June 2, 2016 ①

US-Mexico NC alg & Rep Thy conference @ USC

No Quantum Symmetry

based on joint work with Pavel Etingof & Juan Cuadra

$k = \bar{k}$, char $k = 0$

Goal: Study symmetries of algebras = Hopf actions on algebras

Why Hopf algebras?

① **Geometry** (Aff/k) $\xleftrightarrow{\text{contravariant}}$ **Algebra** (ComAlg/k)

$G \times X \longrightarrow X$ (group action) $\qquad \mathcal{O}(X) \longrightarrow \mathcal{O}(X) \otimes \mathcal{O}(G)$ (Hopf coaction) & com alg.

w/ morphisms in Aff/k

$m: G \times G \longrightarrow G$ mult $\qquad \Delta: \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ w/

$i: G \longrightarrow G$ inversion $\qquad S: \mathcal{O}(G) \to \mathcal{O}(G)$

$e \in G$ identity object $\qquad \varepsilon: \mathcal{O}(G) \longrightarrow \mathcal{O}(G)$, $f \mapsto f(e)$

Satisfying group axioms $\qquad$ Satisfying Hopf alg axioms

$\forall \sigma \in G$

ⓐ $m(\sigma, e) = m(e, \sigma) = \sigma$ $\qquad$ ⓐ $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id$

ⓑ $m(\sigma, i(\sigma)) = m(i(\sigma), \sigma) = e$ $\qquad$ ⓑ antipode axiom

ⓒ associativity $\qquad$ ⓒ coassociativity

"quantize"

Noncom Geometry

Quantum group action on "Quantum Variety" $\qquad \mathcal{O}_q(X) \to \mathcal{O}_q(X) \otimes \mathcal{O}_q(G)$ (coaction of noncom Hopf alg)

② Similar motivation from

$U(\mathfrak{g}) \curvearrowright C^\infty(M)$

cocom Hopf alg $\qquad$ com alg

action of noncom Hopf algebra ...

But there are lots of noncom. algs that don't arise as an alteration of a com. alg **!**

⇝ So actions of arbitrary Hopf algs on arbitrary algs is an interesting subject in its own right

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Say $\boxed{\text{a Hopf algebra } H \text{ acts on an algebra } A}$ if A is an algebra in H-mod.

(A is an H-module, and m_A & u_A are H-maps)

Q: When does this boil down to classical actions?

Given class of { Hopf algebras $\mathcal{H}$ and algebras $\mathcal{A}$, say $\exists$ $\boxed{\text{No Quantum Symmetry}}$ NQS

{ fin. dim'l Hopf algs $\qquad \exists$ No finite QS NFQS

$\Uparrow$

{ semisimple Hopf algs $\qquad \exists$ No semisimple QS NSFQS

if for any $H \in \mathcal{H}$ and $A \in \mathcal{A}$: $H \curvearrowright A$ factors through action of a $\boxed{\text{cocom. Hopf alg}}$ (a classical action)

$U(\mathfrak{g}) \# kG$, $G \curvearrowright \mathfrak{g}$, char $k = 0$

group (algebra)

group (algebra)

Focus here: NSFQS results & the techniques to achieve these

I. No Quantum Symmetry when H = semisimple Hopf algs, A = commutative domains

Assume $H \curvearrowright A$ comdomain, ss

WLOG (*) action doesn't factor through action of proper quotient Hopf alg.

(**) A is finitely generated.

Want to show that H^* commutative (+ finite dim'l $\Rightarrow$ H = group algebra)

Use coideal subalgebras of H^*: A right coideal subalg B of H^* is a subalg of H^* so that $\Delta(B) \subseteq B \otimes H^*$, $\Delta(B) \subseteq H^* \otimes H^*$

Ex. $H \times A \to A$ action $\rightsquigarrow$ $A \xrightarrow{\rho} A \otimes H^*$ coaction

Take $x: A \to k$ character. Then for $\rho_x: A \xrightarrow{\rho} A \otimes H^* \xrightarrow{x \otimes id} H^*$, $\rho_x(A) =: A_x$ is a right coideal subalg of H^*

Theorem [EW] A semisimple Hopf algebra has finitely many coideal subalgebras.

By (**) Take $X = \mathrm{Spec}(A)$ & $\gamma: X \to \coprod_d \underbrace{CS_d(H^*)}$ — variety of dim d coideal subalgs of H^*

points of X $\leftrightarrow$ char. x

$x \mapsto A_x$

$d_0 = \max\{\dim_k A_x\}$ $\quad X_0 = \{x \in X \mid \dim_k A_x = d_0\}$

Consider $\gamma|_{X_0}: X_0 \longrightarrow CS_{d_0}(H^*)$

regular; irreducible ($\emptyset$, open, dense in X); finite (by Thm above) $\Rightarrow$ $\gamma|_{X_0}$ constant

$\forall x \in X_0$ $\gamma|_{X_0}(x) = B$ for some coideal subalg. B of H^* of dim d_0

Argue that $A \xrightarrow{\rho} A \otimes H^*$ (coaction) restricts to $A \to A \otimes \langle B \rangle$ coaction of Hopf subalg generated by B.

By (*), $H^* = \langle B \rangle$. But $B = A_x = \rho_x(A)$ is commutative (image of com. alg)

$\therefore$ H^* commutative, as desired.

Next, extend to NSFQS for quantizations of com. domains

Holds in positive characteristic if H is semisimple and cosemisimple

II. No Quantum Symmetry when H = semisimple Hopf algs, A = Weyl algebras

Assume $\underset{ss}{H} \curvearrowright A_n(k)$ doesn't factor through action of a proper quotient Hopf alg

<u>want</u> H cocommutative. <u>Pf follows in 3 steps</u>

Reduce modulo p, $p >> 0$

- $\exists$ fin. gen. subring R of k so that $H_R \curvearrowright A_n(R)$ (action doesn't factor properly)

  "R-order" R-subalg $[H \simeq k \otimes_R H_R]$

- $\exists$ homom. $\psi: R \to \overline{\mathbb{F}_p}$ to define $H_{\psi,p} = H_R \otimes_R \overline{\mathbb{F}_p} =: H_p$

  & get $\underset{ss,\ coss}{H_p} \curvearrowright A_n(\overline{\mathbb{F}_p})$ (action doesn't factor properly)

  PI domain

Localize: use result on Hopf actions on Div Algs to get result in char p

- Get (Skryabin–van Oystaeyen) $\underset{ss,\ coss}{H_p} \curvearrowright D_p$ (action doesn't factor properly) $(*)$

  quot. div. alg of $A_n(\overline{\mathbb{F}_p})$

<u>Technical lemma</u>: H_F Hopf alg of dim d / F = alg closed field of arb. char. Say a division algebra D of degree N over $Z(D)$ admits action of H_F. If $\gcd(d!, N) = 1$, then $D = Z(D) D^{H_F}$ and $Z(D)$ is H_F-stable

have to be able to control PI degree of A_p = degree of D_p.

Have degree $D_p = p^m$ for some m. <u>Now</u> ($p >> 0$) $\overset{\text{Tech. lemma}}{\Rightarrow}$ $D_p = Z(D_p) D_p^{H_p}$ $(\dagger)$ & $Z(D_p)$ is H_p-stable $(\ddagger)$

So take any Hopf ideal I of H_p so that $I Z(D_p) = 0$

$\overset{(\dagger)}{\Longrightarrow} I D_p = 0 \overset{(*)}{\Longrightarrow} I = 0 \overset{(\ddagger)}{\Longrightarrow} H_p \curvearrowright$ a field $Z(D_p)$, action doesn't factor properly

com. domain

$\overset{\text{Result I}}{\Longrightarrow}$ H_p cocommutative.

Pass back to char 0

Com Alg lemma $\Rightarrow R \hookrightarrow$ product of fields

$\Rightarrow H_R$ cocommutative $\to H$ cocom. ✓

III. No Quantum Symmetry when H-semisimple Hopf alg, A = large class of skew poly'l rings

Same ideas as pf of Result II, but with a modification

$A = k_q[x_1, \ldots, x_n]$ w/ relations $x_i x_j = q_{ij} x_j x_i$, $q = (q_{ij})$ multip'ly antisym.

Reduce modulo p

- $\exists$ fin. generated subring $R \subseteq k$ s.t. $H_R \curvearrowright R_q[x_1, \ldots, x_n]$ (action doesn't factor properly)

We don't necessarily get that PI degree of A modulo p (as before) is a prime power.

Need new way to reduce modulo p, so that the PI degree of $k_q[x_1, \ldots, x_n]$ is controlled.

Go through number field K: given homom. $\xi : R \to K$, $R' := \operatorname{im} \xi$

- Get $H_{R'} \curvearrowright R'_{\xi(q)}[x_1, \ldots, x_n]$ doesn't factor properly.

If $\left| G_q / G_q^\circ \right| \left(= \left| G_{\xi(q)} / G_{\xi(q)}^\circ \right| \right)$ is coprime to $(\dim H)!$

G_q: Zariski closure of $\langle q \rangle \subseteq (k^\times)^{(?)}$

G_q°: connected component of identity of G_q

the condition (!!) on skew parameters of A

then by a result of A. Perucca: $\exists\, \infty$ many primes p with prime ideals $\mathfrak{p}$ of R' lying over p

so that for a generic homom $\psi' : R' \to \overline{\mathbb{F}_p}$ annihilating $\mathfrak{p}$:

order of $\psi'(\xi(q)) = N$ is finite & coprime to $(\dim H)!$ (N depends on p)

Now form $A_p = R'_q[x_1, \ldots, x_n] \otimes_{R'} \overline{\mathbb{F}_p}$ via ψ' and get PI deg A_p divides N^n

Localize: Hopf actions on div algs to get result in char p

- Get $H_p \curvearrowright D_p$ (action doesn't factor properly)
  - D_p: full localization of A_p

- We can now control degree D_p so that for $p \gg 0$, get $\gcd(\deg D_p, (\dim H_p)!) = 1$.

Technical Lemma from II. $\Longrightarrow$ H_p cocommutative.

Pass back to char 0 $\longrightarrow$ H is cocommutative ✓

Here's the precise result

<u>Theorem</u>: Let H be a semisimple Hopf algebra of dimension d.

If the $\underbrace{\text{order of } G_q/G_q^\circ}$ is coprime to $d!$, then any H-action on $k_q[x_1,\dots,x_n]$ factors through a group action.

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* If each skew parameter q_{ij} is a root of unity of order r_{ij}, then $|G_q/G_q^\circ| = \mathrm{lcm}\{r_{ij}\}_{i<j}$.

* If the set of q_{ij} are multiplicatively independent, then $|G_q/G_q^\circ| = 1$ (and hypothesis of theorem is satisfied).

* Eg. Take $n=2$: $A = k\langle x,y\rangle/(xy - qyx)$.

Then $|G_q/G_q^\circ|$ is coprime to $d!$ $\iff$ order of q is coprime to $d!$
— or —
order of q is infinite

* Eg. Take $n=3$: $A = k\langle x,y,z\rangle \Big/ \begin{pmatrix} xy - \zeta_{11} yx \\ yz - \zeta_7 zy \\ zx - \zeta_{13} xz \end{pmatrix}$ for ζ_j = primitive jth root of 1

Get NSFQS for $H \curvearrowright A$ if $((\dim H)!, 7\cdot 11\cdot 13) = 1$

(eg actions of semisimple Hopf algebras of dim 45 must factor through a group algebra)

Summary of No Quantum Symmetry results

* w/ Etingof ** w/ Cuadra + Etingof *** w/ Etingof, Goswami, + Mandal

(see works of Banica, Bhowmick, Chirvasitu, Das, Goswami, Joarder, Wang, ... for NQS results in the analytic setting)

ArXiv #	Module algebras A	NSFQS (H = semisimple)	NFQS (H = finite dim'l)
* 1301.4161	Commutative domains	✓	
*** 1507.08486	com. fin. gen alg w/ no homog deg 2 relations	✓	
** 1409.1644	Weyl algs $A_n(k[x_1,\dots,x_s])$	✓	
** 1509.01165	Weyl algs $A_n(k)$ Rings of diff'l ops $D(X)$		$n \geq 1$ ✓
* 1602.00532	quantum (formal) deformations of com. domains A_0 filtered (PBW) deformations of com. domains.	✓	✓ if Poisson center of $Q(A_0)$ is trivial ✓ " — " & if filtration is preserved
* 1605.00560	filtered deformation $\mathcal{A}$, is PI module alg Skew poly'l rings twisted homog. coordinate rings 3-dim'l Sklyanin algebras	✓ ✓ — for generic $A \in \mathcal{A}$ ✓ — ✓ —	✓ under a nondeg. condition on action ✓ " — "

There are nice Hopf actions that do not factor through cocom Hopf algebras (e.g. "genuine Quantum Symmetry") but that's another talk…