CONSTRUCTING NON-SEMISIMPLE MODULAR TENSOR CATEGORIES

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MODULAR TENSOR CATEGORIES
ARE QUITE IMPORTANT
AS YOU JUST HEARD IN SIMON LENTNER'S & JULIA PLAVNIK'S TALKS

GOAL: CONSTRUCT MORE!
... ESP. IN NONSS SETTING

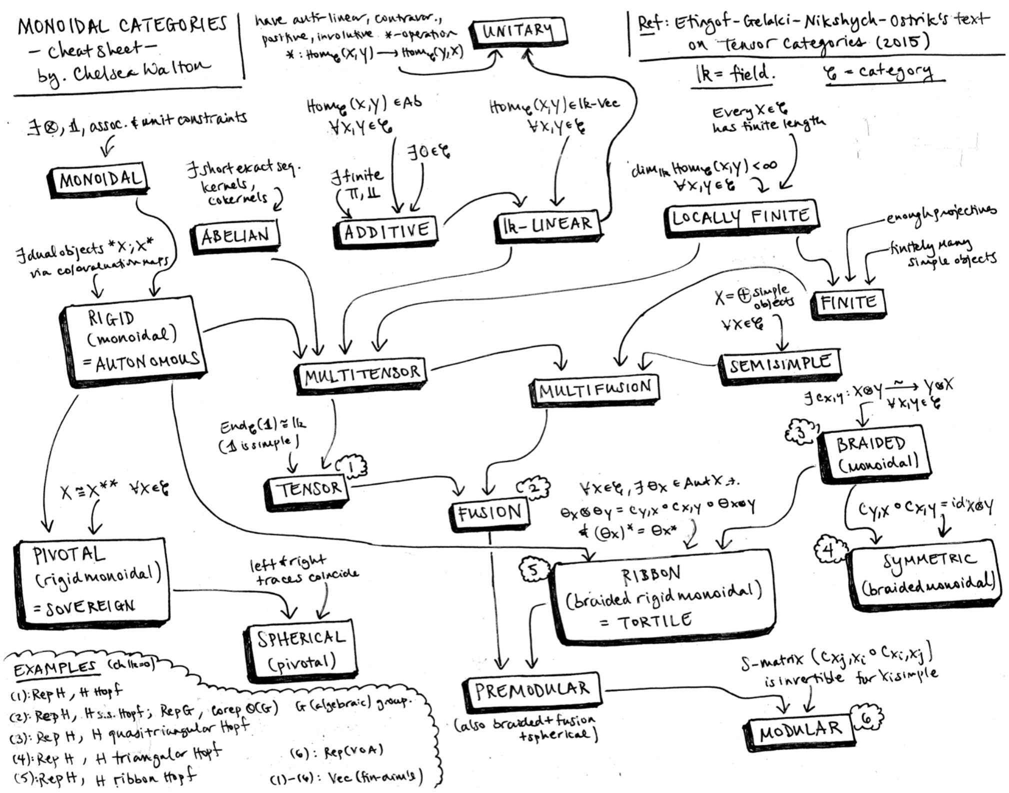
CLASSIFICATION OF 3D-TQFTS $\rightsquigarrow$ INVARIANTS OF 3-MANIFOLDS

REPRESENTATION THEORY OF VOAS $\rightsquigarrow$ FEATURES OF 2-D CFTS

APPLICATIONS IN
- TOPOLOGICAL QUANTUM COMPUTING
- TOPOLOGICAL PHASES OF MATTER

$\vdots$

THE DEFINITION IS A WEE BIT INVOLVED...



MONOIDAL CATEGORIES "CHEAT SHEET" FOUND ON MY SITE

BUT LET'S REVIEW

A CATEGORY $\mathcal{C}$ IS MODULAR IF... / $\Bbbk$ ALG. CLOSED FIELD

LIKE $\mathrm{Vec}_{\Bbbk}$

ABELIAN
∃ SHORT EX. SEQS.

$\Bbbk$-LINEAR
∃ FINITE $\Pi, \amalg$, 0
HOM SETS ARE $\Bbbk$-VSPACES

HAS FINITENESS CONDITIONS

LOCALLY FINITE
FINITE LENGTH OBJECTS
HOM SETS ARE F.D. $\Bbbk$-VSPACES

FINITE
ENOUGH PROJECTIVES
FIN. MANY SIMPLES

LIKE $\mathrm{Rep}(H)$, F.D. HOPF ALG

MONOIDAL
$\exists \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
$\mathbb{1} \in \mathcal{C}$ ∋.
$(\mathcal{C}, \otimes, \mathbb{1}) \equiv$ "MONOID"

SIMPLE UNIT
$\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk$

RIGID
∃ DUAL OBJECTS
$X^*, {}^*X$
$\forall X \in \mathcal{C}$

FINITE TENSOR CATEGORY

TRADITION

LIKE $\mathrm{Rep}(H)$, SS HOPF ALG

SEMISIMPLE

$\forall X \in \mathcal{C}: X \cong \bigoplus$ SIMPLES

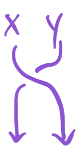
FUSION CATEGORY

A CATEGORY $\mathcal{C}$ IS MODULAR IF...

FINITE TENSOR CATEGORY —OR— FUSION CATEGORY

SEMISIMPLE FINITE TENSOR CAT. $\equiv$ FUSION CATEGORY

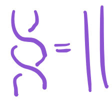
& BRAIDED



$\exists$ ISOS $X \otimes Y \xrightarrow{\sim} Y \otimes X$

COMPATIBLE WITH MONOIDAL STRUC.

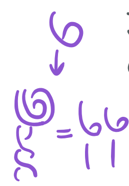
& NONDEGENERATE



$X \otimes Y \xrightarrow{\sim} Y \otimes X \xrightarrow{\sim} X \otimes Y$ $= \mathrm{id}$ $\forall X \in \mathcal{C}$

$\Rightarrow Y = \mathbb{1}$ $\Rightarrow Y \cong \mathbb{1}$

& RIBBON



$\exists$ ISOS $X \xrightarrow{\sim} X$

COMPATIBLE WITH BRAIDING & RIGIDITY

LIKE $\mathrm{Rep}^{f.d.}(H)$ H QUASITRIANGULAR

LIKE $\mathrm{Rep}^{f.d.}(H)$ H FACTORIZABLE

LIKE $\mathrm{Rep}^{f.d.}(H)$ H RIBBON

$\Rightarrow$ $\mathcal{C}$ IS A MODULAR TENSOR CATEGORY (MTC)

$\Rightarrow$ $\mathcal{C}$ IS A MODULAR FUSION CATEGORY (MFC)

!! TRADITIONALLY MTCs ARE SEMISIMPLE

AS IN PLAVNIK'S TALK

& "MFC" IS NOT USED

BUT THE DISTINCTION SHOULD BE MADE AS APPLICATIONS ARE ARISING IN THE NONSEMISIMPLE SETTING

AS IN LENTNER'S TALK

EXAMPLES

	MFCs	MTCs
FROM DOUBLES —	$D(G)-\mathrm{Mod}^{f.d.}$ ch $\Bbbk \nmid \lvert G \rvert$ DRINFELD DOUBLE	$D(G)-\mathrm{Mod}^{f.d.}$ ch $\Bbbk$ ARBITRARY
FROM QUANTUM GROUPS —	$\mathcal{C}_q(\mathfrak{g})$ VERLINDE MODULAR CAT. $U_q^{\text{Lusztig}}(\mathfrak{g})$ AT ROOT OF UNITY	$u_q(\mathfrak{g})-\mathrm{Mod}^{f.d.}$ SMALL QUANTUM GROUP AT ROOT OF UNITY

↪ SUBCATEGORY OF $U_q^{\text{Lusztig}}(\mathfrak{g})-\mathrm{Mod}$ CONSISTING OF TILTING MODULES

↪ QUOTIENT BY SUBCATEGORY OF NEGLIGIBLE MODULES

$\equiv$ "SEMISIMPLE PART" OF $U_q^{\text{Lusztig}}(\mathfrak{g})-\mathrm{Mod}$

(... I'M A BIT PARTIAL TO THE NONSS SETTING HERE)

EXAMPLES

	MFCs	MTCs
FROM DOUBLES —	$D(G)-\mathrm{Mod}^{f.d.}$ ch $\Bbbk \nmid \lvert G \rvert$	$D(G)-\mathrm{Mod}^{f.d.}$ ch $\Bbbk$ ARBITRARY

MORE GENERALLY FROM MONOIDAL CENTERS

$\mathcal{Z}(\Bbbk G\text{-Mod}^{f.d.}) \cong D(G)\text{-Mod}^{f.d.}$

LEFT & RIGHT TRACES OF OBJECTS COINCIDE

$\mathcal{C}$ (TRACE) SPHERICAL FUSION CATEGORY

$\Downarrow$ MÜGER

$\mathcal{Z}(\mathcal{C})$ MFC

TECHNICAL NOTION DUE TO DOUGLAS, SCHOMMER-PRIES, SNYDER

$\mathcal{C}$ (DSPS) SPHERICAL FINITE TENSOR CATEGORY

$\Downarrow$ SHIMIZU

$\mathcal{Z}(\mathcal{C})$ MTC

FROM QUANTUM-GROUPS

$\exists$ SIMILAR FRAMEWORK TO HANDLE $\longrightarrow$ $u_q(\mathfrak{g})\text{-Mod}^{f.d.}$

& AND MORE

SMALL QUANTUM GROUP AT ROOT OF UNITY

RELATIVE MONOIDAL CENTERS DUE TO LAUGWITZ

TAKE $\mathcal{B}$ BRAIDED TENSOR CATEGORY & $\mathcal{C}$ TENS CATEG.

& $G : \mathcal{B} \longrightarrow \mathcal{Z}(\mathcal{C})$ FAITHFUL BRAIDED TENS. FUNC.

$\rightsquigarrow$ $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) = C_{\mathcal{Z}(\mathcal{C})}(G(\mathcal{B}))$ MÜGER CENTRALIZER

BRAIDED TENS. CATEG.

OBJECTS OF $\mathcal{Z}(\mathcal{C})$ THAT BRAIDED COMMUTE WITH OBJECTS OF $G(\mathcal{B})$

Ex. $\mathcal{B} = \mathrm{Vec}_{\Bbbk}$: $\mathcal{Z}_{\mathrm{Vec}_{\Bbbk}}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{C})$.

Ex. $\mathcal{B} = K\text{-Mod}$: $\mathcal{Z}_{K\text{-Mod}}(H\text{-Mod}(K\text{-Mod})) \simeq \mathrm{Drin}_K(H)\text{-Mod}$

K QUASI TRIANGULAR

H HOPF ALG IN K-Mod

BRAIDED DRINFELD DOUBLE (DOUBLE BOSONIZATION)

Ex. $u_q(\mathfrak{g}) \cong \mathrm{Drin}_K(H)$ FOR $K = u_q(\mathfrak{h})$ $H = u_q(\mathfrak{n}^+)$

RECALL

FROM MONOIDAL CENTERS

	MFCs	MTCs
	$\mathcal{C}$ (TRACE) SPHERICAL FUSION CATEGORY	$\mathcal{C}$ (DSPS) SPHERICAL FINITE TENSOR CATEGORY
	$\Downarrow$ MÜGER	$\Downarrow$ SHIMIZU
	$\mathcal{Z}(\mathcal{C})$ MFC	$\mathcal{Z}(\mathcal{C})$ MTC

THEOREM [LAUGWITZ-W]

FROM RELATIVE MONOIDAL CENTERS

$\mathcal{B}$ MTC + $\mathcal{C}$ (DSPS) SPHERICAL FINITE TENSOR CATEGORY + $\exists\, G : \overline{\mathcal{B}} \longrightarrow \mathcal{Z}(\mathcal{C})$ FAITHFUL BRAIDED TENSOR FUNCTOR

$\Rightarrow$ $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ MTC

PROPOSITION [LAUGWITZ-W]

RECALL $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) = C_{\mathcal{Z}(\mathcal{C})}(G(\overline{\mathcal{B}}))$

FURTHER, $\mathcal{Z}(\mathcal{C}) \simeq \overline{\mathcal{B}} \boxtimes \mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ AS MTCs

"MÜGER DECOMPOSITION"

Ex. $\mathcal{B} = \mathsf{K}\text{-Mod}$ K QUASI Δ + H HOPF ALG IN $\mathcal{B}$ $\rightsquigarrow$ $\mathcal{C} \simeq H \rtimes K\text{-mod}$

$$\underbrace{\mathrm{Drin}(H \rtimes K)}_{H \otimes K \otimes K^* \otimes H^*}\text{-mod} \simeq \underbrace{K}_{K}\text{-mod} \boxtimes \underbrace{\mathrm{Drin}_K(H)}_{H \otimes K \otimes H^* \,\cong\, K^*}\text{-mod}$$

SO FAR —

CONSTRUCTING NON-SEMISIMPLE MODULAR TENSOR CATEGORIES

VIA RELATIVE MONOIDAL CENTERS

$\underset{\substack{\uparrow \\ \text{MTC}}}{\mathcal{B}}$ in $Z_{\mathcal{B}}(\underset{\substack{\uparrow \\ \text{SPHERICAL}}}{\mathcal{C}})$

Ex. $U_q(\mathfrak{g})$-mod IS MTC

VIA DELIGNE PRODUCTS (KNOWN)

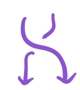
$\underset{\text{MTC}}{Z(\mathcal{C})} \simeq \underset{\text{MTC}}{\bar{\mathcal{B}}} \boxtimes \underset{\text{MTC}}{Z_{\mathcal{B}}(\mathcal{C})}$

Ex. $D(u_q(\mathfrak{b}^+))$-mod $\simeq u_q(\mathfrak{g})$-mod $\boxtimes$ $u_q(\mathfrak{g})$-mod

LET'S DISCUSS ONE MORE WAY

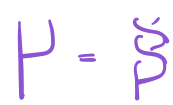
MFCs FROM LOCAL MODULES

TAKE BRAIDED FINITE TENSOR CATEGORY

$(\mathcal{C}, \{c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X\})$ 

TAKE COMMUTATIVE ALGEBRA A IN $\mathcal{C}$

A RIGHT A-MODULE $(V, \rho_V : V \otimes A \to V)$ IS LOCAL

IF $\rho_V = \rho_V \circ c_{A,V} \circ c_{V,A}$ 

$\underset{\text{PAREIGIS}}{\Longrightarrow}$ $\mathrm{LocMod}_{\mathcal{C}}(A)$ BRAIDED MONOIDAL

$\mathcal{C}$ MFC + A RIGID $\mathcal{C}$-ALGEBRA $\Longrightarrow$ $\mathrm{LocMod}_{\mathcal{C}}(A)$ MFC

CONNECTED, NONZERO DIM

$\exists\, \varepsilon : A \to \mathbb{1}$ SATISFYING CONDS...

KIRILLOV-OSTRIK

$\boxed{\text{THEOREM [LAUGWITZ-W]}}$

$A = (A, m, u, \Delta, \varepsilon)$

$\mathcal{C}$ MTC + A RIGID FROBENIUS $\Longrightarrow$ $\mathrm{LocMod}_{\mathcal{C}}(A)$ MTC

CONNECTED, COMMUTATIVE,

SPECIAL FROBENIUS

$m\Delta = \beta_A \, \mathrm{id}_A$, $\varepsilon u = \beta_{\mathbb{1}} \, \mathrm{id}_{\mathbb{1}}$ FOR $\beta_A, \beta_{\mathbb{1}} \in \Bbbk^\times$

EXAMPLES IN PAPER RELATED TO DOUBLES / MONOIDAL CENTERS.

REGARDING QUANTUM GROUPS / RELATIVE MONOIDAL CENTERS —

$\boxed{\text{CONJECTURE [LAUGWITZ-W]}}$ $\mathrm{ch}\,\Bbbk = 0$

A RIGID FROBENIUS IN $\mathcal{C} = u_q(\mathfrak{sl}_2)$-mod $\Rightarrow$ $A \cong \mathbb{1}_{\mathcal{C}}$

(IN CONVERSATIONS WITH SCHWEIGERT — SUSPECT THAT "SPECIAL" IS TOO STRONG NONSEMISIMPLE SETTING)

CONSTRUCTING NON-SEMISIMPLE MODULAR TENSOR CATEGORIES

VIA RELATIVE MONOIDAL CENTERS

$$\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$$

MTC SPHERICAL

VIA DELIGNE PRODUCTS (KNOWN)

$$\underset{\text{MTC}}{\mathcal{Z}(\mathcal{C})} \simeq \underset{\text{MTC}}{\overline{\mathcal{B}}} \boxtimes \underset{\text{MTC}}{\mathcal{Z}_{\mathcal{B}}(\mathcal{C})}$$

VIA LOCAL MODULES

$$\mathrm{LocMod}_{\mathcal{C}}(A)$$

MTC RIGID FROBENIUS

Thanks for listening!

& Thanks for organizing!