},\mathbb{1}}_{(2)} \circ F^{(0)} = F^{(2)}_{\mathbb{1},\mathbb{1}} \circ (\check{F} \otimes' \check{F})$.

(c) The following are true for each $X, Y \in \mathcal{C}$:

  (i) $\widehat{F}_X \circ \widehat{F}_X = \mathrm{id}_{F(X)}$;

  (ii) $\widehat{F}_{\mathbb{1}\otimes X} \circ F^{(2)}_{\mathbb{1},X} \circ (\check{F} \otimes' \mathrm{id}_{F(X)}) = F^{(2)}_{\mathbb{1},X} \circ (\check{F} \otimes' \mathrm{id}_{F(X)})$;

  (iii) $F^{(2)}_{X,Y} \circ (\widehat{F}_X \otimes' \mathrm{id}_{F(Y)}) \circ F^{X,Y}_{(2)} = F^{(2)}_{X\otimes Y,\mathbb{1}} \circ (\widehat{F}_{X\otimes Y} \otimes' \mathrm{id}_{F(\mathbb{1})}) \circ F^{X\otimes Y,\mathbb{1}}_{(2)}$.

Extended Frobenius monoidal functors are plentiful. Specifically, we have the following result; compare to Proposition 4.2.5.

Proposition 4.3.2. *A separable Frobenius monoidal functor admits the structure of an extended Frobenius monoidal functor.*

Proof. Let $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)})$ be a Frobenius monoidal functor. Recall that it is separable if $F^{(2)} \circ F_{(2)} = \mathrm{id}$ (see Definition 2.5.1). Then, take $\widehat{F} = \mathrm{Id}_F$ and $\check{F} = F^{(0)}$. It is then straightforward to verify that these choices of $\widehat{F}$ and $\check{F}$ extend the Frobenius monoidal structure on F. $\square$

Example 4.3.3. Strong monoidal functors are separable with $F_{(2)} := F^{(-2)}$ and $F_{(0)} := F^{(-0)}$, so they are also extended Frobenius monoidal functors.

The next result is the desired extension of Proposition 2.5.4. See Appendix A.1 for proof.

Proposition 4.3.4. *Let $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}, \widehat{F}, \check{F}) : \mathcal{C} \to \mathcal{C}'$ be an extended Frobenius monoidal functor. This induces a functor $\mathsf{ExtFrobAlg}(\mathcal{C}) \to \mathsf{ExtFrobAlg}(\mathcal{C}')$. Specifically, for $A \in \mathsf{ExtFrobAlg}(\mathcal{C})$, we have that $F(A) \in \mathsf{ExtFrobAlg}(\mathcal{C}')$ with $m_{F(A)}$, $u_{F(A)}$, $\Delta_{F(A)}$, $\varepsilon_{F(A)}$ as in Proposition 2.5.4(a,b), with $\phi_{F(A)} = F(\phi_A)\,\widehat{F}_A$, and with $\theta_{F(A)} = F(\theta_A)\,\check{F}$.* $\square$

We also have the following extension of Proposition 2.5.5 to extended Frobenius monoidal functors. The proof of this theorem can be found in Appendix A.2.

Theorem 4.3.5. *The composition of two extended Frobenius monoidal functors is again an extended Frobenius monoidal functor.* $\square$

Remark 4.3.6. One can also obtain Proposition 4.3.4 as a consequence of Theorem 4.3.5. Take the monoidal category $\overline{\mathbb{1}}$ consisting of a single object $\mathbb{1}$ and morphism $\mathrm{id}_{\mathbb{1}}$. Then, a Frobenius monoidal functor $(E, E^{(2)}, E^{(0)}, E_{(2)}, E_{(0)}) : \overline{\mathbb{1}} \to \mathcal{C}$ is extendable if and only if $E(\mathbb{1}) \in \mathsf{ExtFrobAlg}(\mathcal{C})$. So, when $A \in \mathsf{ExtFrobAlg}(\mathcal{C})$, the functor $A^{\#} : \overline{\mathbb{1}} \to \mathcal{C}$ with $A^{\#}(\mathbb{1}) := A$ is extended Frobenius monoidal. Now if $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}, \widehat{F}, \check{F}) : \mathcal{C} \to \mathcal{C}'$ is extended Frobenius monoidal, Theorem 4.3.5 implies that the functor $FA^{\#} : \overline{\mathbb{1}} \to \mathcal{C}'$ is also extended Frobenius monoidal. Hence, $F(A)$ is an extended Frobenius algebra in $\mathcal{C}'$ as in the proof of Proposition 4.3.4. Compare to [DP08, Corollary 5].

4.3.2 Examples

Building upon Propositions 4.2.7 and 4.2.8, consider the examples of extended Frobenius monoidal functors below.

Example 4.3.7. Let $(\mathcal{C}, \otimes, \mathbb{1}, c)$ be symmetric monoidal with $B \in \mathsf{ExtFrobAlg}(\mathcal{C})$. Then, the functor $- \otimes B : \mathcal{C} \to \mathcal{C}$ is extended Frobenius monoidal with

$$(- \otimes B)^{(2)}_{X,Y} := (\mathrm{id}_{X\otimes Y} \otimes m_B)(\mathrm{id}_X \otimes c_{B,Y} \otimes \mathrm{id}_B),$$

$$(- \otimes B)^{X,Y}_{(2)} := (\mathrm{id}_X \otimes c_{Y,B} \otimes \mathrm{id}_B)(\mathrm{id}_{X\otimes Y} \otimes \Delta_B),$$

$$(- \otimes B)^{(0)} := u_B, \qquad (- \otimes B)_{(0)} := \varepsilon_B,$$

$$\widehat{(- \otimes B)}_X := \mathrm{id}_X \otimes \phi_B, \qquad \text{and} \qquad \widetilde{(- \otimes B)} := \theta_B,$$

for any $X, Y \in \mathcal{C}$. We note further that when B is not a separable Frobenius algebra, the Frobenius functor defined above is not separable.

Example 4.3.8. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be additive monoidal with $B \in \mathsf{ExtFrobAlg}(\mathcal{C})$. Then, the functor $- \sqcup B : \mathcal{C} \to \mathcal{C}$ is extended Frobenius monoidal with

$$(- \sqcup B)^{(2)}_{X,Y} := \pi_{X\otimes Y} \sqcup (m_B \circ \pi_{B\otimes B}), \qquad (- \sqcup B)^{X,Y}_{(2)} := \iota_{X\otimes Y} \sqcup (\iota_{B\otimes B} \circ \Delta_B),$$

$$(- \sqcup B)^{(0)} := \mathrm{id}_{\mathbb{1}} \sqcup u_B, \qquad (- \sqcup B)_{(0)} := \mathrm{id}_{\mathbb{1}} \sqcup \varepsilon_B,$$

$$\widehat{(- \sqcup B)}_X := \pi_X \sqcup (\phi_B \circ \pi_B), \qquad \text{and} \qquad \widetilde{(- \sqcup B)} := \mathrm{id}_{\mathbb{1}} \sqcup \theta_B,$$

for any $X, Y \in \mathcal{C}$. Again, when B is not a separable Frobenius algebra, the Frobenius functor defined above is not separable.

4.4 Connection to Hopf algebras

In this section, we explore the extension of Frobenius algebra structures inherited from Hopf algebras. We begin with the induced Frobenius structure on integral Hopf algebras in §4.4.1. In §4.4.2, we define an extended Hopf structure that guarantees extendability of the inherited Frobenius structure. We introduce functorial constructions preserving integral and extended Hopf algebras in §4.4.3 and §4.4.4, respectively.

4.4.1 Frobenius algebras from Hopf algebras

Take a symmetric monoidal category $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, c)$ and consider the following structures on a Hopf algebra $(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S)$ in $\mathcal{C}$ (see Definition 2.3.3).

Definition 4.4.1. (a) A *left integral* for a Hopf algebra $(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S)$ is a morphism $\Lambda : \mathbb{1} \to H$ which satisfies $m(\mathrm{id}_H \otimes \Lambda) = \Lambda\underline{\varepsilon}$. A *right cointegral* for the Hopf algebra $(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S)$ is a morphism $\lambda : H \to \mathbb{1}$ satisfying $(\lambda \otimes \mathrm{id}_H)\underline{\Delta} = u\lambda$. If Λ and λ further satisfy $\lambda\Lambda = \mathrm{id}_{\mathbb{1}}$, then Λ and λ are said to be *normalized*. A Hopf algebra equipped with a normalized (co)integral pair is called an *integral Hopf algebra*.

See Figures B.1-B.4 in Appendix B for a graphical depiction of this definition.

(b) A *morphism of integral Hopf algebras* $f : H \to K$ is a morphism, which is both an algebra and coalgebra morphism, and which satisfies $f\Lambda_H = \Lambda_K$ and $\lambda_K f = \lambda_H$.

(c) We organize the above into a category, $\mathsf{IntHopfAlg}(\mathcal{C})$, whose objects are integral Hopf algebras and whose morphisms are morphisms of integral Hopf algebras as defined above.

For further information on the objects in the above definition, see [Rad12, Chapter 10] and the references within for the case when $\mathcal{C} = \mathsf{Vec}$.

Remark 4.4.2. If a Hopf algebra is equipped with a normalized integral and cointegral, then its antipode is invertible; see, e.g., [CD20, Lemma 3.5].

We also have the following proposition saying that integral Hopf algebras in $\mathcal{C}$ admit the structure of Frobenius algebras in $\mathcal{C}$. This is proved in Appendix B using a graphical argument due to Yadav. A similar, nongraphical argument can be found in [FS10, Appendix A.2].

Proposition 4.4.3. *We have that*

$$\Psi : \mathsf{IntHopfAlg}(\mathcal{C}) \to \mathsf{FrobAlg}(\mathcal{C})$$
$$(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1}, \Lambda, \lambda) \mapsto (H, m, u,\ \Delta := (m \otimes S)(\mathrm{id}_H \otimes \underline{\Delta}\Lambda),\ \varepsilon := \lambda)$$

is a well-defined functor, which acts as the identity on morphisms.

Example 4.4.4. Let G be any finite group. The group algebra $\Bbbk G$ is a finite-dimensional Hopf algebra with $\underline{\Delta}(g) = g \otimes g$, $\underline{\varepsilon}(g) = 1$, and $S(g) = g^{-1}$, for all $g \in G$. This Hopf algebra admits a normalized (co)integral pair given by $\Lambda := \sum_{h \in G} h$ and $\lambda(g) := \delta_{e,g} 1_{\Bbbk}$. Applying Ψ to this integral Hopf algebra, we obtain the Frobenius algebra structure on $\Bbbk G$ described in Example 4.1.6 using equation (4.1.3), where $\Delta(g) := \sum_{h \in G} gh \otimes h^{-1}$ and $\varepsilon(g) := \lambda(g) = \delta_{e,g} 1_{\Bbbk}$, for all $g \in G$.

Proposition 4.4.5. *If $H \in \mathsf{IntHopfAlg}(\mathcal{C})$ is equipped with $\theta : \mathbb{1} \to H \in \mathcal{C}$ such that $m(\theta \otimes \theta) = u\underline{\varepsilon}\Lambda$, then the Frobenius algebra $\Psi(H)$ from Proposition 4.4.3 is extendable.*

In particular, when $\mathcal{C} = \mathsf{Vec}$, the Frobenius algebra $\Psi(H)$ over $\Bbbk$ is always extendable with $\phi = \mathrm{id}_{\Psi(H)}$ and $\theta = \pm\sqrt{\underline{\varepsilon}(\Lambda(1_{\Bbbk}))}\, u$.

Proof. Suppose that the morphism $\theta : \mathbb{1} \to H$ as in the statement exists. Then, taking $\phi = \mathrm{id}_{\Psi(H)}$, and using this θ, we extend the Frobenius structure. To verify the axioms of Definition 4.2.1(b), notice that conditions (i) and (ii) hold trivially. Condition (iii) is verified in Figure 4.12; using notation and axioms from Appendix B. The last statement on the case when $\mathcal{C} = \mathsf{Vec}$ is clear. $\square$

Example 4.4.6. Let G be a finite group, and recall the induced Frobenius-from-Hopf algebra strcuture on $\Bbbk G$ described in Example 4.4.4. In this case, we have that

$$u\underline{\varepsilon}(\Lambda) = u\left(\underline{\varepsilon}\left(\textstyle\sum_{h \in G} h\right)\right) = u\left(\textstyle\sum_{h \in G} 1_{\Bbbk}\right) = |G| \cdot u(1_{\Bbbk}) = |G| \cdot e_G.$$

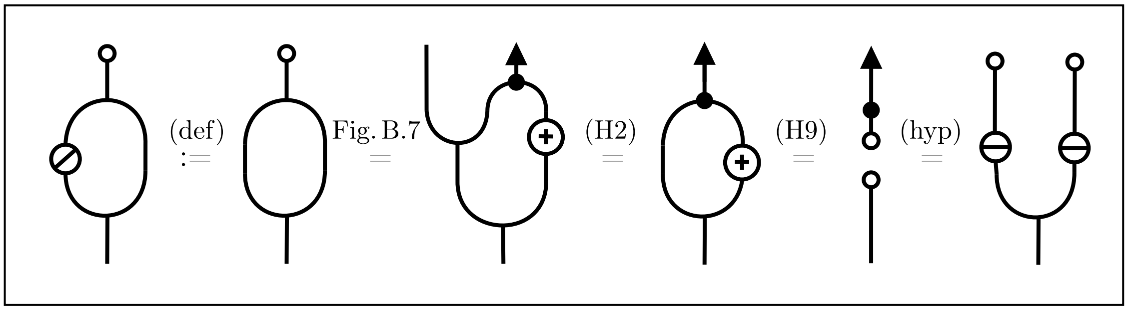

Figure 4.12 : Proof of Definition 4.2.1(b)(iii) for Proposition 4.4.5.

The above proposition then tells us that the choice $\phi = \mathrm{id}_{\Bbbk G}$ and $\theta = \pm\sqrt{|G|} \cdot e_G$ extends the induced Frobenius algebra structure on $\Bbbk G$. Note that this is the same extended Frobenius structure introduced in Example 4.1.6.

4.4.2 Extended Hopf algebras

Continue to let $\mathcal{C}$ be a symmetric monoidal category. Here, we introduce extended Hopf algebras in $\mathcal{C}$ as a way to obtain extensions of Frobenius-from-Hopf algebra structures.

Definition 4.4.7. An integral Hopf algebra $(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1}, \Lambda, \lambda)$ is called *extended* if it is equipped with two morphisms $\phi : H \to H$ and $\theta : \mathbb{1} \to H$ in $\mathcal{C}$ satisfying the following axioms:

(i) ϕ is a morphism of integral Hopf algebras such that $\phi^2 = \mathrm{id}_H$;

(ii) $\phi m(\theta \otimes \mathrm{id}_H) = m(\theta \otimes \mathrm{id}_H)$;

(iii) $m(\phi \otimes S)\underline{\Delta}\Lambda = m(\theta \otimes \theta)$.

A *morphism of extended Hopf algebras* $f : (H, \phi, \theta) \to (H', \phi', \theta')$ is a morphism of integral Hopf algebras in $\mathcal{C}$ which also satisfies $f\phi = \phi' f$ and $f\theta = \theta'$.

With the above, we define a category $\mathsf{ExtHopfAlg}(\mathcal{C})$ and obtain a forgetful functor,

$$U : \mathsf{ExtHopfAlg}(\mathcal{C}) \to \mathsf{IntHopfAlg}(\mathcal{C})$$
$$(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1}, \Lambda, \lambda, \phi, \theta) \mapsto (H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1}, \Lambda, \lambda).$$

We have the following result.

Proposition 4.4.8. *Take $H \in \mathsf{ExtHopfAlg}(\mathcal{C})$. Then, the Frobenius algebra $\Psi U(H)$ in $\mathcal{C}$ from Proposition 4.4.3 is extendable via the morphisms ϕ and θ.*

Proof. To verify that ϕ and θ extend the Frobenius algebra $\Psi U(H)$, we check the axioms of Definition 4.2.1(b). Since $\phi : (H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, \Lambda, \lambda) \to (H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, \Lambda, \lambda)$ is a morphism of integral Hopf algebras, the functoriality of Ψ and U gives that $\phi : (H, m, u, \Delta, \varepsilon) \to (H, m, u, \Delta, \varepsilon)$ is a Frobenius algebra morphism. Moreover, we have that $\phi^2 = \mathrm{id}_H$ by Definition 4.4.7(i). So, condition (i) of Definition 4.2.1(b) holds. Condition (ii) of Definition 4.2.1(b) also holds by Definition 4.4.7(ii) since the multiplication morphism is the same for both the Hopf and Frobenius structures on H. Towards condition (iii) of Definition 4.2.1(b), we compute:

$$m(\phi \otimes \mathrm{id}_H)\Delta u \;=\; m(\phi \otimes S)(m \otimes \mathrm{id}_H)(u \otimes \underline{\Delta})\Lambda \;=\; m(\theta \otimes \theta),$$

where the first equality is the definition of Δ and a level exchange, and the second equality is by the unitality of m and u and Definition 4.4.7(iii). □

The consequence below is straight-forward.

Corollary 4.4.9. *There is a functor $\underline{\Psi} : \mathsf{ExtHopfAlg}(\mathcal{C}) \to \mathsf{ExtFrobAlg}(\mathcal{C})$ which sends an extended Hopf algebra $(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, S^{-1}, \Lambda, \lambda, \phi, \theta)$ to the extended Frobenius algebra $(H, m, u, \Delta, \varepsilon, \phi, \theta)$, with Δ and ε defined as in Proposition 4.4.3, and which acts as the identity on morphisms.* □

Remark 4.4.10. While the above result tells us that every extended Hopf algebra gives rise to an extended Frobenius algebra via the same ϕ and θ, the converse is not true. In particular, given $H \in \mathsf{IntHopfAlg}(\mathcal{C})$, we get that $\Psi(H) \in \mathsf{FrobAlg}(\mathcal{C})$.

But, even if $\Psi(H)$ is extendable via $\phi_{\Psi(H)}$ and $\theta_{\Psi(H)}$, it is not necessarily true that $(H, \phi_{\Psi(H)}, \theta_{\Psi(H)})$ is an extended Hopf algebra in $\mathcal{C}$.

For instance, consider the Frobenius algebra structure on $\Bbbk C_2$, induced by the Hopf structure, as described in Example 4.4.4. This Frobenius structure can be extended by taking $\phi(g) = -g$ (where g is a generator of C_2) and $\theta = 0$, as in Proposition 4.1.14(b). However, this choice of ϕ and θ does not extend the integral Hopf structure on $\Bbbk C_2$, since ϕ is not comultiplicative with respect to $\underline{\Delta}$.

4.4.3 Integral Hopf monoidal functors

We continue the extension of Proposition 2.5.4 and Proposition 2.5.5 by defining types of monoidal functors that preserve integral and extended Hopf algebras.

Definition 4.4.11. A Hopf monoidal functor $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}, \Upsilon) : \mathcal{C} \to \mathcal{C}'$ is called an *integral Hopf monoidal functor* if it comes equipped with two morphisms $\overline{F} : \mathbb{1}' \to F(\mathbb{1})$ and $\underline{F} : F(\mathbb{1}) \to \mathbb{1}'$ satisfying $\underline{F} \circ \overline{F} = \mathrm{id}_{\mathbb{1}'}$ and the two diagrams in Figure 4.13.

$$
\begin{tikzcd}
F(\mathbb{1}) \arrow[rr, "F_{(0)}"] \arrow[d, equal, "r^{-1}_{F(\mathbb{1})}"'] & & \mathbb{1}' \arrow[dd, "\overline{F}"] \\
F(\mathbb{1}) \otimes' \mathbb{1}' \arrow[dd, "\mathrm{id}_{F(\mathbb{1})} \otimes' \overline{F}"'] & & \\
& & F(\mathbb{1}) \arrow[d, equal, "F(r_{\mathbb{1}}^{-1})"] \\
F(\mathbb{1}) \otimes' F(\mathbb{1}) \arrow[rr, "F^{(2)}_{\mathbb{1},\mathbb{1}}"'] & & F(\mathbb{1} \otimes \mathbb{1})
\end{tikzcd}
\qquad
\begin{tikzcd}
F(\mathbb{1}) \arrow[rr, "\underline{F}"] \arrow[d, equal, "F(\ell_{\mathbb{1}}^{-1})"'] & & \mathbb{1}' \arrow[dd, "F^{(0)}"] \\
F(\mathbb{1} \otimes \mathbb{1}) \arrow[dd, "F^{\mathbb{1},\mathbb{1}}_{(2)}"'] & & \\
& & F(\mathbb{1}) \arrow[d, equal, "\ell^{-1}_{F(\mathbb{1})}"] \\
F(\mathbb{1}) \otimes' F(\mathbb{1}) \arrow[rr, "\underline{F} \otimes' \mathrm{id}_{F(\mathbb{1})}"'] & & \mathbb{1}' \otimes' F(\mathbb{1})
\end{tikzcd}
$$

Figure 4.13 : Axioms for an integral Hopf monoidal functor.

With this definition, we obtain the following.

Proposition 4.4.12. *Let* $(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}, \Upsilon, \overline{F}, \underline{F}) : \mathcal{C} \to \mathcal{C}'$ *be an integral Hopf monoidal functor. This induces a functor* $\mathsf{IntHopfAlg}(\mathcal{C}) \to \mathsf{IntHopfAlg}(\mathcal{C}')$*. Specifically, for* $H \in \mathsf{IntHopfAlg}(\mathcal{C})$*, we have that* $F(H) \in \mathsf{IntHopfAlg}(\mathcal{C}')$*, with Hopf algebra structure given by Proposition 2.5.4(e), with integral* $\Lambda_{F(H)} := F(\Lambda_H) \circ \overline{F}$*, and with cointegral* $\lambda_{F(H)} := \underline{F} \circ F(\lambda_H)$*.*

Proof. Proposition 2.5.4(e) gives that $F(H)$ is a Hopf algebra in $\mathcal{C}'$, so it only remains to show that $\Lambda_{F(H)}$ and $\lambda_{F(H)}$ form a normalized(co)integral pair for this Hopf algebra structure.

First, we see that $\Lambda_{F(H)}$ and $\lambda_{F(H)}$ are normalized, since

$$\lambda_{F(H)} \circ \Lambda_{F(H)} = \underline{F} F(\lambda_H) \circ F(\Lambda_H) \overline{F} = \underline{F}\,\overline{F} = \mathrm{id}_{F(H)},$$

where the first equality is by definition, the second is from the fact that Λ_H and λ_H are normalized, and the last equality is by the normalization condition of $\overline{F}$ and $\underline{F}$ from Definition 4.4.11.

Next, Figure 4.14 gives that $\Lambda_{F(H)}$ is indeed an integral for $F(H)$. Here, regions (1), (5), (8), and (11) commute by definition, regions (2), (6), (7), and (9) by naturality, region (3) is a level exchange, region (4) is the condition on $\overline{F}$ from Definition 4.4.11 and Figure 4.13, and region (10) follows from the fact that Λ_H is an integral for H.

Figure 4.15 gives that $\lambda_{F(H)}$ is a cointegral for $F(H)$, where regions (1), (5), (10), and (11) commute by definition, regions (3), (6), (7), and (9) by naturality, region (8) is a level exchange, region (4) is the condition on $\underline{F}$ from Definition 4.4.11 and Figure 4.13, and region (2) follows from the fact that λ_H is a cointegral for H.

Finally, we must check that given a morphism $f : (H, \Lambda_H, \lambda_H) \to (H', \Lambda_{H'}, \lambda_{H'})$ of integral Hopf algebras in $\mathcal{C}$, its image $F(f) : F(H) \to F(H')$ is a morphism of integral Hopf algebras in $\mathcal{C}'$. Proposition 2.5.4(e) gives that $F(f)$ is a morphism of Hopf algebras, and so it only remains to check that $F(f)$ respects the (co)integrals of $F(H)$ and $F(H')$.

Indeed, we have that

$$F(f)\Lambda_{F(H)} = F(f)F(\Lambda_H)\overline{F} = F(\Lambda_{H'})\overline{F} = \Lambda_{H'};$$
$$\lambda_{F(H)}F(f) = \underline{F}F(\lambda_H)F(f) = \underline{F}F(\lambda_{H'}) = \lambda_{F(H')},$$

where the first and last equalities are the definitions of $\Lambda_{F(-)}$ and $\lambda_{F(-)}$ and the middle equalities are because $f : H \to H'$ is a morphism integral Hopf algebras and hence respects the (co)integrals of H and H'. This completes the proof. $\square$

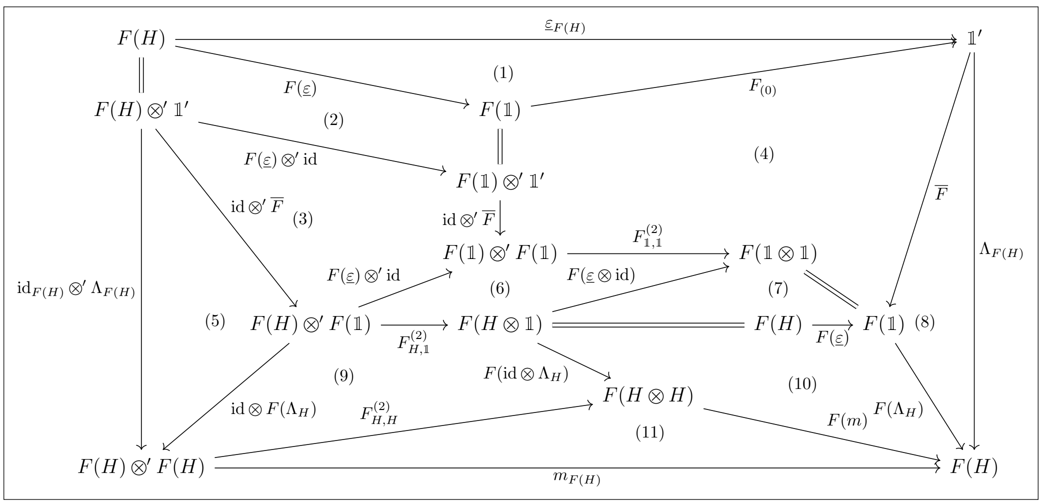

Figure 4.14 : Proof that $\Lambda_{F(H)}$ is an integral for $F(H)$.

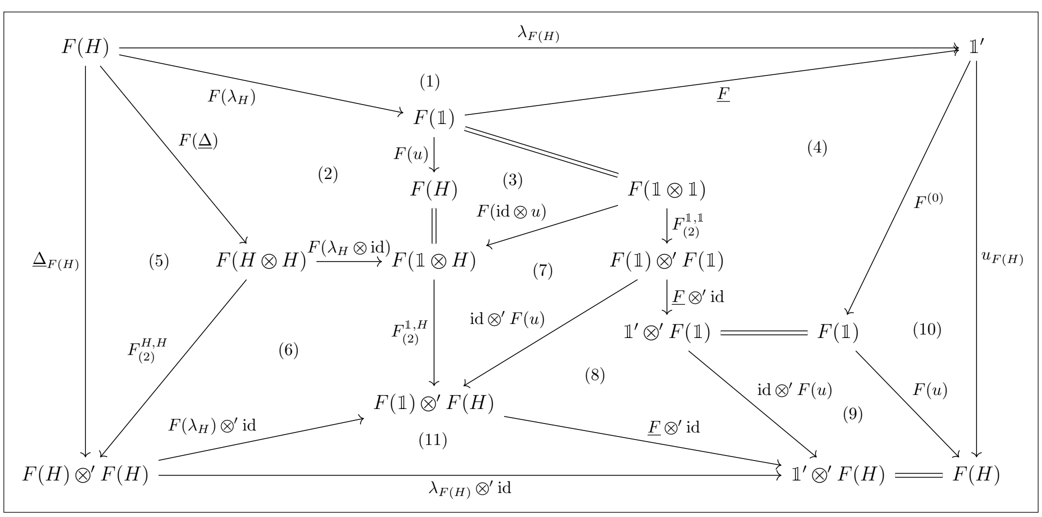

Figure 4.15 : Proof that $\lambda_{F(H)}$ is a cointegral for $F(H)$.

The following result extends Proposition 2.5.5 to integral Hopf monoidal functors.

Proposition 4.4.13. *Given two integral Hopf monoidal functors* $(F, \overline{F}, \underline{F}) : \mathcal{C} \to \mathcal{C}'$ *and* $(G, \overline{G}, \underline{G}) : \mathcal{C}' \to \mathcal{C}''$*, the composition* $GF : \mathcal{C} \to \mathcal{C}''$ *is also integral Hopf monoidal.*

Proof. We first note that GF is Hopf monoidal by Theorem 2.5.5(e), so we only need to show that GF admits an integral structure. To see this, define

$$\overline{GF} := G(\overline{F}) \circ \overline{G} : \mathbb{1}'' \to GF(\mathbb{1}) \qquad \text{and} \qquad \underline{GF} := \underline{G} \circ G(\underline{F}) : GF(\mathbb{1}) \to \mathbb{1}''.$$

Now, the conditions of Definition 4.4.11 must be verified. We first note that

$$\underline{GF} \circ \overline{GF} = \underline{G} \circ G(\underline{F}) \circ G(\overline{F}) \circ \overline{G} = \underline{G} \circ G(\underline{F}\overline{F}) \circ \overline{G} = \underline{G}\overline{G} = \mathrm{id}_{\mathbb{1}''},$$

since $(\overline{F}, \underline{F})$ and $(\overline{G}, \underline{G})$ are integral structures for F and G, respectively. Next, we check the two axioms of Figure 4.13. For the strict case, this is done in Figure 4.16, where regions (1), (5), (6), and (11) commute by definition, regions (2), (7), (8), and (9) by naturality, region (3) is a level exchange, and regions (4) and (10) are the condition from Definition 4.4.11 Figure 4.13 for $\overline{G}$ and $\overline{F}$, respectively.

The verification that $\underline{GF}$ satisfies the condition of Definition 4.4.11 and Figure 4.13 is analogous, so we omit it. This completes the proof. □

Analogously to how morphisms of integral Hopf algebras preserve integrals and cointegrals, we require that natural transformations of integral Hopf monoidal functors also behave nicely with the functorial analogues of integrals and cointegrals.

Definition 4.4.14. A natural transformation $\phi : (F, \overline{F}, \underline{F}) \Rightarrow (G, \overline{G}, \underline{G})$ between integral Hopf monoidal functors is an *integral Hopf monoidal natural transformation* if the following conditions are satisfied.

(a) ϕ is simultaneously monoidal and comonoidal;

(b) $\phi_{\mathbb{1}} \circ \overline{F} = \overline{G}$;

(c) $\underline{G} \circ \phi_{\mathbb{1}} = \underline{F}$.

We think of the second condition above as saying that ϕ "respects integrals" and the third as saying that ϕ "respects cointegrals."

$$
\begin{tikzcd}
GF(\mathbb{1}) \arrow[rrrrr, "(GF)_{(0)}"] \arrow[drr, "G(F_{(0)})"'] \arrow[dd, equal] & & {} \arrow[d, phantom, "(1)"] & & & \mathbb{1}'' \arrow[ddl, "\overline{G}"'] \arrow[ddddd, "\overline{GF}"] \\
 & (2) & G(\mathbb{1}') \arrow[urrr, "G_{(0)}"'] \arrow[d, equal] & & & \\
GF(\mathbb{1}) \otimes'' \mathbb{1}'' \arrow[rr, "G(F_{(0)}) \otimes'' \mathrm{id}_{\mathbb{1}''}"] \arrow[ddr, "\mathrm{id}_{GF(\mathbb{1})} \otimes'' \overline{G}"] \arrow[ddddd, "\mathrm{id}_{GF(\mathbb{1})} \otimes'' \overline{GF}"'] & & G(\mathbb{1}') \otimes'' \mathbb{1}'' \arrow[d, "\mathrm{id}_{G(\mathbb{1}')} \otimes'' \overline{G}"] & (4) & G(\mathbb{1}') \arrow[ddl, equal] \arrow[dddr, "G(\overline{F})"] \arrow[r, phantom, "(5)"] & {} \\
 & (3) & G(\mathbb{1}') \otimes'' G(\mathbb{1}') \arrow[dr, "G^{(2)}_{\mathbb{1}',\mathbb{1}'}"] & & & \\
{} \arrow[r, phantom, "(6)"] & GF(\mathbb{1}) \otimes'' G(\mathbb{1}') \arrow[ur, "G(F_{(0)}) \otimes'' \mathrm{id}_{G(\mathbb{1}')}"] \arrow[dr, "G^{(2)}_{F(\mathbb{1}),\mathbb{1}'}"'] \arrow[dddl, "\mathrm{id}_{GF(\mathbb{1})} \otimes'' G(\overline{F})"] & (7) & G(\mathbb{1}' \otimes' \mathbb{1}') & & \\
 & & G(F(\mathbb{1}) \otimes' \mathbb{1}') \arrow[ur, "G(F_{(0)} \otimes' \mathrm{id}_{\mathbb{1}'})"] \arrow[rr, equal] \arrow[d, "G(\mathrm{id}_{F(\mathbb{1})} \otimes' \overline{F})"] & (8) & GF(\mathbb{1}) \arrow[uuu, "G(F_{(0)})"] & GF(\mathbb{1}) \arrow[dd, equal] \\
 & (9) & G(F(\mathbb{1}) \otimes' F(\mathbb{1})) \arrow[drrr, "G(F^{(2)}_{\mathbb{1},\mathbb{1}})"] \arrow[d, phantom, "(11)"] & & (10) & \\
GF(\mathbb{1}) \otimes'' GF(\mathbb{1}) \arrow[urr, "G^{(2)}_{F(\mathbb{1}),F(\mathbb{1})}"] \arrow[rrrrr, "(GF)^{(2)}_{\mathbb{1},\mathbb{1}}"'] & & {} & & & GF(\mathbb{1} \otimes \mathbb{1})
\end{tikzcd}
$$

Figure 4.16 : $\overline{GF}$ satisfies the condition of Definition 4.4.11 and Figure 4.13.

4.4.4 Extended Hopf monoidal functors

With integral Hopf monoidal functors and natural transformations defined, it is now easy to define extended Hopf monoidal functors in a manner similar to the creation of extended Frobenius monoidal functors from Frobenius monoidal functors. Because of this, the proofs of results in this section will be analogous to those in Appendix A, and so we omit them here.

Definition 4.4.15. An integral Hopf monoidal functor $(F, \overline{F}, \underline{F}) : \mathcal{C} \to \mathcal{C}'$ is called an *extended Hopf monoidal functor*, or is *extendable*, if it can be equipped with a natural transformation $\widehat{F} : F \Rightarrow F$ and a morphism $\check{F} : \mathbb{1}' \to F(\mathbb{1}) \in \mathcal{C}'$ such that the conditions below hold.

(a) $\widehat{F}$ is an integral Hopf monoidal natural transformation.

(b) $F^{(2)}_{\mathbb{1},\mathbb{1}} \circ (\widehat{F}_{\mathbb{1}} \otimes' \Upsilon_{\mathbb{1}}) \circ F^{\mathbb{1},\mathbb{1}}_{(2)} \circ \overline{F} = F^{(2)}_{\mathbb{1},\mathbb{1}} \circ (\check{F} \otimes' \check{F})$.

(c) The following are true for each $X, Y \in \mathcal{C}$:

(i) $\widehat{F}_X \circ \widehat{F}_X = \mathrm{id}_{F(X)}$;

(ii) $\widehat{F}_{\mathbb{1}\otimes X} \circ F^{(2)}_{\mathbb{1},X} \circ (\check{F} \otimes' \mathrm{id}_{F(X)}) = F^{(2)}_{\mathbb{1},X}(\check{F} \otimes' \mathrm{id}_{F(X)})$;

(iii) $F^{(2)}_{X,Y} \circ (\widehat{F}_X \otimes' \Upsilon_Y) \circ F^{X,Y}_{(2)} = F^{(2)}_{X\otimes Y,\mathbb{1}} \circ (\widehat{F}_{X\otimes Y} \otimes' \Upsilon_{\mathbb{1}}) \circ F^{X\otimes Y,\mathbb{1}}_{(2)}$.

Again, we now extend Propositions 2.5.4 and 2.5.5 to the extended Hopf case.

Proposition 4.4.16. *An extended Hopf monoidal functor* $(F, \widehat{F}, \check{F}) : \mathcal{C} \to \mathcal{C}'$ *induces a functor* $\mathsf{ExtHopfAlg}(\mathcal{C}) \to \mathsf{ExtHopfAlg}(\mathcal{C}')$. *Specifically, for* $H \in \mathsf{ExtHopfAlg}(\mathcal{C})$, *we have that* $F(H) \in \mathsf{ExtHopfAlg}(\mathcal{C}')$, *with integral Hopf algebra structure given by Proposition 4.4.12, and with extended structure given by* $\phi_{F(H)} := F(\phi_H)\widehat{F}_H$ *and* $\theta_{F(H)} := F(\theta_H)\check{F}$.

Proof. This proof is analogous to the one given in Appendix A.1, with only minor modifications. As was the case with extended Frobenius monoidal functors, Definition 4.4.15(a),(c)(i) allow us to conclude that $\phi_{F(H)}$ is an integral Hopf convolution, Definition 4.4.15(c)(ii) is used to show that $F(H)$ satisfies condition (ii) of Definition 4.4.7, and finally Definition 4.4.15(b),(c)(iii) are necessary in proving that $F(H)$ satisfies condition (iii) of Definition 4.4.7. □

Proposition 4.4.17. *Let* $(F, \widehat{F}, \check{F}) : \mathcal{C} \to \mathcal{C}'$ *and* $(G, \widehat{G}, \check{G}) : \mathcal{C}' \to \mathcal{C}''$ *be two extended Hopf monoidal functors. The composition* $GF : \mathcal{C} \to \mathcal{C}''$ *is also an extended Hopf monoidal functor, with integral Hopf monoidal structure given in Proposition 4.4.13, and with extended structure defined by* $\widehat{GF}_X := G(\widehat{F}_X) \circ \widehat{G}_{F(X)}$ *for all* $X \in \mathcal{C}$ *and* $\widecheck{GF} := G(\check{F}) \circ \check{G}$.

Proof. Again, the proof of this statement is analogous to the one given in Appendix A.2. The only additional thing that must be checked is that $\widehat{GF}$ is an integral monoidal natural transformation, but this is straightforward from naturality and the fact that both $\widehat{G}$ and $\widehat{F}$ are integral monoidal natural transformations. □

4.5 Discussion

There are a number of possible directions for future research based upon the material presented above. We discuss some here for the interested reader.

4.5.1 On forgetful functors

In searching for examples of extended Frobenius monoidal functors, we came across [BT15, Theorem 6.2], which states that if A is an algebra in a monoidal category $\mathcal{C}$, then the forgetful functor ${}_A\mathfrak{U}_A : A\text{-}\mathsf{Bimod}(\mathcal{C}) \to \mathcal{C}$, equipped with the trivial monoidal structure, is Frobenius monoidal if and only if A is a Frobenius algebra. We wonder how this could be modified to the cases of integral Hopf, extended Frobenius, and extended Hopf, giving a new example of these types of monoidal functors.

4.5.2 On pullbacks

It is natural to want $\mathsf{ExtHopfAlg}(\mathcal{C})$ to be the pullback of the functor $\Psi : \mathsf{IntHopfAlg}(\mathcal{C})$ from Proposition 4.4.3 and the forgetful functor $U : \mathsf{ExtFrobAlg}(\mathcal{C}) \to \mathsf{FrobAlg}(\mathcal{C})$, but Remark 4.4.10 shows that this is not the case. If, however we restrict to only ϕ-trivial extended Hopf and Frobenius algebras, then we do obtain this result. This leads to questions concerning how the pullback subcategory and the subcategory $\mathsf{ExtHopfAlg}(\mathcal{C})$ of $\mathcal{C}$ relate, how these two types of objects can be used in different applications and scenarios, and whether there are any conditions on $\mathcal{C}$ or on extended Hopf algebras that could produce special cases of this result.

4.5.3 On a functorial version of Frobenius-from-Hopf structures

In §4.4.1, we saw that integral Hopf algebras all admit a Frobenius structure. We inquire whether a functorial analogue of this result could be obtained. There are many results that showcase relationships between Hopf monoidal functors, Hopf adjunctions, Frobenius monoidal functors, and Frobenius functors, see for example

[Bal17, Sar21, FLP24, Yad24, FLP25, JY25], but none have considered the functorial analogue of integral Hopf algebras. The initial example $- \otimes A : \mathcal{C} \to \mathcal{C}$ on a symmetric monoidal category $\mathcal{C}$ looks promising, as if H is an integral Hopf algebra, then $-\otimes H$ obtains both an integral Hopf monoidal and Frobenius monoidal structure from the corresponding algebra structures on H. However, it is not clear whether this could be generalized, as it may be the case that for general integral Hopf monoidal functors, only Frobenius algebras in the image of Ψ are preserved.

Appendix A

Proofs of selected results in Section 4.3

Following [CKQW24], Proposition 4.3.4 is proved in Section A.1 and Proposition 4.3.5 in Section A.2.

A.1 Proof of Proposition 4.3.4

Given $(A, m_A, u_A, \Delta_A, \varepsilon_A, \phi_A, \theta_A) \in \mathsf{ExtFrobAlg}(\mathcal{C})$, we first define an extended Frobenius algebra structure on $F(A)$. Let $m_{F(A)}$, $u_{F(A)}$, $\Delta_{F(A)}$, and $\varepsilon_{F(A)}$ be as in Proposition 2.5.4(a,b). By Proposition 2.5.4(c), this makes $F(A)$ a Frobenius algebra in $\mathcal{C}'$. Define

$$\phi_{F(A)} := F(\phi_A)\,\widehat{F}_A, \qquad \theta_{F(A)} := F(\theta_A)\,\check{F},$$

and note that by naturality, $\phi_{F(A)} := F(\phi_A)\,\widehat{F}_A = \widehat{F}_A\,F(\phi_A)$. We will now show that $\phi_{F(A)}$ and $\theta_{F(A)}$ satisfy the conditions in Definition 4.2.1(b).

To verify Definition 4.2.1(b)(i) for $F(A)$, we first show that $\phi_{F(A)}$ is a Frobenius algebra morphism. Commutativity of the diagram in Figure A.1 verifies that $m_{F(A)}(\phi_{F(A)} \otimes' \phi_{F(A)}) = \phi_{F(A)}\,m_{F(A)}$. Regions (1), (2), (5), and (8) commute by definition, (3) by monoidality of $\widehat{F}$, (4) and (6) by naturality, and (7) by multiplicativity of ϕ_A. Likewise, comonoidality of $\widehat{F}$ gives $(\phi_{F(A)} \otimes' \phi_{F(A)})\Delta_{F(A)} = \Delta_{F(A)}\,\phi_{F(A)}$.

Commutativity of the diagram in Figure A.2 shows that $u_{F(A)} = \phi_{F(A)}\,u_{F(A)}$. Regions (1), (4), and (6) commute by definition, (2) by monoidality of $\widehat{F}$, (3) by ϕ_A being an algebra morphism, and (5) by naturality. Comonoidality of $\widehat{F}$ analogously gives that $\varepsilon_{F(A)} = \varepsilon_{F(A)}\,\phi_{F(A)}$, concluding the proof that $\phi_{F(A)} \in \mathsf{FrobAlg}(\mathcal{C}')$.

$$
\begin{tikzcd}
F(A) \otimes' F(A) \arrow[r, "F^{(2)}_{A,A}"] \arrow[rr, bend left, "m_{F(A)}", "(1)"'] \arrow[d, "\widehat{F}_A \otimes' \widehat{F}_A"] \arrow[dd, bend right=60, "\phi_{F(A)} \otimes' \phi_{F(A)}"', "(2)"] & F(A \otimes A) \arrow[r, "F(m_A)"] \arrow[d, "\widehat{F}_{A \otimes A}"] \arrow[dl, phantom, "(3)"] & F(A) \arrow[d, "\widehat{F}_A"'] \arrow[dd, bend left=60, "\phi_{F(A)}", "(5)"'] \arrow[dl, phantom, "(4)"] \\
F(A) \otimes' F(A) \arrow[r, "F^{(2)}_{A,A}"] \arrow[d, "F(\phi_A) \otimes' F(\phi_A)"] & F(A \otimes A) \arrow[r, "F(m_A)"] \arrow[d, "F(\phi_A \otimes \phi_A)"] \arrow[dl, phantom, "(6)"] & F(A) \arrow[d, "F(\phi_A)"'] \arrow[dl, phantom, "(7)"] \\
F(A) \otimes' F(A) \arrow[r, "F^{(2)}_{A,A}"] \arrow[rr, bend right, "m_{F(A)}"', "(8)"] & F(A \otimes A) \arrow[r, "F(m_A)"] & F(A)
\end{tikzcd}
$$

Figure A.1 : $\phi_{F(A)}$ is multiplicative.

$$
\begin{tikzcd}
\mathbb{1}' \arrow[r, "F^{(0)}"] \arrow[rr, bend left, "u_{F(A)}", "(1)"'] \arrow[dr, "F^{(0)}"', "(2)"] \arrow[ddrr, bend right, "u_{F(A)}"', "(4)"] & F(\mathbb{1}) \arrow[r, "F(u_A)"] \arrow[d, "\widehat{F}_{\mathbb{1}}"] \arrow[dr, "F(u_A)"', "(3)"] & F(A) \arrow[d, "F(\phi_A)"'] \arrow[dd, bend left=60, "\phi_{F(A)}", "(6)"'] \\
& F(\mathbb{1}) \arrow[dr, "F(u_A)"'] \arrow[r, phantom, "(5)"] & F(A) \arrow[d, "\widehat{F}_A"'] \\
& & F(A)
\end{tikzcd}
$$

Figure A.2 : $\phi_{F(A)}$ is unital.

To see that $\phi_{F(A)}$ is an involution, note that

$$\phi_{F(A)} \circ \phi_{F(A)} = F(\phi_A) \circ \widehat{F}_A \circ \widehat{F}_A \circ F(\phi_A) = F(\phi_A \circ \phi_A) = \mathrm{id}_{F(A)},$$

where we use that $\phi_{F(A)} := F(\phi_A)\,\widehat{F}_A = \widehat{F}_A\,F(\phi_A)$, Definition 4.3.1(c)(i), and the fact that ϕ_A is an involution.

Definition 4.2.1(b)(ii) for $F(A)$ follows from the diagram in Figure A.3. Regions (1), (2), (5), (9), and (11) commute by definition, (4), (6), (7), and (10) by naturality, (3) by Definition 4.3.1(c)(ii), and (8) by Definition 4.2.1(b)(ii) for A.

Lastly, Definition 4.2.1(b)(iii) for $F(A)$ holds by Figure A.6, where regions (1), (2), (3), (8), (20), and (21) commute by definition, (5), (6), and (9)-(18) by naturality, (4) by Definition 4.3.1(b), (7) by Definition 4.3.1(c)(iii), and (19) by Definition 4.2.1(b)(iii) for A. This completes the proof that $F(A) \in \mathsf{ExtFrobAlg}(\mathcal{C}')$.

$$\begin{tikzcd}
\mathbb{1}' \otimes' F(A) \arrow[rrr, "\check{F} \otimes' \mathrm{id}_{F(A)}"'] \arrow[rrrrr, bend left=10, "\theta_{F(A)} \otimes' \mathrm{id}_{F(A)}"] \arrow[rrrrr, phantom, bend left=5, "(1)"] \arrow[dr, "\check{F} \otimes' \mathrm{id}_{F(A)}"] \arrow[dddr, bend right, "\theta_{F(A)} \otimes' \mathrm{id}_{F(A)}"'] \arrow[dddr, phantom, "(2)"] & & & F(\mathbb{1}) \otimes' F(A) \arrow[rr, "F(\theta_A) \otimes' \mathrm{id}_{F(A)}"'] \arrow[d, "F^{(2)}_{\mathbb{1},A}"] \arrow[ddr, phantom, "(4)"] & & F(A) \otimes' F(A) \arrow[ddl, "F^{(2)}_{A,A}"'] \arrow[ddd, "m_{F(A)}"] \arrow[dddl, phantom, "(5)"] \\
& F(\mathbb{1}) \otimes' F(A) \arrow[dd, "F(\theta_A) \otimes' \mathrm{id}_{F(A)}"'] \arrow[dr, "F^{(2)}_{\mathbb{1},A}"] \arrow[rr, phantom, "(3)"] \arrow[ddr, phantom, "(6)"] & & F(\mathbb{1} \otimes A) \arrow[dl, "\hat{F}_{\mathbb{1} \otimes A}"'] \arrow[dd, "F(\theta_A \otimes \mathrm{id}_A)"'] \arrow[dr, "F(\theta_A \otimes \mathrm{id}_A)"] \arrow[ddl, phantom, "(7)"] \arrow[ddr, phantom, "(8)"] & & \\
& & F(\mathbb{1} \otimes A) \arrow[d, "F(\theta_A \otimes \mathrm{id}_A)"] & & F(A \otimes A) \arrow[dr, "F(m_A)"] & \\
& F(A) \otimes' F(A) \arrow[r, "F^{(2)}_{A,A}"] \arrow[drrr, bend right, "m_{F(A)}"'] & F(A \otimes A) \arrow[drr, "F(m_A)"] \arrow[drr, phantom, shift right=4, "(9)"] & F(A \otimes A) \arrow[l, "\hat{F}_{A \otimes A}"'] \arrow[r, "F(m_A)"] \arrow[dr, phantom, "(10)"] & F(A) \arrow[d, "\hat{F}_A"] & F(A) \arrow[l, "F(\phi_A)"'] \arrow[dl, "\phi_{F(A)}"] \arrow[dl, phantom, shift right=4, "(11)"] \\
& & & & F(A) &
\end{tikzcd}$$

Figure A.3 : $F(A)$ satisfies Definition 4.2.1(b)(ii).

It remains to show that morphisms of extended Frobenius algebras are also preserved. Specifically, if $f : (A, m_A, u_A, \Delta_A, \varepsilon_A, \phi_A, \theta_A) \to (B, m_B, u_B, \Delta_B, \varepsilon_B, \phi_B, \theta_B)$ is a morphism of extended Frobenius algebras in $\mathcal{C}$, then $F(f) : F(A) \to F(B)$ is a morphism of extended Frobenius algebras in $\mathcal{C}'$. By Proposition 2.5.4(c), $F(f)$ is a morphism of Frobenius algebras in $\mathcal{C}'$, so it suffices to verify that $F(f)\,\phi_{F(A)} = \phi_{F(B)}\,F(f)$ and $F(f)\,\theta_{F(A)} = \theta_{F(B)}$ in $\mathcal{C}'$. The first equation follows from Figure A.4, where regions (1) and (4) commute by definition of $\phi_{F(-)}$, (2) by naturality of $\widehat{F}$, and (3) because f is a morphism of extended Frobenius algebras in $\mathcal{C}$. For the second equation, observe that regions (1) and (3) in Figure A.5 commute by definition of $\theta_{F(-)}$, and (2) commutes because f is a morphism of extended Frobenius algebras in $\mathcal{C}$. $\square$

$$\begin{tikzcd}
F(A) \arrow[r, "\hat{F}_A"] \arrow[rr, bend left, "\phi_{F(A)}"] \arrow[rr, phantom, bend left=15, "(1)"] \arrow[d, "F(f)"] \arrow[dr, phantom, "(2)"] & F(A) \arrow[r, "F(\phi_A)"] \arrow[d, "F(f)"] \arrow[dr, phantom, "(3)"] & F(A) \arrow[d, "F(f)"] \\
F(B) \arrow[r, "\hat{F}_B"'] \arrow[rr, bend right, "\phi_{F(B)}"'] \arrow[rr, phantom, bend right=15, "(4)"] & F(B) \arrow[r, "F(\phi_B)"'] & F(B)
\end{tikzcd}$$

Figure A.4 : $F(f)$ respects ϕ.

$$\begin{tikzcd}
& & F(A) \arrow[dd, "F(f)"] \\
\mathbb{1}' \arrow[r, "\check{F}"] \arrow[urr, bend left, "\theta_{F(A)}"] \arrow[urr, phantom, "(1)"] \arrow[drr, bend right, "\theta_{F(B)}"'] \arrow[drr, phantom, "(3)"] & F(\mathbb{1}) \arrow[ur, "F(\theta_A)"] \arrow[dr, "F(\theta_B)"'] \arrow[r, phantom, "(2)"] & \\
& & F(B)
\end{tikzcd}$$

Figure A.5 : $F(f)$ respects θ.

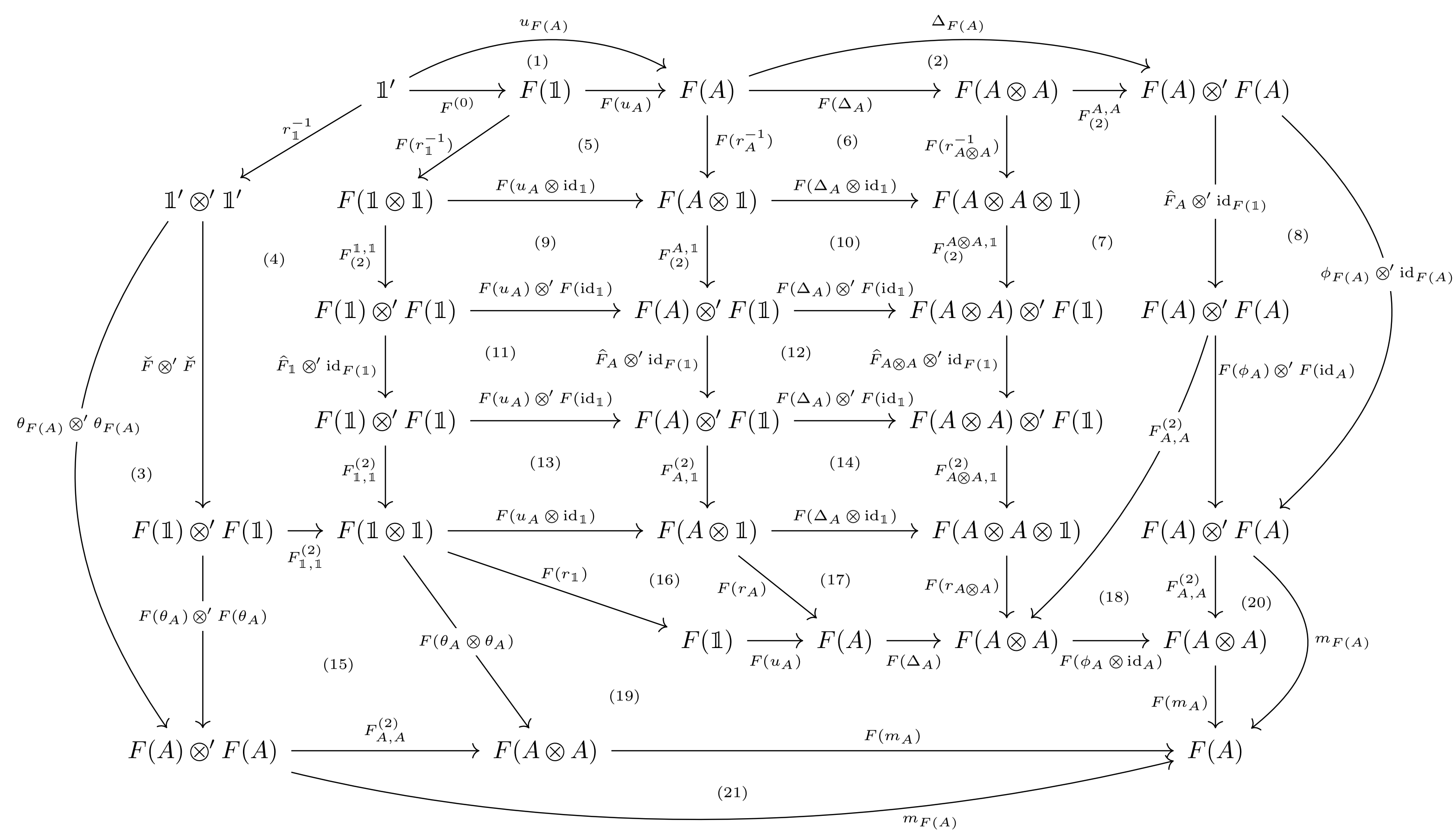

Figure A.6 : $F(A)$ satisfies Definition 4.2.1(b)(iii).

A.2 Proof of Theorem 4.3.5

Let

$$(F, F^{(2)}, F^{(0)}, F_{(2)}, F_{(0)}, \widehat{F}, \check{F}) : (\mathcal{C}, \otimes, \mathbb{1}) \to (\mathcal{C}', \otimes', \mathbb{1}');$$

$$(G, G^{(2)}, G^{(0)}, G_{(2)}, G_{(0)}, \widehat{G}, \check{G}) : (\mathcal{C}', \otimes', \mathbb{1}') \to (\mathcal{C}'', \otimes'', \mathbb{1}'')$$

be two extended Frobenius monoidal functors. To show that the composition

$$GF : (\mathcal{C}, \otimes, \mathbb{1}) \to (\mathcal{C}'', \otimes'', \mathbb{1}'')$$

admits the structure of an extended Frobenius monoidal functor, let $(GF)^{(2)}$, $(GF)^{(0)}$, $(GF)_{(2)}$, and $(GF)_{(0)}$ be as in Proposition 2.5.5(a,b). Proposition 2.5.5(c) gives that this makes GF into a Frobenius monoidal functor. Now, define $\widehat{GF} : GF \Rightarrow GF$ by $\widehat{GF}_X := G(\widehat{F}_X) \circ \widehat{G}_{F(X)}$ for all $X \in \mathcal{C}$, and define $\widecheck{GF} := G(\check{F}) \circ \check{G} : \mathbb{1}'' \to GF(\mathbb{1})$. We need to show that $\widehat{GF}$ and $\widecheck{GF}$ extend the Frobenius monoidal structure on GF.

Note first that the composition of (co)monoidal natural transformations is again (co)monoidal, so $\widehat{GF}$ is simultaneously a monoidal and comonoidal natural transformation. So, Definition 4.3.1(a) holds for GF.

That Definition 4.3.1(b) is satisfied by GF follows from commutativity of the diagram in Figure A.9: regions (1), (2), (8), (18), (25), and (26) commute by definition, (4)-(6), (9)-(17), and (19)-(23) by naturality, (3) and (24) by Definition 4.3.1(b) for G and F respectively, and (7) by Definition 4.3.1(c)(iii) for G.

To see that Definition 4.3.1(c)(i) holds for GF, see Figure A.7. Regions (1) and (3) commute by definition of $\widehat{GF}$, and regions (2) and (4) commute by Definition 4.3.1(c)(i) for F and G respectively.

$$\begin{tikzcd}
 & & GF(X) \arrow[dr, "G(\widehat{F}_X)"] \arrow[drr, bend left, "\widehat{GF}_X"] & & \\
GF(X) \arrow[r, "\widehat{G}_{F(X)}"] \arrow[urr, bend left, "\widehat{GF}_X"] \arrow[rrrr, bend right, "\mathrm{id}_{GF(X)}"'] & GF(X) \arrow[ur, "G(\widehat{F}_X)"] \arrow[rr, "\mathrm{id}_{GF(X)}"] & & GF(X) \arrow[r, "\widehat{G}_{F(X)}"] & GF(X)
\end{tikzcd}$$

Figure A.7 : GF satisfies Definition 4.3.1(c)(i).

Next, GF satisfies Definition 4.3.1(c)(ii) by Figure A.8: regions (1), (4), (7), (8), and (11) commute by definition; (3), (5), (6), and (9) by naturality; and (2) and (10) by Definition 4.3.1(c)(ii) for G and F respectively.

Finally, Definition 4.3.1(c)(iii) is satisfied by GF via Figure A.10: regions (1), (2), (5), (6), (25), and (26) commute by definition; (4), (7)-(11), and (14)-(24) by naturality; and (3), (12), and (13) by Definition 4.3.1(c)(iii) for F and G respectively.

This concludes the proof of Theorem 4.3.5 □

$$
\begin{tikzcd}
\mathbb{1}'' \otimes'' GF(X) \arrow[rr, "\check{G} \otimes'' GF(\mathrm{id}_X)"] \arrow[rrr, bend left, "\widetilde{GF} \otimes'' GF(\mathrm{id}_X)"] \arrow[rrr, phantom, bend left=15, "(1)"] \arrow[ddd, "\widetilde{GF} \otimes'' GF(\mathrm{id}_X)"'] \arrow[ddr, "\check{G} \otimes'' GF(\mathrm{id}_X)"] & & G(\mathbb{1}') \otimes'' GF(X) \arrow[r, "G(\check{F}) \otimes'' GF(\mathrm{id}_X)"'] \arrow[d, "G^{(2)}_{\mathbb{1}', F(X)}"'] \arrow[dr, phantom, "(3)"] & GF(\mathbb{1}) \otimes'' GF(X) \arrow[r, "(GF)^{(2)}_{\mathbb{1}, X}"] \arrow[d, "G^{(2)}_{F(\mathbb{1}), F(X)}"] \arrow[dr, phantom, "(4)"] & GF(\mathbb{1} \otimes X) \arrow[dd, "\hat{G}_{F(\mathbb{1} \otimes X)}"'] \arrow[ddd, bend left, "\widehat{GF}_{\mathbb{1} \otimes X}"] \arrow[ddd, phantom, bend left=20, "(7)"] \\
 & (2) & G(\mathbb{1}' \otimes' F(X)) \arrow[r, "G(\check{F} \otimes' F(\mathrm{id}_X))"'] \arrow[d, "\hat{G}_{\mathbb{1}' \otimes' F(X)}"'] \arrow[dr, phantom, "(5)"] & G(F(\mathbb{1}) \otimes' F(X)) \arrow[ur, bend right, "G(F^{(2)}_{\mathbb{1}, X})"'] \arrow[d, "\hat{G}_{F(\mathbb{1}) \otimes' F(X)}"] \arrow[dr, phantom, "(6)"] & \\
(8) & G(\mathbb{1}') \otimes'' GF(X) \arrow[r, "G^{(2)}_{\mathbb{1}', F(X)}"'] \arrow[dl, "G(\check{F}) \otimes'' GF(\mathrm{id}_X)"] \arrow[d, phantom, "(9)"] & G(\mathbb{1}' \otimes' F(X)) \arrow[r, "G(\check{F} \otimes' F(\mathrm{id}_X))"'] \arrow[d, "G(\check{F} \otimes' F(\mathrm{id}_X))"] & G(F(\mathbb{1}) \otimes' F(X)) \arrow[r, "G(F^{(2)}_{\mathbb{1}, X})"'] \arrow[d, phantom, "(10)"] & GF(\mathbb{1} \otimes X) \arrow[d, "G(\hat{F}_{\mathbb{1} \otimes X})"'] \\
GF(\mathbb{1}) \otimes'' GF(X) \arrow[rr, "G^{(2)}_{F(\mathbb{1}), F(X)}" description] \arrow[rrrr, bend right, "(GF)^{(2)}_{\mathbb{1}, X}"'] \arrow[rrrr, phantom, bend right=15, "(11)"] & & G(F(\mathbb{1}) \otimes' F(X)) \arrow[rr, "G(F^{(2)}_{\mathbb{1}, X})" description] & & GF(\mathbb{1} \otimes X)
\end{tikzcd}
$$

Figure A.8 : GF satisfies Definition 4.3.1(c)(ii).

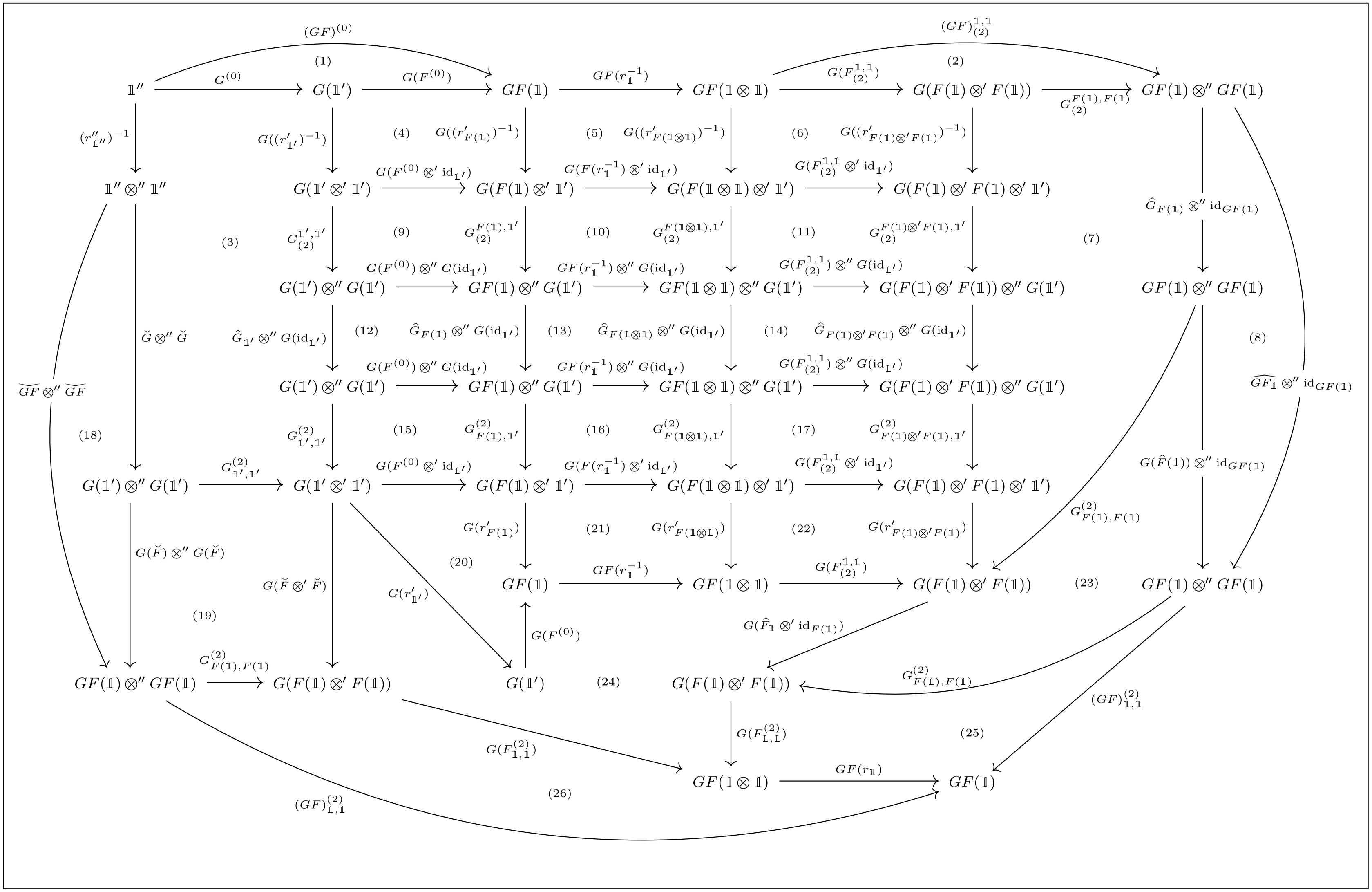

Figure A.9 : GF satisfies Definition 4.3.1(b).

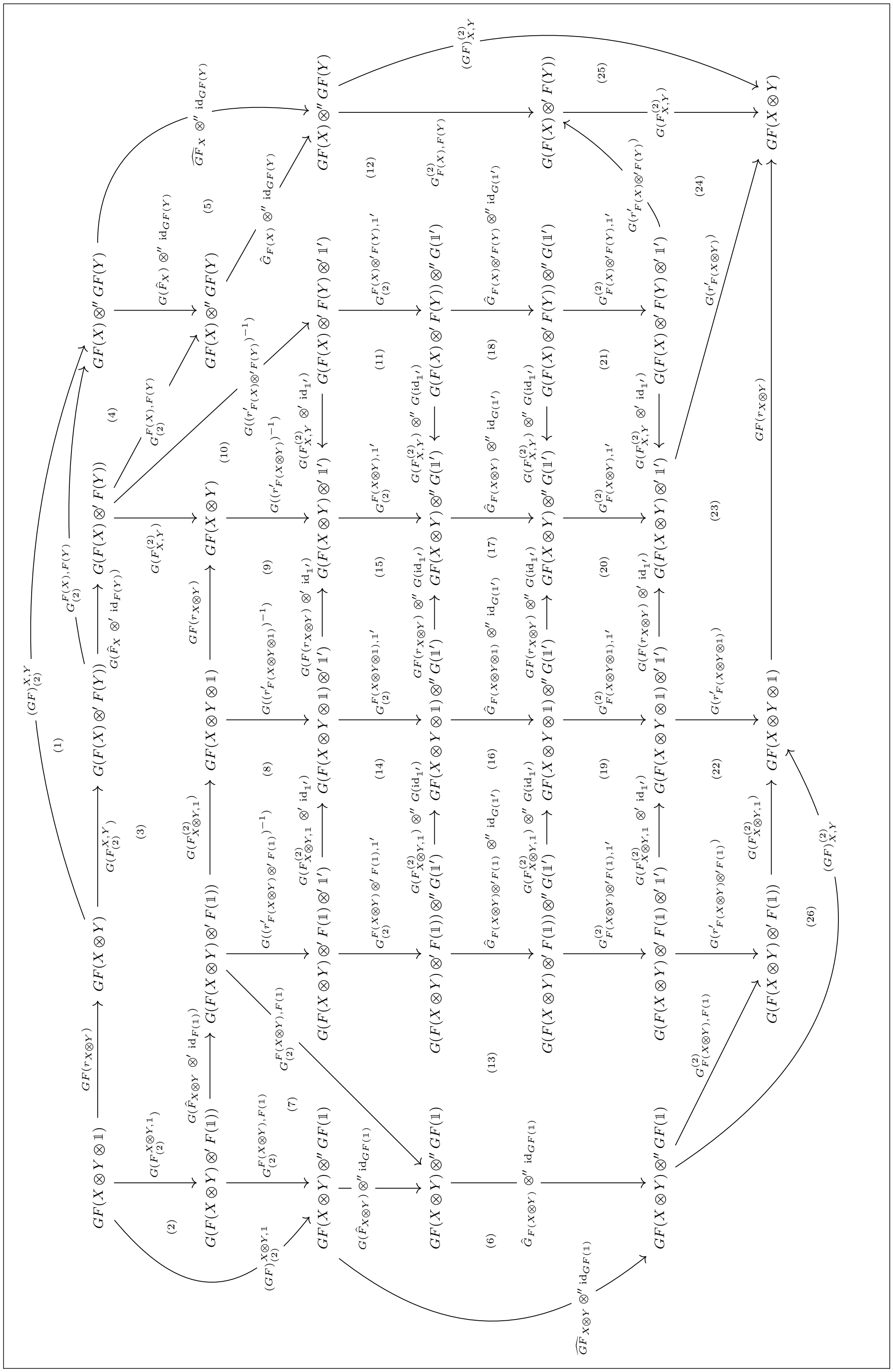

Figure A.10 : GF satisfies Definition 4.3.1(c)(iii).

Appendix B

Graphical proof that integral Hopf implies Frobenius

In this appendix, which is connected to Section 4.4 and also comes from the paper [CKQW24], we give a graphical proof of Proposition 4.4.3, showing that an integral Hopf algebra in a symmetric monoidal category $\mathcal{C}$ is a Frobenius algebra in $\mathcal{C}$. This result can be found in [FS10, Appendix A.2], and the graphical proof is due to Harshit Yadav. Recall axioms (S1)–(S5) from Figure 2.4 in §2.2.4.

B.1 Diagrams for integral Hopf algebras

Recall from Definition 2.3.3 that a Hopf algebra in $\mathcal{C}$ is an object $H \in \mathcal{C}$ equipped with morphisms $m : H \otimes H \to H$, $u : \mathbb{1} \to H$, $\underline{\Delta} : H \to H \otimes H$, $\underline{\varepsilon} : H \to \mathbb{1}$, $S : H \to H$ satisfying specific conditions. In this section, we consider Hopf algebras equipped with a normalized (co)integral pair, given by morphisms $\Lambda : \mathbb{1} \to H$ and $\lambda : H \to \mathbb{1}$ which satisfy a number of axioms (see Definition 4.4.1). Note that this also means that the antipode S is invertible, with inverse S^{-1} (see Remark 4.4.2).

Graphical representations of the structure morphisms for a Hopf algebra with invertible antipode are given in Figure B.1, and the axioms they satisfy are in Figures B.2 and B.3. The normalized integral and a cointegral of a Hopf algebra H are given graphically in Figure B.4, together with the axioms they satisfy.

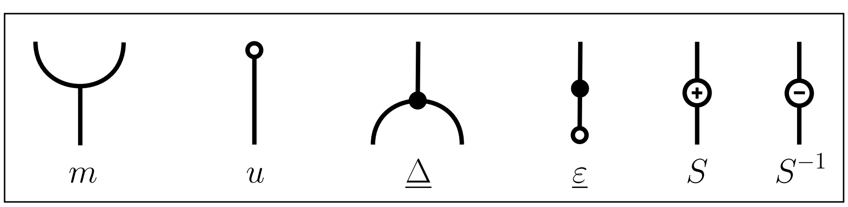

Figure B.1 : Structure morphisms for a Hopf algebra in $\mathcal{C}$.

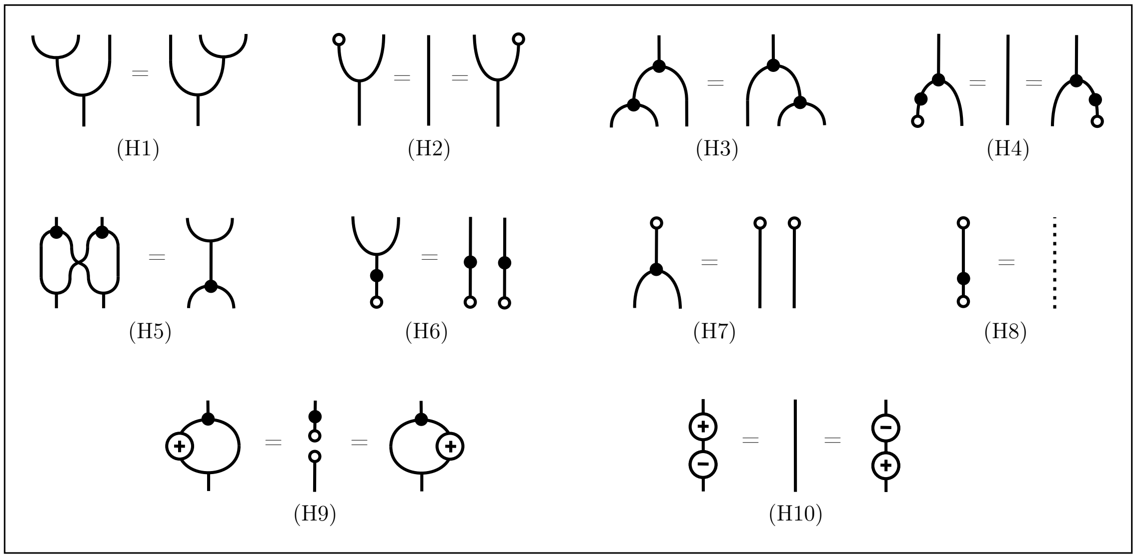

Figure B.2 : Axioms for a Hopf algebra with invertible antipode in $\mathcal{C}$.

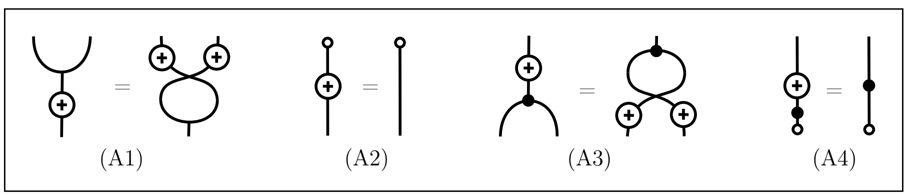

Figure B.3 : Identities for a Hopf algebra in $\mathcal{C}$.

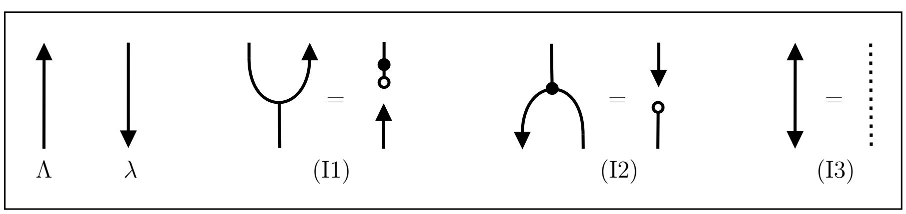

Figure B.4 : Normalized (co)integral for a Hopf algebra in $\mathcal{C}$.

Lemma B.1.1. *We have the following identities.*

(a) $(m \otimes S)(\mathrm{id}_H \otimes \underline{\Delta}\Lambda) = (\mathrm{id}_H \otimes m)(\mathrm{id}_H \otimes S \otimes \mathrm{id}_H)(\underline{\Delta}\, m \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \Lambda \otimes \mathrm{id}_H)\underline{\Delta}$.

(b) $\lambda S \Lambda = \mathrm{id}_{\mathbb{1}}$.

Proof. Part (a) is proved in Figure B.5, and part (b) is proved in Figure B.6. References to Figures 2.4, B.2, B.3, and B.4 are made throughout. $\square$

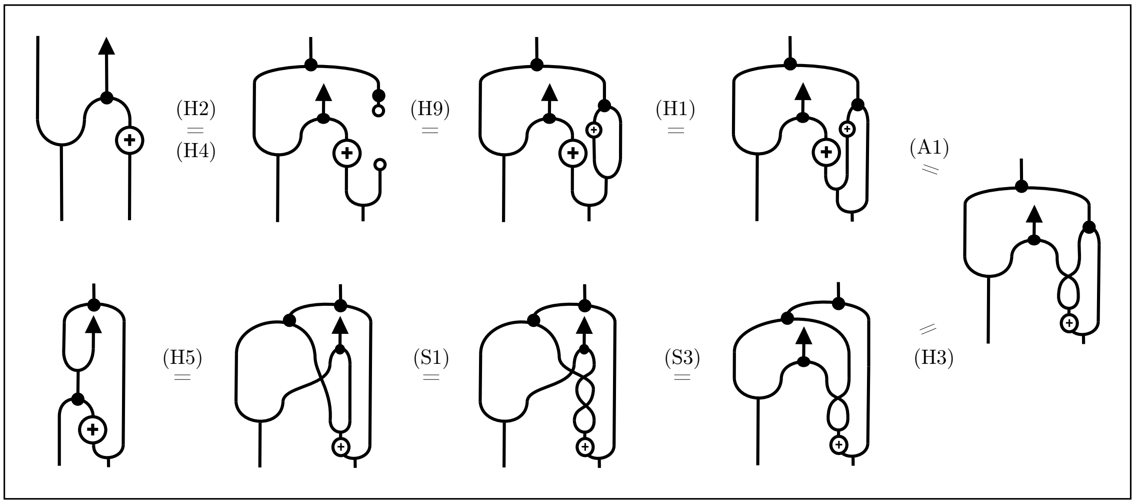

Figure B.5 : Proof of Lemma B.1.1(a).

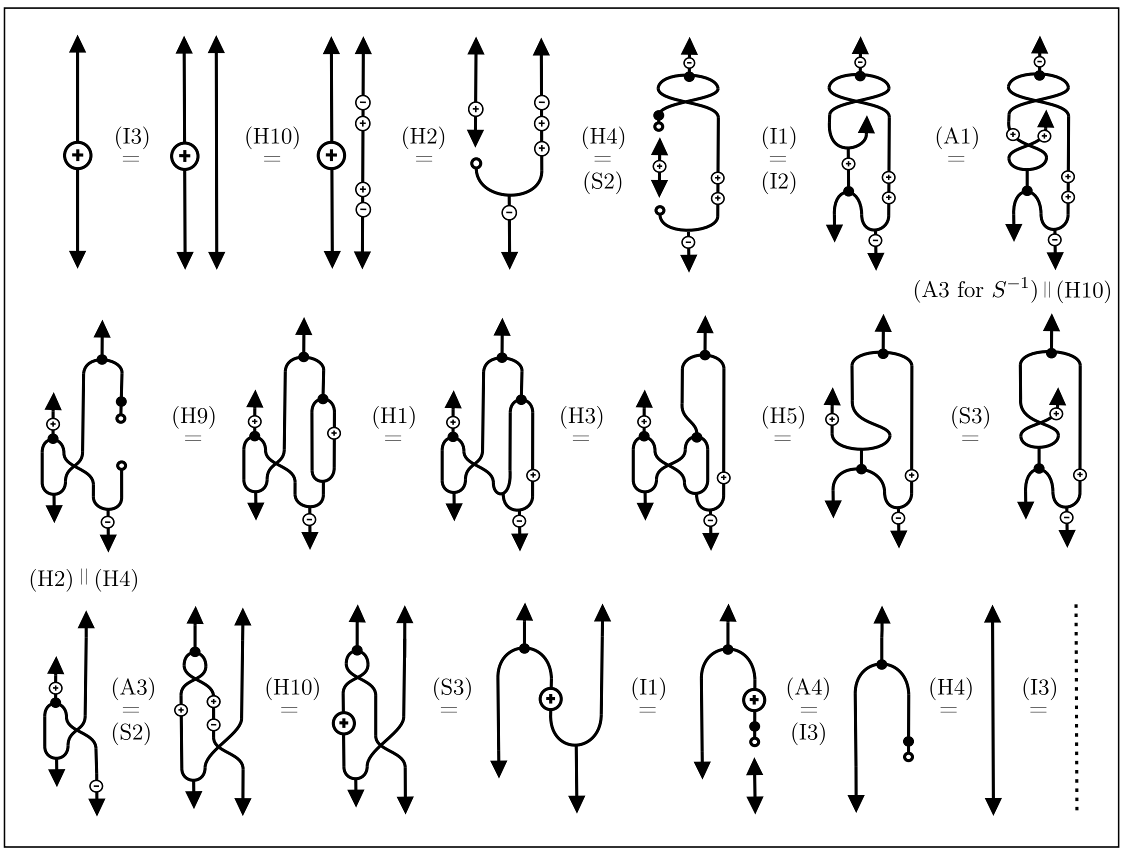

Figure B.6 : Proof of Lemma B.1.1(b).

B.2 Proof of Proposition 4.4.3

We aim to show that

$$\Psi : \mathsf{IntHopfAlg}(\mathcal{C}) \to \mathsf{FrobAlg}(\mathcal{C})$$
$$(H, m, u, \underline{\Delta}, \underline{\varepsilon}, S, \Lambda, \lambda) \mapsto (H, m, u, \Delta := (m \otimes S)(\mathrm{id}_H \otimes \underline{\Delta}\Lambda),\ \varepsilon := \lambda)$$

is a well-defined functor, which acts as the identity on morphisms.

For the assignment of objects under the functor Ψ, we depict graphically the coproduct Δ and counit ε in Figure B.7. Coassociativity is verified in Figure B.8, counitality is in Figure B.9, and the Frobenius laws are established in Figure B.10. References to Figures B.2–B.6 are made throughout.

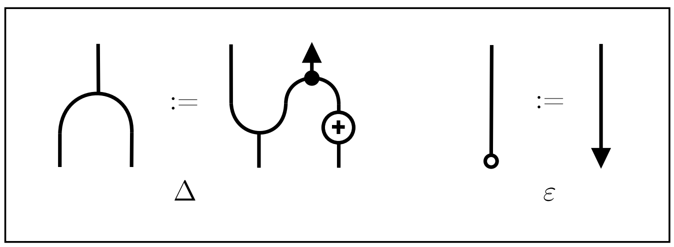

Figure B.7 : Coproduct and counit for the Frobenius-from-Hopf structure in $\mathcal{C}$.

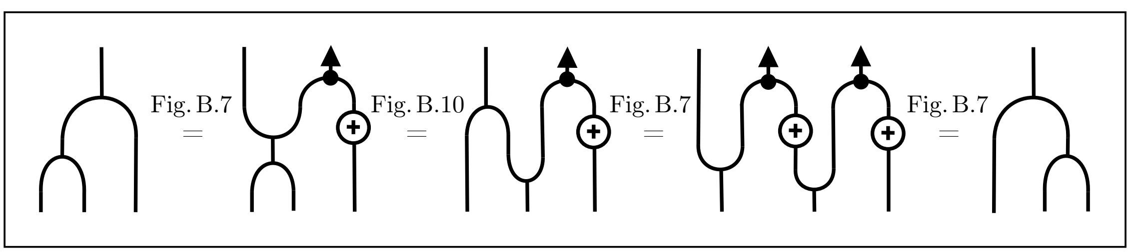

Figure B.8 : Proof of coassociativity for the Frobenius-from-Hopf structure in $\mathcal{C}$.

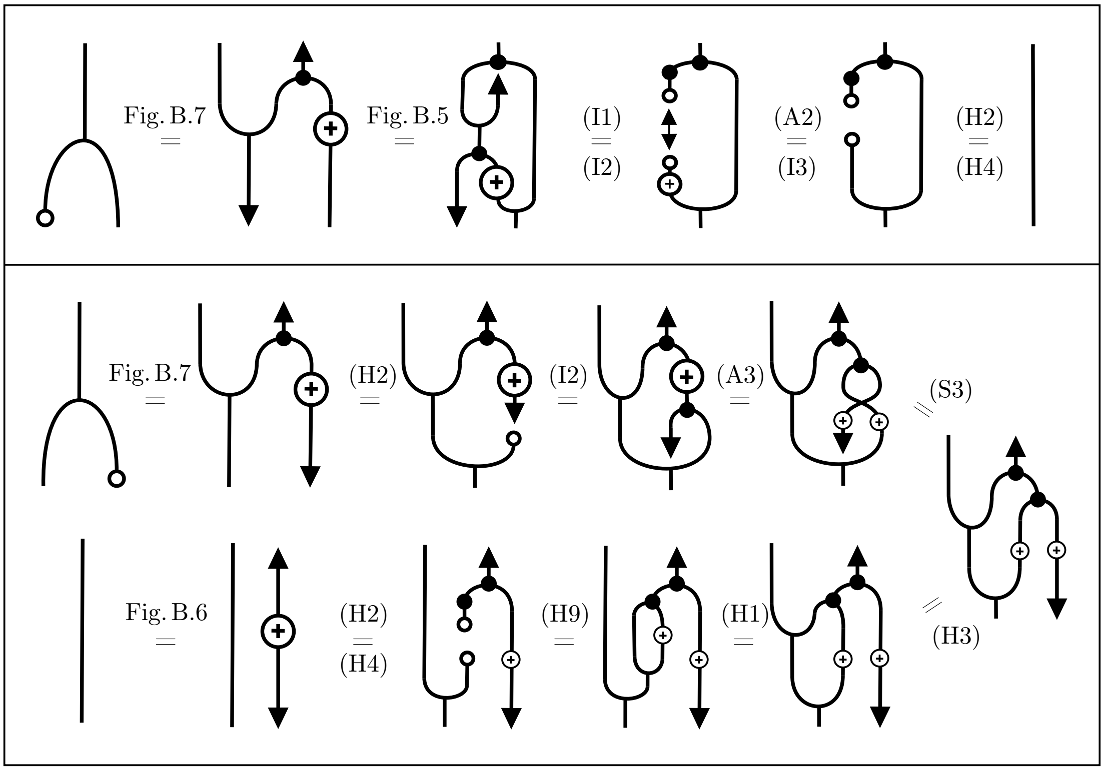

Figure B.9 : Proof of counitality for the Frobenius-from-Hopf structure in $\mathcal{C}$.

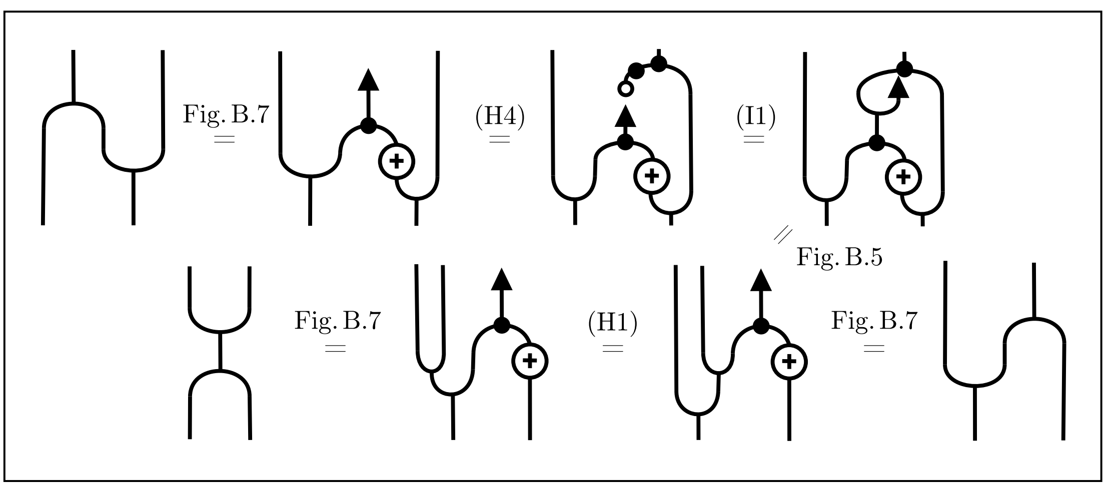

Figure B.10 : Proof of Frobenius laws for the Frobenius-from-Hopf structure in $\mathcal{C}$.

For the assignment of morphisms under Ψ, take a morphism of integral Hopf algebras

$$f : (H, m_H, u_H, \underline{\Delta}_H, \underline{\varepsilon}_H, \Lambda_H, \lambda_H) \to (K, m_K, u_K, \underline{\Delta}_K, \underline{\varepsilon}_K, \Lambda_K, \lambda_K).$$

We will verify that $\Psi(f) := f : (H, m_H, u_H, \Delta_H, \varepsilon_H) \to (K, m_K, u_K, \Delta_K, \varepsilon_K)$ is a morphism of Frobenius algebras. We have multiplicativity and unitality for free, since the Hopf multiplications and units on H and K are the same as the Frobenius multiplications and units on H and K.

Next, because the Frobenius counits of H and K are given by $\varepsilon_H = \lambda_H$ and $\varepsilon_K = \lambda_K$, we get Frobenius counitality immediately from the fact that f is compatible with the cointegrals of H and K.

Finally, we have that Frobenius comultiplicativity holds via the commutative diagram in Figure B.11. Specifically, the regions (2) and (4) commute by definition of Δ_H and Δ_K, respectively. Region (1) commutes because f is compatible with the integrals of H and K, region (5) because f is an algebra map and is compatible with the antipodes of H and K, and region (3) because f is a coalgebra map between the Hopf algebras H and K. $\square$

$$\begin{tikzcd}
H \arrow[rrr, "f"] \arrow[dr, "\mathrm{id}_H \otimes \Lambda_H"] \arrow[ddd, "\Delta_H"'] & & & K \arrow[dl, "\mathrm{id}_K \otimes \Lambda_K"'] \arrow[ddd, "\Delta_K"] \\
& H \otimes H \arrow[r, "f \otimes f"] \arrow[d, "\mathrm{id}_H \otimes \underline{\Delta}_H"] & K \otimes K \arrow[d, "\mathrm{id}_K \otimes \underline{\Delta}_K"] & \\
& H \otimes H \otimes H \arrow[r, "f \otimes f \otimes f"'] \arrow[dl, "m_H \otimes S_H"] & K \otimes K \otimes K \arrow[dr, "m_K \otimes S_K"'] & \\
H \otimes H \arrow[rrr, "f \otimes f"'] & & & K \otimes K
\arrow[from=1-2, to=2-3, phantom, "(1)"]
\arrow[from=2-1, to=3-2, phantom, "(2)"]
\arrow[from=2-2, to=3-3, phantom, "(3)"]
\arrow[from=2-3, to=3-4, phantom, "(4)"]
\arrow[from=3-2, to=4-3, phantom, "(5)"]
\end{tikzcd}$$

Figure B.11 : Frobenius comultiplicativity for $\Psi(f) := f$.

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