CONSTRUCTING NONSEMISIMPLE MODULAR TENSOR CATEGORIES
(VIA RELATIVE CENTERS)

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OUTLINE

/$\Bbbk$ ALG CLOSED
CHAR 0

I. MODULAR TENSOR CATEGORIES
(DEFINITION & WHY CARE?)

II. FOUNDATIONAL RESULTS

III. OUR RESULTS

IV. OUR APPLICATIONS (= EXAMPLES)
+ RECENT WORK OF OTHERS

MTCs ARE MONOIDAL CATEGORIES WITH LOTS OF STRUCTURE—

MODULAR TENSOR CATEGORY (MTC)

≡ BRAIDED FINITE TENSOR CATEGORY THAT IS RIBBON & NONDEGENERATE

MODULAR FUSION CATEGORY (MFC)

≡ SEMISIMPLE MTC

MTCs ARE MONOIDAL CATEGORIES WITH LOTS OF STRUCTURE—

/$\Bbbk$ FIELD

MODULAR TENSOR CATEGORY (MTC)

≡ BRAIDED FINITE TENSOR CATEGORY THAT IS RIBBON & NONDEGENERATE

EX.

$\mathrm{Vec}_{\Bbbk}^{\text{f.d.}}$

$H\text{-mod}_{\Bbbk}$

↑

f.d. Hopf alg.

A CATEGORY $\mathcal{C}$ THAT'S • ABELIAN • ADDITIVE • $\Bbbk$-LINEAR & LOCALLY FINITE (($\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS A FIN. DIM. $\Bbbk$-VS) • FINITE (... FINITELY MANY SIMPLES)

EQUIPPED WITH $\otimes$, $\mathbb{1}$ $\rightsquigarrow$ $(\mathcal{C}, \otimes, \mathbb{1})$ IS MONOIDAL SO THAT

- $\mathbb{1}$ IS SIMPLE
- $\exists$ DUAL OBJECTS X^*, ${}^*X$ $\forall X \in \mathcal{C}$. ("RIGID")

MTCs ARE MONOIDAL CATEGORIES WITH LOTS OF STRUCTURE—

MODULAR TENSOR CATEGORY (MTC)

≡ BRAIDED FINITE TENSOR CATEGORY THAT IS RIBBON & NONDEGENERATE

A TENSOR CATEGORY $(\mathcal{C}, \otimes, \mathbb{1})$ IS **BRAIDED** IF EQUIPPED WITH NATURAL ISOMORPHISMS

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X \quad \forall X, Y \in \mathcal{C}$$

(SATISFYING COMPATIBILITY CONDITIONS...).

EX.
H-mod$_{\Bbbk}$
↑
f.d. quasitriangular Hopf alg.
· quantized enveloping alg.
· quantum/Drinfeld double...

FURTHER, IT IS **RIBBON** IF EQUIPPED WITH NAT'L ISOM $\theta_X : X \xrightarrow{\sim} X \quad \forall X \in \mathcal{C}$

WHERE $\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y)\, c_{Y,X}\, c_{X,Y}$ & $(\theta_X)^* = \theta_{X^*} \quad \forall X, Y \in \mathcal{C}$

EX.
H-mod$_{\Bbbk}$
↑
f.d. ribbon Hopf alg

MTCs ARE MONOIDAL CATEGORIES WITH LOTS OF STRUCTURE—

MODULAR TENSOR CATEGORY (MTC)

≡ BRAIDED FINITE TENSOR CATEGORY THAT IS RIBBON & NONDEGENERATE

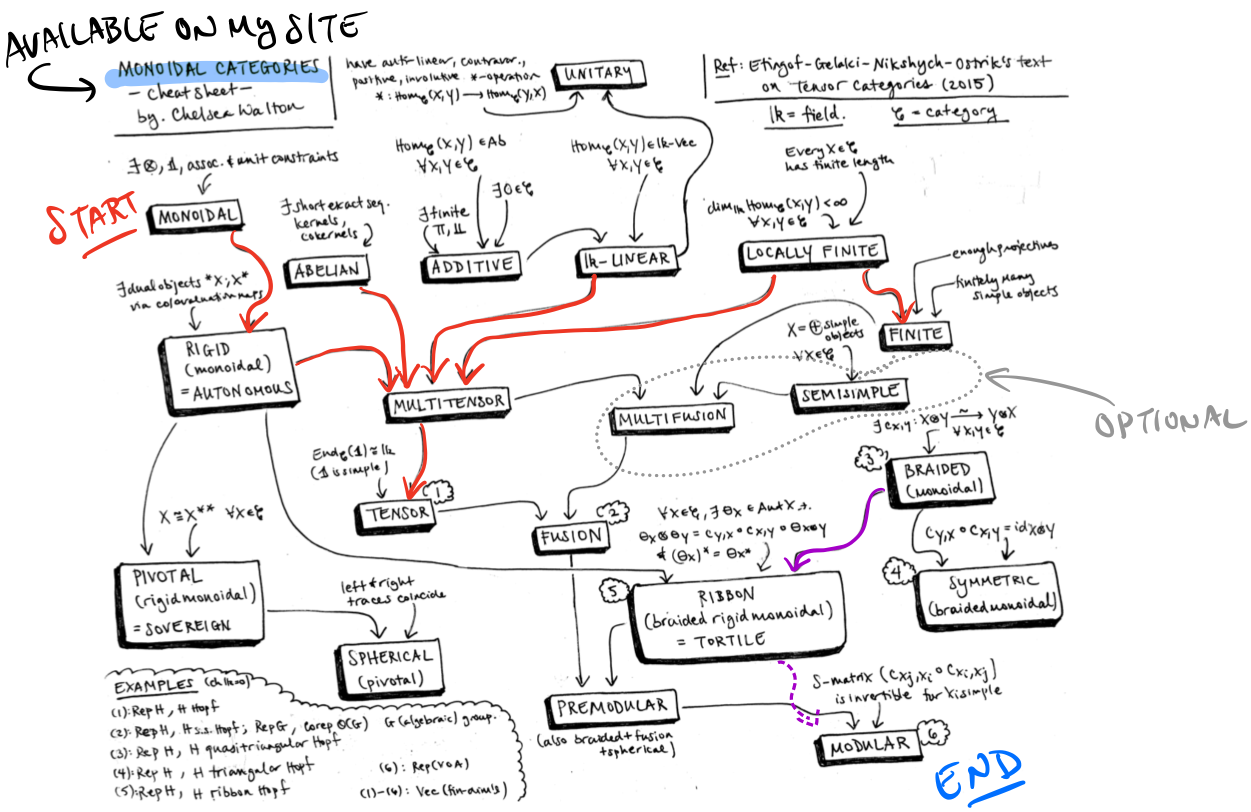
A BRAIDED TENSOR CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS **NONDEGENERATE** IF ITS "MÜGER CENTER"

$$\mathcal{C}' = \{ X \in \mathcal{C} \mid c_{Y,X}\, c_{X,Y} = \mathrm{Id}_{X \otimes Y} \;\; \forall Y \in \mathcal{C} \}$$

IS EQUIV. TO $\mathrm{Vec}_{\Bbbk}$.

EX.
H-mod$_{\Bbbk}$
↑
f.d. factorizable Hopf alg.
· quantum/Drinfeld double...

(OTHER CHARACTERIZATIONS INVOLVE "INVERTIBILITY OF S-MATRIX" (IN SS CASE), "FACTORIZABILITY", INVERTIBILITY OF CERTAIN MAP INDUCED BY HOPF PAIRING)



MODULAR TENSOR CATEGORY (MTC)

≡ BRAIDED FINITE TENSOR CATEGORY THAT IS RIBBON & NONDEGENERATE

MODULAR FUSION CATEGORY (MFC)

≡ SEMISIMPLE MTC

WHY CARE ABOUT MFCs & MTCs?

CONNECTIONS TO A MYRIAD OF FIELDS, INCLUDING —

LOW-DIMENSIONAL TOPOLOGY /
TOPOLOGICAL QUANTUM FIELD THEORY

MATHEMATICAL PHYSICS /
VARIOUS CONFORMAL FIELD THEORIES

SUBFACTOR THEORY

... A LOT OF PPL MAKE THEIR LIVING STUDYING SUCH GADGETS...

GOAL: CONSTRUCT MORE!

KEY CONSTRUCTION OF AN MFC

FOR $\mathcal{C}$ $\otimes$ CATEGORY,

$\mathcal{Z}(\mathcal{C})$ = CENTER OF $\mathcal{C}$ WITH OBJECTS $(V, \{c_{V,X} : V \otimes X \to X \otimes V\}_{X \in \mathcal{C}})$

OBJ IN $\mathcal{C}$

"HALF BRAIDING" NAT'L TRANSF. IN $\mathcal{C}$

SATISFYING CERTAIN COMP. CONDITIONS

Ex. $\mathcal{C} = H\text{-mod} \rightsquigarrow \mathcal{Z}(\mathcal{C}) \sim \mathrm{Drin}(H)\text{-mod}$

F.D HOPF ALG

DRINFELD DOUBLE OF H

SEMISIMPLE FINITE TENSOR CATEG.

THEOREM (MÜGER) IF $\mathcal{C}$ IS A FUSION CATEGORY THAT IS

$\exists \{j_X : X \xrightarrow{\sim} X^{**}\}_{X \in \mathcal{C}}$ SAT. CERT. CONDS. (PIVOTALITY)

(TRACE-) SPHERICAL $\equiv$ FOR $\dim_j(X) = ev_{X^*}(j_X \otimes id_{X^*})\, coev_X : \mathbb{1} \to \mathbb{1} \in \mathcal{C}$

GET $\dim_j(X) = \dim_j(X^*)$ $\forall X \in \mathcal{C}$

THEN $\mathcal{Z}(\mathcal{C})$ IS A MODULAR FUSION CATEGORY.

KEY CONSTRUCTION OF AN MTC

NEED GENERALIZED NOTION OF SPHERICALITY —

FOR $\mathcal{C}$ FIN TENS. CAT.,

TAKE DISTING. INV. OBJ $D \in \mathcal{C}$ & CAN'L NAT'L $\otimes$ ISOM $\xi_D(X) : D \otimes X \xrightarrow{\sim} X^{****} \otimes D$

$\mathrm{Sqrt}(D, \xi_D) :=$ {EQ. CLASSES OF INV OBJ (V, σ_V) IN $\mathcal{Z}(\mathrm{Id}_{\mathcal{C}}, (-)^{**})$ SO THAT $\exists$ ISOM $\nu : V^{**} \otimes V \to D$ WHERE

$$\begin{tikzcd} V^{**} \otimes V \otimes X \arrow[r, "\mathrm{id} \otimes \sigma_V"] \arrow[d, "\nu \otimes \mathrm{id}"'] & V^{**} \otimes X^{**} \otimes V \arrow[r, "\sigma_V^{**} \otimes \mathrm{id}"] \arrow[d, phantom, "\circlearrowright"] & X^{****} \otimes V^{**} \otimes V \arrow[d, "\mathrm{id} \otimes \nu"] \\ D \otimes X \arrow[rr, "\xi_D"] & & X^{****} \otimes D \end{tikzcd}$$

THIS SET BEING NONEMPTY TIED TO THIS RIBBONALITY OF $\mathcal{Z}(\mathcal{C})$

A PIVOTAL FIN. $\otimes$ CAT $(\mathcal{C}, j)$ IS DSPS-SPHERICAL IF $\exists \nu : \mathbb{1} \xrightarrow{\sim} D$ s.t.

$$\begin{tikzcd} X \arrow[r, "j^{**} j"] \arrow[d, "\nu \otimes \mathrm{id}"'] \arrow[dr, phantom, "\circlearrowright"] & X^{****} \arrow[d, "\mathrm{id} \otimes \nu"] \\ D \otimes X \arrow[r, "\xi_D"] & X^{****} \otimes D \end{tikzcd}$$

DSPS-SPHERICAL + SEMISIMPLE $\Downarrow$ TRACE-SPHERICAL

DSPS-SPHERICAL $\Rightarrow (\mathbb{1}, j) \in \mathrm{Sqrt}(D, \xi_D) \Rightarrow \mathrm{Sqrt}(D, \xi_D) \neq \emptyset$

THEOREM (SHIMIZU) IF $\mathcal{C}$ IS A TENSOR CATEGORY FOR WHICH

$\mathrm{Sqrt}(D, \xi_D) \neq \emptyset$ (OR IS (DSPS-)SPHERICAL)

THEN $\mathcal{Z}(\mathcal{C})$ IS A MODULAR TENSOR CATEGORY

MODULARITY OF CENTERS

SEMISIMPLE SETTING: $\mathcal{C}$ TRACE-SPHERICAL $\Rightarrow$ $\mathcal{Z}(\mathcal{C})$ MFC [Müger]

NONSEMISIMPLE SETTING: $\mathcal{C}$ DSPS-SPHERICAL $\Downarrow$ $\mathrm{Sqrt}(\mathcal{D},\xi_{\mathcal{D}}) \neq \emptyset$ $\Rightarrow$ $\mathcal{Z}(\mathcal{C})$ MTC [Shimizu]

RELATIVE NONSEMISIMPLE SETTING (OVER NONDEG. BRAID. $\otimes$ CATEG. $\mathcal{B}$): $\mathcal{C}$ DSPS-SPHERICAL $\Downarrow$ $\mathrm{Sqrt}(\mathcal{D},\xi_{\mathcal{D}}) \neq \emptyset$ $\Rightarrow$ $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ MTC [Laugwitz-W] SEEN AS FOLLOWS

RELATIVE MONOIDAL CENTER: DEFINITION

$(\mathcal{B},\psi)$ BRAIDED TENSOR CATEGORY $\qquad \overline{\mathcal{B}} := (\mathcal{B},\psi^{-1})$

A TENSOR CATEGORY $\mathcal{C}$ IS $\mathcal{B}$-CENTRAL IF
$\exists$ FAITHFUL BRAIDED TENSOR FUNCTOR $G: \overline{\mathcal{B}} \longrightarrow \mathcal{Z}(\mathcal{C})$

THE RELATIVE MONOIDAL CENTER $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ FOR A $\mathcal{B}$-CENTRAL TENSOR CATEGORY IS (FOR SIMPLICITY HERE)
$C_{\mathcal{Z}(\mathcal{C})}(G(\overline{\mathcal{B}}))$ (MÜGER CENTRALIZER)

IF $(\mathcal{D},c)$ IS A BRAIDED $\otimes$ CATEG., $\mathcal{E} \subseteq \mathcal{D}$ SUBSET OF OBJECTS
$C_{\mathcal{D}}(\mathcal{E})$ = FULL SUBCAT. OF $\mathcal{D}$ W/ OBJECTS $\{X \in \mathcal{D} \mid c_{Y,X}\, c_{X,Y} = \mathrm{Id}_{X\otimes Y} \ \forall Y \in \mathcal{E}\}$

RELATIVE MONOIDAL CENTER: EXAMPLES

$(\mathcal{B}, \psi)$ BRAIDED $\otimes$ CATEG.

$\mathcal{C}$ = $\mathcal{B}$-CENTRAL $\otimes$ CATEG. W/ FAITHFUL BR. $\otimes$ FUNC. $G: \mathcal{B} \to \mathcal{Z}(\mathcal{C})$

$\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) = C_{\mathcal{Z}(\mathcal{C})}(G(\mathcal{B}))$

EX 1. $\mathcal{B} = K$-MOD (K: F.D. QT HOPF ALG) $\quad \mathcal{C} = H$-MOD (K-MOD) (H: F.D. HOPF ALG IN $\mathcal{B}$) $\quad \rightsquigarrow \quad \mathcal{Z}_{\mathcal{B}}(\mathcal{C}) = \mathrm{Drin}_K(H, H^*)$-MOD ($\mathrm{Drin}_K$: BRAIDED DRIN. DOUBLE)

EX 2. $\mathcal{B} = \mathrm{Vec}_{\Bbbk}$ $(K = \Bbbk)$ $\quad \mathcal{C} = H$-MOD (H: F.D. HOPF ALG/$\Bbbk$) $\quad \rightsquigarrow \quad \mathcal{Z}_{\mathcal{B}}(\mathcal{C}) = \mathcal{Z}(\mathcal{C}) = \mathrm{Drin}(H)$-MOD ($\mathrm{Drin}$: DRINFELD DOUBLE)

EX. 3 $\mathcal{B} = U(\mathfrak{h})$-MOD ($\mathfrak{h}$: CARTAN PART OF SS LIE ALG. $\mathfrak{g}$) (HERE, $\psi = \psi_q$) $\quad \mathcal{C} = U_q(\mathfrak{n})$-MOD $(\mathcal{B})$ ($\mathfrak{n}$: NILP. PART OF SS LIE ALG. $\mathfrak{g}$) $\quad \rightsquigarrow \quad \mathcal{Z}_{\mathcal{B}}(\mathcal{C}) = u_q(\mathfrak{g})$-MOD ($u_q(\mathfrak{g})$: SMALL QUANTUM GROUP)

MAIN RESULT: MODULARITY OF MÜGER CENTRALIZERS

RECALL: FOR $\mathcal{C}$ BRAIDED $\otimes$ CATEG,

$\mathcal{C}' := \{X \in \mathcal{C} \mid c_{Y,X} c_{X,Y} = \mathrm{Id}_{X \otimes Y} \ \forall Y \in \mathcal{C}\} = C_{\mathcal{C}}(\mathcal{C})$ MÜGER CENTER

$\mathcal{C}$ NONDEG $\Leftrightarrow$ $\mathcal{C}' = \mathrm{Vec}_{\Bbbk}$.

THEOREM 1 LET $\mathcal{D}$ BE AN MTC (BRAIDED FIN. TENSOR CATEG. THAT'S NONDEG + RIBBON), $\mathcal{E}$ A TOPOLOGIZING SUBCAT OF $\mathcal{D}$ (FULL, CLOSED UNDER FINITE DIRECT SUMS & SUBQUOTIENTS)

THEN $\quad C_{\mathcal{D}}(\mathcal{E})' \simeq \mathcal{E}'$.

SO, $C_{\mathcal{D}}(\mathcal{E})$ IS MODULAR $\Leftrightarrow$ $\mathcal{E}$ IS MODULAR

($C_{\mathcal{D}}(\mathcal{E})$, $\mathcal{E}$ ARE RIBBON SUBCATEGS. OF $\mathcal{D}$)

GENERALIZING MÜGER, PLMS 2003 IN SS SETTING

MAIN RESULT: MODULARITY OF RELATIVE ⊗ CENTERS

THEOREM 2 LET $\mathcal{B}$ BE A NONDEG. BRAIDED FIN. ⊗ CATEGORY.
LET $\mathcal{C}$ BE A $\mathcal{B}$-CENTRAL ⊗ CATEGORY W/ FAITHFUL BR. ⊗ FUNC $G: \mathcal{B} \to \mathcal{Z}(\mathcal{C})$.
ASSUME $G(\mathcal{B}) \subseteq \mathcal{Z}(\mathcal{C})$ IS TOPOLOGIZING & $\underbrace{\mathrm{Sqrt}_{\mathcal{C}}(\mathcal{D}, \mathcal{E}_{\mathcal{D}}) \neq \emptyset}_{(*)}$.
THEN $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ IS AN MTC.

PF/ RECALL $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) = C_{\mathcal{Z}(\mathcal{C})}(G(\mathcal{B}))$. APPLY THM 1 TO $\mathcal{D} = \mathcal{Z}(\mathcal{C})$ & $\mathcal{E} = G(\mathcal{B})$.

SHIMIZU: MTC BY (*)

GENERALIZING
MÜGER, JPAA 2003 IN SS SETTING
SHIMIZU, 2018 IN NONSS SETTING
TO THE RELATIVE SETTING

APPLICATION: MODULARITY OF REP. CATEGORIES OF SMALL QUANTUM GROUPS & OTHER POINTED HOPF ALGS.

EX. SHOW THAT $U_q(\mathfrak{g})$-MOD IS AN MTC VIA THEOREM 2.

ODD ORDER   SS LIE ALG

ALSO HAVE CONDITIONS WHEN REP CAT. OF BRAIDED DRINFELD DOUBLE

$$\mathrm{Drin}_{\Bbbk G}(B(V), B(V)^*)\text{-MOD} \quad \text{IS AN MTC.}$$

FIN. ABELIAN   CERTAIN NICHOLS ALGS / G

SUCH NICHOLS ALGS FALL INTO THREE TYPES

- CARTAN ← MODULARITY WELL STUDIED
- SUPER ← ONE MTC EXAMPLE IN PAPER. OTHERS IN UPCOMING WORK OF LAUGWITZ – SANMARCO
- UFO ← STILL A MYSTERY

MAIN RESULT : NONSS DECOMPOSITION THEOREM

THEOREM 3 LET $\mathcal{D}$ BE AN MTC,
$\mathcal{E}$ TOPOLOGIZING NONDEG. BRAIDED TENSOR SUBCATEGORY. THEN:

$$\mathcal{D} \simeq \mathcal{E} \boxtimes C_{\mathcal{D}}(\mathcal{E})$$

AS RIBBON CATEGORIES.

GENERALIZING
MÜGER, PLMS 2003 IN SS SETTING

SPECIAL CASES :

- UNDER HYP. OF THM 2: $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{B} \boxtimes \mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ (DECOMPOSITION OF MTC)

- $\mathcal{B} = K\text{-MOD}$, $\mathcal{C} = H\text{-MOD}(K\text{-MOD})$. GET:

  f.d quasiΔ Hopf alg. f.d Hopf alg. $\in \mathcal{B}$

$$\mathrm{Drin}(H \rtimes K)\text{-mod} \simeq K\text{-mod} \boxtimes \mathrm{Drin}_K(H, H^*)\text{-mod}$$

- $\mathcal{B} = \mathbb{Z}n\text{-mod}$, with braiding ψ_q , $\mathcal{C} = \Bbbk[x]/_{(x^n)}\text{-mod}(\mathcal{B}) \sim T_n(q^{-2})\text{-mod}$

  Taft alg.

  GET:

$$D(T_n(q^{-2}))\text{-mod} \sim \mathbb{Z}n\text{-mod} \boxtimes u_q(\mathfrak{sl}_2)\text{-mod}$$

QUESTION : WHEN IS $\mathrm{Drin}_K(H, H^*)$-MOD PRIME AS A MTC?

DONE IN
GROUP-THEORETICAL
CASE

EVERY TOP. NONDEG. BRAIDED $\otimes$ SUBCAT
IS EQUIV TO $\mathrm{Vec}_{\Bbbk}$ OR ITSELF?

CONSTRUCTING NONSEMISIMPLE MODULAR TENSOR CATEGORIES
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Thanks for listening!

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