Quantum Symmetries

of algebras related to quivers

Rice U. AGNT seminar
Feb. 16, 2022

≡ Questions welcome throughout ≡

I. Background on Hopf stuff
II. Background on quiver stuff
III. Results in explicit form
IV. Results in (tensor) categorical form

I. Hopf stuff

$\mathbb{k}$: field of characteristic 0

Def: A Hopf algebra is an associative $\mathbb{k}$-algebra H with additional structure:

$\Delta : H \longrightarrow H \otimes H$ (comultiplication)

$\varepsilon : H \longrightarrow \mathbb{k}$ (counit)

satisfying some axioms which are not important in this talk.

Intuition: For a noncommutative ring R and $M, N \in R\text{-Mod}$, there is no canonical way to form "$M \otimes N$" in $R\text{-Mod}$.

The extra structure of a Hopf algebra fixes this:

- $M, N \in H\text{-Mod} \xRightarrow{\text{use } \Delta} M \otimes_{\mathbb{k}} N \in H\text{-Mod}$
- $\xRightarrow{\text{use } \varepsilon} \mathbb{k} \in H\text{-Mod}$

Ex: G : group, $H = \Bbbk G$ group algebra.

$$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \quad \forall g \in G.$$

- Then $M, N \in \Bbbk G\text{-Mod} \Rightarrow M \otimes_{\Bbbk} N \in \Bbbk G\text{-Mod}$ via $g \cdot (m \otimes n) = (g \cdot m) \otimes (g \cdot n)$
- G (or $\Bbbk G$) acts on $\Bbbk$ "trivially" via
$$g \cdot \alpha = \varepsilon(g)\alpha = \alpha \qquad \forall g \in G \quad \forall \alpha \in \Bbbk$$

Ex: $\mathfrak{g}$: Lie algebra, $H = U(\mathfrak{g})$ envel. algebra

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \varepsilon(x) = 0 \qquad \forall x \in \mathfrak{g}$$

- Then $M, N \in U(\mathfrak{g})\text{-Mod} \Rightarrow M \otimes_{\Bbbk} N \in U(\mathfrak{g})\text{-Mod}$ via $x \cdot (m \otimes n) = m \otimes (x \cdot n) + (x \cdot m) \otimes n$
- $\mathfrak{g}$ (or $U(\mathfrak{g})$) acts on $\Bbbk$ "trivially" via
$$x \cdot \alpha = \varepsilon(x)\alpha = 0 \qquad \forall x \in \mathfrak{g}$$

- "Symmetries" of an algebra can be formalized by having a group act on that algebra.

<u>Ex.</u> Symmetric group S_n acts on poly ring $\mathbb{k}[x_1, x_2, \ldots, x_n]$.

- This generalizes nicely to Hopf algebras:

<u>Def</u>: Let A be an assoc. $\mathbb{k}$-alg. An action of a Hopf algebra H on A is an:

- H-Module structure on A (the action, how $h \cdot a \in A$ $\forall h \in H, a \in A$)

↳ which "respects" multiplication: $A \otimes A \to A$ is H-Mod. homom.

↳ acts trivially on scalars: $\mathbb{k} \to A$, $1_{\mathbb{k}} \mapsto 1_A$ is H-Mod hom

Main takeaway of this section:
"Extra structure" on a $\mathbb{k}$-algebra A can often be formalized as a Hopf algebra action.

Ex: Action of group G on A by automorphisms

$\Updownarrow$ equiv. data

Hopf Action of $\mathbb{k}G$ on A

Ex: Action of Lie algebra $\mathfrak{g}$ on A by derivations

$\Updownarrow$ equiv data

Hopf action of $U(\mathfrak{g})$ on A

Ex: Grading of A by finite group G

$\Updownarrow$ equiv. data

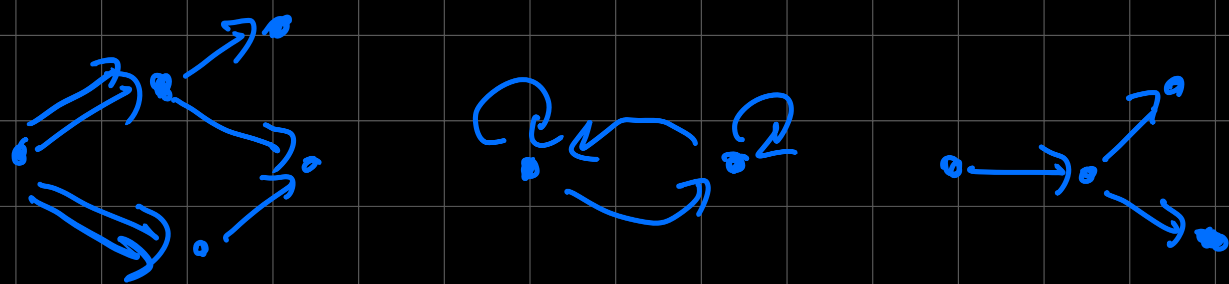
Hopf action of $(\mathbb{k}G)^*$ on A

One can take fancier Hopf algebras like quantum groups $U_q(\mathfrak{g})$ ...

"Quantum symmetries" of an algebra in this talk will mean an action of a Hopf algebra.

II. Quiver stuff

Def: A <u>quiver</u> Q is a (finite) directed graph, allowing loops, parallel edges, etc.



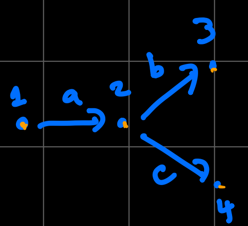
They are interesting in algebra / rep theory because:

Def: The <u>path algebra</u> $\Bbbk Q$ of a quiver Q is:

- vector space with basis $\{p \mid p \text{ a path in } Q \text{ of length} \geq 0\}$
- (associative) multiplication

$$p \cdot q = \begin{cases} pq & \text{if } \overset{p}{\leadsto}\cdot\overset{q}{\leadsto} \text{ is path in } Q \\ 0 & \text{otherwise} \end{cases}$$

Ex:



$\dim_{\Bbbk} \Bbbk Q = 9$ with basis

$$\left\{ \overbrace{e_1, e_2, e_3, e_4}^{\text{length 0 paths}},\ a, b, c,\ ab, ac \right\}$$

$a \cdot b = ab$, $b \cdot c = 0$,
$a \cdot e_2 = a$, $a \cdot e_1 = 0$, $e_1^2 = e_1$ ...

These are more than toy examples.
↳ In fact, universal in following sense:

<u>Theorem</u> (Gabriel, 60s). Let A be any $\Bbbk$-algebra of finite $\Bbbk$-dimension, & assume $\Bbbk = \overline{\Bbbk}$. Then $\exists$ quiver Q & 2-sided ideal $I \subset \Bbbk Q$ s.t. $A\text{-Mod} \approx (\Bbbk Q / I)\text{-Mod}$.

<u>Intuition</u>: Quiver path algebras play a role in the noncommutative fin. dim. world analogous to what polynomial rings play in affine alg. geom.

Reformulation of $\Bbbk Q$ without graphs:
For a ring S & S-S-bimodule E, form:
<u>tensor algebra</u>: $T_S(E) = S \oplus E \oplus (E \otimes_S E) \oplus (E \otimes_S E \otimes_S E) \oplus \cdots$

Given Q with n vertices, let $S \simeq \Bbbk^n \subset \Bbbk Q$ be the subalgebra of length 0 paths.
Let $E \subset \Bbbk Q$ be the $\Bbbk$-span of the arrows.
$\Longrightarrow \Bbbk Q \simeq T_S(E)$ from defs above.

III. Results in explicit form

Many Hopf algebras are generated by subalgebras isomorphic to the following:

Def: Let $q \in \Bbbk \setminus \{-1,0,1\}$ & define a $\Bbbk$-algebra $U_q(\mathfrak{b})$ generated by g, g^{-1}, x subject to:

$$gg^{-1} = g^{-1}g = 1, \quad gxg^{-1} = qx.$$

It is a Hopf algebra via:

$$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g.$$
$$\varepsilon(g) = 1 \qquad \varepsilon(x) = 0.$$

If q is primitive r^{th} root of 1, $n \in \mathbb{Z}_{>0}r$, the generalized Taft algebra is:

$$T(r,n) := U_q(\mathfrak{b}) / \langle g^n - 1, x^r \rangle$$

a fin. dim. Hopf algebra, & $T(n) := T(n,n)$ is a Taft algebra.

$U_q(\mathfrak{sl}_2)$ & its quotient $u_q(\mathfrak{sl}_2)$ are generated by subalgebras isomorphic to $U_q(\mathfrak{b})$ & $T(n)$...

- The study of Hopf algebra actions on quivers was initiated in [K-Walton, '16] for actions of $T(n)$ and $u_q(\mathfrak{sl}_2)$ on quivers without loops nor parallel arrows.
- Extended to arbitrary quivers for $T(n)$ in part of Ph.D. thesis of Ana Berrizbeitia, 2018.
- Extended to actions of $U_q(\mathfrak{b})$ & $T(r,n)$ as well as $U_q(\mathfrak{sl}_2)$ & $u_q(\mathfrak{sl}_2)$ in [K-Oswald, '21, §3].

Here's what describing Hopf actions in "explicit form" means:

<u>Thm</u> [K-O]. Let Q be a quiver with vertex set Q_0 and arrow set Q_1. Then the following data determines an action of $U_q(\mathfrak{b})$ on $\Bbbk Q$, & every (filtered) action is of this form:

(i) A permutation action of $\langle g\rangle \cong \mathbb{Z}$ on the set Q_0

(ii) A collection of scalars $(\gamma_i \in \Bbbk)_{i \in Q_0}$ s.t.

$$\gamma_{g \cdot i} = q^{-1}\gamma_i \qquad \forall i \in Q_0$$

(iii) A linear representation of $\langle g\rangle \cong \mathbb{Z}$ on $\Bbbk Q_1$ which preserves sources & targets

(iv) A $\Bbbk$-linear map $\sigma : \Bbbk Q_0 \oplus \Bbbk Q_1 \to \Bbbk Q_0 \oplus \Bbbk Q_1$ s.t. $\sigma(\Bbbk Q_0) = 0$, $\sigma(g \cdot a) = q^{-1} g \cdot \sigma(a)$ $\forall a \in Q_1$, and for any $a \in Q_1$, $\sigma(a)$ is a linear combination of arrows from source(a) to $g \cdot$target(a)

$$\begin{tikzcd} \mathrm{source}(a) \arrow[d, "a"'] \arrow[drr, "\sigma(a)"] & & g\cdot \mathrm{source}(a) \arrow[d, "g\cdot a"] \\ \mathrm{target}(a) & & g\cdot \mathrm{target}(a) \end{tikzcd}$$

The action from this data is explicitly:

$$x \cdot e_i = \gamma_i e_i - \gamma_i q^{-1} e_{g \cdot i}$$

$$x \cdot a = \gamma_{\mathrm{target}(a)}\, a - \gamma_{\mathrm{source}(a)}\, q^{-1} (g \cdot a) + \sigma(a)$$

$\forall i \in Q_0$ and $\forall a \in Q_1$, extending uniquely to the rest of $\Bbbk Q$.

- We give explicit criteria for when an action descends to the quotient $T(r,n)$, & analogous results for $U_q(\mathfrak{sl}_2)$ & $u_q(\mathfrak{sl}_2)$.
- Summary: explicit descriptions are satisfying & useful for creating/analyzing examples.

↳ But they don't give global insight...

Example Question: Given algebras A, A' with actions of Hopf algebra H, does there exist $A \cong A'$ as algebras AND H-modules? List of such isoclasses?

IV. Results in $\otimes$-categorical form

More abstract view gives approach to the last question.

"Def." A tensor category $\mathcal{C}$ is formed by taking a category of reps of a fin. dim.* algebra & endowing it abstractly with an additional operation $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$.

Ex. $\mathcal{C} = \text{rep}(H)$ for a Hopf algebra H.

[Etingof–K–Walton, '21] reframed the study of actions of Hopf algebras on path algebras via:
$\Bbbk Q \cong T_S(E)$ where $S \cong \Bbbk^m$ & E an S-S-bimod.

It works more generally for:

- S semisimple (product of matrix algebras over $\Bbbk$)

In this language, a graded action of H on $T_S(E)$ is equivalent data to the pair:

(i) S an "algebra in rep(H)"
i.e.
$$\begin{aligned} S \otimes S &\longrightarrow S \\ a \otimes b &\longmapsto ab \end{aligned} \quad \text{and} \quad \begin{aligned} \Bbbk &\longrightarrow S \\ 1 &\longmapsto 1_S \end{aligned} \quad \text{are morphisms in } \mathcal{C} = \cancel{\text{rep}}(H)$$

(ii) E is an "S-S-bimod in rep(H)"
i.e. action map $S \otimes E \otimes S \longrightarrow E$ is morphism in $\mathcal{C} = \cancel{\text{rep}}(H)$

& more generally ...

What do we get from this new perspective?

☆ Every such S, E have unique decompositions in $\mathcal{C}$:

$$S = S_1 \oplus \cdots \oplus S_m \quad , \quad E = E_1 \oplus \cdots \oplus E_r$$

into indecomposables (essentially Krull-Schmidt thm).

$\Longrightarrow$ Indecomposable algebras & indecomposable bimodules in $\mathcal{C}$ are the "building blocks" which uniquely construct an arbitrary $T_S(E)$ in $\mathcal{C} = \mathrm{rep}(H)$.

☆ "Classify" these first!

There are a few general results related to this in [EKW], but overall few explicit answers. A brief survey of works where these building blocks are described or classified:
(probably incomplete!)

① [Etingof-Ostrik] exact algebras S for $H = T(n)$.

② [Ostrik] & [Natale] s.s. algebras S in $\mathcal{C} = \mathrm{Vec}_G^\omega$

③ [Morales-Müller-Plavnik-Ros Camacho-Tabiri-Walton] s.s. algebras in $\mathcal{C}$ a group-theoretical fusion category

④ [EKW] s.s algebras for $H = H_8$, Kac-Paljutkin alg.

(5) [K-Oswald] bimodules over algebras of the form $S = \Bbbk^m$ for $H = U_q(\mathfrak{b})$, $T(r,n)$, $U_q(\mathfrak{sl}_2)$, $u_q(\mathfrak{sl}_2)$

↳ translation of earlier theorem, implies classification of indecomposables is "wild" (intractable) problem except for $H = T(r,n)$, & gives solution there.

Last point: All these results only help understand building blocks. Understanding when $T_S(E) \cong T_{S'}(E')$ in rep(H) or $\mathcal{C}$ in general requires more research in a broader framework involving directed graph isomorphisms & conjugacy of bimodules. A helpful start is:

☆ Recent theorem of Oswald decomposes

$$E \otimes_{S_2} F \quad \text{as} \quad S_1\text{-}S_3\text{-bimodule} \quad \text{when}$$

E is S_1-S_2-bimod & F is S_2-S_3-bimod.

↳ Will help with the conjugacy part.

A final summary: