Constructing Nonsemisimple Modular Categories

by Chelsea Walton

for Seattle Noncommutative Algebra Day

Dec 2020

$\Bbbk$ = ground field, alg closed;

ch 0 for explicit examples

joint with Robert Laugwitz

Arxiv: 2010.11872

GAME

launched in early 1990s by Reshetikin, Turaev, Witten

CATEGORY THEORY: MODULAR CATEGORY $\mathcal{C}$ $\rightsquigarrow$ QUANTUM FIELD THEORY: TQFT $\rightsquigarrow$ TOPOLOGY: INVARIANTS of LOW DIM'L MANIFOLDS & REPS of MAPPING CLASS GROUPS of SURFACES

And why may care?

QUANTUM ALGEBRA

certain Hopf algebra H

via reps

Outline of Talk

I. More on the game: category theory to topology

II. Focus on Quantum Algebra & Category Theory

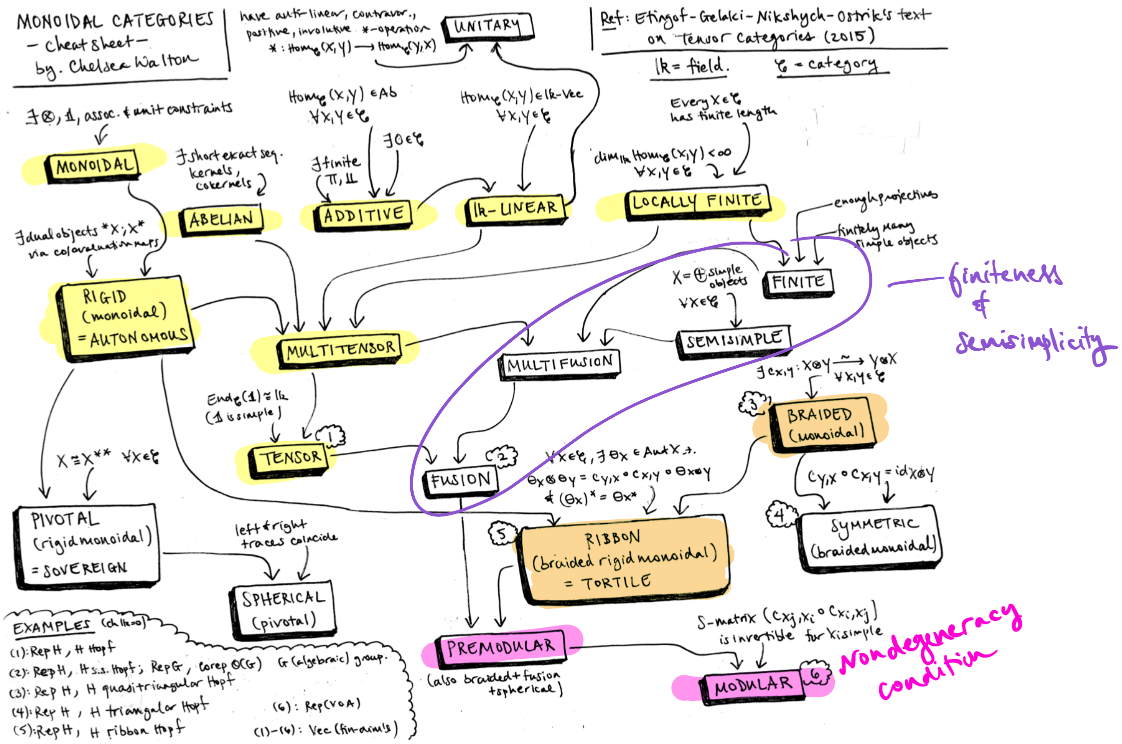
III. Results

IV. Future Directions

I | CATEGORY THEORY
MODULAR CATEGORY $\mathcal{C}$

introduced by Turaev in the early 1990's for the game above

The structure is quite complicated...



MTC $\mathcal{C}$ = fusion + ribbon + nondegenerate monoidal category

fusion = finite semisimple

Main Example: semisimple quotient of $Rep(U_q^{res}(sl_2))$ for q a root of 1

semisimple quotient ← a bit complicated.

$Rep(U_q^{res}(sl_2))$ ← nonsemisimple category

CATEGORY THEORY: MODULAR CATEGORY $\mathcal{C}$ $\rightsquigarrow$ QUANTUM FIELD THEORY: TQFT

due to Reshetikin-Turaev (1991) inspired by work of Witten (1988)

TQFT ← functor depending on $\mathcal{C}$

$F_{\mathcal{C}} : R_{\mathcal{C}} \longrightarrow \mathcal{C}$ "RT-functor"

$R_{\mathcal{C}}$ ← category of "ribbon graphs" of $\mathcal{C}$

here $F_{\mathcal{C}}(\text{closed manifold}) \in End_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})$

a scalar

QUANTUM FIELD THEORY

TQFT $\rightsquigarrow$

TOPOLOGY

INVARIANTS of LOW DIM'L MANIFOLDS & REPS of MAPPING CLASS GROUPS of SURFACES $\leftarrow$ for 3-manifolds M in particular get "RT-invariants"

$$RT_{\mathcal{C}}\left(M, \text{"ribbon graph of } M\text{"}\right) =$$

a certain scalar multiple of $F_{\mathcal{C}}$(closed components of M) (scalar)

CATEGORY THEORY

MODULAR CATEGORY $\mathcal{C}$ $\rightsquigarrow$

QUANTUM FIELD THEORY

TQFT $F_{\mathcal{C}}$ $\rightsquigarrow$

TOPOLOGY

INVARIANTS of LOW DIM'L MANIFOLDS & REPS of MAPPING CLASS GROUPS of SURFACES

* See recent survey of Blanchet & DeRenzi : 2011.12931 for more.

* Also people use language of higher category theory to get TQFTs (now functors between higher categories). modular categories are still used to study invariants of 3-manifolds in this framework

  See Freed's expository paper on the Cobordism Hypothesis 1210.5100

  Bakalov-Kirillov's book (2001) on Tensor Categories & modular Functors

  & Lurie's foundational paper 0905.0465 for more details

II|

QUANTUM ALGEBRA

certain Hopf algebra H $\longrightarrow$

CATEGORY THEORY

MODULAR CATEGORY $\mathcal{C} = H\text{-mod}$

$\equiv$ fusion + ribbon + nondegeneracy

(fusion = finite, semisimple)

<u>Main Example</u>: $U_q^{res}(\mathfrak{sl}_2)$ q root q 1 — a certain semisimple quotient !!! of $U_q^{res}(\mathfrak{sl}_2)$-mod.

Towards examples <u>without</u> using categorical quotients...

H finite dim'l Hopf algebra + semisimple — H-mod is a $\underbrace{\text{finite tensor}}_{\text{fusion}}$ category

If, H is also quasitriangular, ribbon, factorizable — H-mod is braided, ribbon, nondegenerate — Modular

<u>Example</u> Drinfeld double $D(H)$ of a semisimple Hopf algebra

= semisimple (⇒ f.d.), quasitriangular, factorizable

$\mathcal{C} = D(H)$-mod = fusion + braided + nondegenerate

Kauffman–Radford (1993) classify ribbon structures on $D(H)$ — + know when ribbon

Other natural examples of factorizable finite-dim'l Hopf algebras include <u>small</u> quantum groups $u_q(\mathfrak{g})$ — nonsemisimple !!!

GAME
launched in
early 1990s
by Reshetikhin,
Turaev,
Witten

CATEGORY THEORY

MODULAR
CATEGORY
$\mathcal{C}$? ~~semisimple~~ ? ? ?

⇝

QUANTUM
FIELD THEORY

TQFT

⇝

TOPOLOGY

INVARIANTS of LOW
DIM'L MANIFOLDS
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via
reps

QUANTUM ALGEBRA

certain
Hopf algebra H ? ~~semisimple~~ ? ? ?

CAN THIS GAME
BE PLAYED WHEN
SEMISIMPLICITY
IS REMOVED ?

FORTUNATELY, yes!

NEW GAME
launched in
mid 1990s
by Hennings,
Lyubashenko
& carried on by DeRenzi
-Gainutdinov-Geer
-Patureau-Runkel

CATEGORY THEORY

MODULAR
CATEGORY
$\mathcal{C}$ just finite,
not nec. semisimple

⇝

QUANTUM
FIELD THEORY

TQFT
See De Renzi
talks here! :)

https://youtu.be/K-tBzUJmbJE

⇝

TOPOLOGY

INVARIANTS of LOW
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QUANTUM ALGEBRA

certain
Hopf algebra H not nec.
semisimple

GOAL: TO CONSTRUCT
NONSEMISIMPLE MTCs by

(1) GENERALIZING
SEMISIMPLE CONSTRUCTIONS

(2) SUPPLYING EXPLICIT
EXAMPLES USING SMALL
QUANTUM GROUPS & VARIANTS

See Gainutdinov–Lentner–Ohrmann 1809.02116

& Negron 1812.02277 for related work.

From now on, we'll use Lyubashenko's notion of an MTC:

Def'n
MTC $\mathcal{C}$ = ~~fusion~~ + ribbon + nondegenerate monoidal category
finite tensor / ~~semisimple~~

Example Drinfeld double $D(H)$ of a finitedim'l ~~Semisimple~~ Hopf algebra H

= quasitriangular, factorizable

Kauffman–Radford (1993) classify ribbon structures on $D(H)$

$\mathcal{C} = D(H)$–mod
= ~~fusion~~ finite tensor
+ braided + nondegenerate

+ know when ribbon

GOAL: To construct nonsemisimple MTCs by
(1) generalizing semisimple constructions

Categorically, consider "monoidal centers" $\mathcal{Z}(\mathcal{C})$ of finite tensor category $\mathcal{C}$.

Here, objects of $\mathcal{Z}(\mathcal{C})$ are pairs:

$$\Big(X \in \mathcal{C},\ \underbrace{c_{X,-} : X \otimes - \xrightarrow{\sim} - \otimes X}_{\text{"half braiding"}}\Big)$$

finite tensor category
+ braided + nondegenerate

Example $\mathcal{C} = H$–mod
finite dim'l Hopf algebra

Get $\mathcal{Z}(\mathcal{C}) \sim D(H)$–mod

Shimizu [1707.09691] generalized Kauffman–Radford's result to get ribbon structure on $\mathcal{Z}(\mathcal{C})$.

Theorem (using [Kauffman–Radford, Shimizu–2017])

① Let H be a finite dim'l Hopf algebra subject to ribbon condition in [KR].
Then $D(H)$-mod is an MTC.

② Let $\mathcal{C}$ be a finite tensor category subject to ribbon condition in [Shimizu-2017].
Then $\mathcal{Z}(\mathcal{C})$ is an MTC.

GOAL: TO CONSTRUCT NONSEMISIMPLE MTCs by
(1) GENERALIZING SEMISIMPLE CONSTRUCTIONS
(2) SUPPLYING EXPLICIT EXAMPLES USING SMALL QUANTUM GROUPS & VARIANTS

We will generalize this by

$D(H) \rightsquigarrow$ "braided Drinfeld doubles" $D_K(H)$
including small quantum groups & variants

$\mathcal{Z}(\mathcal{C}) \rightsquigarrow$ "relative monoidal centers" $\mathcal{Z}_\mathcal{B}(\mathcal{C})$
including reps of $D_K(H)$.

III

For H a finite-dim'l Hopf algebra

$D(H)$ = gens of H & of H^*
Drinfeld double
$\left\{\begin{array}{l}\text{relations of } H \text{ \& of } H^* \\ \text{relations interchanging gens of } H \text{ \& } H^* \text{ according to pairing } \langle , \rangle : H^* \otimes H \to \Bbbk\end{array}\right.$

K quasitriangular fin.dim'l Hopf algebra
H a Hopf alg in braided fin.ten.categ. K-mod.

$D_K(H)$ = gens of H, of H^*, & of K.
braided Drinfeld double
$\left\{\begin{array}{l}\text{relations of } K \\ \text{relations of } H, H^* \text{ in } K\text{-mod} \\ \text{relations interchanging gens of } H \text{ \& } H^* \text{ according to pairing } \langle , \rangle \text{ in } K\text{-mod.}\end{array}\right.$

Examples of braided Drinfeld doubles $D_K(H)$

(1) $K = \Bbbk$ is a quasiΔ finite dim'l Hopf algebra, with R-matrix $1 \otimes 1$

$\Bbbk$-mod $= \mathrm{Vec}_{\Bbbk}^{f.d.}$, a braided finite tensor category (trivial braiding)

Get $D_{\Bbbk}(H) \cong D(H)$, for any Hopf algebra H in $\mathrm{Vec}_{\Bbbk}^{f.d.}$

(2) $K = \Bbbk\mathbb{Z}_n$, for $n \geq 3$, quasiΔ fin. dim'l Hopf algebra, with R-matrix depending on n-th root of unity q

K-mod is a braided finite tens. categ. (braiding)

$H = \Bbbk[x]/(x^n)$, Hopf algebra in K-mod. Here, $H \rtimes K \cong T_n(q)$, Taft algebra.

Get $D_K(H) \cong u_q(\mathfrak{sl}_2)$, Frobenius-Lusztig kernel (small quantum group)

For a finite tensor category $\mathcal{C}$

$\mathcal{Z}(\mathcal{C})$ $\equiv$ consists of objects

monoidal center $(X \in \mathcal{C},\ c_{X,-} : X \otimes - \xrightarrow{\sim} - \otimes X)$

half-braidings

For a braided finite tensor category $\mathcal{B}$ & $\mathcal{C}$ a "$\mathcal{B}$-central" finite tens. category

$\exists$ nice functor $\mathcal{B}^{op} \to \mathcal{Z}(\mathcal{C})$

$\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$ $\equiv$ consists of objects

relative monoidal center $(X \in \mathcal{C},\ c_{X,-})$

compatible with $\mathcal{B}$-central structure

Examples of relative monoidal centers $\mathcal{Z}_{\mathcal{B}}(\mathcal{C})$

(1) $\mathcal{B} = \mathrm{Vec}_{\Bbbk}^{f.d.}$ & $\mathcal{C} = H$-mod for H a finite-dim'l Hopf alg./$\Bbbk$

Get $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) \sim \mathcal{Z}(\mathcal{C})$ (ordinary monoidal center),

also get $\sim D(H)$-mod $\sim {}^H_H\mathcal{YD}(\mathrm{Vec}_{\Bbbk}^{f.d.})$ (categ. of H-Yetter Drinfeld mod.)

(2) $\mathcal{B} = (\Bbbk\mathbb{Z}_n$-mod, c_q non-trivial braiding depending on n^{th} root of unity $q)$

$\mathcal{C} = T_n(q)$-mod, which is $\mathcal{B}$-central. Here, $H = \Bbbk[x]/(x^n) \in \mathrm{HopfAlg}(\mathcal{B})$.

Get $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) \sim u_q(\mathfrak{sl}_2)$-mod

(3) $\mathcal{B} = (K\text{-mod}, R)$ for (K, R) a finite dim'l quasi Δ Hopf algebra

$\mathcal{C} = H\text{-mod}$, for $H \in \mathrm{HopfAlg}(\mathcal{B})$

Get $Z_{\mathcal{B}}(\mathcal{C}) \sim D_K(H)\text{-mod} \sim {}^H_H\mathcal{YD}(\mathcal{B})$ (H-YD mods in $\mathcal{B}$)

Recall result that we'll generalize

Theorem (using [Kauffman-Radford, Shimizu-2017])

① Let H be a finite dim'l Hopf algebra

subject to ribbon condition in [KR].

Then $D(H)$-mod is an MTC.

② Let $\mathcal{C}$ be a finite tensor category

subject to ribbon condition in [Shimizu-2017].

Then $Z(\mathcal{C})$ is an MTC.

Theorem [Laugwitz-W, 2020]

① Let K be a fin-dim'l quasitriangular factorizable Hopf algebra

Let H be a finite dim'l Hopf algebra, in K-mod,

subject to ribbon condition in [KR-1993]

Then $D_K(H)$-mod is an MTC.

② Let $\mathcal{B}$ be a nondegenerate braided finite tensor category

Let $\mathcal{C}$ be a finite tensor category, that is $\mathcal{B}$-central,

subject to ribbon condition in [Shimizu-2017].

Then $Z_{\mathcal{B}}(\mathcal{C})$ is an MTC.

Applies to Examples above when K (or $\mathcal{B}$) is factorizable (or nondeg.)
eg. $(\Bbbk$, trivial braiding$)$, $(\Bbbk\mathbb{Z}_n$, eq non-trivial braiding$)$

Proposition [LW, 2020] Consider $\mathcal{B} = {}^K_K\mathcal{YD}$ for K a finite abelian group,
& $B(V) \in \mathrm{HopfAlg}(\mathcal{B})$, Nichols algebra of "diagonal type"
Then $\mathcal{Z}_{\mathcal{B}}(B(V)\text{-mod}(\mathcal{B})) \sim D_K(B(V))\text{-mod}$ is modular when
(i) the (canonical) symmetric bilinear form b on K is nondegenerate
(ii) certain conditions involving elements of the top degree of $B(V)$ & an R-matrix of K are satisfied.

Examples (1) $U_q(\mathfrak{g})$-mod is modular. , for $\mathfrak{g}$ semisimple Lie alg.
Here, the corresponding Nichols alg is of "Cartan type"

(2) $\exists\, D_K(B(V))$ not of Cartan type so that
$D_K(B(V))$-mod is modular
checked explicitly for one of "super-type".

IV. This prompts the following investigation:
Both Examples (1) & (2) above are pointed finite-dim'l
Hopf algebras attached to Nichols algebras of "diagonal type"

There's a thorough study and classification of such Hopf algs.

See Andruskiewitsch-Angiono 1707.08387 (182 pages!)
Fall into three types

* "Cartan type": small quantum groups & their deformations / "liftings"

* "Super type": related to super Lie algebras

* "UFO type": exceptional cases unrelated to Lie theory

have a family of such modular Hopf algebras (Example 5.17 in [LW])

have an example of such a modular Hopf algebra (Example 5.18 in [LW])

QUESTION: What about the rest of the pointed fin. dim'l Hopf algebras H above? When is H-mod modular?

Could use Proposition above in explicit computations to produce not nec. MTCs.

QUESTION The "modular Nichols algebra" Proposition above doesn't overlap completely the related results of

GLO [1809.02116] concerns quasi-Hopf algs, $\mathcal{B} = \mathrm{Vec}_G$ and Vec_G^ω, G fin. abelian

Negron [1812.02277] at even roots of unity, concerns quasi-Hopf algs

Is there a unifying approach?

GAME

launched in mid 1990s by Hennings, Lyubashenko

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certain Hopf algebra H not nec. semisimple

CONSTRUCTED NONSEMISIMPLE MTCs by

(1) GENERALIZING SEMISIMPLE CONSTRUCTIONS $\quad Z_B(\mathcal{C})$

(2) SUPPLYING EXPLICIT EXAMPLES USING SMALL QUANTUM GROUPS & VARIANTS $\quad D_K(H)$

Thanks for Listening!

POST-TALK: COMMENTS ON MAIN PROOFS

Theorem [Laugwitz-W, 2020]

① Let K be a fin-dim'l quasitriangular factorizable Hopf algebra.
Let H be a finite dim'l Hopf algebra in K-mod,
subject to ribbon condition in [KR-1993].
Then $D_K(H)$-mod is an MTC.

② Let $\mathcal{B}$ be a nondegenerate braided finite tensor category.
Let $\mathcal{C}$ be a finite tensor category, that is $\mathcal{B}$-central,
subject to ribbon condition in [Shimizu-2017].
Then $Z_{\mathcal{B}}(\mathcal{C})$ is an MTC.

Pf/ ② $\Rightarrow$ ①. And we made connection between ribbon conditions very explicit

Have $\mathcal{Z}_{\mathcal{B}}(\mathcal{C}) = C_{\mathcal{B}^{op}}(\mathcal{Z}(\mathcal{C}))$ is a Müger centralizer.

We generalized Müger's result on modular centralizers to the nonsemisimple case: $\mathcal{B}$ nondeg + $\mathcal{Z}(\mathcal{C})$ modular $\Rightarrow$ $C_{\mathcal{B}^{op}}(\mathcal{Z}(\mathcal{C}))$ modular

Key tool: Shimizu's Double Centralizer Theorem in nonsemisimple case.

Proposition [LW, 2020] Consider $\mathcal{B} = {}^{K}_{K}\mathcal{YD}$ for K a finite abelian group, & $B(V) \in \mathrm{HopfAlg}(\mathcal{B})$, Nichols algebra of "diagonal type"

Then $\mathcal{Z}_{\mathcal{B}}(B(V)\text{-mod}(\mathcal{B})) \sim D_K(B(V))\text{-mod}$ is modular when:

(i) the (canonical) symmetric bilinear form b on K is nondegenerate

(ii) certain conditions involving elements of the top degree of $B(V)$ & an R-matrix of K are satisfied.

Pf/ (i) $\Rightarrow$ $\mathcal{B}$ is nondegenerate

(ii) $\Rightarrow$ $D_K(B(V))$ is ribbon (via Shimizu's ribbon condition)

Question Could inquire about converses holding.