CMS meeting - Ottawa

December 7, 2013

(2:00)

$$\boxed{\text{PBW deformations of } \underset{\uparrow}{B} \# H \leftarrow \text{Hopf alg}}$$

Koszul alg

joint w/ Sarah Witherspoon
Arxiv 1308.6011

Given a graded algebra $A = \bigoplus_{i\in\mathbb{N}} A_i$

and a filtered algebra $D = \bigcup_{i\in\mathbb{N}} F_i$, $\{0\}\subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq D$

say D is a <u>PBW deformation</u> of A if $\underset{\underset{\bigoplus_{i\in\mathbb{N}} F_i/F_{i-1}}{\parallel}}{\mathrm{gr}_F D} \cong A$ as algebras

Examples

- Weyl algebra $A_1(\mathbb{C}) = \mathbb{C}\langle v,w\rangle / (vw-wv-1)$ is a PBW deformation of $\mathbb{C}[v,w]$

- V = finite dim'l $\mathbb{C}$-v.s.

$D_K := T(V) / (vw-wv-K(v,w))_{v,w\in V}$, $K: \underset{\underset{V\otimes V}{\cup\!\!\!\!\mid}}{\mathrm{Span}_{\mathbb{C}}\{vw-wv\}_{v,w\in V}} \to V$
$\mathbb{C}$-bilinear map.

is a PBW deformation of $S(V) = T(V)/(vw-wv)_{v,w\in V}$

PBW Theorem $\Uparrow$

$\Updownarrow$

* K is skew symmetric
* K satisfies Jacobi identity

i.e. $D_K = U(V)$
univ env alg of Lie alg V
w/ $[v,w] =: K(v,w)$

Properties preserved under PBW deformation

proved in [MR]
- integral domain
- prime
- (right) noetherian

$GK(A) \le GK(D)$
$K\dim(D) \le K\dim(A)$
$\mathrm{gl}\dim(D) \le \mathrm{gl}\dim(A)$

CY property investigated [Berger-Taillefer 2007], [Wu-Zhu, 2013]

Representation thy of PBW deform'ns
has been of great (recent) interest

eg: symplectic reflection algs
rational Cherednik algs
various types of Hecke algs

$\}\to$ of smash product algs $B\#\mathbb{C}G$ $(= B\rtimes G)$

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<u>Some results</u> on PBW deformations of $B\#H$, $B = T(V)/R$ quadratic algebra

Have nec & suff conditions for $\circledast$

$$\boxed{\mathcal{D}_\kappa = \overset{\text{deg } 0}{T(V)\#H} \Big/ (r-\kappa(r))_{r\in R} \qquad \kappa: R\to H \ \mathbb{C}\text{-linear}}$$

to be a PBW def of $B\#H$

Some of these results and our thm holds w/

$\kappa: R\longrightarrow H\oplus(V\otimes H)$

B	H	$\circledast$	Ref
$\mathbb{C}[v,w]$	$\Gamma\subseteq SL_2(\mathbb{C})$ finite	$\kappa(r)\in Z(\Gamma)$	Crawley-Boevey Holland [CBH, 1998]
$S(V)$	various G	Various results	Drinfeld, Ram-Shepler [D] [L], [RS], [SW] Lusztig, Shepler-Witherspoon
$S(V)$	$U(\mathfrak{g})$		Etingof-Ginzburg [EG], [KT] Khare-Tikaradze
$S_q(V)$	G		[LS], [NW] Levandovskyy-Shepler, Naidu-Witherspoon
$S_q(V)$	$U_q(\mathfrak{sl}_2)$		[GK] Gan-Khare
all Koszul algs	all Hopf algebras		

Theorem [W-Witherspoon] $B = T(V)/R$ Koszul algebra $(R\subseteq V\otimes V)$
Let H be a Hopf algebra (w/ bijective antipode)
that acts on B (preserving grading)

Then
$\mathcal{D}_\kappa$ is a PBW deformation of $B\#H$ $\Longleftrightarrow$
- κ is H-invariant $\left[\kappa(h\cdot r) = h\cdot\kappa(r) \ {}^{h\in H}_{r\in R}\right]$
- $\underset{\uparrow \text{ mult of } B}{m}\circ(\kappa\otimes id - id\otimes\kappa) \equiv 0$ $\uparrow$ as maps on $(R\otimes V)\cap(V\otimes R)$

need two more conditions

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— $\mathbb{k}=\mathbb{C}$.

<u>Outline of proof</u> (following Braverman–Gaitsgory)

($\Rightarrow$) easier direction

($\Leftarrow$) STS $\quad \underset{\underset{\deg t}{\uparrow}}{\mathcal{D}_{\kappa,t}} = T(V)\#H \big/ (r-\kappa(r)t^2)_{r\in R}$ is a graded def of $\underbrace{B\#H}_{A}$ over $\mathbb{C}[t]$

$\Updownarrow$

$\mathcal{D}_{\kappa,t} \cong A[t]$ as $\mathbb{C}$-vs.

new mult:

$a_1 * a_2 = \sum \mu_i(a_1,a_2)\, t^i$

$\mu_i : A\otimes A \to A$ i^{th} mult map

associativity of $*$ imposes conditions on $\{\mu_i\}$

e.g. μ_1 must be a Hochschild 2-cocycle on bar resolution of A $\Big)$ $(\dagger)$

Construct bimodule resolution $X_\bullet$ of $A=B\#H$ $=$ (Koszul resolution of B) $\otimes$ (bar resolution of H) (adapted from Guccione–Guccione 2002)

Extend $\kappa : R \to H$ to map on $X_\bullet$ $\rightsquigarrow$ used to construct μ

conditions on κ $\Rightarrow$ $(\dagger)$

Construct appropriate μ_i using $\mu_1, \dots$ ///

(2.18)

New Examples of PBW deformations of smash product algebras from Hopf actions on Koszul algs

in particular noncocommutative $\downarrow$

(actions from joint w/ K.Chan, E.Kirkman, J.Zhang)

① $H=H_8$ 8-dim'l semisimple

$\hookrightarrow B = \mathbb{C}\langle v,w\rangle \big/ (\underbrace{vw+wv}_{r})$

Get $\mathcal{D}_\kappa = \mathbb{C}\langle v,w\rangle \# H_8 \big/ (r-\kappa(r))$ is a PBW def of $B\#H_8$

Get 5 parameter family

for $\kappa(r) = p(\gamma_1,\dots,\gamma_5) \in H_8$ $\quad \gamma_i \in \mathbb{C}$

② $H=H_{16}$ 16-dim'l semisimple

$\hookrightarrow B = \mathbb{C}\langle v_1,\dots,v_4\rangle \big/ (r_1 = v_1v_2 - v_2v_1,\ r_2, \dots, r_6)$

$\mathcal{D}_\kappa = \mathbb{C}\langle v_1,\dots,v_4\rangle \# H_{16} \big/ (r-\kappa(r))_{r\in R}$ is a PBW def of $B\#H_{16}$

Get 2 parameter family

for $\kappa(r_1) = p(\gamma_1,\gamma_2) \in H_{16}$

$\kappa(r_i) = 0$ for $i\neq 1$

<u>Future work</u>: Study PBW deformations of $B \# H$ where

	<u>B</u>	<u>H</u>	<u>Ref</u>
1)	N-Koszul	H	
	[N-Koszul	G	Cassidy-Shelton]
2) ["quantized sym. oscillator alg of rank n-1"]	$S_q(V_{\rho_{\dim n}})$	$U_q(\mathfrak{sl}_n)$	
	$S_q(V_{\rho_{\dim 2}})$	$U_q(\mathfrak{sl}_2)$	[WW]]

3) Ties to recent work of Ding-Tsymbaliuk, Losev-Tsymbaliuk, Tikaradze, in general: "infinitesimal Hecke algebras" of Etingof-Gan-Ginzburg.

$S(V)$	$U(\mathfrak{g})$
$V = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$	$\mathfrak{gl}_n$
$V = \mathbb{C}^{2n}$	$\mathfrak{sp}_{2n}$