

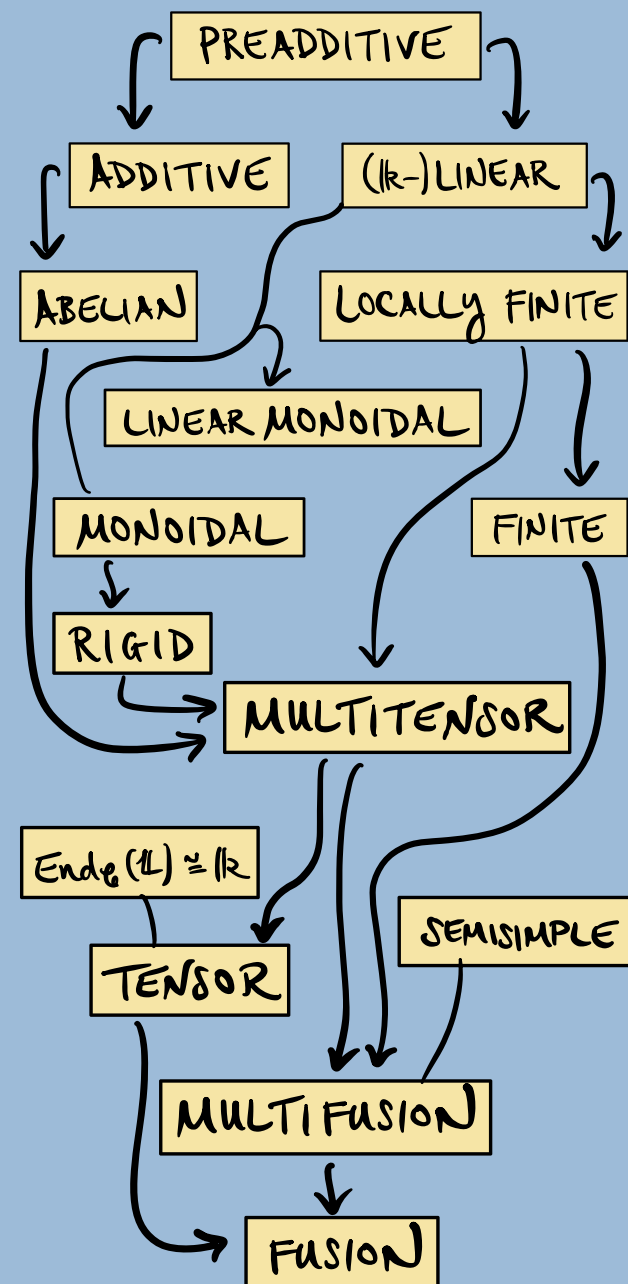
AN INTRODUCTION TO TENSOR CATEGORIES

Hour 2: ON THE ROAD TO MODULARITY

CHELSEA WALTON
RICE UNIVERSITY
NOTLAW@RICE.EDU

LAST TIME: THE ROAD TO FUSION

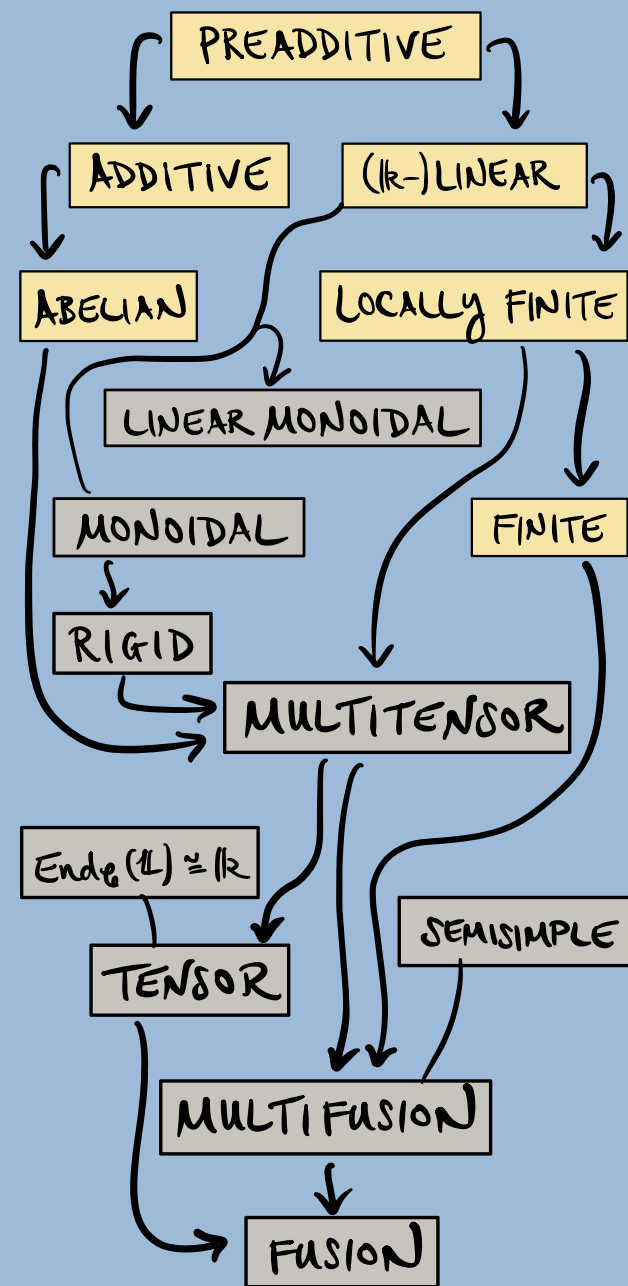
A CATEGORY $\mathcal{C}$ IS ...



LAST TIME: THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

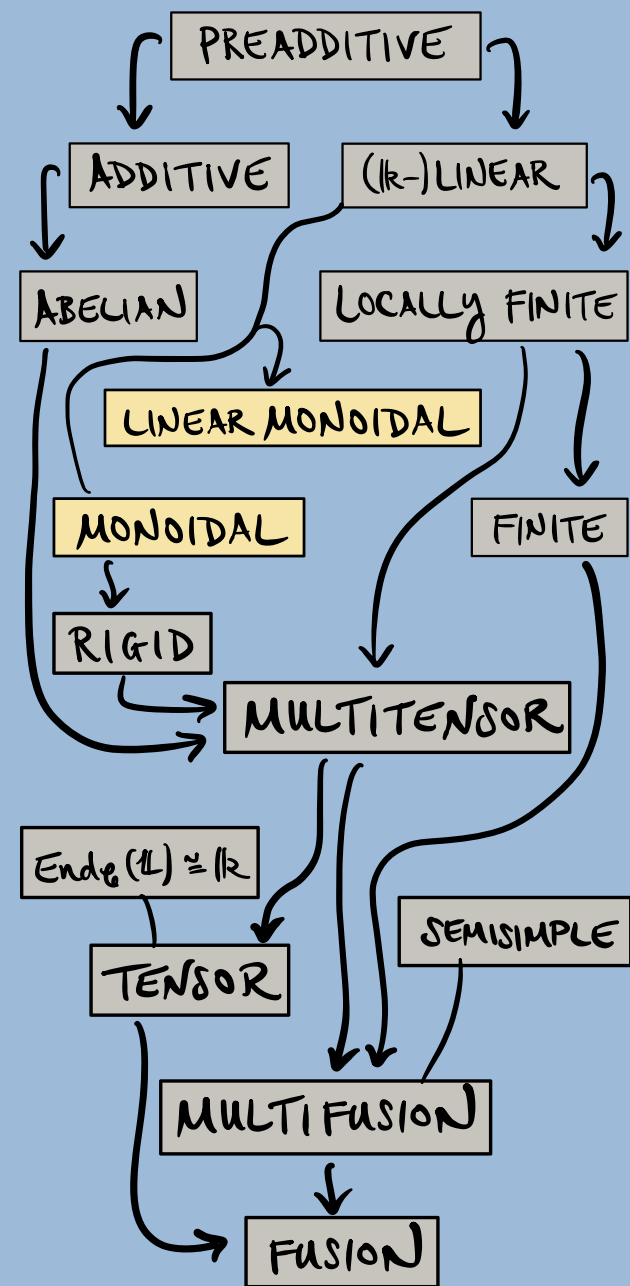
HOMOLOGICAL
CONDITIONS



LAST TIME: THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

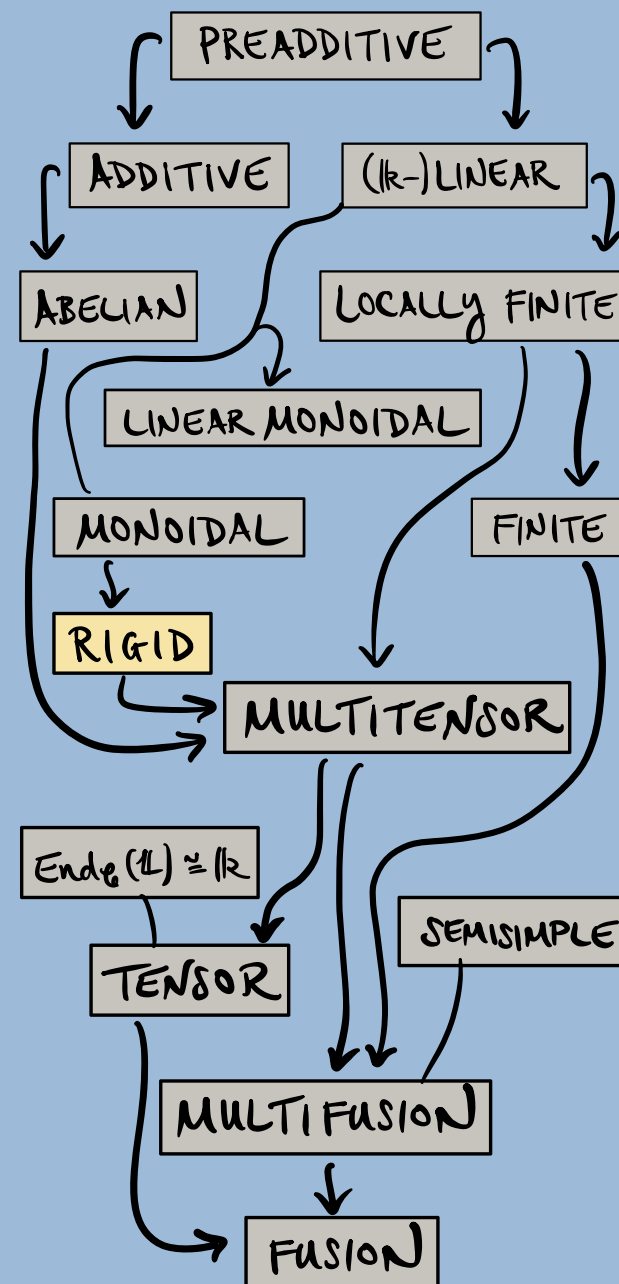
AN OPERATION $\otimes$



LAST TIME: THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

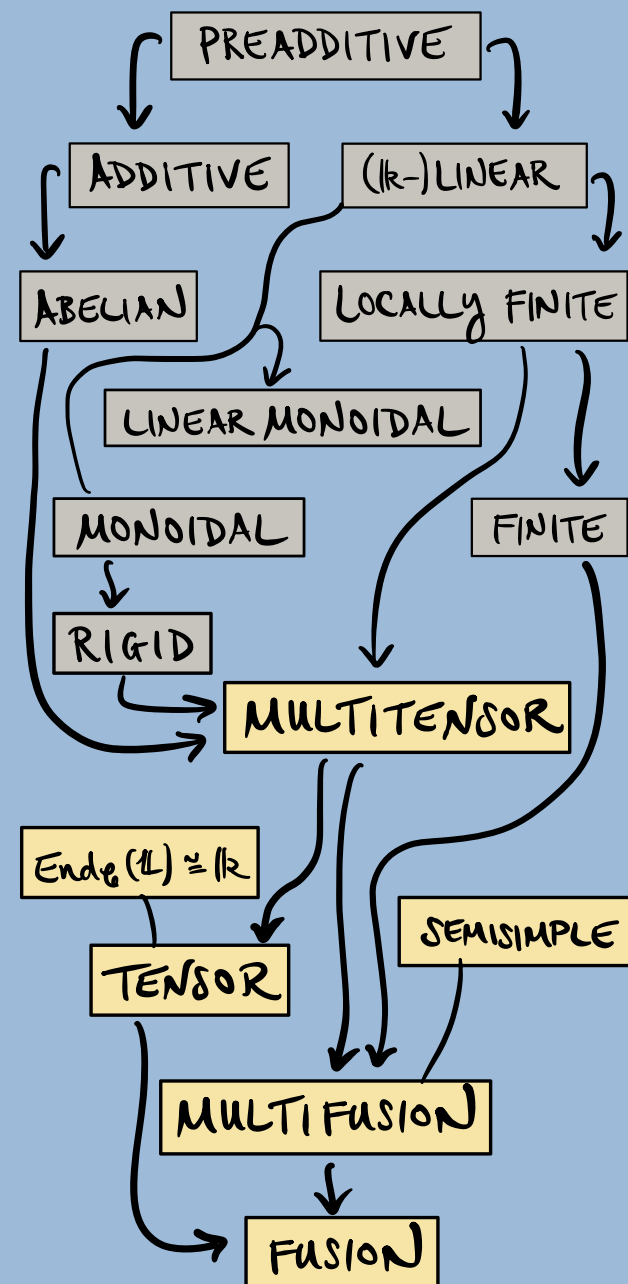
DUALITY WITH RESP TO $\otimes$



LAST TIME: THE ROAD TO FUSION

A CATEGORY $\mathcal{C}$ IS ...

COMBINATIONS OF
THE ABOVE WITH SOME
EXTRA FEATURES



LAST TIME: THE ROAD TO FUSION

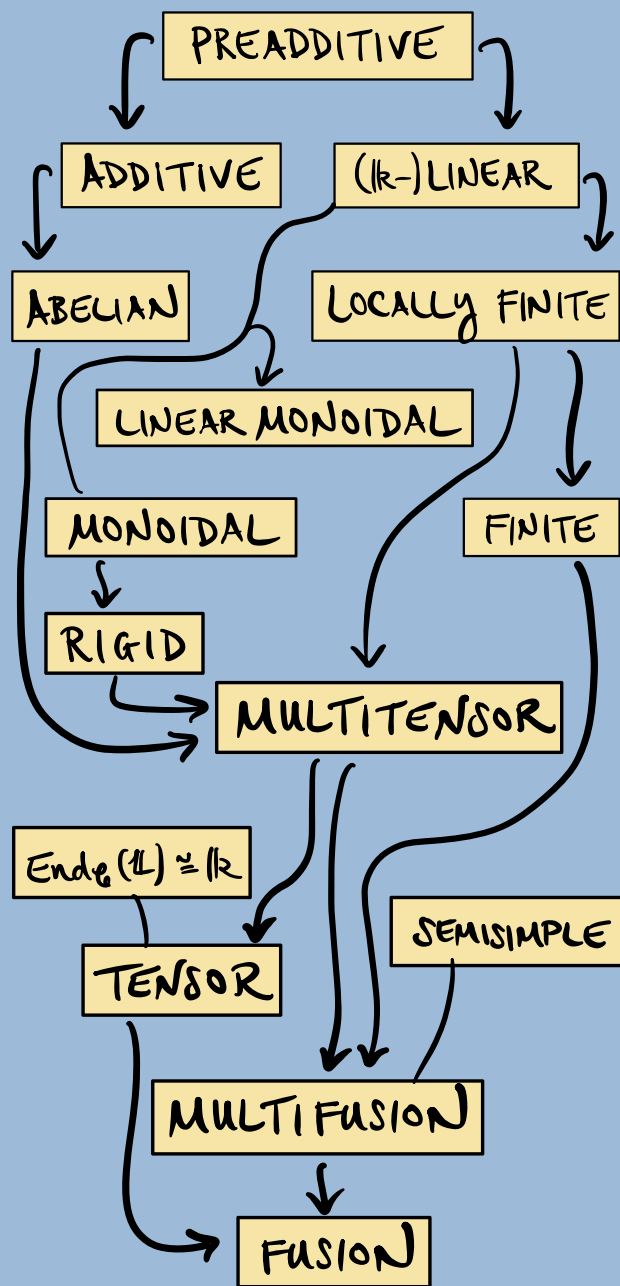
A CATEGORY $\mathcal{C}$ IS ...

HOMOLOGICAL
CONDITIONS

AN OPERATION $\otimes$

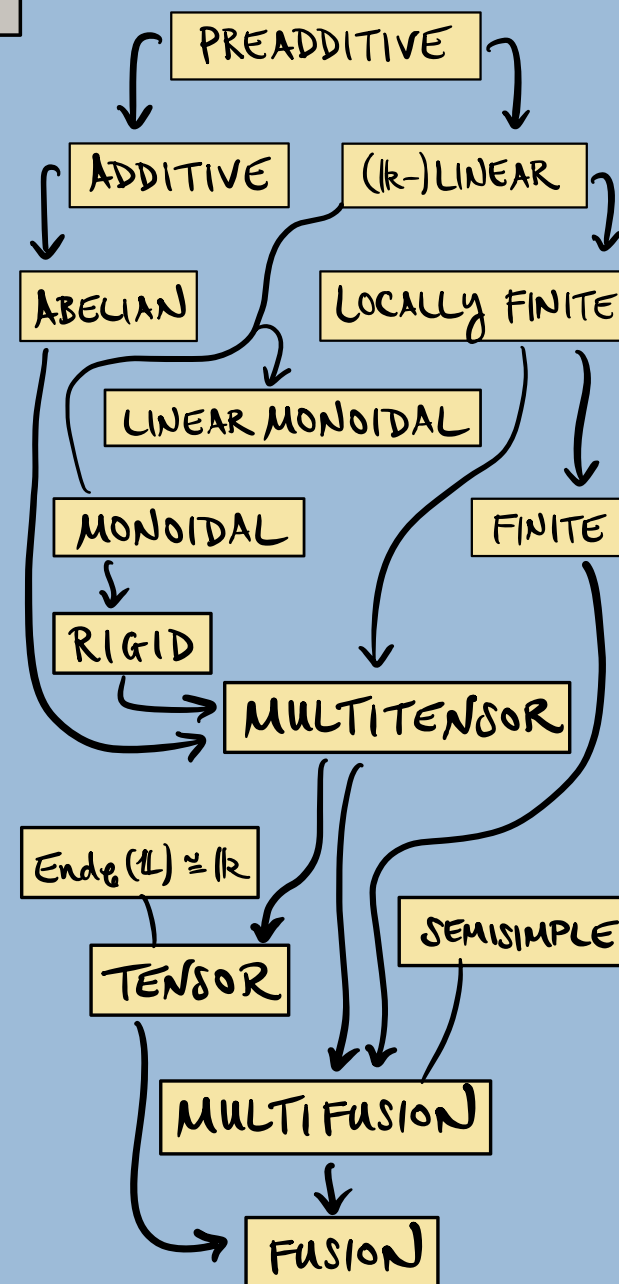
DUALITY WITH RESP TO $\otimes$

COMBINATIONS OF
THE ABOVE WITH SOME
EXTRA FEATURES



LAST TIME: THE ROAD TO FUSION & DETOURS

A CATEGORY $\mathcal{C}$ IS ...



LAST TIME: THE ROAD TO FUSION & DETOURS

A CATEGORY $\mathcal{C}$ IS ...

(CO) COMPLETE

EXACT
(IN THE SENSE OF QUILLÉN)

TRIANGULATED

PREADDITIVE

ADDITIVE

($\mathbb{K}$ -)LINEAR

ABELIAN

LOCALLY FINITE

LINEAR MONOIDAL

MONOIDAL

FINITE

RIGID

MULTITENSOR

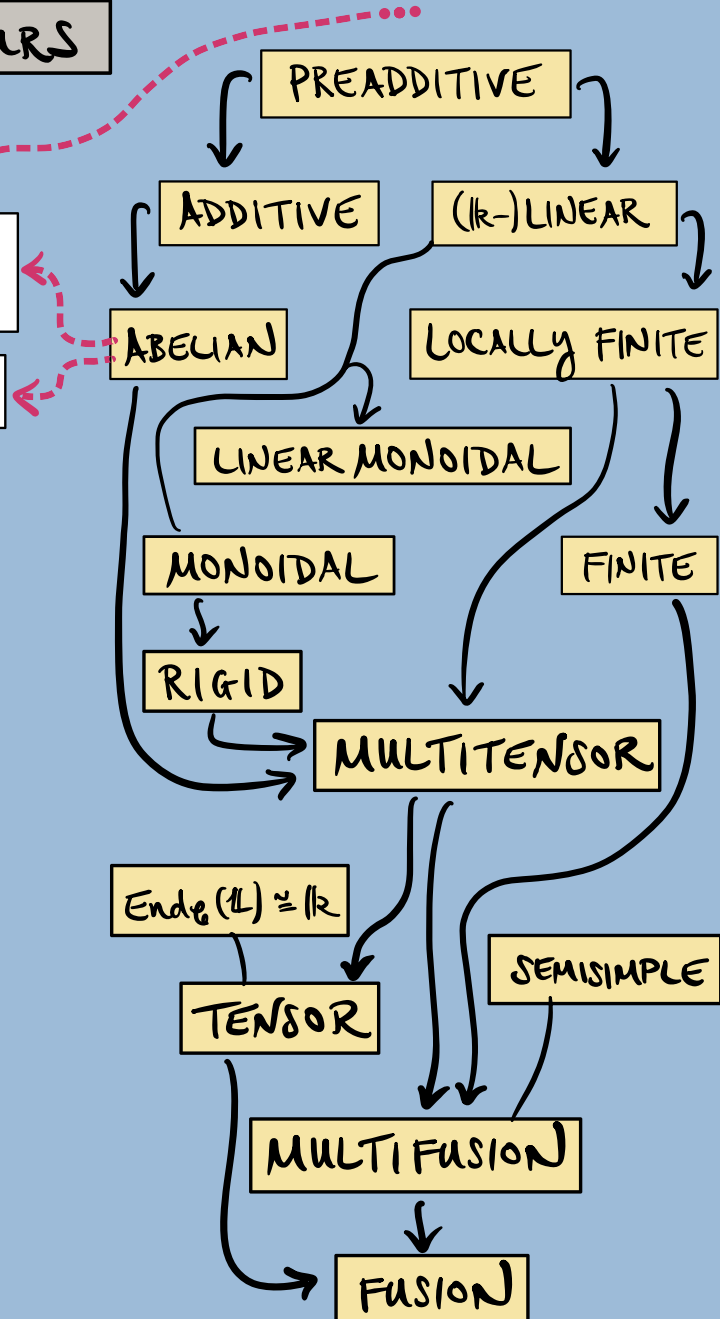
$\text{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{K}$

TENSOR

SEMISIMPLE

MULTIFUSION

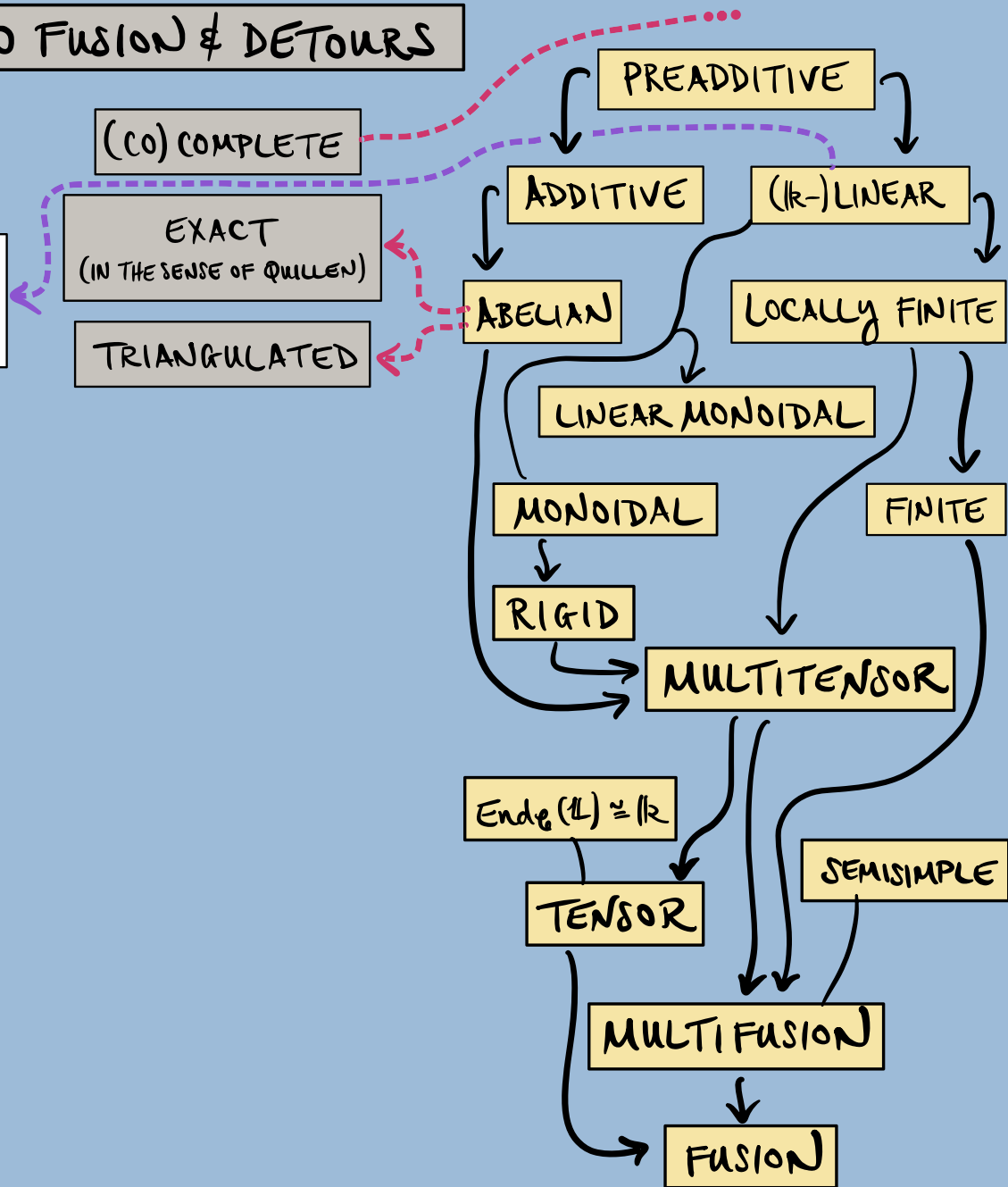
FUSION



LAST TIME: THE ROAD TO FUSION & DETOURS

A CATEGORY $\mathcal{C}$ IS ...

CATEGORY $\mathcal{A}$ ENRICHED OVER
A MONOIDAL CATEGORY $\mathcal{V}$



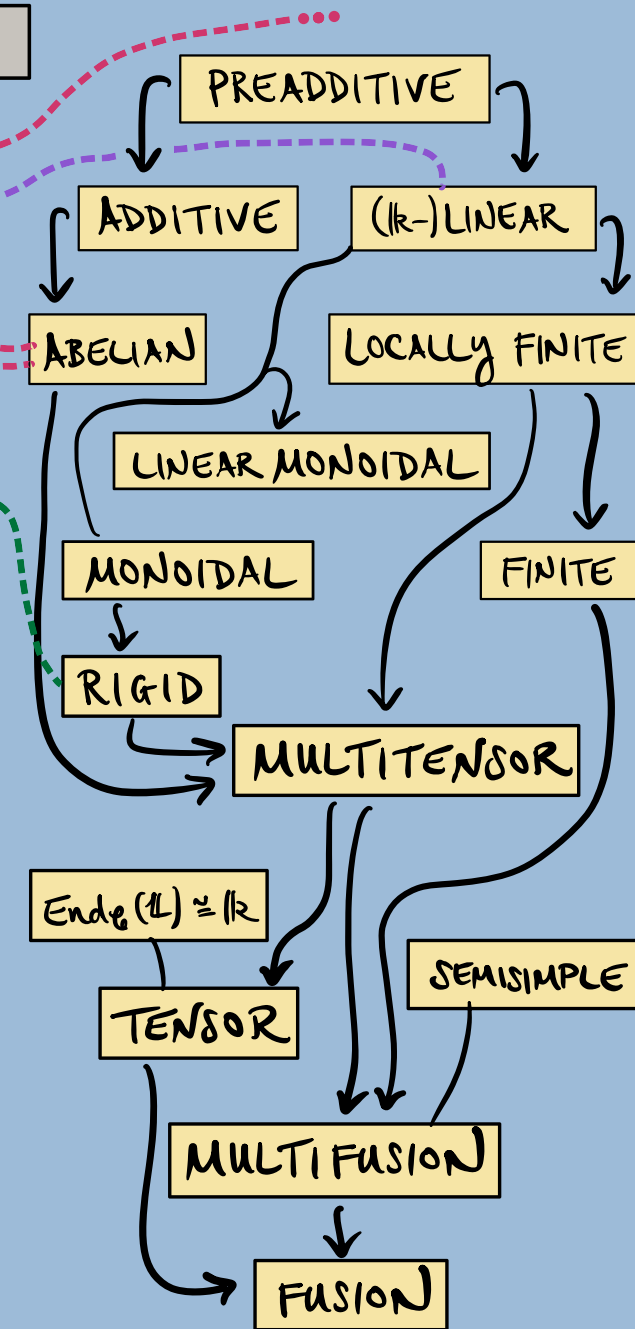
LAST TIME: THE ROAD TO FUSION & DETOURS

A CATEGORY $\mathcal{C}$ IS ...

CATEGORY $\mathcal{A}$ ENRICHED OVER
A MONOIDAL CATEGORY $\mathcal{V}$

(CO) COMPLETE
EXACT
(IN THE SENSE OF QUILLÉN)
TRIANGULATED

A GV-CATEGORY

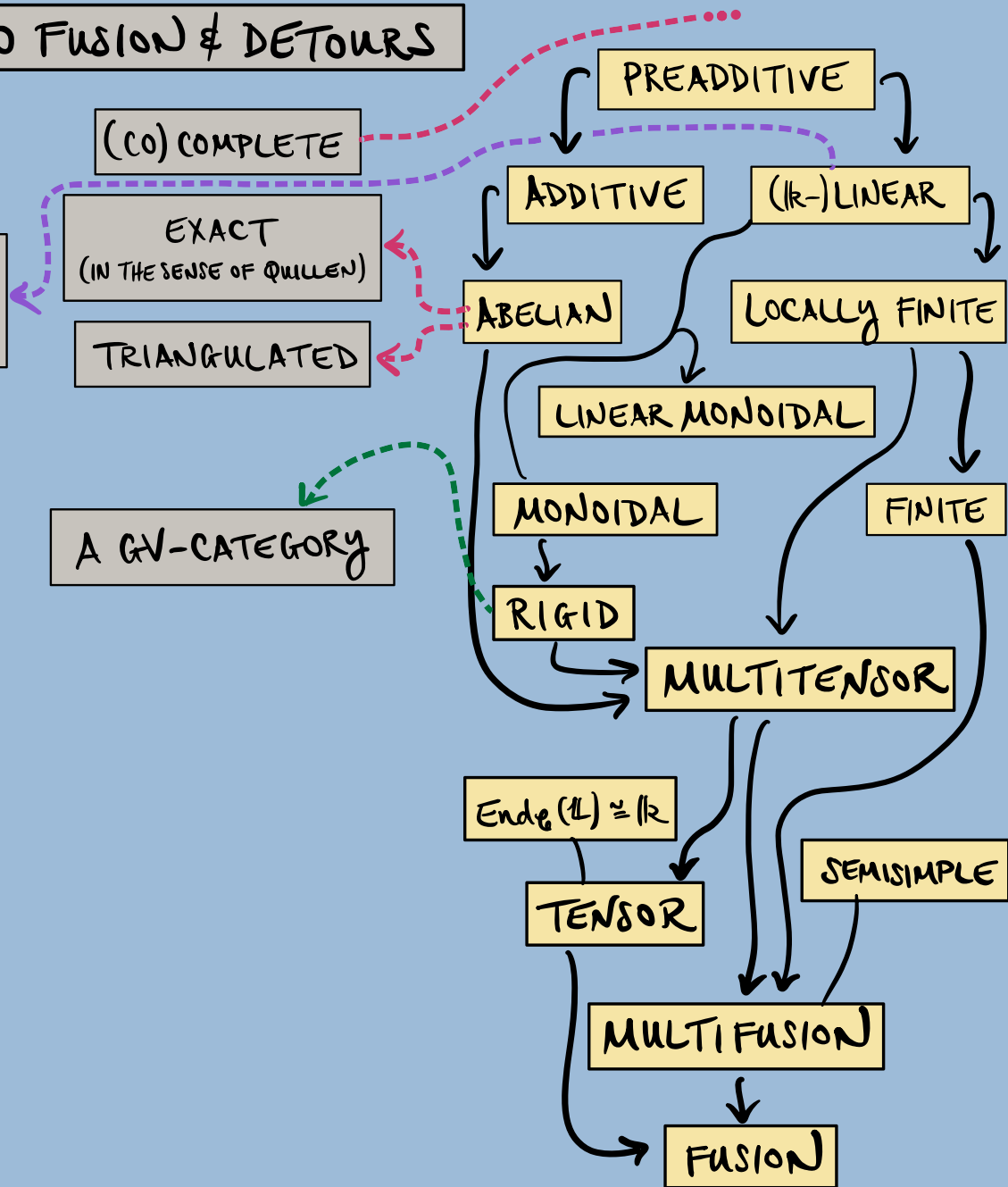


LAST TIME: THE ROAD TO FUSION & DETOURS

A CATEGORY $\mathcal{C}$ IS ...

CATEGORY $\mathcal{A}$ ENRICHED OVER
A MONOIDAL CATEGORY $\mathcal{V}$

USEFUL CONSTRUCTION:
COENDS & ENDS



LAST TIME: THE ROAD TO FUSION & DETOURS

A CATEGORY $\mathcal{C}$ IS ...

CATEGORY $\mathcal{A}$ ENRICHED OVER
A MONOIDAL CATEGORY $\mathcal{V}$

USEFUL CONSTRUCTION:
COENDS & ENDS

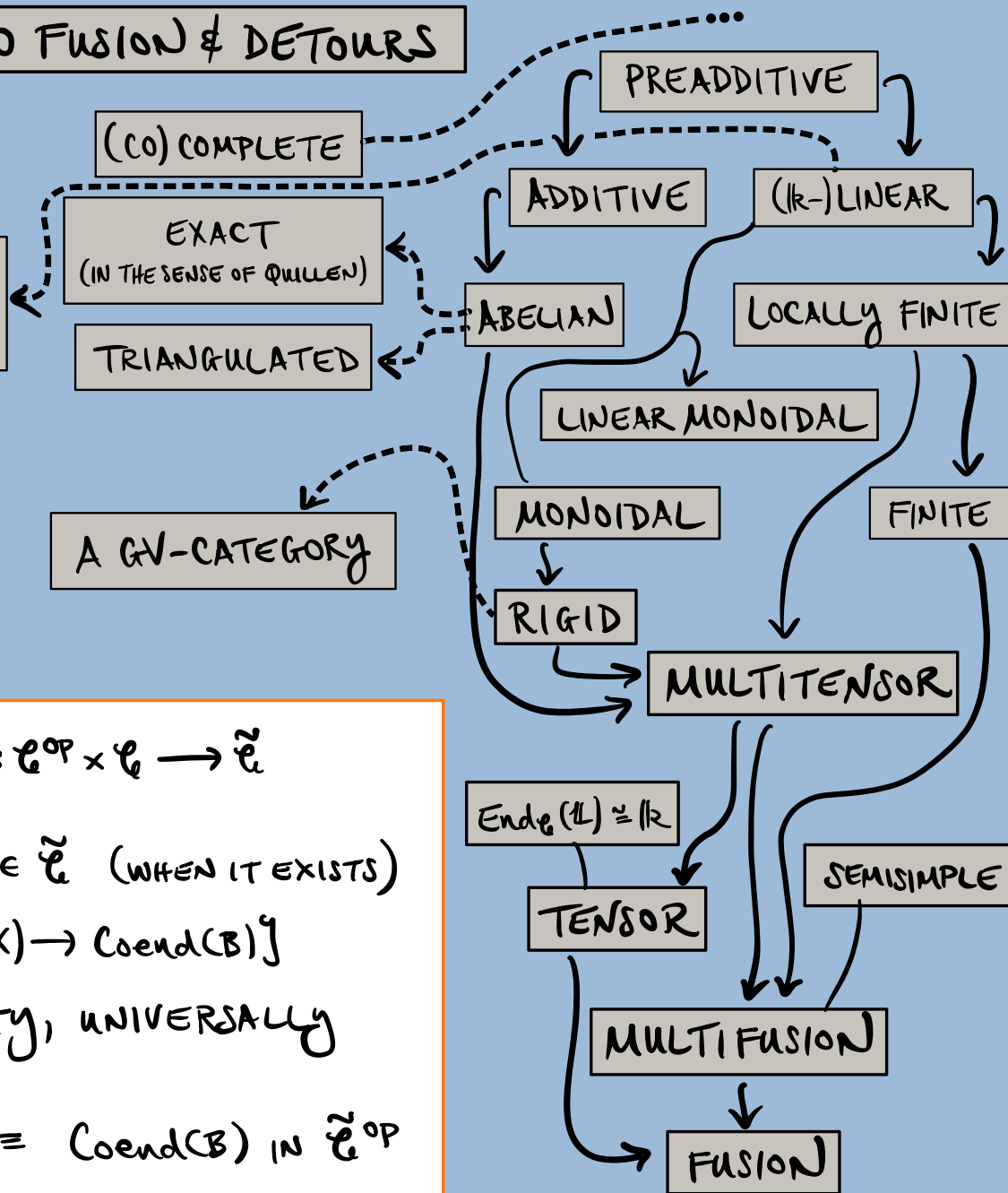
TAKE BIFUNCTOR $B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}}$

$\text{Coend}(B) := \int^{X \in \mathcal{C}} B(X, X) \in \tilde{\mathcal{C}}$ (WHEN IT EXISTS)

EQUIPPED W/ $\{ \iota_X: B(X, X) \rightarrow \text{Coend}(B) \}$

SATISFYING DINATURALITY, UNIVERSALLY

$\text{End}(B) := \int_{X \in \mathcal{C}} B(X, X) = \text{Coend}(B) \text{ IN } \mathcal{C}^{\text{op}}$



LAST TIME: THE ROAD TO FUSION & DETOURS

EX. TAKE $\tilde{\mathcal{C}} = \text{FdVec}$.

FOR A $\mathbb{K}$ -COALGEBRA C :

$$\int_{V \in \text{FdComod-}C} V^* \otimes V \cong C \text{ AS } \mathbb{K}\text{-COALGS}$$

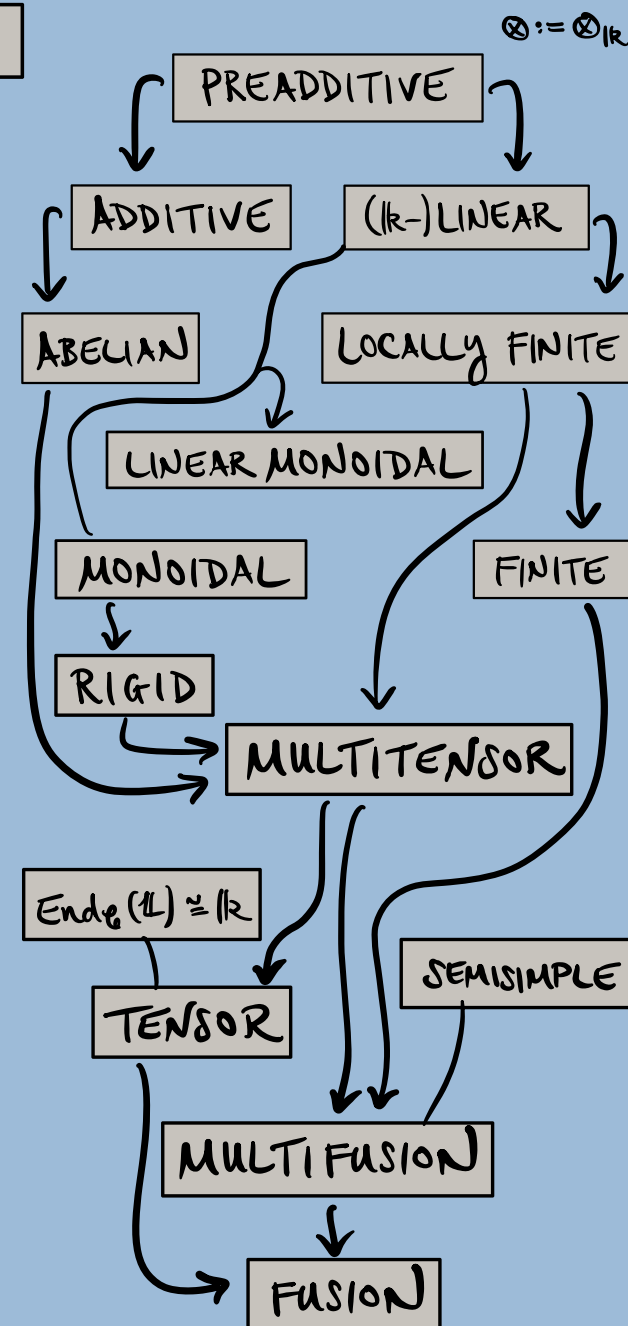
TAKE BIFUNCTOR $B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}}$

$\text{Coend}(B) := \int^{X \in \mathcal{C}} B(X, X) \in \tilde{\mathcal{C}}$ (WHEN IT EXISTS)

EQUIPPED W/ $\{\iota_X: B(X, X) \rightarrow \text{Coend}(B)\}$

SATISFYING DINATURALITY, UNIVERSALLY

$\text{End}(B) := \int_{X \in \mathcal{C}} B(X, X) = \text{Coend}(B) \text{ IN } \mathcal{C}^{\text{op}}$



LAST TIME: THE ROAD TO FUSION & DETOURS

EX. TAKE $\tilde{\mathcal{C}} = \text{FdVec}$.

FOR A $\mathbb{K}$ -COALGEBRA C :

$$\int_{V \in \text{FdComod-}C} V^* \otimes V \cong C \text{ AS } \mathbb{K}\text{-COALGS}$$

FOR A F.D. $\mathbb{K}$ -ALGEBRA A :

$$\int_{V \in A\text{-FdMod}} V \otimes V^* \cong A \text{ AS } \mathbb{K}\text{-ALGS}$$

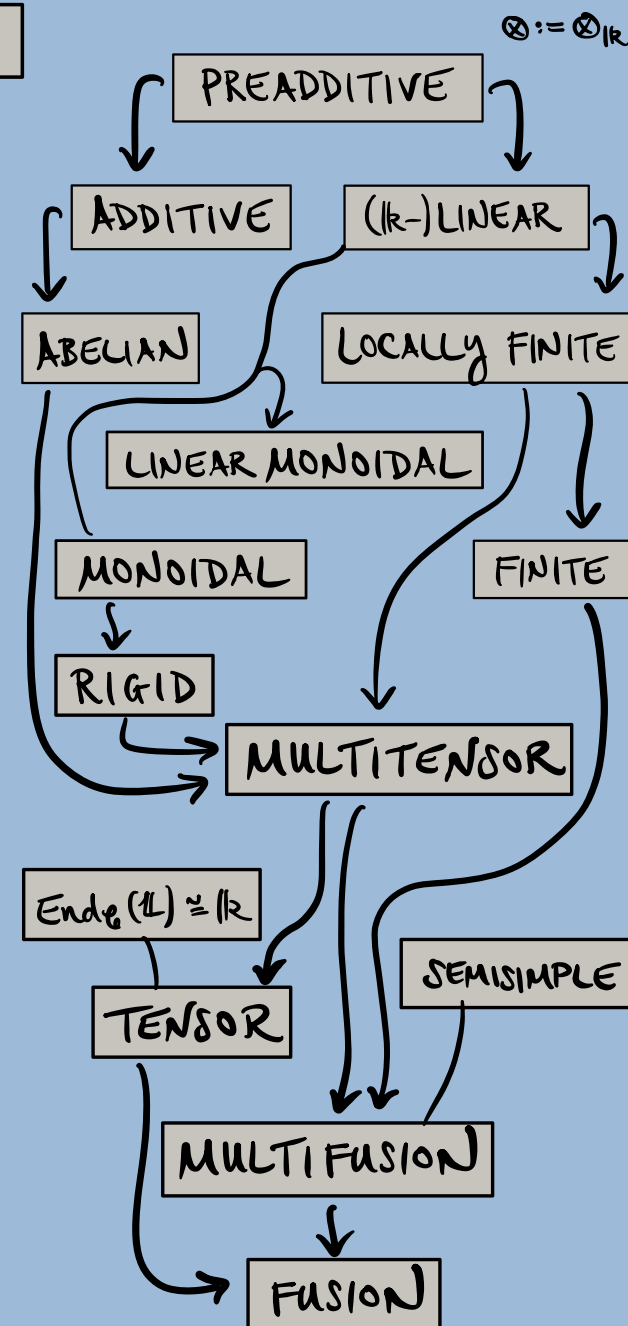
TAKE BIFUNCTOR $B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}}$

$$\text{Coend}(B) := \int^{X \in \mathcal{C}} B(X, X) \in \tilde{\mathcal{C}} \text{ (WHEN IT EXISTS)}$$

EQUIPPED W/ $\{\iota_X: B(X, X) \rightarrow \text{Coend}(B)\}$

SATISFYING DINATURALITY, UNIVERSALLY

$$\text{End}(B) := \int_{X \in \mathcal{C}} B(X, X) = \text{Coend}(B) \text{ IN } \tilde{\mathcal{C}}^{\text{op}}$$



LAST TIME: THE ROAD TO FUSION & DETOURS

EX. TAKE $\tilde{\mathcal{C}} = \text{FdVec}$.

FOR A $\mathbb{K}$ -COALGEBRA C :

$$\int_{V \in \text{FdComod-}C} V^* \otimes V \cong C \text{ AS } \mathbb{K}\text{-COALGS}$$

FOR A F.D. $\mathbb{K}$ -ALGEBRA A :

$$\int_{V \in A\text{-FdMod}} V \otimes V^* \cong A \text{ AS } \mathbb{K}\text{-ALGS}$$

RECOVER (CO)ALG.
FROM ITS CATEG.
OF (CO)MODS.

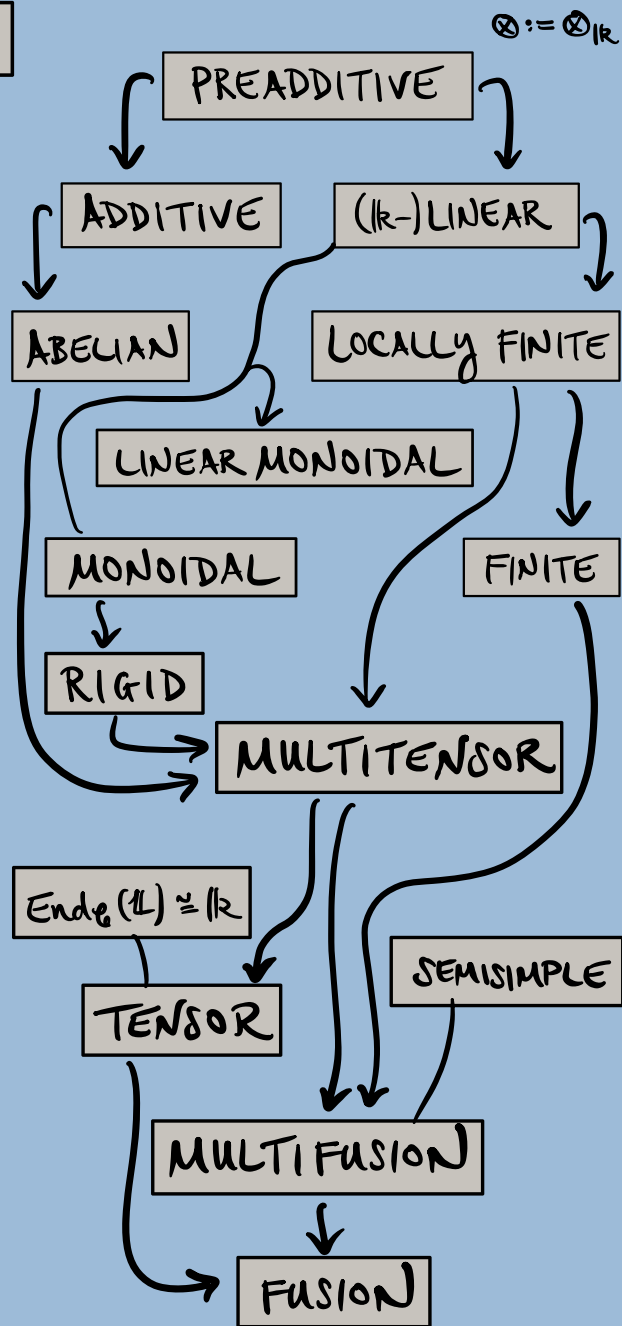
TAKE BIFUNCTOR $B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}}$

$$\text{Coend}(B) := \int^{X \in \mathcal{C}} B(X, X) \in \tilde{\mathcal{C}} \text{ (WHEN IT EXISTS)}$$

EQUIPPED W/ $\{\iota_X: B(X, X) \rightarrow \text{Coend}(B)\}$

SATISFYING DINATURALITY, UNIVERSALLY

$$\text{End}(B) := \int_{X \in \mathcal{C}} B(X, X) = \text{Coend}(B) \text{ IN } \tilde{\mathcal{C}}^{\text{op}}$$



RECONSTRUCTION THEORY

EX. TAKE $\tilde{\mathcal{C}} = \text{FdVec}$.

FOR A $\mathbb{K}$ -COALGEBRA C :

$$\int_{V \in \text{FdComod-}C} V^* \otimes V \cong C \text{ AS } \mathbb{K}\text{-COALGS}$$

FOR A F.D. $\mathbb{K}$ -ALGEBRA A :

$$\int_{V \in A\text{-FdMod}} V \otimes V^* \cong A \text{ AS } \mathbb{K}\text{-ALGS}$$

RECOVER (CO)ALG.
FROM ITS CATEG.
OF (CO)MODS.

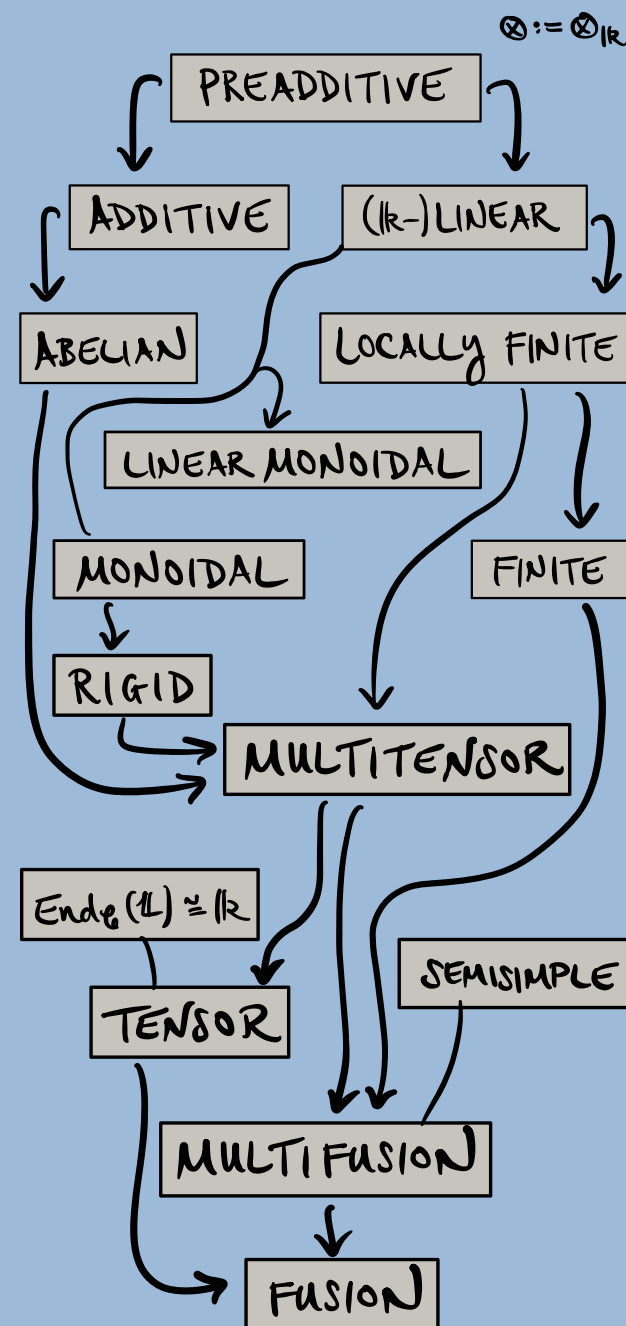
TAKE BIFUNCTOR $B: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \tilde{\mathcal{C}}$

$$\text{Coend}(B) := \int^{X \in \mathcal{C}} B(X, X) \in \tilde{\mathcal{C}} \text{ (WHEN IT EXISTS)}$$

EQUIPPED W/ $\{\iota_X: B(X, X) \rightarrow \text{Coend}(B)\}$

SATISFYING DINATURALITY, UNIVERSALLY

$$\text{End}(B) := \int_{X \in \mathcal{C}} B(X, X) \cong \text{Coend}(B) \text{ IN } \tilde{\mathcal{C}}^{\text{op}}$$



RECONSTRUCTION THEORY

EX. TAKE $\tilde{\mathcal{C}} = \text{FdVec}$.

FOR A $\mathbb{K}$ -COALGEBRA C :

$$\int_{V \in \text{FdComod-}C} V^* \otimes V \cong C \text{ AS } \mathbb{K}\text{-COALGS}$$

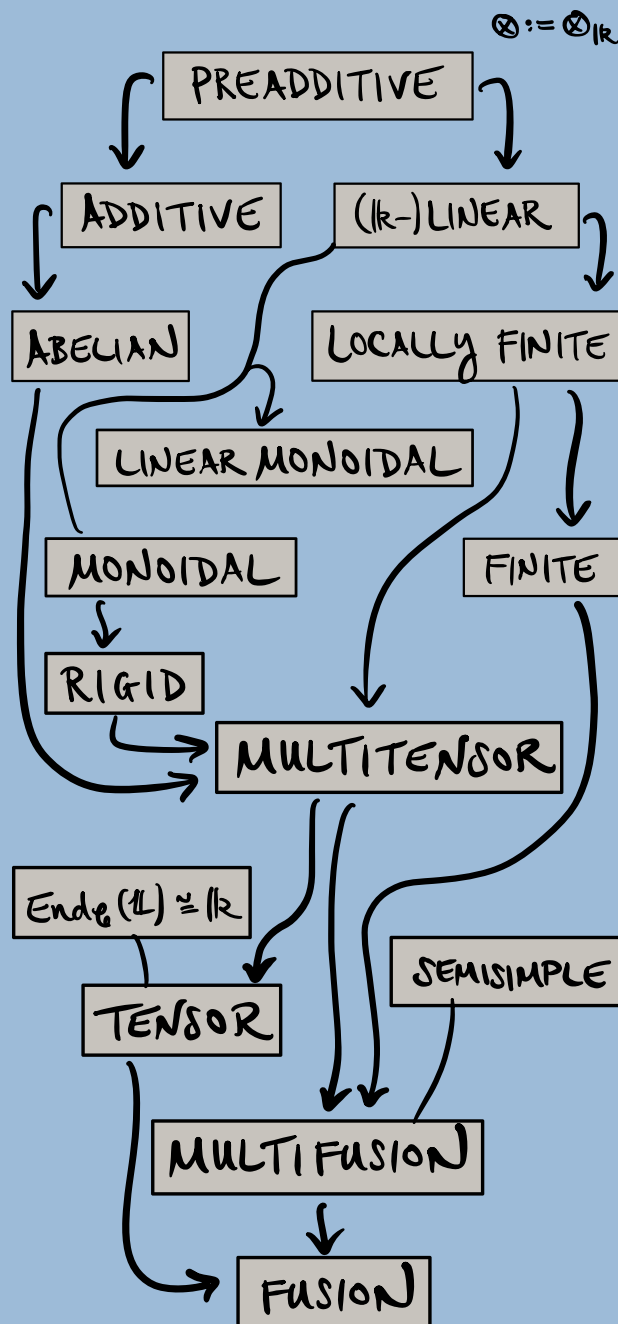
FOR A F.D. $\mathbb{K}$ -ALGEBRA A :

$$\int_{V \in A\text{-FdMod}} V \otimes V^* \cong A \text{ AS } \mathbb{K}\text{-ALGS}$$

RECOVER (CO)ALG.
FROM ITS CATEG.
OF (CO)MODS.

QUESTION:

TAKE A (CO)ALGEBRAIC STRUCTURE $\mathcal{S}$
 $\rightarrow$ A CATEGORY OF (CO)MODULES OVER $\mathcal{S}$
 WITH NICE FEATURES ★



RECONSTRUCTION THEORY

EX. TAKE $\tilde{\mathcal{C}} = \text{FdVec}$.

FOR A $\mathbb{K}$ -COALGEBRA C :

$$\int_{V \in \text{FdComod-}C} V^* \otimes V \cong C \text{ AS } \mathbb{K}\text{-COALGS}$$

FOR A F.D. $\mathbb{K}$ -ALGEBRA A :

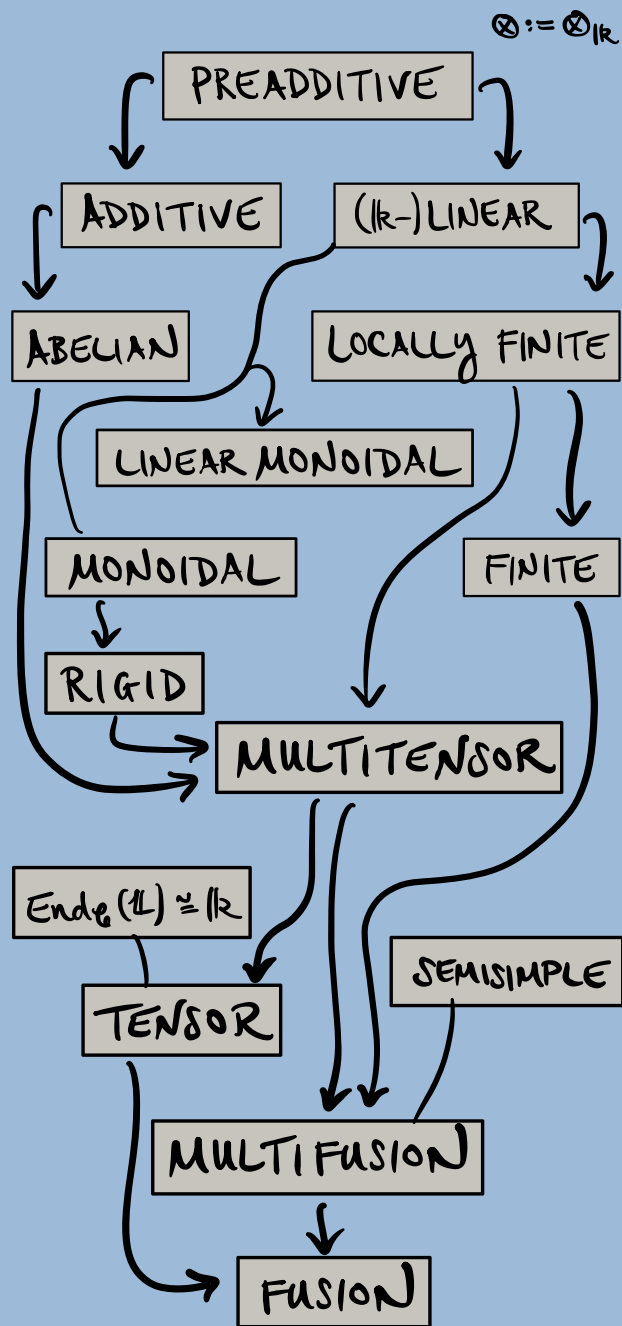
$$\int_{V \in A\text{-FdMod}} V \otimes V^* \cong A \text{ AS } \mathbb{K}\text{-ALGS}$$

RECOVER (CO)ALG.
FROM ITS CATEG.
OF (CO)MODS.

QUESTION:

TAKE A (CO)ALGEBRAIC STRUCTURE $\mathcal{S}$
 $\rightarrow$ A CATEGORY OF (CO)MODULES OVER $\mathcal{S}$
 WITH NICE FEATURES ★

DOES A CATEG. $\mathcal{C}$ SATISFYING ★
 $\Rightarrow \mathcal{C} \cong (\text{CO})\text{MODS OVER } \mathcal{S}$?



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{co})\text{MODS}$

EX. TAKE $\tilde{\mathcal{C}} = \text{FdVec.}$

FOR A $\mathbb{K}$ -COALGEBRA C :

$$\int_{V \in \text{FdComod-}C} V^* \otimes V \simeq C \text{ AS } \mathbb{K}\text{-COALGS}$$

FOR A F.D. $\mathbb{K}$ -ALGEBRA A :

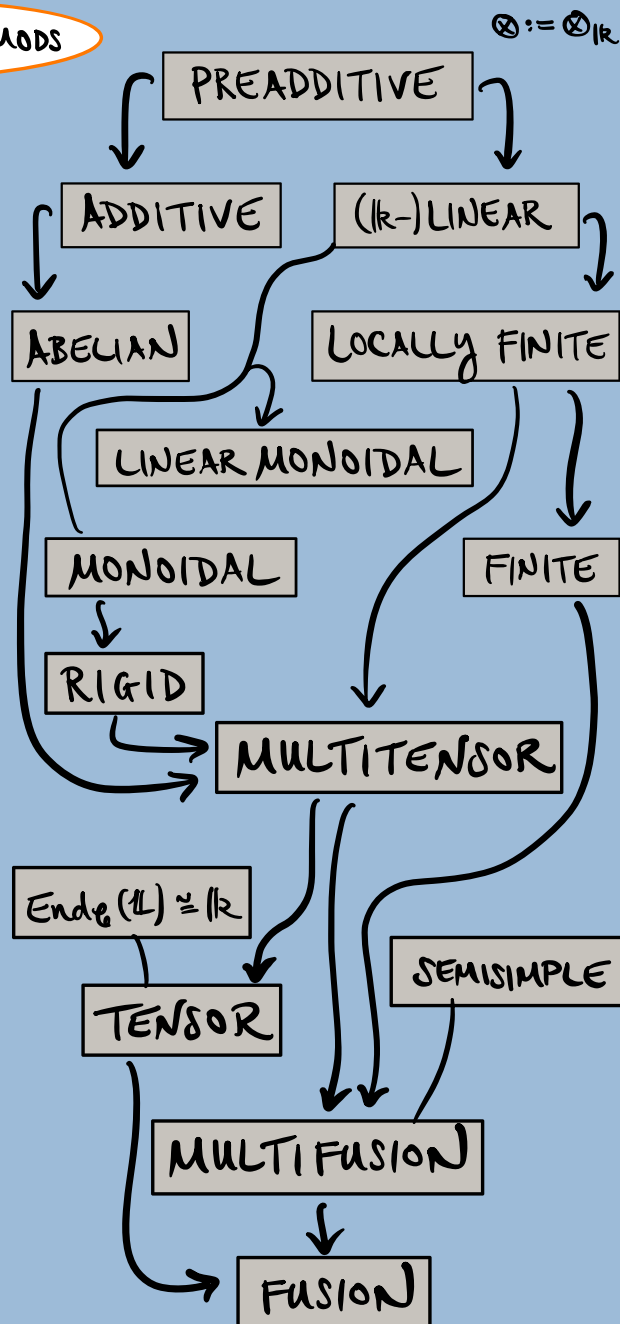
$$\int_{V \in A\text{-FdMod}} V \otimes V^* \simeq A \text{ AS } \mathbb{K}\text{-ALGS}$$

RECOVER $(\text{co})\text{ALG.}$
FROM ITS CATEG.
OF $(\text{co})\text{MODS.}$

QUESTION:

TAKE A $(\text{co})\text{ALGEBRAIC STRUCTURE } \mathcal{S}$
 $\rightarrow$ A CATEGORY OF $(\text{co})\text{MODULES OVER } \mathcal{S}$
WITH NICE FEATURES ★

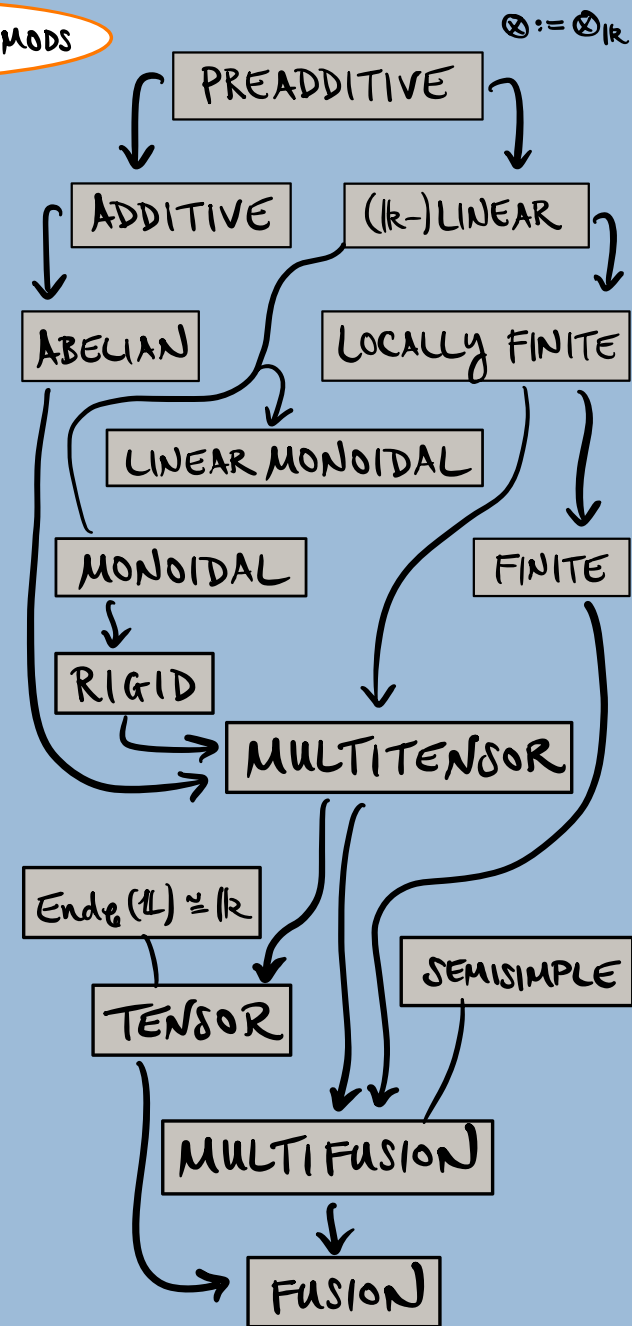
DOES A CATEG. $\mathcal{C}$ SATISFYING ★
 $\Rightarrow \mathcal{C} \simeq (\text{co})\text{MODS OVER } \mathcal{S}?$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{co})\text{MODS}$

RECIPE -

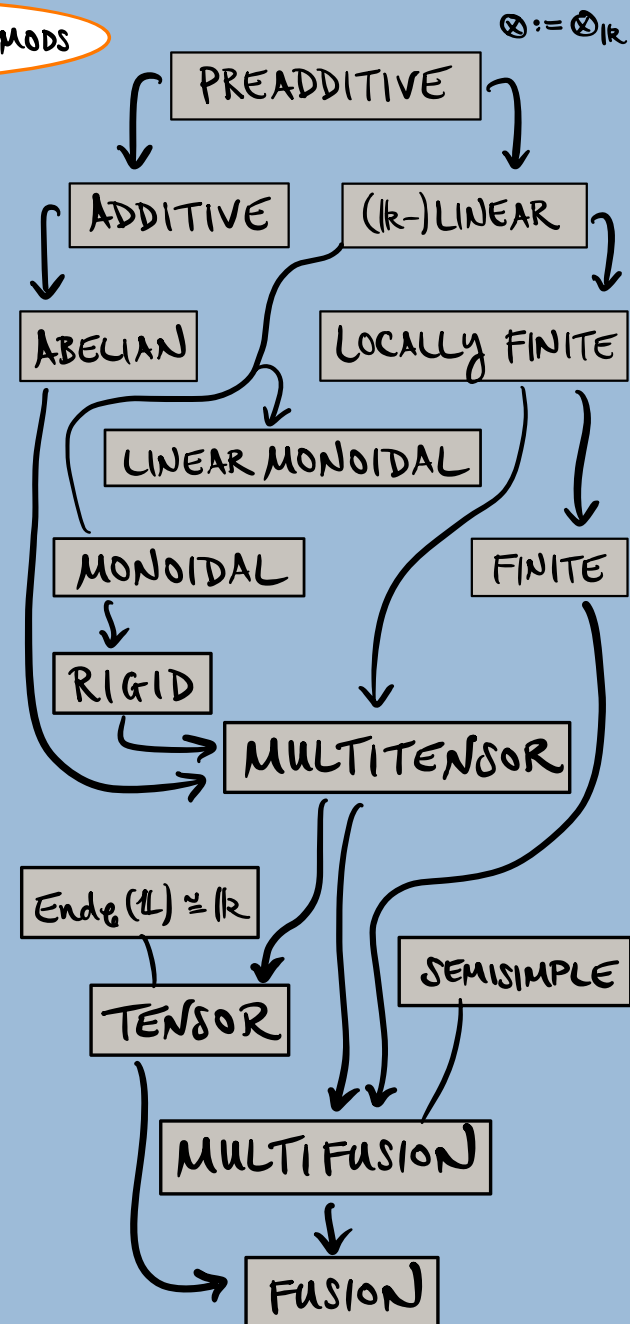


RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{co})\text{MODS}$

RECIPE -

- TAKE A CATEGORY $\mathcal{C}$ W/ NICE FEATURES



RECONSTRUCTION THEORY

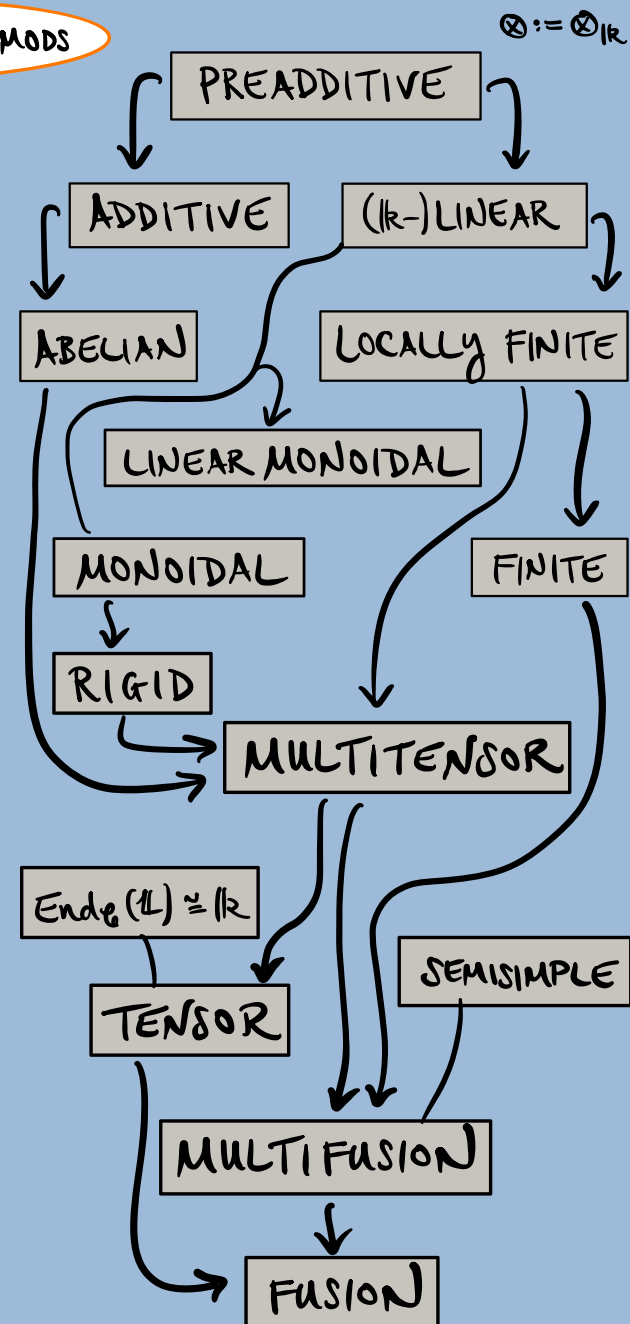
$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

RECIPE -

- TAKE A CATEGORY $\mathcal{C}$ W/ NICE FEATURES

- EQUIP IT W/ A NICE FUNCTOR:

$$U: \mathcal{C} \rightarrow (\text{Fd})\text{Vec}$$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

RECIPE -

- TAKE A CATEGORY $\mathcal{C}$ W/ NICE FEATURES
- EQUIP IT W/ A NICE FUNCTOR:

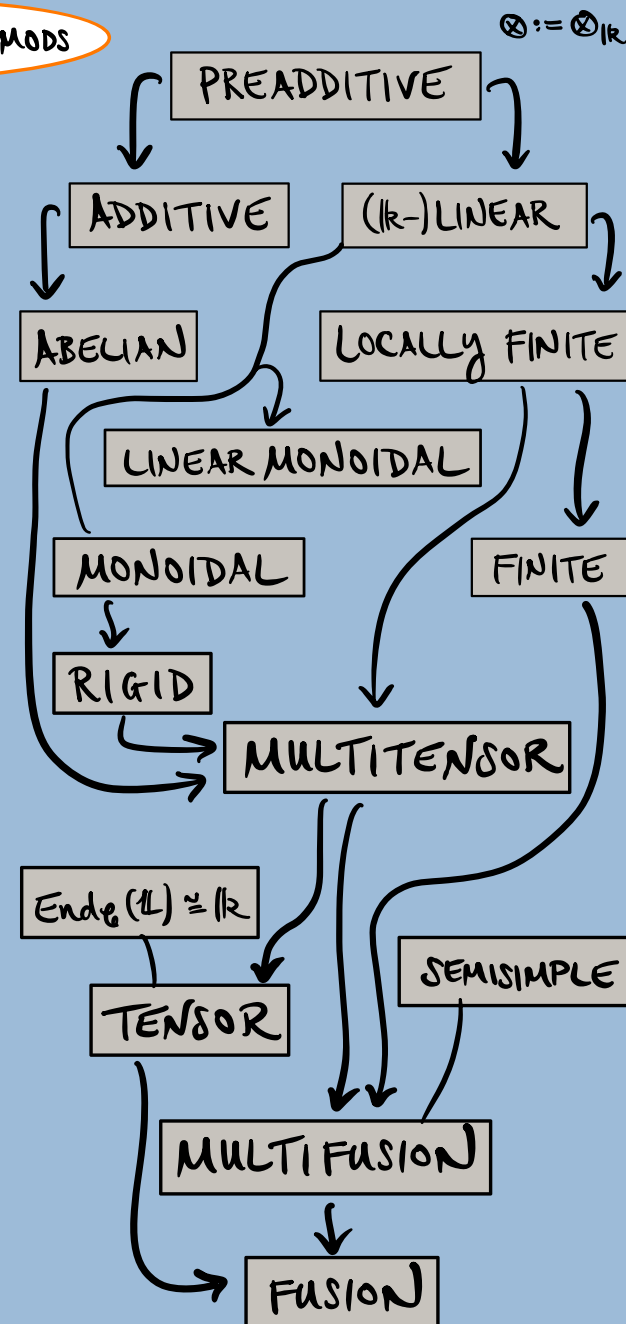
$$U: \mathcal{C} \rightarrow (\text{Fd})\text{Vec}$$

- EITHER: FORM THE COEND -

$$\text{Coend}_{\mathcal{C}}(U) := \int^{X \in \mathcal{C}} U(X)^* \otimes U(X) \in (\text{Fd})\text{Vec}$$

(WHEN IT EXISTS, E.G. $\mathcal{C} \simeq \text{SMALL CATEG.}$)

‡ GET THAT IT IS A NICE COALG. $_{/\mathbb{K}}$...



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

RECIPE -

- TAKE A CATEGORY $\mathcal{C}$ W/ NICE FEATURES
- EQUIP IT W/ A NICE FUNCTOR:
 $U: \mathcal{C} \rightarrow (\text{Fd})\text{Vec}$

- EITHER: FORM THE COEND -

$$\text{Coend}_{\mathcal{C}}(U) := \int^{X \in \mathcal{C}} U(X)^* \otimes U(X) \in (\text{Fd})\text{Vec}$$

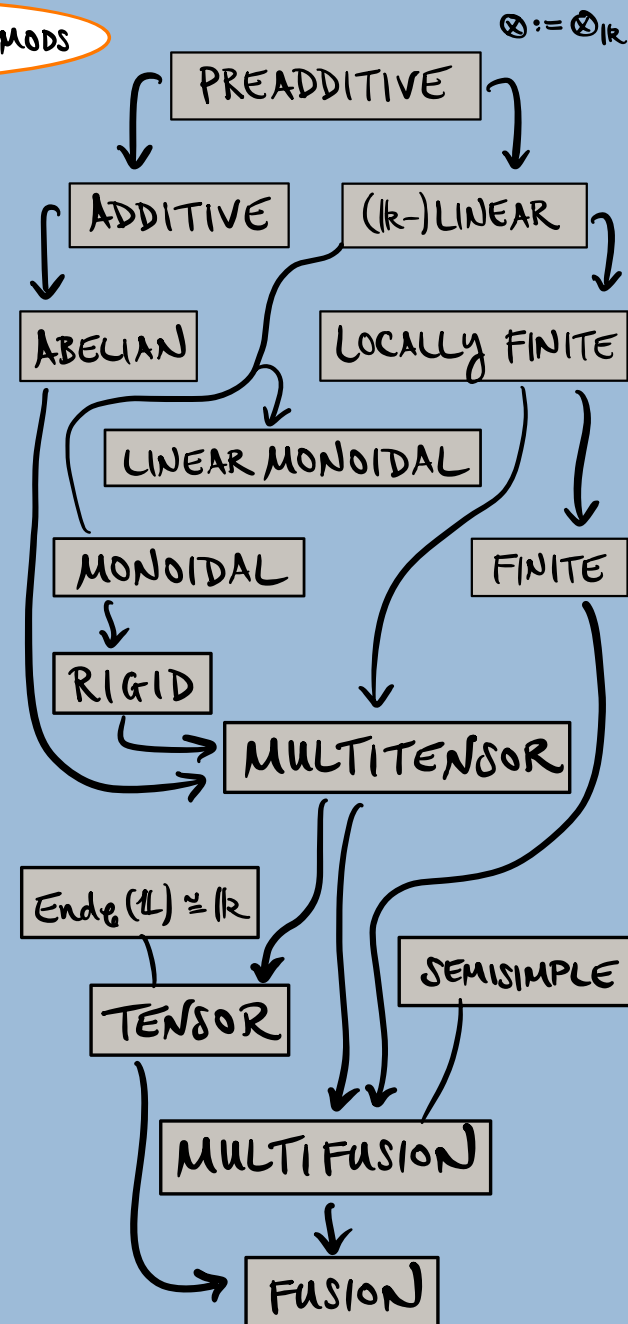
(WHEN IT EXISTS, E.G. $\mathcal{C} \simeq \text{SMALL CATEG.}$)

‡ GET THAT IT IS A NICE COALG./ $\mathbb{K}$...

OR FORM THE END -

$$\text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^* \in (\text{Fd})\text{Vec}$$

‡ GET THAT IT IS A NICE ALG./ $\mathbb{K}$...



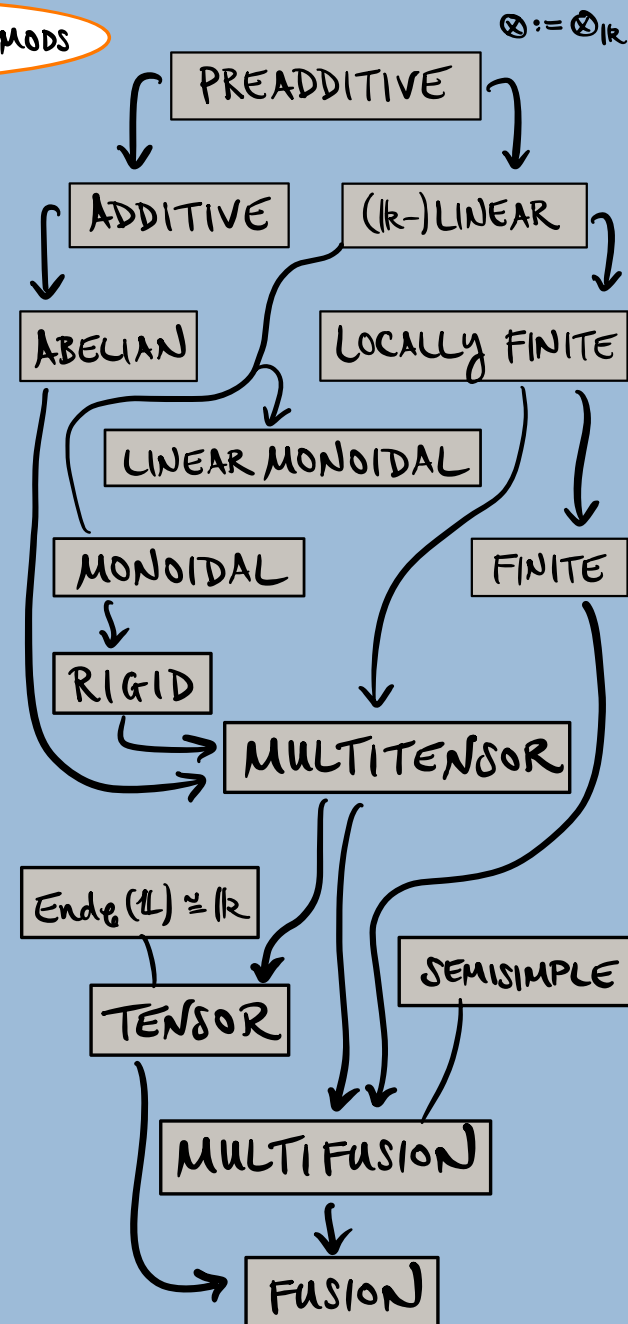
RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow (\text{Fd})\text{Vec}$

$$\text{Coend}_{\mathcal{C}}(U) := \int^{X \in \mathcal{C}} U(X)^* \otimes U(X)$$

$$\text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow (\text{Fd})\text{Vec}$

$$\text{Coend}_{\mathcal{C}}(U) := \int^{X \in \mathcal{C}} U(X)^* \otimes U(X)$$

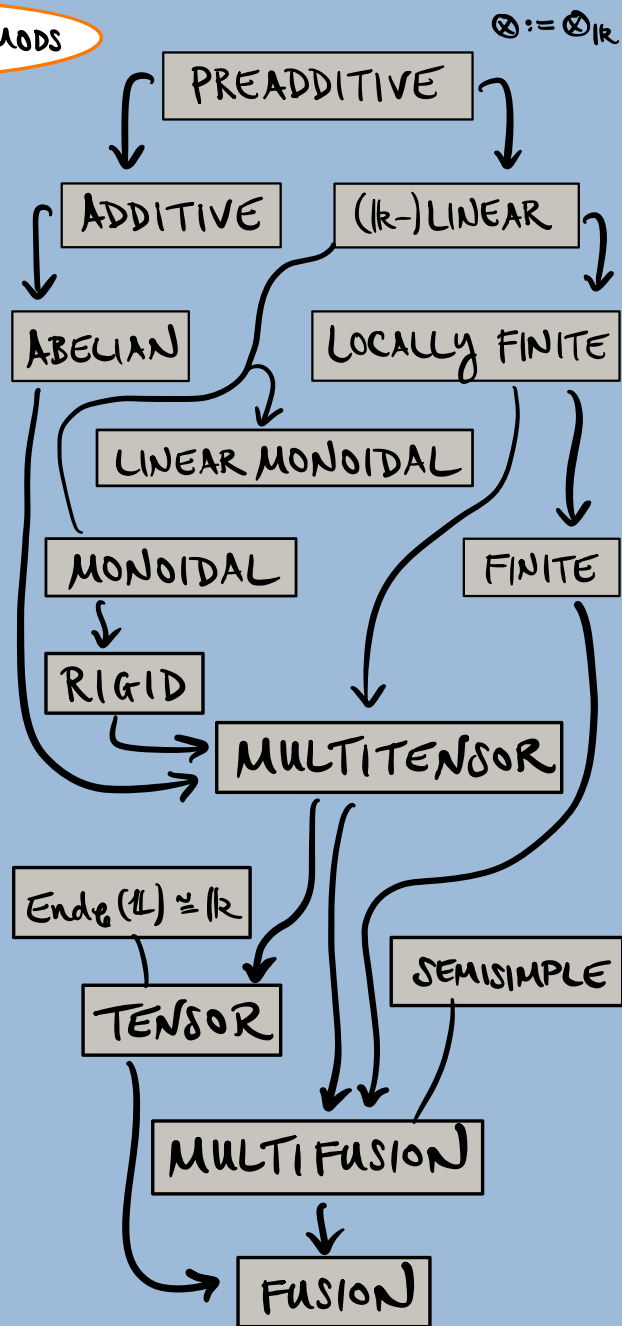
$$\text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

RECONSTRUCTION THEOREM:

"CATEG. OF PAIRS" $\simeq$ "CAT. OF STRUCTURES/ $\mathbb{R}$ "
 $\{(\mathcal{C}, U)\} \simeq \text{Struc}$

$$(\mathcal{C}, U) \longmapsto (\text{Co})\text{End}_{\mathcal{C}}(U)$$

$$((\text{Co})\text{Mod}/s, \text{Forg}) \longleftarrow \mathcal{S}$$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow (\text{Fd})\text{Vec}$

$$\text{Coend}_{\mathcal{C}}(U) := \int^{X \in \mathcal{C}} U(X)^* \otimes U(X)$$

$$\text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

RECONSTRUCTION THEOREM:

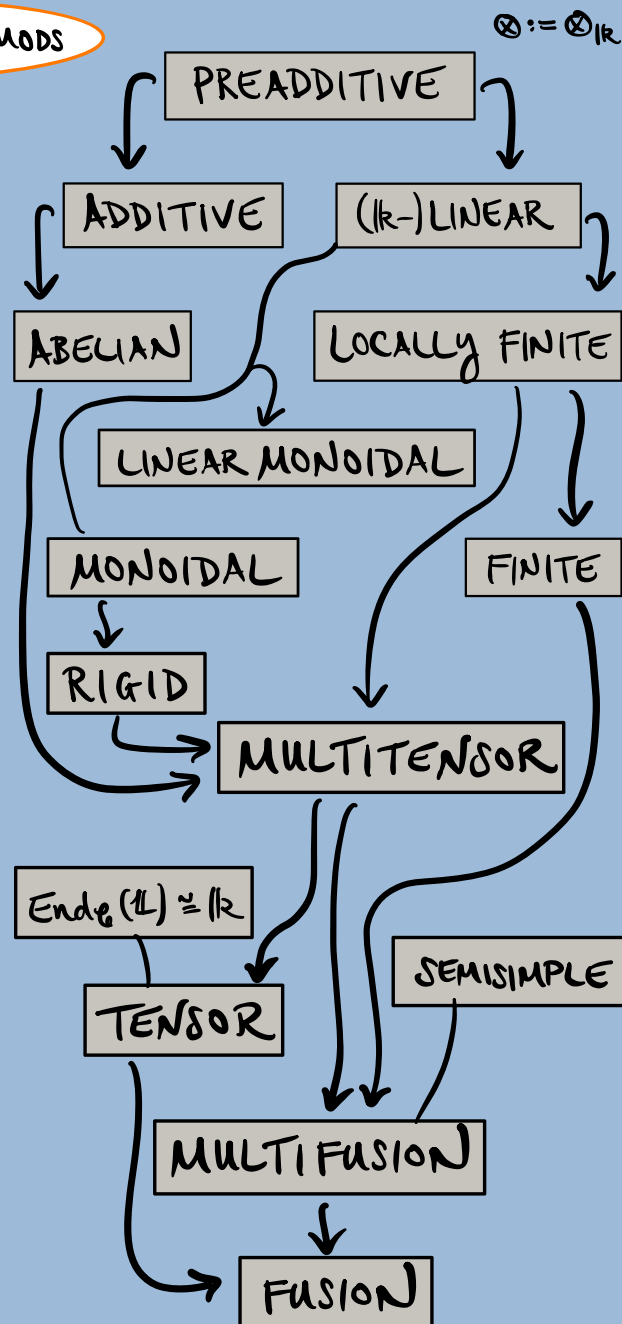
"CATEG. OF PAIRS" $\{(\mathcal{C}, U)\} \simeq$ "CAT. OF STRUCTURES/ $\mathbb{R}$ "
Struc

$$(\mathcal{C}, U) \longmapsto (\text{Co})\text{End}_{\mathcal{C}}(U)$$

$$((\text{Co})\text{Mod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$$

$\therefore \mathcal{C} \simeq (\text{Co})\text{Mod}/_{(\text{Co})\text{End}_{\mathcal{C}}(U)}$ AS CATEGORIES

$\therefore \mathcal{S} \simeq (\text{Co})\text{End}_{(\text{Co})\text{Mod}/\mathcal{S}}(U)$ AS STRUCTURES



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE NICE? $\mathcal{C}$ & NICE? $U: \mathcal{C} \rightarrow (\text{Fd})\text{Vec}$

$$\text{Coend}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X)^* \otimes U(X) / \text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

RECONSTRUCTION THEOREM:

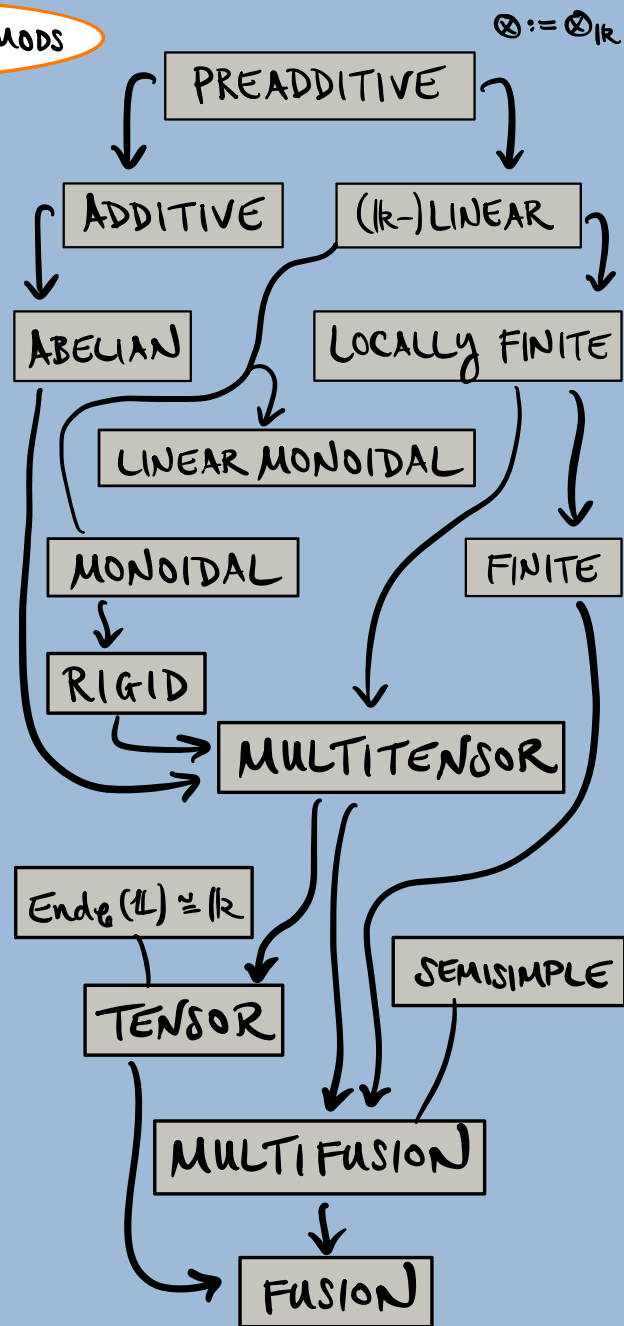
"CATEG. OF PAIRS" $\simeq$ "CAT. OF STRUCTURES/ $\mathbb{R}$ "
 $\{(\mathcal{C}, U)\} \simeq \text{Struc?}$

$$(\mathcal{C}, U) \longmapsto (\text{Co})\text{End}_{\mathcal{C}}(U)$$

$$((\text{Co})\text{Mod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$$

$\therefore \mathcal{C} \simeq (\text{Co})\text{Mod}/_{(\text{Co})\text{End}_{\mathcal{C}}(U)} \text{ AS CATEGORIES}$

$\therefore \mathcal{S} \simeq (\text{Co})\text{End}_{(\text{Co})\text{Mod}/\mathcal{S}}(U) \text{ AS STRUCTURES}$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE? $U: \mathcal{C} \rightarrow (\text{Fd})\text{Vec}$
(SEE HIGHLIGHTED FEATURES)

$$\text{Coend}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X)^* \otimes U(X) / \text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

RECONSTRUCTION THEOREM:

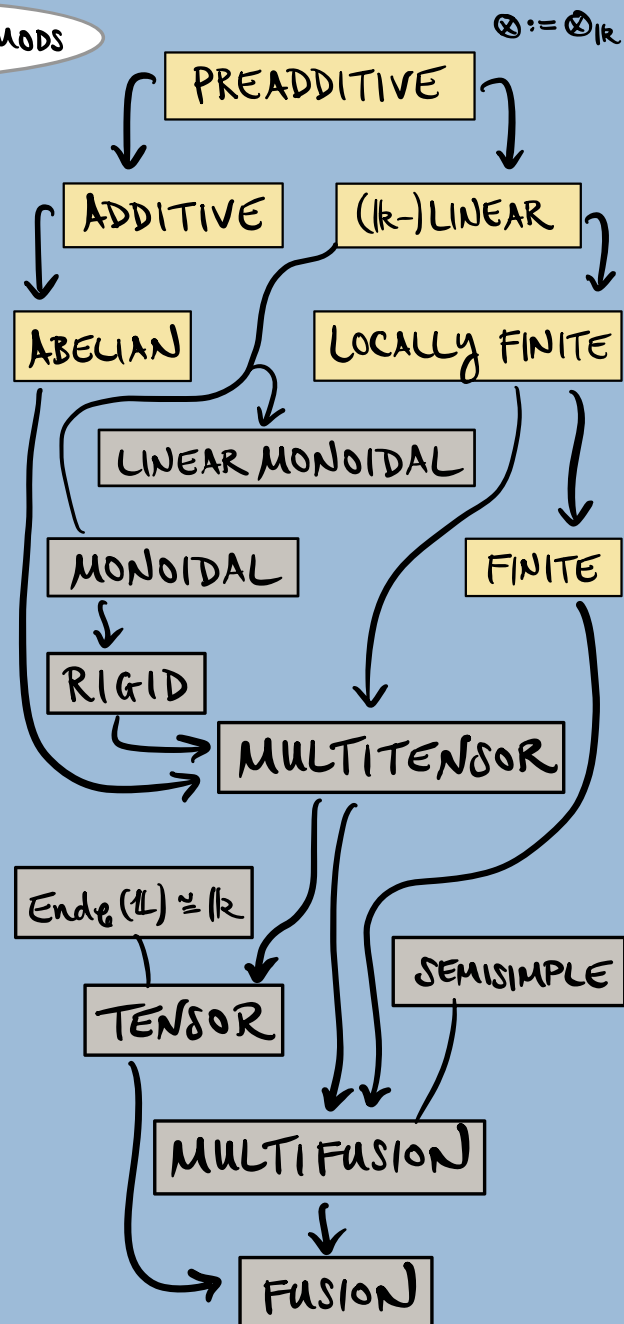
"CATEG. OF PAIRS" $\simeq$ "CAT. OF STRUCTURES/ $\mathbb{R}$ "
 $\{(\mathcal{C}, U)\} \simeq \text{Struc?}$

$$(\mathcal{C}, U) \longmapsto (\text{Co})\text{End}_{\mathcal{C}}(U)$$

$$((\text{Co})\text{Mod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$$

$$\therefore \mathcal{C} \simeq (\text{Co})\text{Mod}/_{(\text{Co})\text{End}_{\mathcal{C}}(U)} \text{ AS CATEGORIES}$$

$$\therefore \mathcal{S} \simeq (\text{Co})\text{End}_{(\text{Co})\text{Mod}/\mathcal{S}}(U) \text{ AS STRUCTURES}$$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE ~~NICE~~ $\mathcal{C}$ & ~~NICE~~ $U: \mathcal{C} \rightarrow \text{Fd Vec}$
 (SEE HIGHLIGHTED FEATURES)
 • LINEAR • EXACT • FAITHFUL

$$\text{Coend}_{\mathcal{C}}(U) := \int^{X \in \mathcal{C}} U(X)^* \otimes U(X) / \text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

RECONSTRUCTION THEOREM:

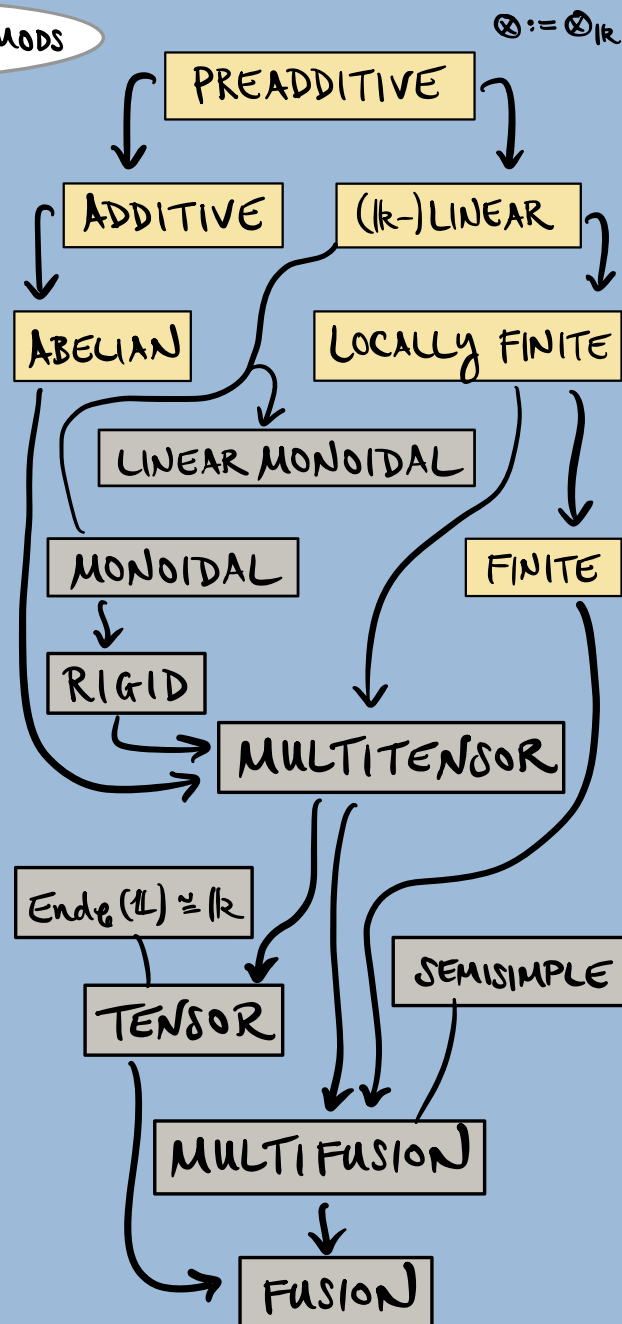
"CATEG. OF PAIRS" $\simeq$ "CAT. OF STRUCTURES/ $\mathbb{R}$ "
 $\{(\mathcal{C}, U)\} \simeq \text{Struc ?}$

$$(\mathcal{C}, U) \longmapsto (\text{Co})\text{End}_{\mathcal{C}}(U)$$

$$((\text{Co})\text{Mod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$$

$$\therefore \mathcal{C} \simeq (\text{Co})\text{Mod}/_{(\text{Co})\text{End}_{\mathcal{C}}(U)} \text{ AS CATEGORIES}$$

$$\therefore \mathcal{S} \simeq (\text{Co})\text{End}_{(\text{Co})\text{Mod}/\mathcal{S}}(U) \text{ AS STRUCTURES}$$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE ~~NICE~~ $\mathcal{C}$ & ~~NICE~~ $U: \mathcal{C} \rightarrow \text{Fd Vec}$

(SEE HIGHLIGHTED FEATURES)

• LINEAR • EXACT • FAITHFUL

$$\text{Coend}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X)^* \otimes U(X) / \text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

RECONSTRUCTION THEOREM:

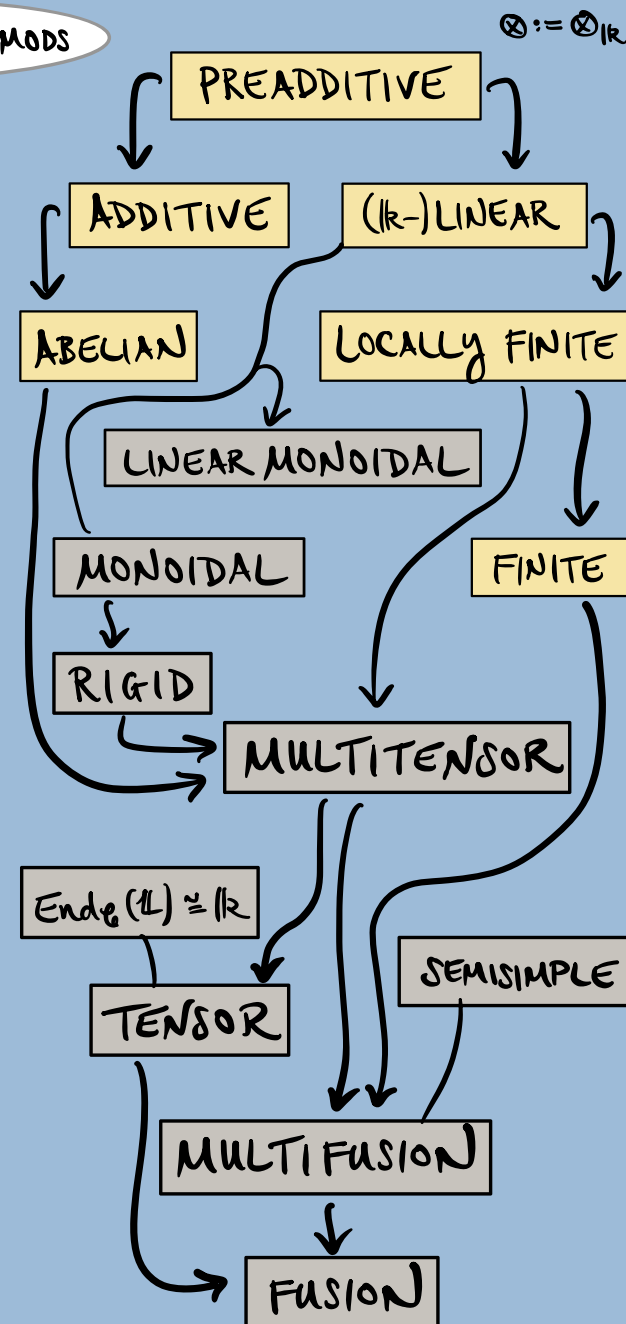
"CATEG. OF PAIRS" $\simeq$ "CAT. OF ~~STRUCTURES~~ ^{F.D. ALGEBRAS} / $\mathbb{K}$ "
 $\{(\mathcal{C}, U)\}$ $\simeq$ FdAlg

$(\mathcal{C}, U) \longleftrightarrow \text{End}_{\mathcal{C}}(U)$

$(\text{Mod}/\mathcal{S}, \text{Forg}) \longleftrightarrow \mathcal{S}$

$\therefore \mathcal{C} \simeq (\text{Co})\text{Mod} / (\text{Co})\text{End}_{\mathcal{C}}(U)$ AS CATEGORIES

$\therefore \mathcal{S} \simeq (\text{Co})\text{End}_{(\text{Co})\text{Mod}/\mathcal{S}}(U)$ AS STRUCTURES



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE ~~NICE~~ $\mathcal{C}$ & ~~NICE~~ $U: \mathcal{C} \rightarrow \text{Fd Vec}$

(SEE HIGHLIGHTED
FEATURES)

• LINEAR • EXACT • FAITHFUL

$$\text{Coend}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X)^* \otimes U(X) / \text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

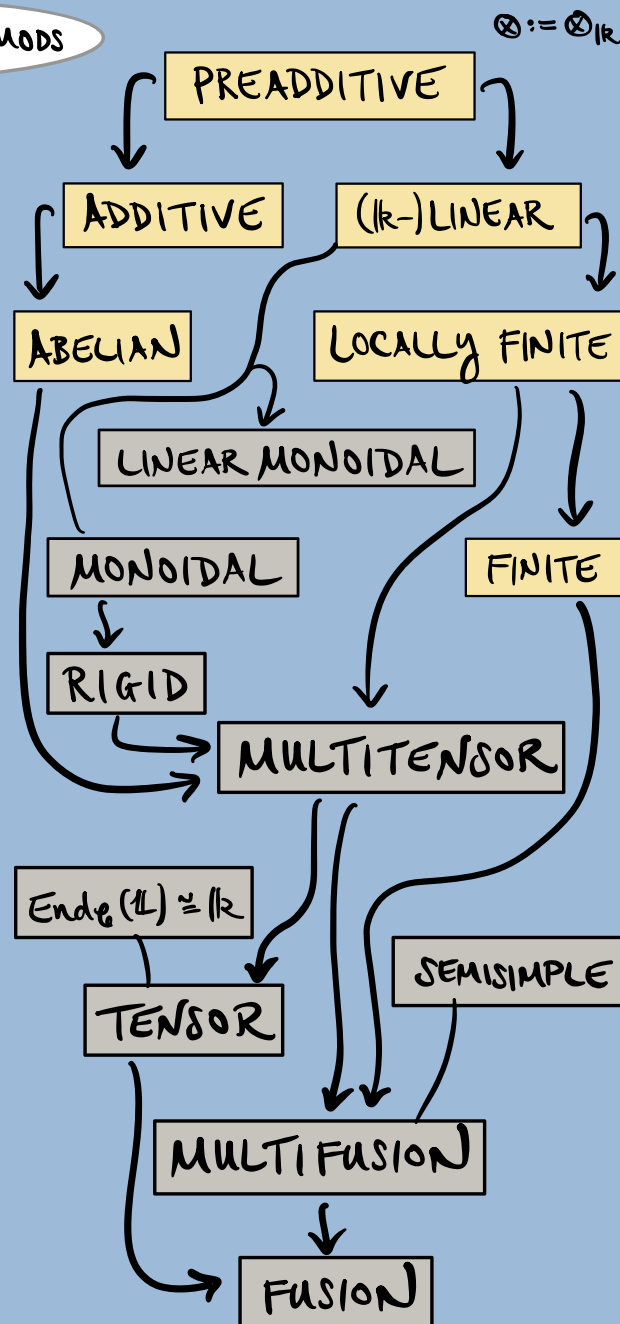
RECONSTRUCTION THEOREM:

"CATEG. OF PAIRS" $\simeq$ "CAT. OF ~~STRUCTURES~~ ^{F.D. ALGEBRAS} / $\mathbb{R}$ "
 $\{(\mathcal{C}, U)\} \simeq \text{FdAlg}$

$(\mathcal{C}, U) \longmapsto \text{End}_{\mathcal{C}}(U)$

$(\text{Mod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$

EVERY FINITE LINEAR ABELIAN CATEGORY
EQUIPPED W/ A LINEAR, EXACT, FAITHFUL U
IS EQUIVALENT TO $A\text{-FdMod}$
FOR SOME F.D. ALG A



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow \text{Fd Vec}$

(SEE HIGHLIGHTED FEATURES)

• LINEAR • EXACT • FAITHFUL

$$U = \text{Hom}_{\mathcal{C}}(\bigoplus_{i \in \text{Irr}(\mathcal{C})} \text{Proj Cov}(X_i), -)$$

$$\text{Coend}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X)^* \otimes U(X) / \text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

RECONSTRUCTION THEOREM:

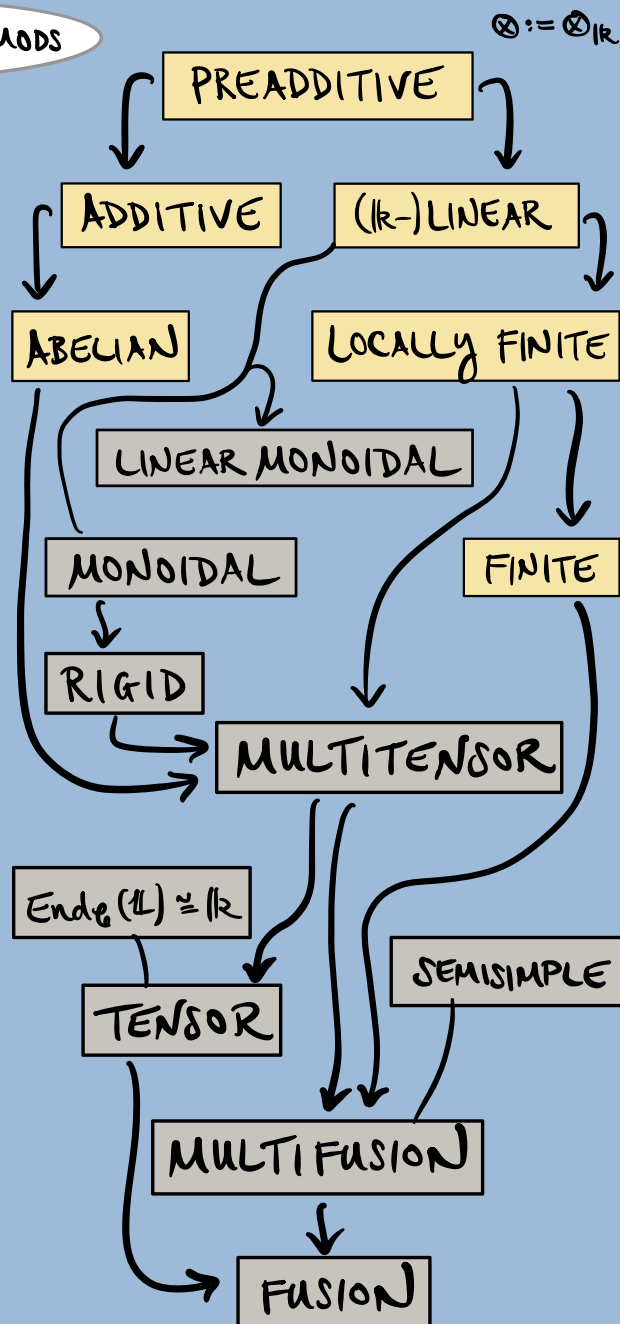
"CATEG. OF PAIRS" $\simeq$ "CAT. OF ~~STRUCTURES~~ / $\mathbb{R}$ "

$\{(\mathcal{C}, U)\}$ $\simeq$ FdAlg

$$(\mathcal{C}, U) \longmapsto \text{End}_{\mathcal{C}}(U)$$

$$(\text{Mod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$$

EVERY FINITE LINEAR ABELIAN CATEGORY
EQUIPPED W/ A LINEAR, EXACT, FAITHFUL U
IS EQUIVALENT TO $A\text{-FdMod}$
FOR SOME F.D. ALG A



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow \text{Fd Vec}$

(SEE HIGHLIGHTED FEATURES)

• LINEAR • EXACT • FAITHFUL

$$U = \text{Hom}_{\mathcal{C}}(\bigoplus_{i \in \text{Irr}(\mathcal{C})} \text{Proj Cov}(X_i), -)$$

$$\text{Coend}_{\mathcal{C}}(U) := \int^{X \in \mathcal{C}} U(X)^* \otimes U(X) / \text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

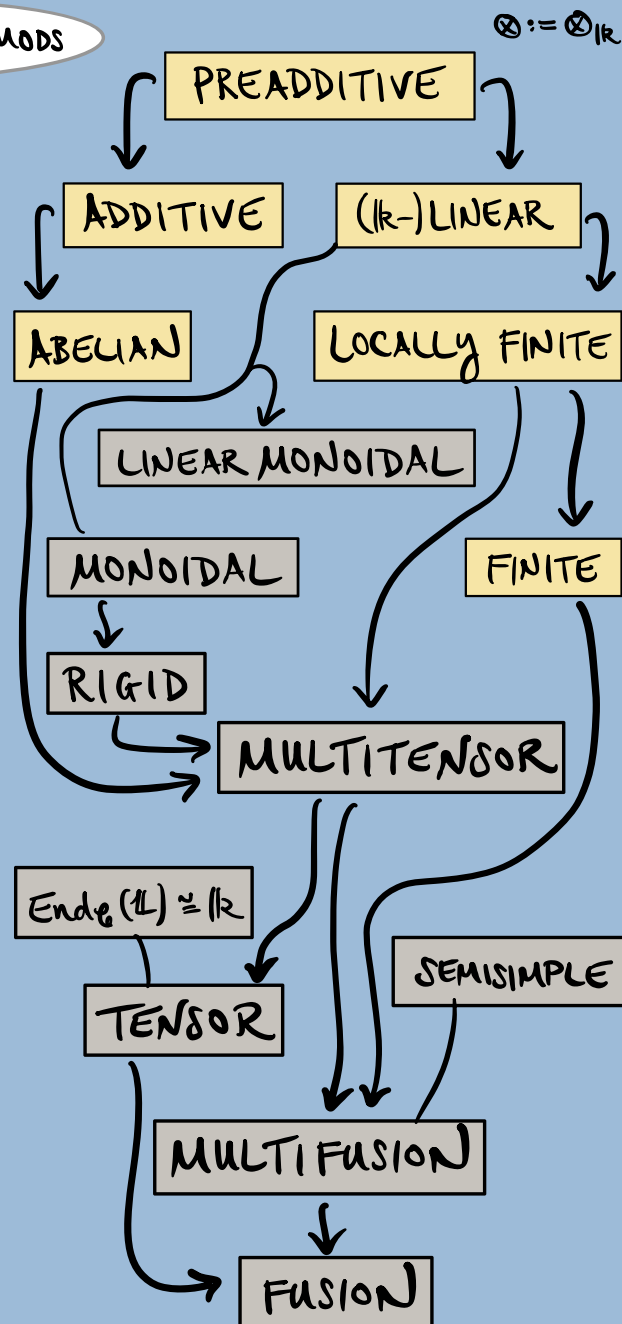
RECONSTRUCTION THEOREM:

"CATEG. OF PAIRS" $\simeq$ "CAT. OF ~~STRUCTURES~~ / $\mathbb{R}$ "
 $\{(\mathcal{C}, U)\}$ $\simeq$ FdAlg

$$(\mathcal{C}, U) \longmapsto \text{End}_{\mathcal{C}}(U)$$

$$(\text{Mod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$$

EVERY FINITE LINEAR ABELIAN CATEGORY
~~EQUIPPED W/ A LINEAR, EXACT, FAITHFUL U~~
 IS EQUIVALENT TO A-FdMod
 FOR SOME F.D. ALG A



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow \text{Fd Vec}$

(SEE HIGHLIGHTED
FEATURES)

• LINEAR • EXACT • FAITHFUL

$$\text{Coend}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X)^* \otimes U(X) / \text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

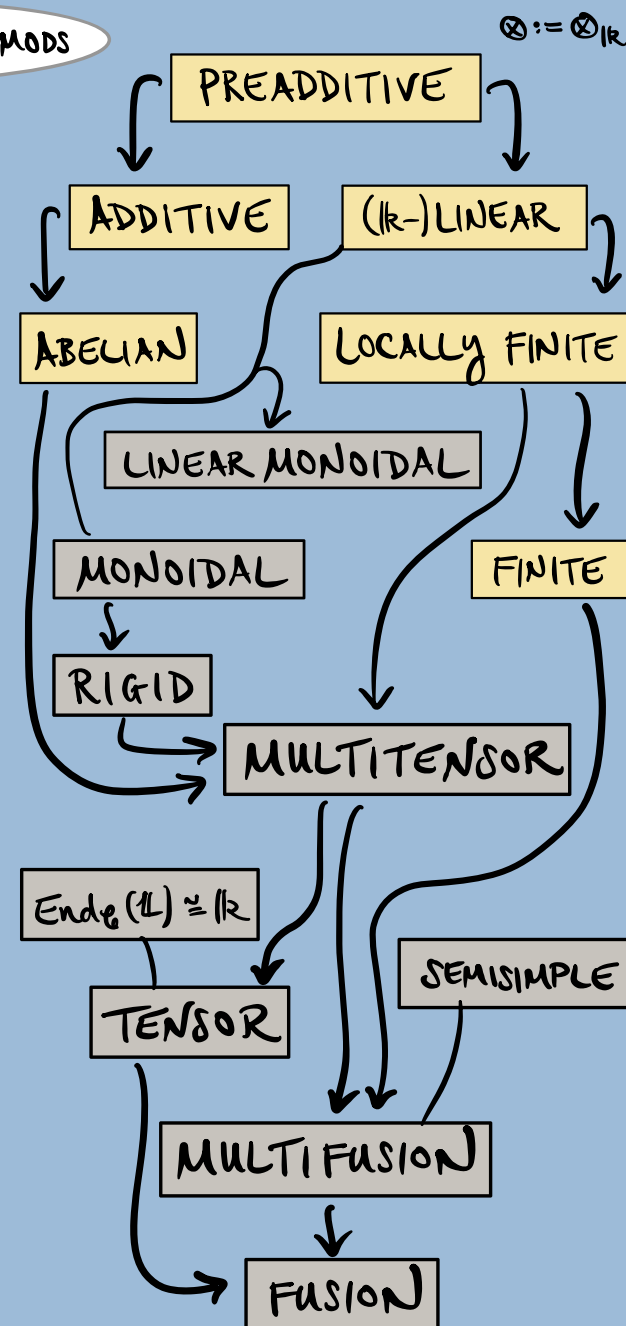
RECONSTRUCTION THEOREM:

"CAT. OF PAIRS" $\simeq$ "CAT. OF ~~STRUCTURES~~ / $\mathbb{R}$ "
 $\{(\mathcal{C}, U)\}$ $\simeq$ FdAlg

$(\mathcal{C}, U) \longmapsto \text{End}_{\mathcal{C}}(U)$

$(\text{Mod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$

EVERY FINITE LINEAR ABELIAN CATEGORY
IS EQUIVALENT TO $A\text{-FdMod}$
FOR SOME F.D. ALG A



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE ~~NICE~~ $\mathcal{C}$ & $\text{HCS } U: \mathcal{C} \rightarrow \text{Fd Vec}$
 (SEE HIGHLIGHTED FEATURES)
 • LINEAR • EXACT • FAITHFUL

$$\text{Coend}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X)^* \otimes U(X) / \text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

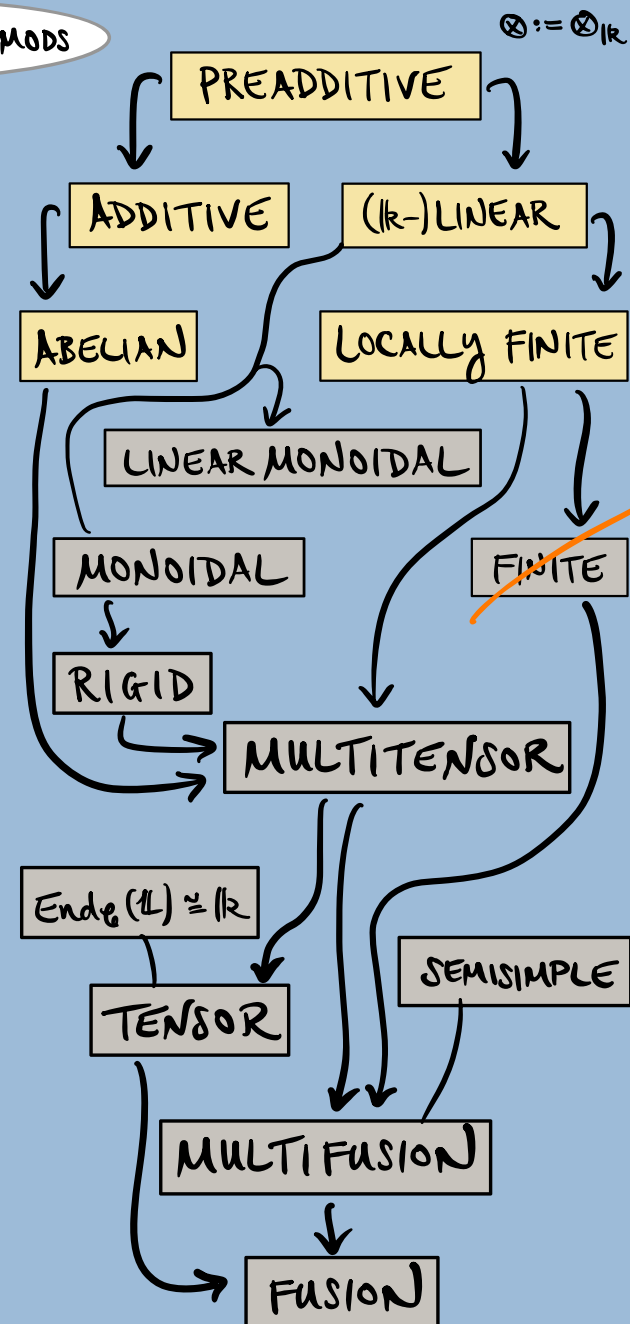
RECONSTRUCTION THEOREM:

"CATEG. OF PAIRS" $\simeq$ "CAT. OF ~~STRUCTURES~~ ^{F.D. ALGEBRAS} / $\mathbb{R}$ "
 $\{(\mathcal{C}, U)\} \simeq \text{FdAlg}$

$(\mathcal{C}, U) \longmapsto \text{End}_{\mathcal{C}}(U)$

$(\text{Mod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$

EVERY FINITE LINEAR ABELIAN CATEGORY
 IS EQUIVALENT TO $A\text{-FdMod}$
 FOR SOME F.D. ALG A



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (co)MODS$

TAKE ~~NICE~~ $\mathcal{C}$ & ~~NICE~~ $U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED
FEATURES)

• LINEAR • EXACT • FAITHFUL

$$\text{Coend}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X)^* \otimes U(X) / \text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

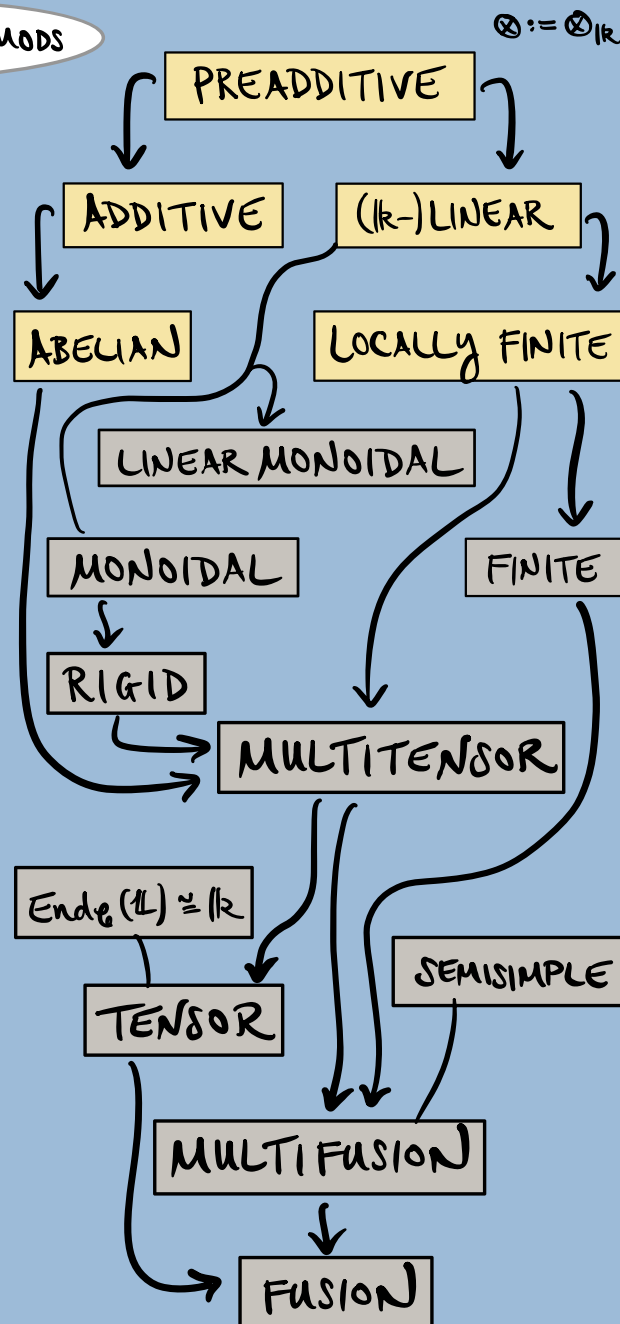
RECONSTRUCTION THEOREM:

"CATEG. OF PAIRS" $\simeq$ "CAT. OF STRUCTURES" $\uparrow$ F.D. ALGEBRAS
 $\{(\mathcal{C}, U)\}$ $\simeq$ $\text{FdAlg} / \mathbb{R}$

$(\mathcal{C}, U) \longmapsto \text{End}_{\mathcal{C}}(U)$

$(\text{Mod}/S, \text{Forg}) \longleftarrow S$

EVERY FINITE LINEAR ABELIAN CATEGORY
IS EQUIVALENT TO $A\text{-FdMod}$
FOR SOME F.D. ALG A



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{co})\text{MODS}$

TAKE ~~NICE~~ $\mathcal{C}$ & ~~NICE~~ $U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED
FEATURES)

• LINEAR • EXACT • FAITHFUL

$$\text{Coend}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X)^* \otimes U(X) / \text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

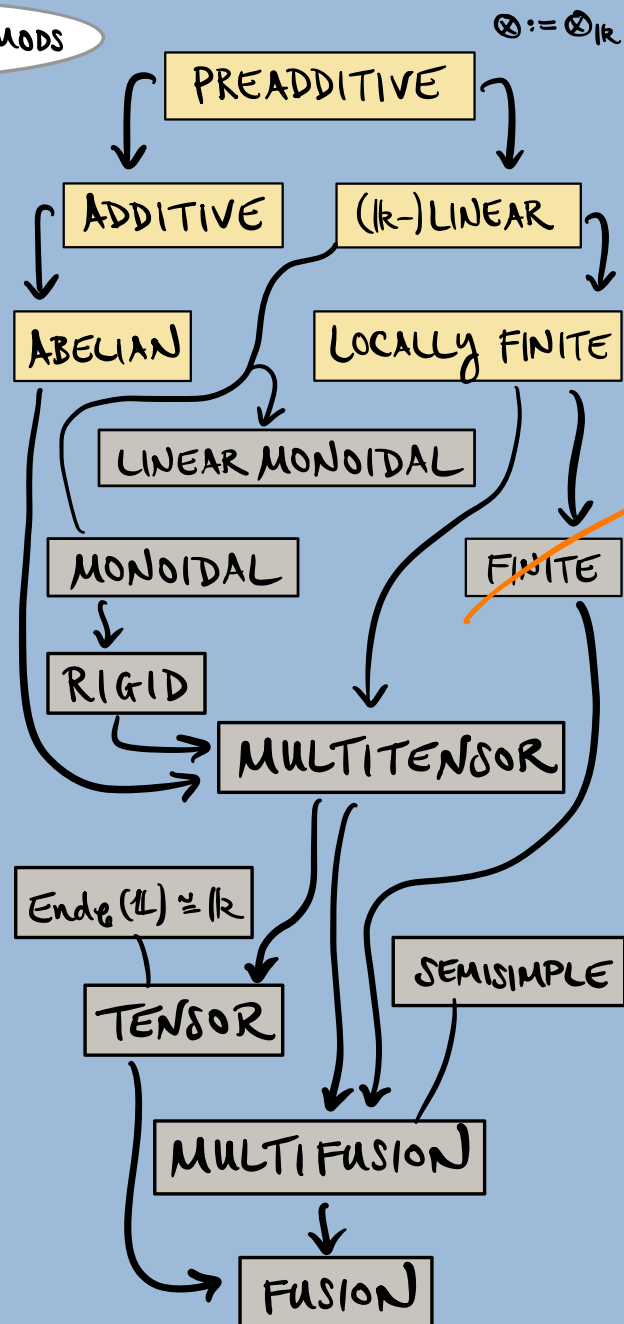
RECONSTRUCTION THEOREM:

"CAT. OF PAIRS" $\simeq$ "CAT. OF ~~STRUCTURES~~ / $\mathbb{R}$ "
 $\{(\mathcal{C}, U)\}$ $\simeq$ $\{\text{COALGEBRAS} / \mathbb{R}\}$
 Coalg

$(\mathcal{C}, U) \longmapsto \text{Coend}_{\mathcal{C}}(U)$

$(\text{coMod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$

EVERY FINITE LINEAR ABELIAN CATEGORY
IS EQUIVALENT TO $A\text{-FdMod}$
FOR SOME F.D. ALG A



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED FEATURES)

• LINEAR • EXACT • FAITHFUL

$$U = \text{Hom}_{\mathcal{C}^{\vee}}(-, \bigoplus_{i \in \text{Irr}(\mathcal{C})} \text{InjFull}(X_i))$$

$$\text{Coend}_{\mathcal{C}}(U) := \int^{X \in \mathcal{C}} U(X)^* \otimes U(X) / \text{End}_{\mathcal{C}}(U) := \int_{X \in \mathcal{C}} U(X) \otimes U(X)^*$$

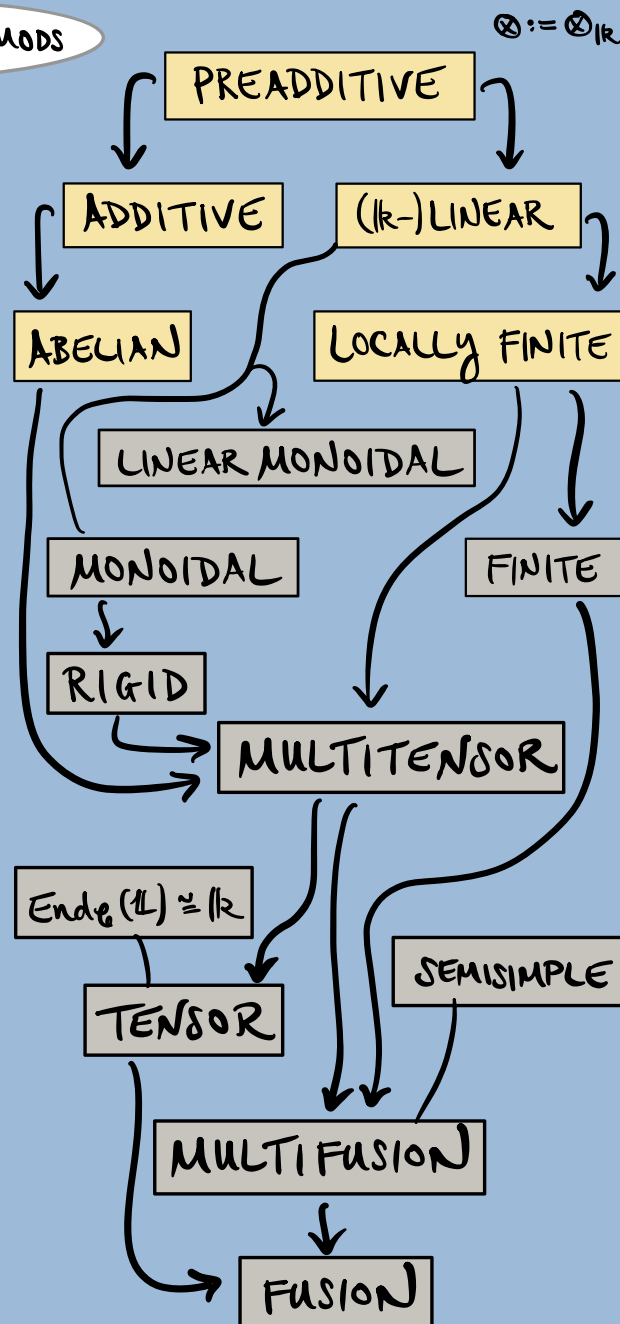
RECONSTRUCTION THEOREM:

"CATEG. OF PAIRS" $\{(\mathcal{C}, U)\} \simeq$ "CAT. OF ~~STRUCTURES~~ ^{COALGEBRAS} _{Coalg} $/ \mathbb{R}$ "

$$(\mathcal{C}, U) \longmapsto \text{Coend}_{\mathcal{C}}(U)$$

$$(\text{coMod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$$

EVERY LOCALLY FINITE, LINEAR ABELIAN CATEG.
IS EQUIVALENT TO $\text{CoMod-}\mathcal{C}$
FOR SOME COALGEBRA $\mathcal{C}$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE ~~NICE~~ $\mathcal{C}$ & $\text{HCS } U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED
FEATURES)

• LINEAR • EXACT • FAITHFUL

$$\text{Coend}_{\mathcal{C}}(U) := \int^{x \in \mathcal{C}} U(x)^* \otimes U(x)$$

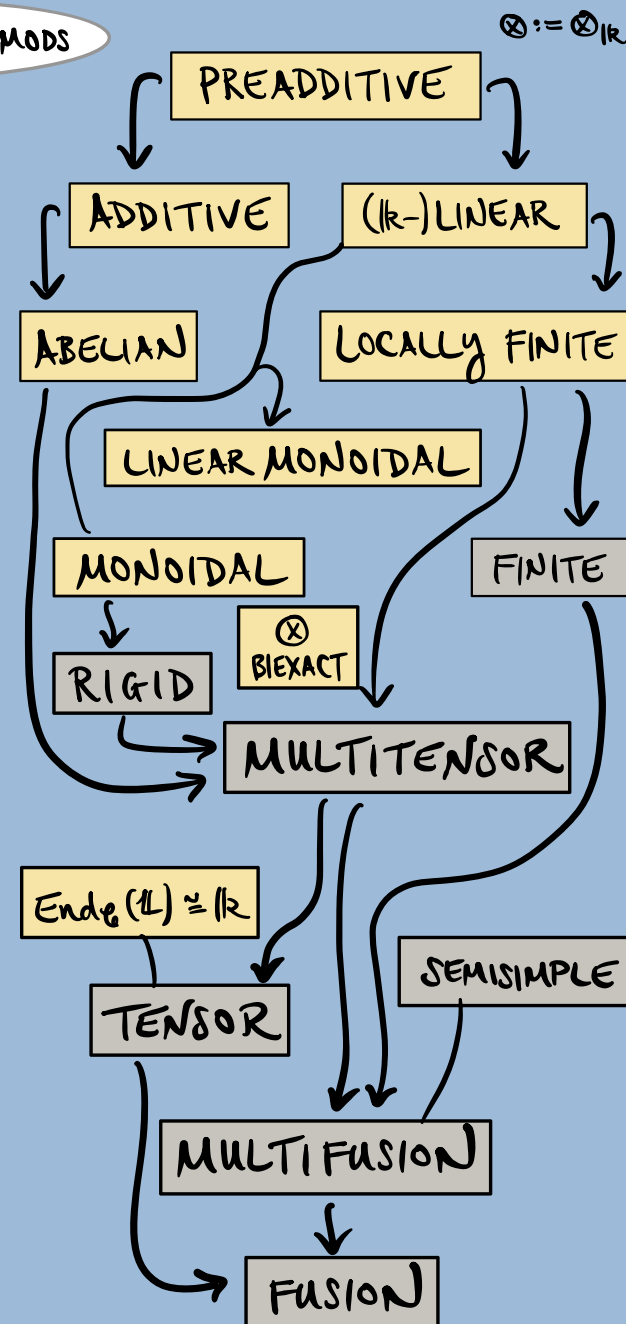
RECONSTRUCTION THEOREM:

"CATEG. OF PAIRS" $\{(\mathcal{C}, U)\} \simeq$ "CAT. OF ~~STRUCTURES~~ ^{COALGEBRAS} _{Coalg} $/ \mathbb{K}$ "

$$(\mathcal{C}, U) \longmapsto \text{Coend}_{\mathcal{C}}(U)$$

$$(\text{CoMod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$$

EVERY LOCALLY FINITE, LINEAR ABELIAN CATEG.
IS EQUIVALENT TO $\text{CoMod-}\mathcal{C}$
FOR SOME COALGEBRA $\mathcal{C}$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE ~~NICE~~ $\mathcal{C}$ & ~~NICE~~ $U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED
FEATURES)

• LINEAR • EXACT • FAITHFUL

• STRONG MONOIDAL:

$$U(-) \otimes_{\mathbb{k}} U(-) \cong U(- \otimes -)$$

$$U(1) \cong \mathbb{k}$$

$$\text{Coend}_{\mathcal{C}}(U) := \int^{x \in \mathcal{C}} U(x)^* \otimes U(x)$$

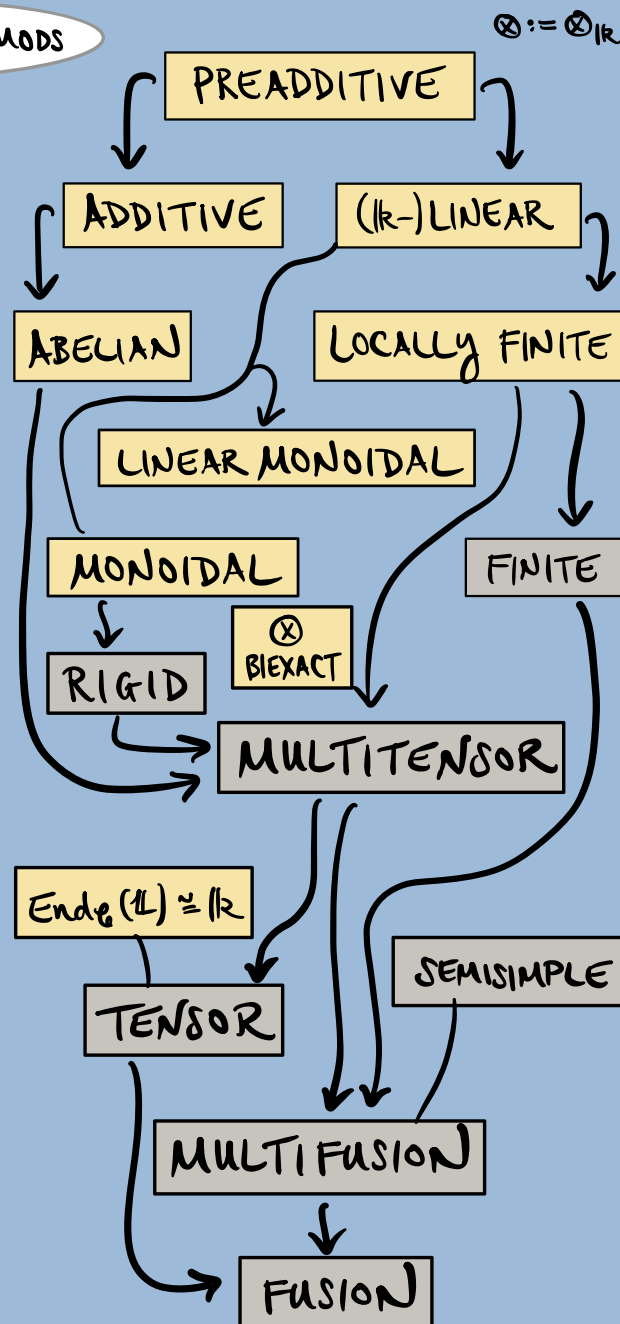
RECONSTRUCTION THEOREM:

"CATEG. OF PAIRS" $\{(\mathcal{C}, U)\} \simeq$ "CAT. OF ~~STRUCTURES~~ ^{COALGEBRAS} $\mathcal{C}/\mathbb{k}$ "
 Coalg

$$(\mathcal{C}, U) \longmapsto \text{Coend}_{\mathcal{C}}(U)$$

$$(\text{CoMod}/\mathcal{C}, \text{Forg}) \longleftarrow \mathcal{S}$$

EVERY LOCALLY FINITE, LINEAR ABELIAN CATEG.
IS EQUIVALENT TO $\text{CoMod-}\mathcal{C}$
FOR SOME COALGEBRA $\mathcal{C}$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{co})\text{MODS}$

TAKE ~~NICE~~ $\mathcal{C}$ & ~~NICE~~ $U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED FEATURES)

• LINEAR • EXACT • FAITHFUL

• STRONG MONOIDAL:

$$U(-) \otimes_{\mathbb{K}} U(-) \cong U(- \otimes -)$$

$$U(1) \cong \mathbb{K}$$

$$\text{Coend}_{\mathcal{C}}(U) := \int^{x \in \mathcal{C}} U(x)^* \otimes U(x)$$

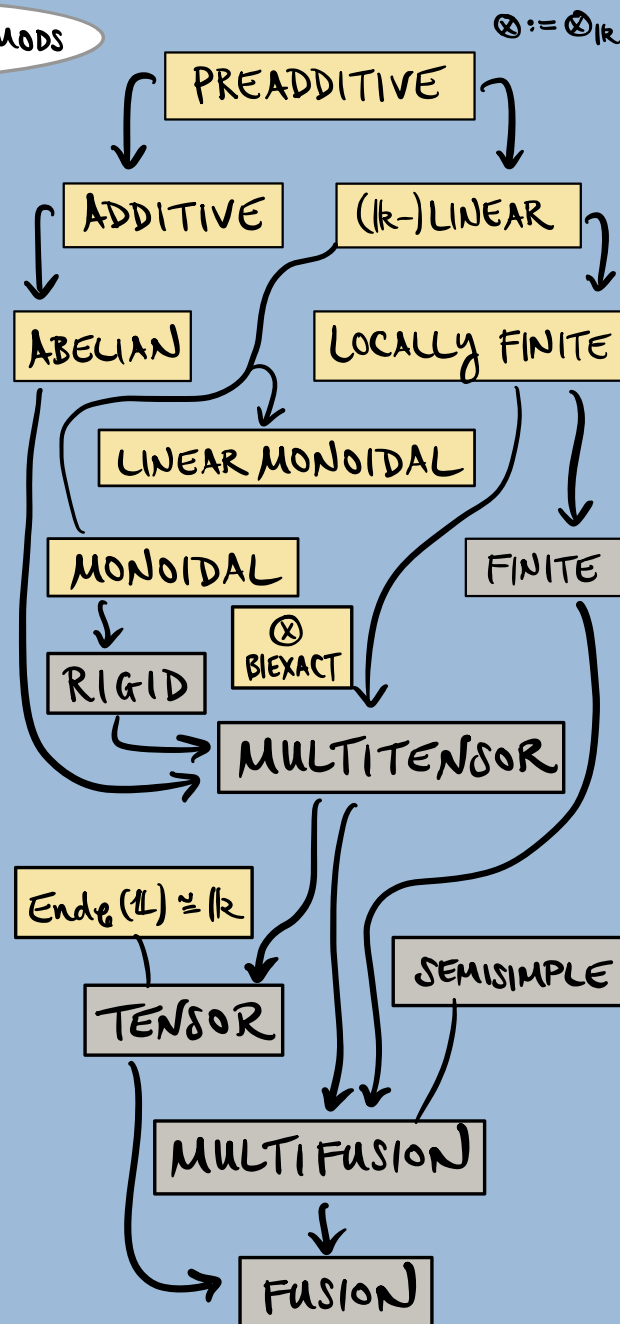
RECONSTRUCTION THEOREM:

"CATEG. OF PAIRS" $\simeq$ "CAT. OF ~~STRUCTURES~~ ^{BIALGEBRAS} / $\mathbb{K}$ "
 $\{(\mathcal{C}, U)\}$ $\simeq$ $\{\text{Bialg}\}$

$$(\mathcal{C}, U) \longmapsto \text{Coend}_{\mathcal{C}}(U)$$

$$(\text{coMod}/\mathcal{C}, \text{Forg}) \longleftarrow \mathcal{S}$$

EVERY LOCALLY FINITE, LINEAR ABELIAN CATEG.
 IS EQUIVALENT TO $\text{CoMod-}\mathcal{C}$
 FOR SOME COALGEBRA $\mathcal{C}$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED FEATURES)

• LINEAR • EXACT • FAITHFUL

• STRONG MONOIDAL:

$$U(-) \otimes_{\mathbb{K}} U(-) \cong U(- \otimes -)$$

$$U(1) \cong \mathbb{K}$$

$$\text{Coend}_{\mathcal{C}}(U) := \int^{x \in \mathcal{C}} U(x)^* \otimes U(x)$$

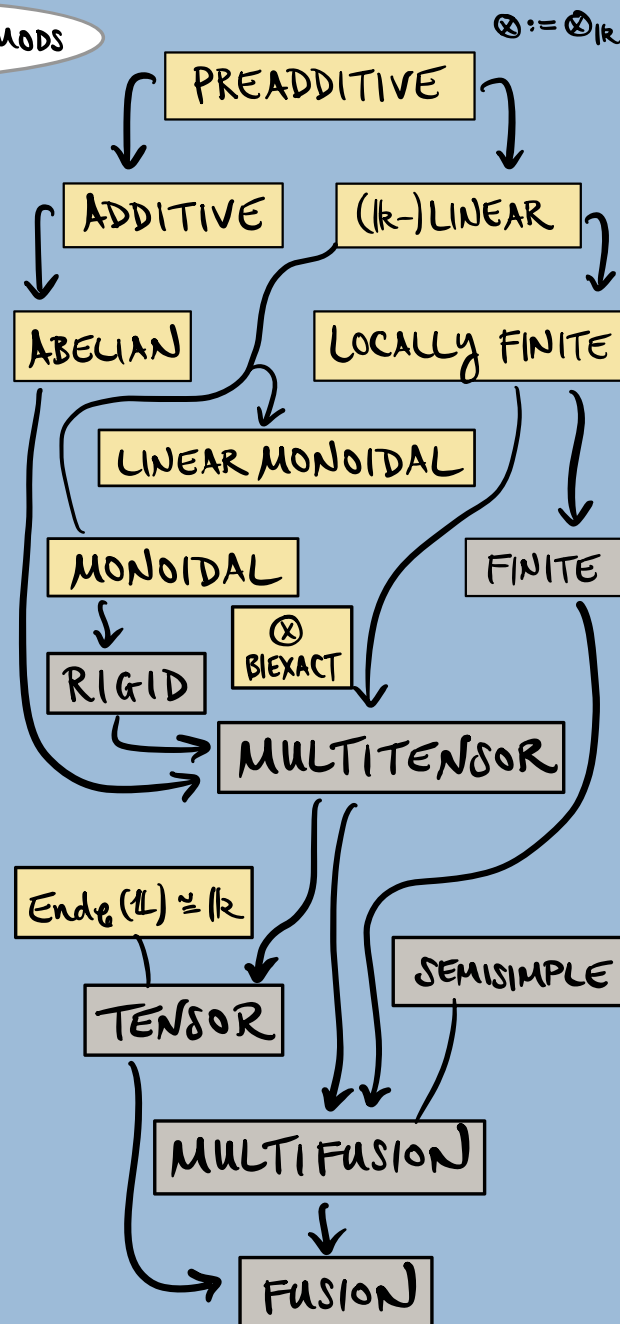
RECONSTRUCTION THEOREM:

"CAT. OF PAIRS" $\{(\mathcal{C}, U)\} \simeq$ "CAT. OF ~~STRUCTURES~~ ^{BIALGEBRAS} $/ \mathbb{K}$ "
 $\{(\mathcal{C}, U)\} \simeq \text{Bialg}$

$$(\mathcal{C}, U) \longmapsto \text{Coend}_{\mathcal{C}}(U)$$

$$(\text{CoMod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$$

EVERY "RING" CAT. EQUIPPED W/ A "FIBER" FUNCTOR
 IS EQUIVALENT TO $\text{CoMod-}\mathcal{B}$
 FOR SOME BIALGEBRA $\mathcal{B}$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED FEATURES)

• LINEAR • EXACT • FAITHFUL

• STRONG MONOIDAL:

$$U(-) \otimes_{\mathbb{K}} U(-) \cong U(- \otimes -)$$

$$U(1) \cong \mathbb{K}$$

$$\text{Coend}_{\mathcal{C}}(U) := \int^{x \in \mathcal{C}} U(x)^* \otimes U(x)$$

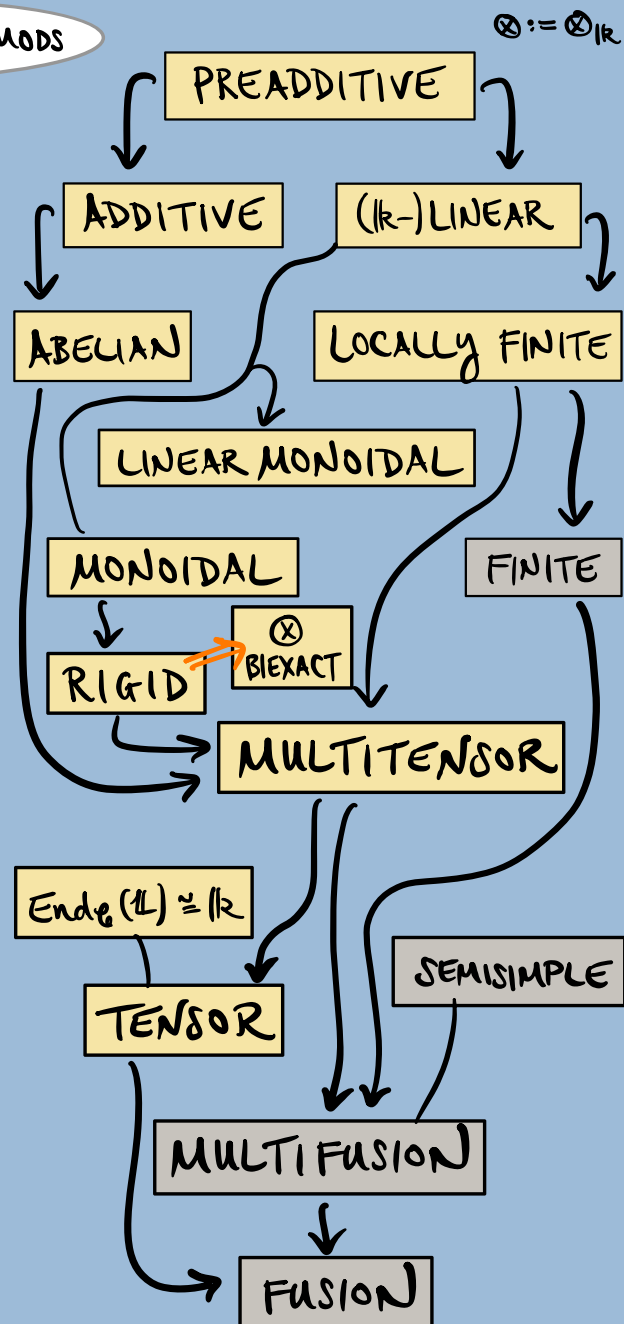
RECONSTRUCTION THEOREM:

"CAT. OF PAIRS" $\{(\mathcal{C}, U)\} \simeq$ "CAT. OF ~~STRUCTURES~~ ^{BIALGEBRAS} $\mathcal{C}/\mathbb{K}$ "
 Bialg

$$(\mathcal{C}, U) \longmapsto \text{Coend}_{\mathcal{C}}(U)$$

$$(\text{CoMod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$$

EVERY "RING" CAT. EQUIPPED W/ A "FIBER" FUNCTOR
 IS EQUIVALENT TO $\text{CoMod-}\mathcal{B}$
 FOR SOME BIALGEBRA $\mathcal{B}$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED FEATURES)

• LINEAR • EXACT • FAITHFUL

• STRONG MONOIDAL:

$$U(-) \otimes_{\mathbb{K}} U(-) \cong U(- \otimes -)$$

$$U(1) \cong \mathbb{K}$$

$$\text{Coend}_{\mathcal{C}}(U) := \int^{x \in \mathcal{C}} U(x)^* \otimes U(x)$$

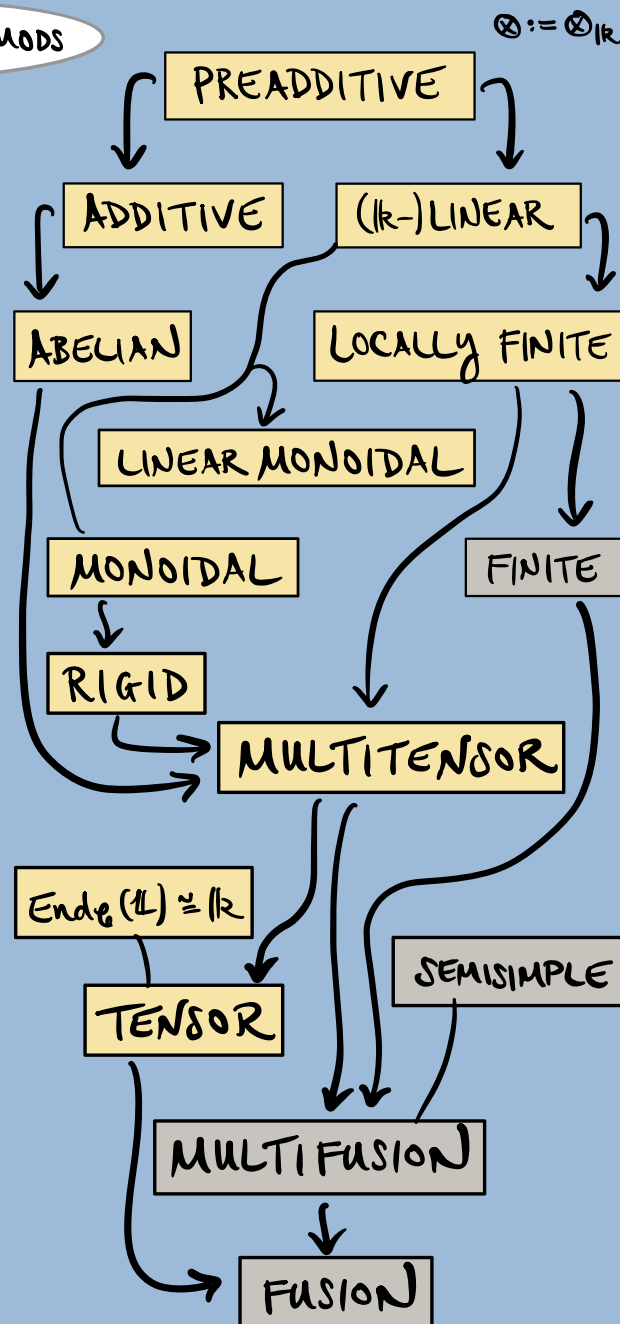
RECONSTRUCTION THEOREM:

"CATEG. OF PAIRS" $\{(\mathcal{C}, U)\} \simeq$ "CAT. OF STRUCTURES" $\xrightarrow{\text{BIALGEBRAS}} \text{Bialg}$

$$(\mathcal{C}, U) \longmapsto \text{Coend}_{\mathcal{C}}(U)$$

$$(\text{CoMod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$$

EVERY "RING" CAT. EQUIPPED W/ A "FIBER" FUNCTOR
IS EQUIVALENT TO $\text{CoMod-}\mathcal{B}$
FOR SOME BIALGEBRA $\mathcal{B}$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED FEATURES)

• LINEAR • EXACT • FAITHFUL

• STRONG MONOIDAL:

$$U(-) \otimes_{\mathbb{K}} U(-) \cong U(- \otimes -)$$

$$U(1) \cong \mathbb{K}$$

$$\text{Coend}_{\mathcal{C}}(U) := \int^{x \in \mathcal{C}} U(x)^* \otimes U(x)$$

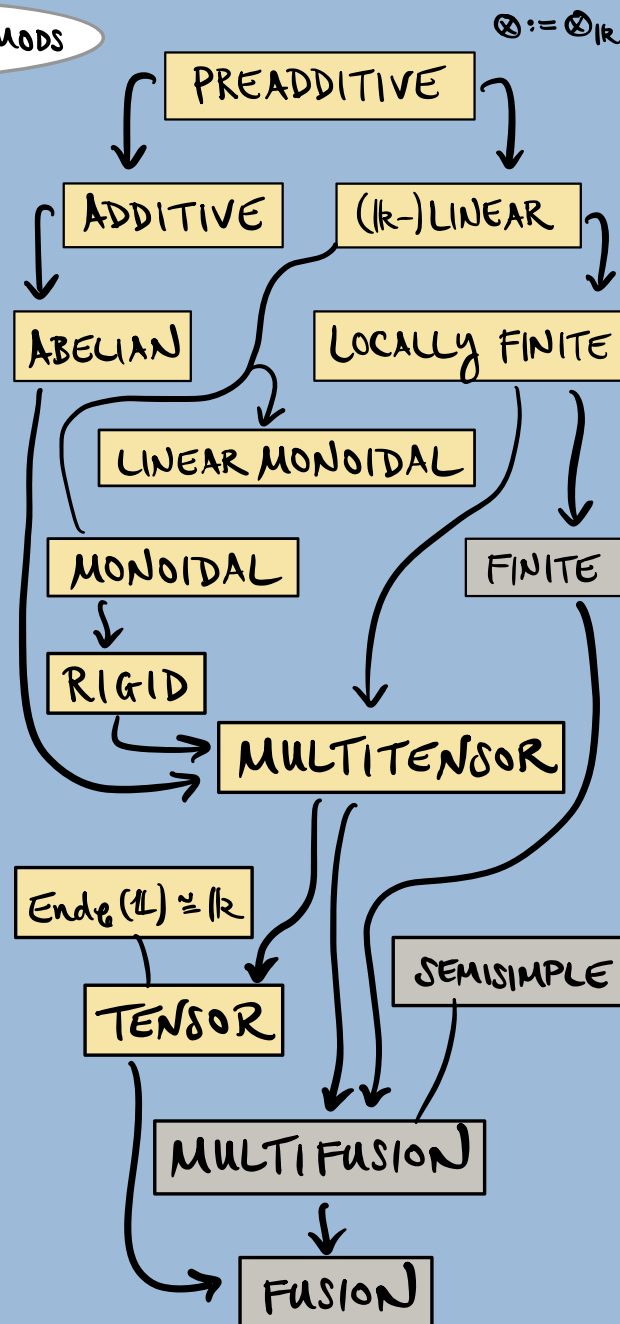
RECONSTRUCTION THEOREM:

"CAT. OF PAIRS" $\{(\mathcal{C}, U)\} \simeq$ "CAT. OF ~~STRUCTURES~~ ^{HOPF ALGEBRAS} / $\mathbb{K}$ "
 $\{(\mathcal{C}, U)\} \simeq \text{Hopf Alg}$

$$(\mathcal{C}, U) \longmapsto \text{Coend}_{\mathcal{C}}(U)$$

$$(\text{CoMod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$$

EVERY "RING" CAT. EQUIPPED W/ A "FIBER" FUNCTOR
 IS EQUIVALENT TO $\text{CoMod-}\mathcal{B}$
 FOR SOME BIALGEBRA $\mathcal{B}$



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED FEATURES)

• LINEAR • EXACT • FAITHFUL

• STRONG MONOIDAL:

$$U(-) \otimes_{\mathbb{K}} U(-) \cong U(- \otimes -)$$

$$U(1) \cong \mathbb{K}$$

$$\text{Coend}_{\mathcal{C}}(U) := \int^{x \in \mathcal{C}} U(x)^* \otimes U(x)$$

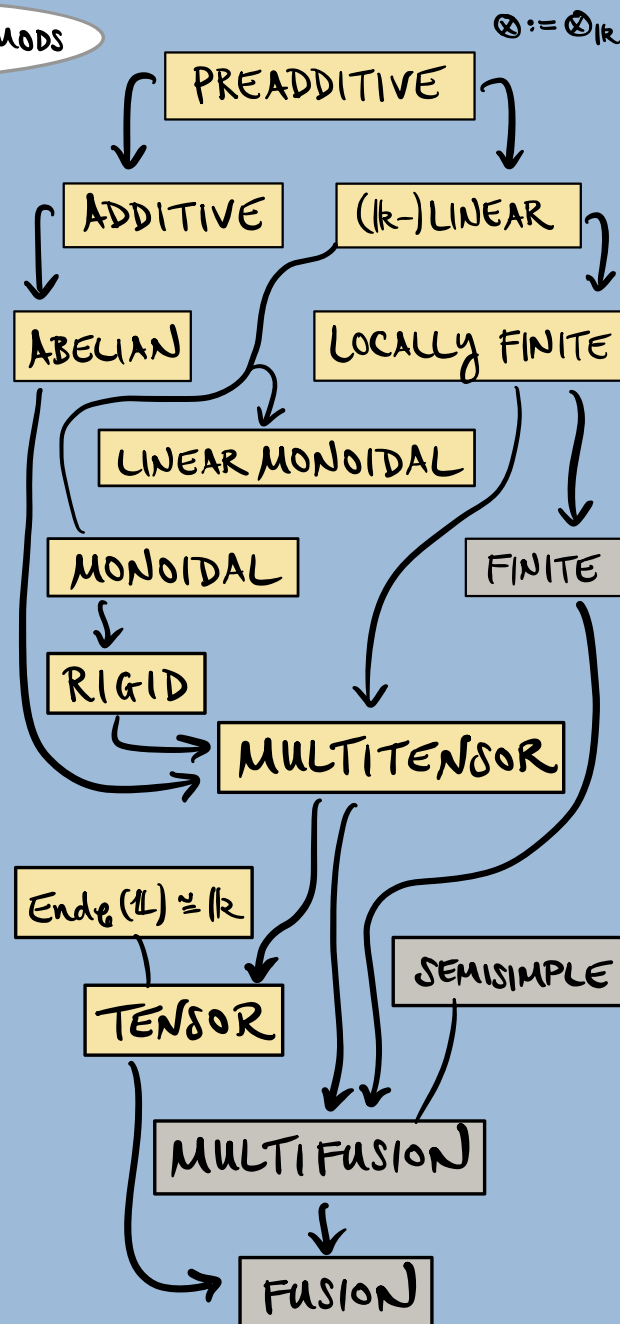
RECONSTRUCTION THEOREM:

"CAT. OF PAIRS" $\{(\mathcal{C}, U)\} \simeq$ "CAT. OF ~~STRUCTURES~~ ^{HOPF ALGEBRAS} / $\mathbb{K}$ "
 $\{(\mathcal{C}, U)\} \simeq \text{Hopf Alg}$

$$(\mathcal{C}, U) \longmapsto \text{Coend}_{\mathcal{C}}(U)$$

$$(\text{CoMod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$$

EVERY TENSOR CAT. EQUIPPED W/ A "FIBER" FUNCTOR
 IS EQUIVALENT TO $\text{CoMod-}H$
 FOR SOME HOPF ALGEBRA H



RECONSTRUCTION THEORY

$\mathcal{C}$ SAT. ?? $\Rightarrow \mathcal{C} \simeq (\text{Co})\text{MODS}$

TAKE NICE $\mathcal{C}$ & NICE $U: \mathcal{C} \rightarrow \text{Vec}$

(SEE HIGHLIGHTED FEATURES)

• LINEAR • EXACT • FAITHFUL

• STRONG MONOIDAL:

$$U(-) \otimes_{\mathbb{K}} U(-) \cong U(- \otimes -)$$

$$U(1) \cong \mathbb{K}$$

$$\text{Coend}_{\mathcal{C}}(U) := \int^{X \in \mathcal{C}} U(X)^* \otimes U(X)$$

RECONSTRUCTION THEOREM:

"CATEG. OF PAIRS" $\xrightarrow{\simeq}$ "CAT. OF STRUCTURES" / $\mathbb{K}$

$\{(\mathcal{C}, U)\}$ $\xrightarrow{\simeq}$ HopfAlg

AND SO ON...

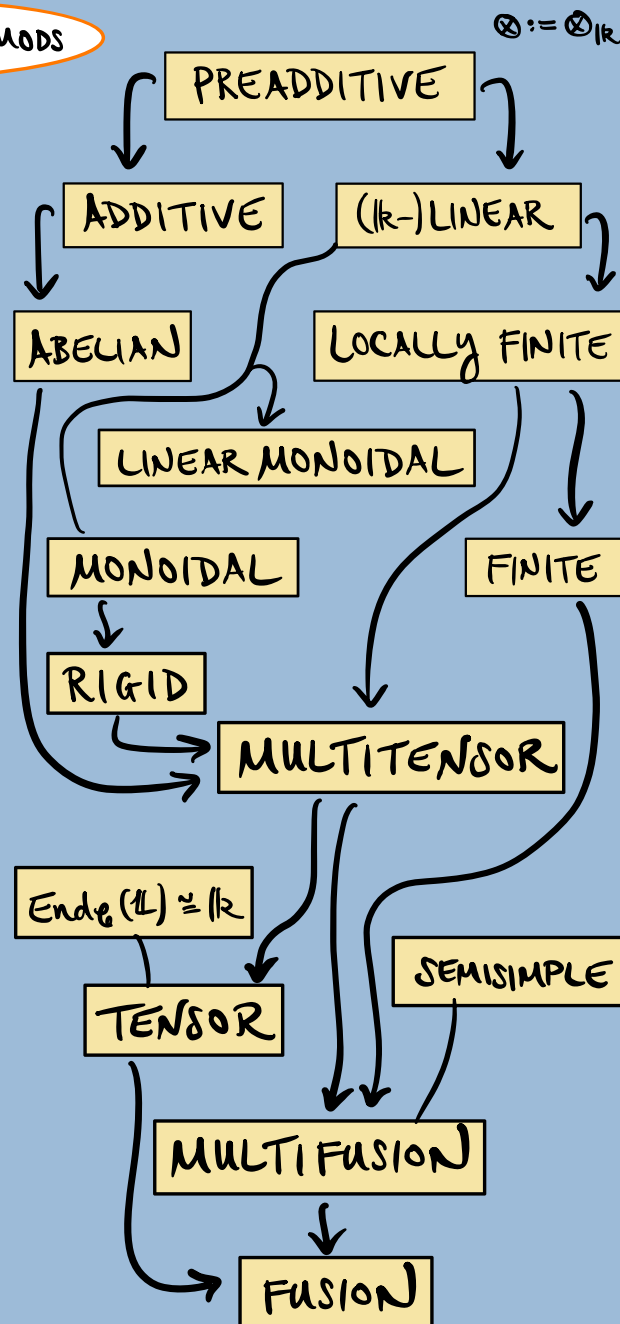
$(\mathcal{C}, U) \xrightarrow{\simeq} \text{Coend}_{\mathcal{C}}(U)$

$(\text{CoMod}/\mathcal{S}, \text{Forg}) \longleftarrow \mathcal{S}$

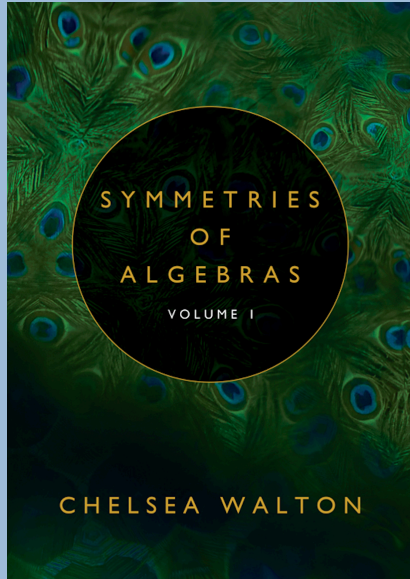
EVERY TENSOR CAT. EQUIPPED W/ A "FIBER" FUNCTOR

IS EQUIVALENT TO $\text{CoMod-}H$

FOR SOME HOPF ALGEBRA H



COMMERCIAL BREAK



"SYMMETRIES OF ALGEBRAS"
3-VOLUME BOOK SERIES
GEARED FOR NEWCOMERS

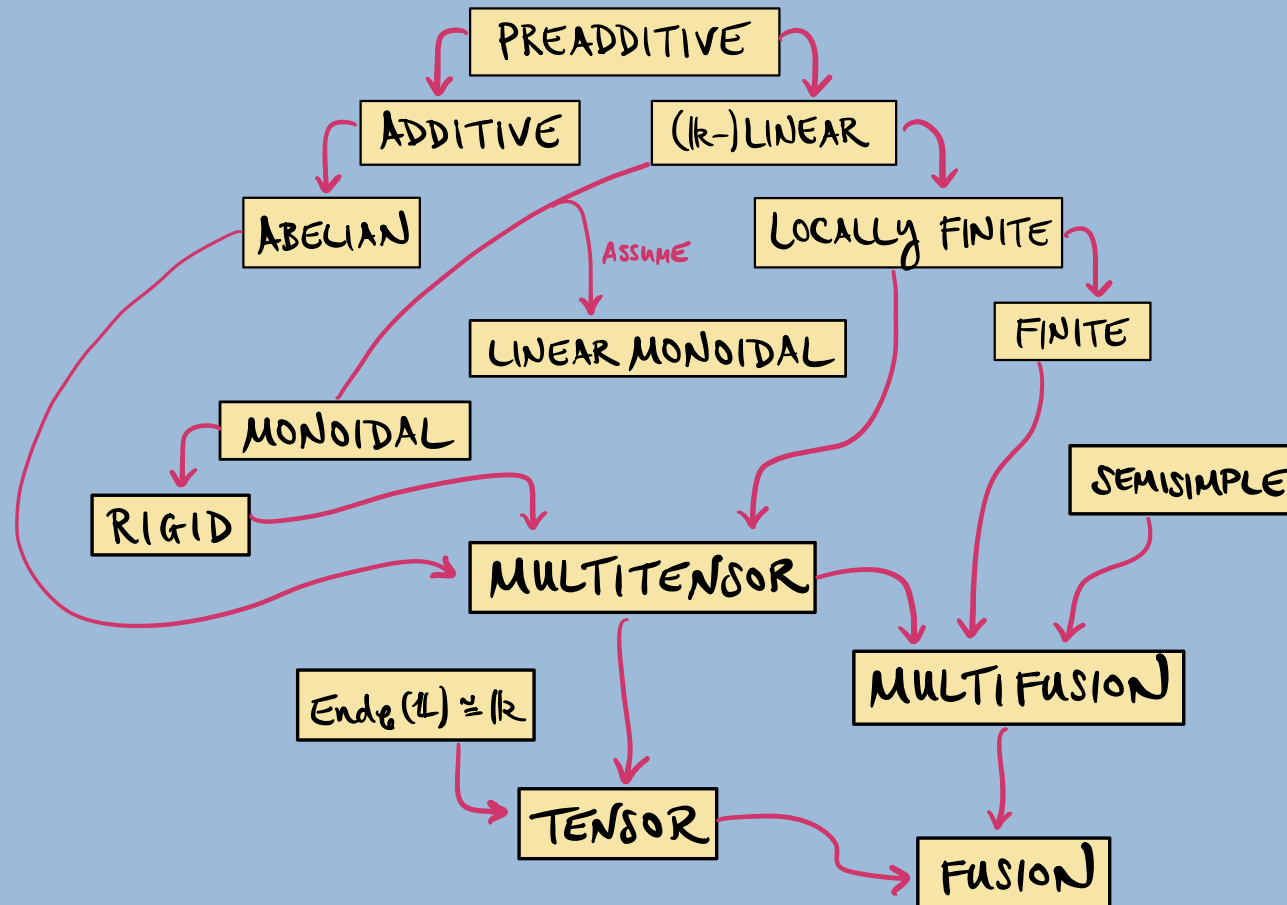
VOL 1: ALGS & MONOIDAL CATS
VOL 2: COALGS & FROBENIUS ALGS
VOL 3: HOPF ALGS & BRAIDINGS

GOAL: TO HAVE PUBLISHED BY
THE END OF 2026

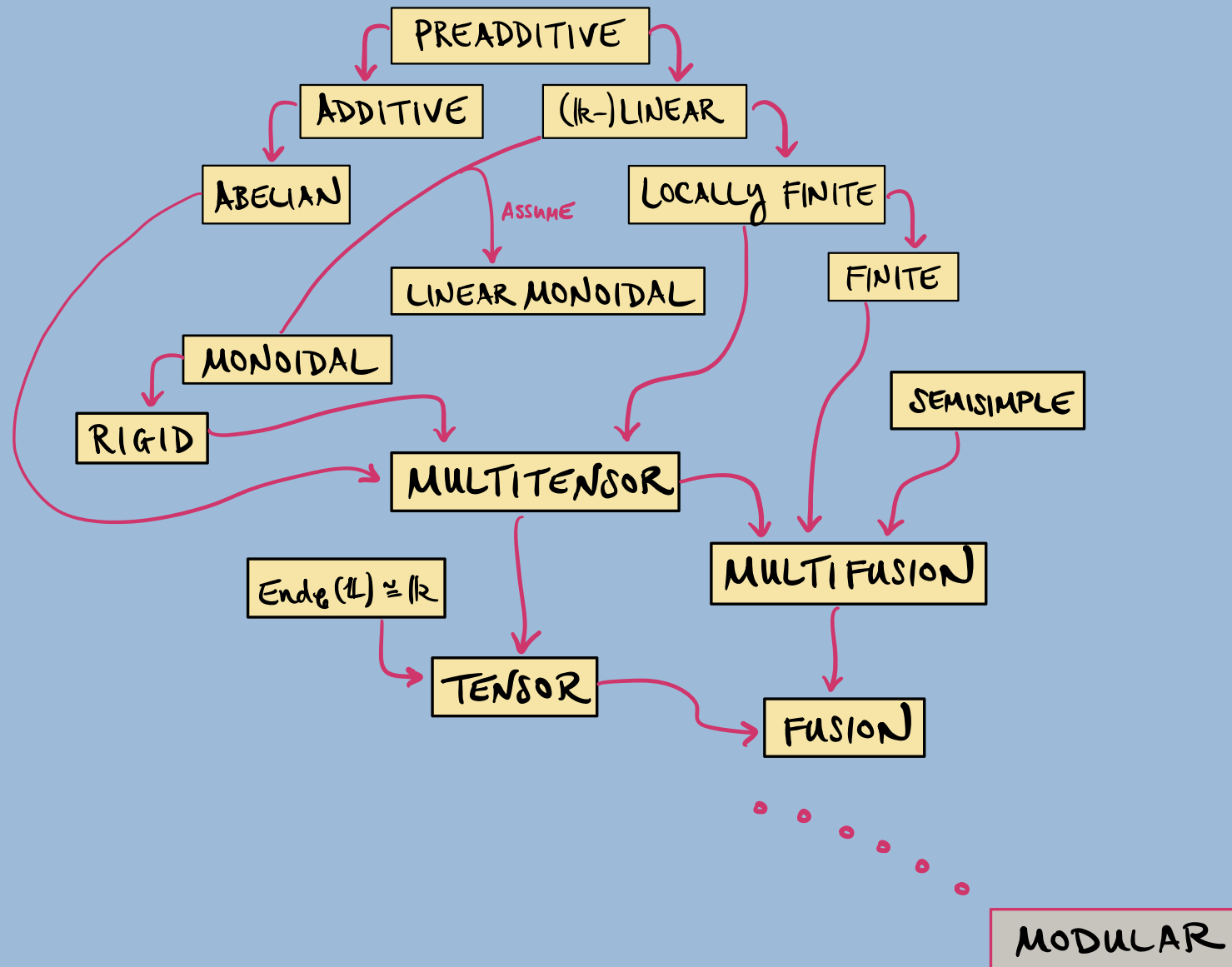
- CHAP 5: COALGS OVER $\mathbb{k}$
- CHAP 6: COALGS IN $\otimes$ CATS
INCLUDING (CO)ENDS
& RECONSTRUCTION THEORY
- CHAP 7: FROB. ALGS. OVER $\mathbb{k}$
- CHAP 8: FROB. ALGS IN $\otimes$ CATS

DONE WITH
DRAFTING CHAPS. 5 & 6
78 EXERCISES & 68 EXERCISES
≡ FEEDBACK WELCOMED ≡
(CAN MEET W/ STUDENTS)

THE ROAD TO MODULARITY



THE ROAD TO MODULARITY



THE ROAD TO MODULARITY

MODULAR FUSION / TENSOR CATEGORIES
ARE AT THE HEART OF
"QUANTUM SYMMETRIES"

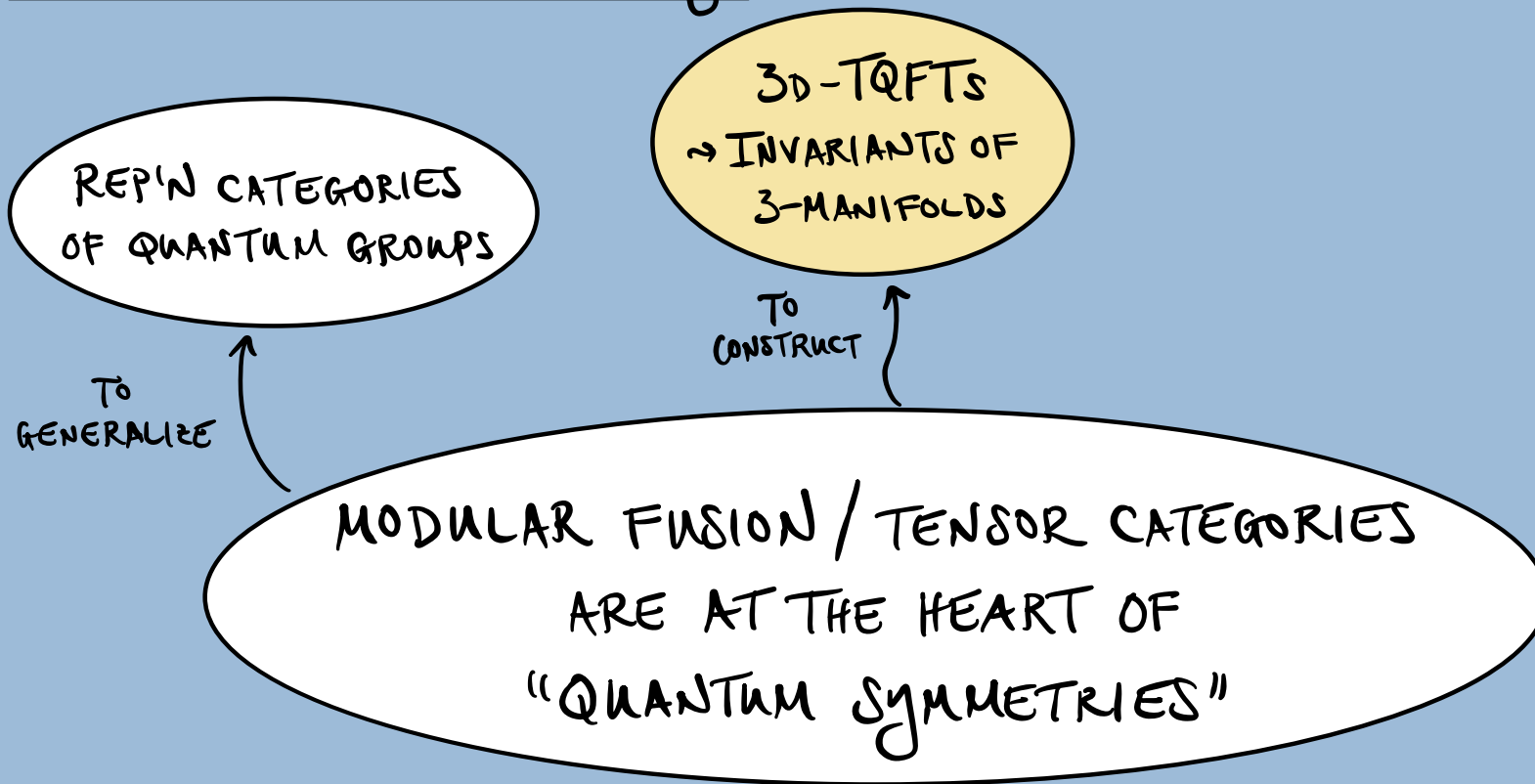
THE ROAD TO MODULARITY

REP'N CATEGORIES
OF QUANTUM GROUPS

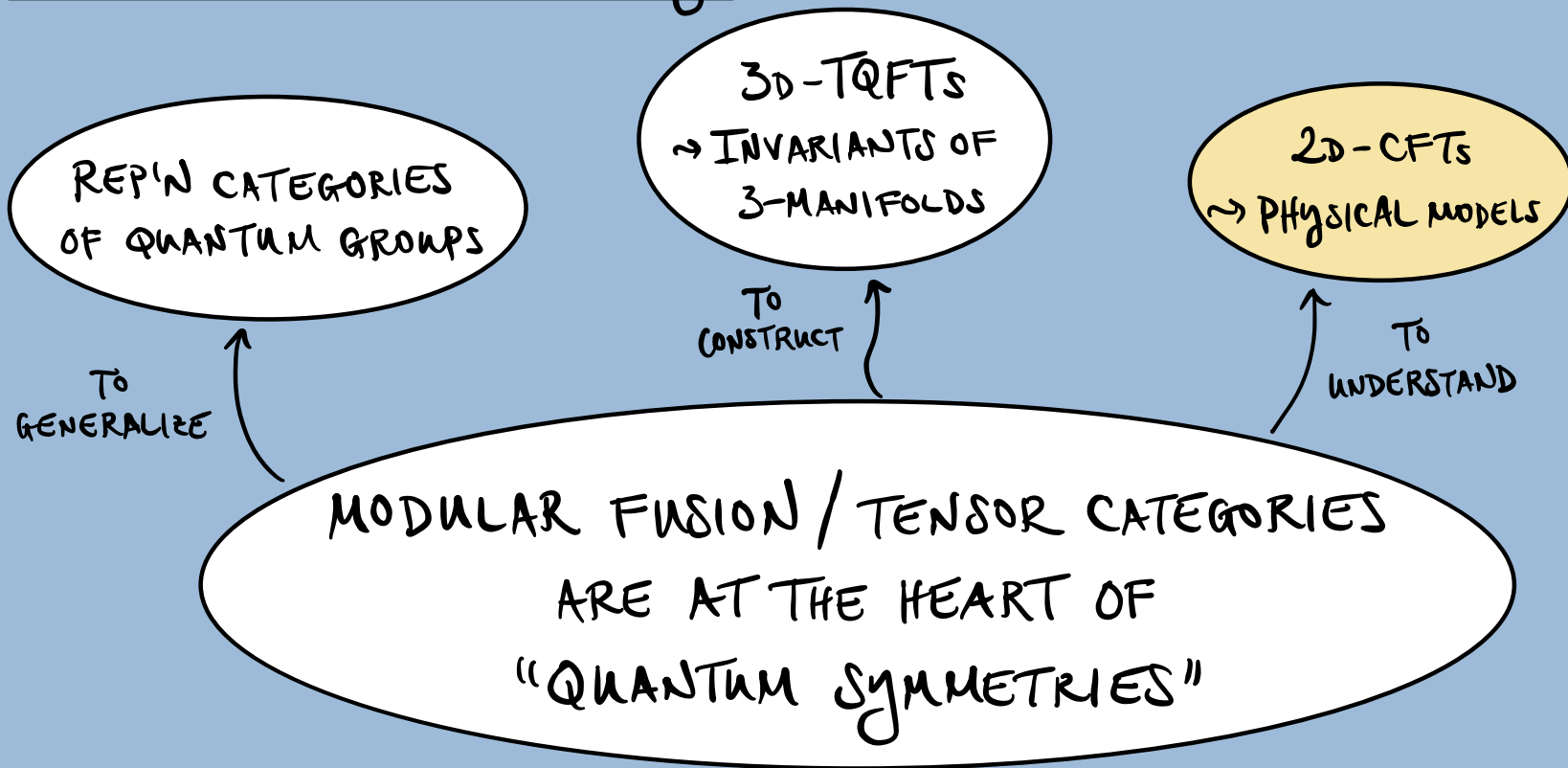
TO
GENERALIZE

MODULAR FUSION / TENSOR CATEGORIES
ARE AT THE HEART OF
"QUANTUM SYMMETRIES"

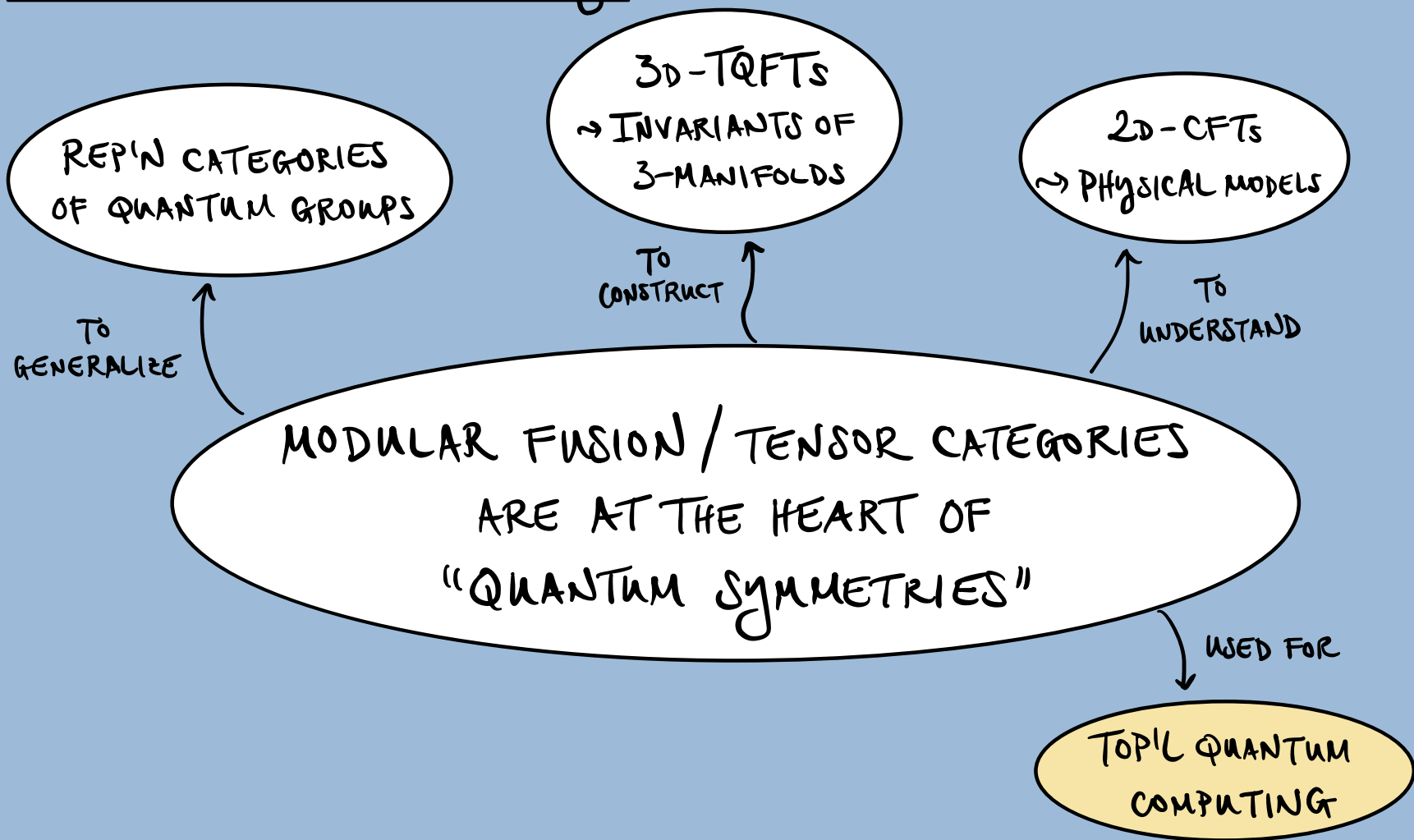
THE ROAD TO MODULARITY



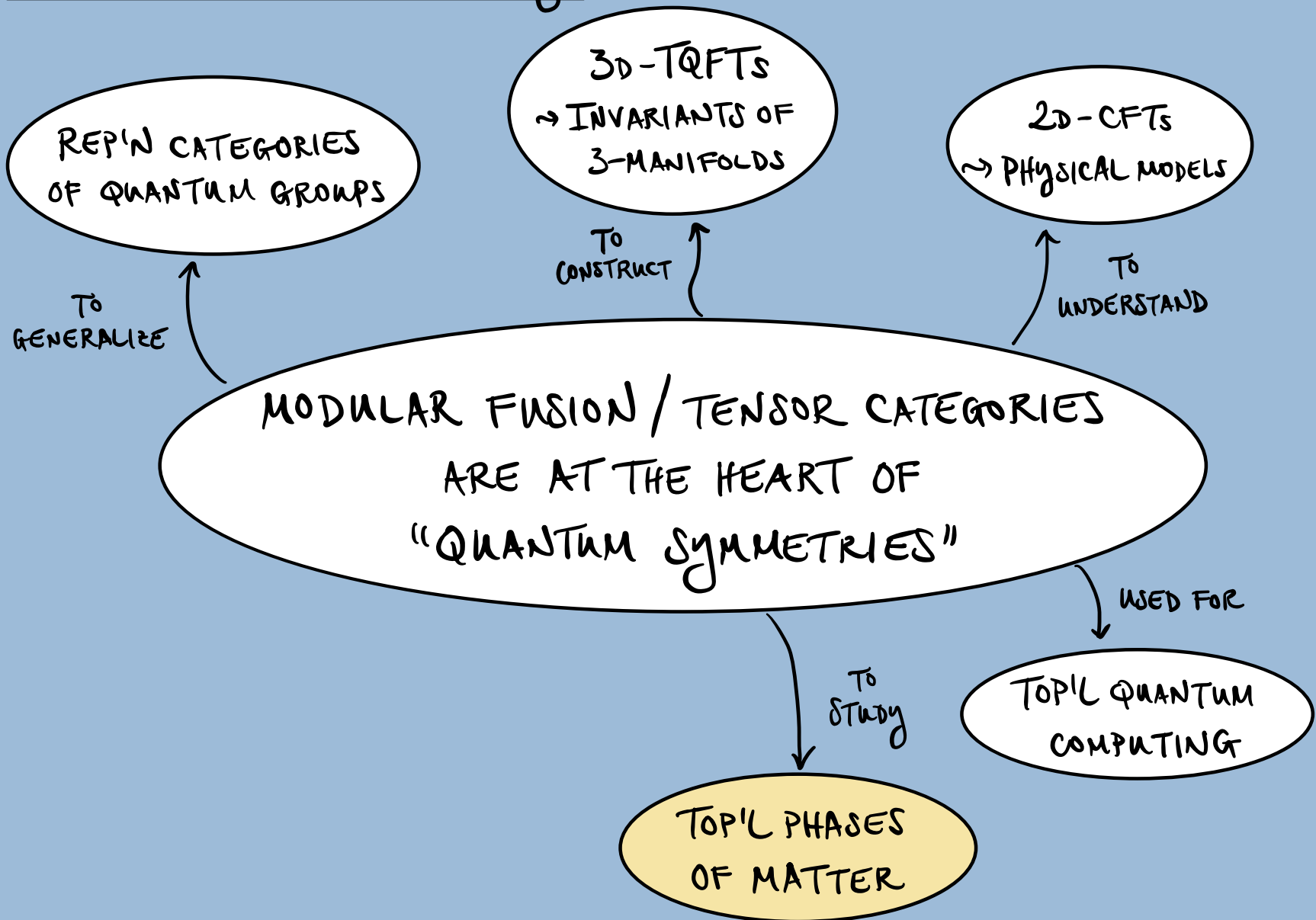
THE ROAD TO MODULARITY



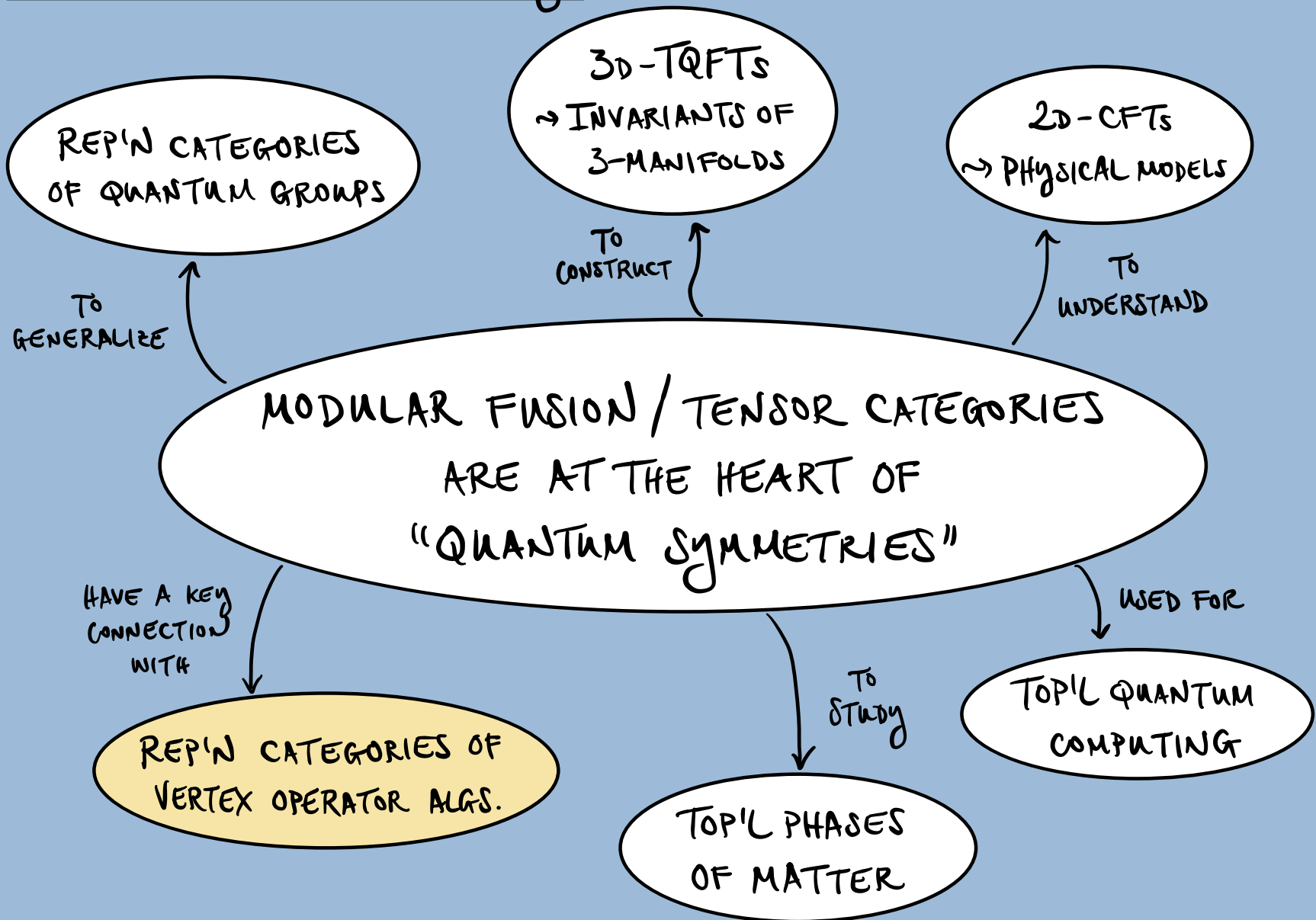
THE ROAD TO MODULARITY



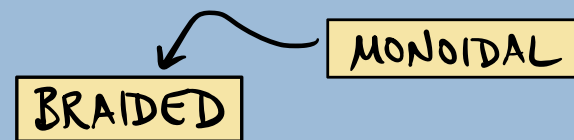
THE ROAD TO MODULARITY



THE ROAD TO MODULARITY



BRAIDINGS



BRAIDINGS

BRAIDED

MONOIDAL

A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{I})^\#$

$\#$ ASSUMING "STRICT" $\equiv$ SUPPRESSING ASSOC, UNITALITY NAT'L ISOMS

BRAIDINGS

BRAIDED

MONOIDAL

A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{I})^\#$
IS "BRAIDED" IF $\exists$ NAT'L ISOMORPHISM
$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$

SUCH THAT

$\#$ ASSUMING "STRICT" $\equiv$ SUPPRESSING ASSOC, UNITALITY NAT'L ISOMS

BRAIDINGS

BRAIDED

MONOIDAL

A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{I})^\#$
IS "BRAIDED" IF $\exists$ NAT'L ISOMORPHISM

$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$

SUCH THAT

$$x \otimes y \otimes z \xrightarrow{c_{x \otimes y, z}} z \otimes x \otimes y$$

$\#$ ASSUMING "STRICT" $\equiv$ SUPPRESSING ASSOC, UNITALITY NAT'L ISOMS

BRAIDINGS

BRAIDED

MONOIDAL

A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{I})^\#$
IS "BRAIDED" IF $\exists$ NAT'L ISOMORPHISM

$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$

SUCH THAT

$$\begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{c_{x \otimes y, z}} & z \otimes x \otimes y \\ \downarrow \text{id} \otimes c_{y,z} & \searrow \quad \nearrow & \uparrow c_{x,z} \otimes \text{id} \\ & x \otimes z \otimes y & \end{array}$$

$\#$ ASSUMING "STRICT" $\equiv$ SUPPRESSING ASSOC, UNITALITY NAT'L ISOMS

BRAIDINGS

BRAIDED

MONOIDAL

A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1})^\#$
IS "BRAIDED" IF $\exists$ NAT'L ISOMORPHISM

$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$

SUCH THAT

$$\begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{c_{x \otimes y, z}} & z \otimes x \otimes y \\ \text{id} \otimes c_{y, z} \searrow & \cong & \nearrow c_{x, z} \otimes \text{id} \\ & x \otimes z \otimes y & \end{array}$$

$$\begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{c_{x, y \otimes z}} & y \otimes z \otimes x \\ c_{x, y} \otimes \text{id} \searrow & \cong & \nearrow \text{id} \otimes c_{x, z} \\ & y \otimes x \otimes z & \end{array}$$

$\#$ ASSUMING "STRICT" $\equiv$ SUPPRESSING ASSOC, UNITALITY NAT'L ISOMS

BRAIDINGS

BRAIDED

MONOIDAL

A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{I})^\ddagger$
IS "BRAIDED" IF $\exists$ NAT'L ISOMORPHISM
 $C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$

SUCH THAT

$$\begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{c_{x,y \otimes z}} & z \otimes x \otimes y \\ \text{id} \otimes c_{y,z} \searrow & \cong & \nearrow c_{x,z} \otimes \text{id} \\ & x \otimes z \otimes y & \end{array}$$

$$\begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{c_{x,y \otimes z}} & y \otimes z \otimes x \\ c_{x,y} \otimes \text{id} \searrow & \cong & \nearrow \text{id} \otimes c_{x,z} \\ & y \otimes x \otimes z & \end{array}$$

VIA $\neq$
GRAPHICAL CALC.
(SAW A BIT LAST TIME)

$$x \otimes y \equiv \begin{array}{c} x \\ | \\ | \\ y \end{array}$$

$$f : x \rightarrow y \equiv \begin{array}{c} x \\ \boxed{f} \\ y \end{array}$$

$$c_{x,y} \equiv \begin{array}{c} x \quad y \\ \searrow \quad \nearrow \\ \nearrow \quad \searrow \end{array}$$

$$c_{y,x}^{-1} \equiv \begin{array}{c} x \quad y \\ \nearrow \quad \searrow \\ \searrow \quad \nearrow \end{array}$$

$\ddagger$ ASSUMING "STRICT" $\equiv$ SUPPRESSING ASSOC, UNITALITY NAT'L ISOMS

BRAIDINGS

BRAIDED

MONOIDAL

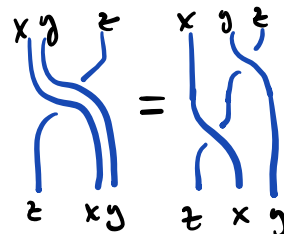
A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{I})^\dagger$
IS "BRAIDED" IF $\exists$ NAT'L ISOMORPHISM

$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$

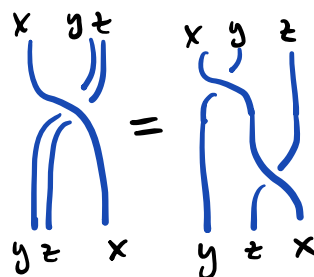


SUCH THAT

$$\begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{c_{x,y} \otimes \text{id}} & y \otimes x \otimes z \\ \text{id} \otimes c_{y,z} \searrow & \cong & \nearrow c_{x,z} \otimes \text{id} \\ & x \otimes z \otimes y & \end{array}$$



$$\begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{c_{x,y} \otimes \text{id}} & y \otimes x \otimes z \\ c_{x,y} \otimes \text{id} \searrow & \cong & \nearrow \text{id} \otimes c_{x,z} \\ & y \otimes x \otimes z & \end{array}$$



VIA $\neq$
GRAPHICAL CALC.
(SAW A BIT LAST TIME)

$$x \otimes y \equiv \begin{array}{c} x \\ | \\ y \end{array}$$

$$f : x \rightarrow y \equiv \begin{array}{c} x \\ \boxed{f} \\ y \end{array}$$

$$c_{x,y} \equiv \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ y \quad x \end{array}$$

$$c_{y,x}^{-1} \equiv \begin{array}{c} x \quad y \\ \diagup \quad \diagdown \\ y \quad x \end{array}$$

$\dagger$ ASSUMING "STRICT" $\equiv$ SUPPRESSING ASSOC, UNITALITY NAT'L ISOMS

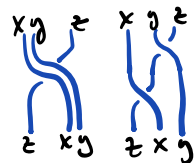
BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{I})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

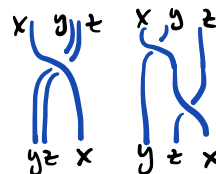
$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH
THAT



$\neq$



BRAIDED

MONOIDAL



REP'NS OF
BRAID GROUPS

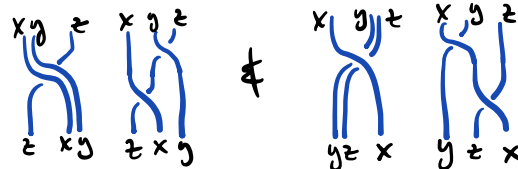
BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{I})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH
THAT



BRAIDED

MONOIDAL



REP'NS OF
BRAID GROUPS

ARTIN BRAID GROUPS OF TYPE A, FOR $n \geq 2$:

$$B_n := \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ FOR } i=1, \dots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{array} \right\rangle$$

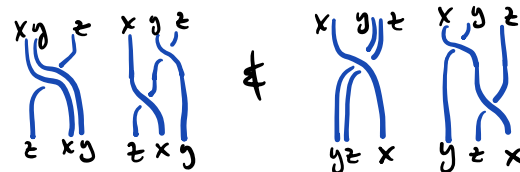
BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH
THAT



BRAIDED

MONOIDAL

REP'NS OF
BRAID GROUPS

ARTIN BRAID GROUPS OF TYPE A, FOR $n \geq 2$:

$$B_n := \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ FOR } i=1, \dots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{array} \right\rangle$$

TAKE AN OBJECT X IN $(\mathcal{C}, \otimes, \mathbb{1}, c)$.

$\rightarrow$ FORM GROUP OF SYMMETRIES OF $\text{Aut}_{\mathcal{C}}(X^{\otimes n})$

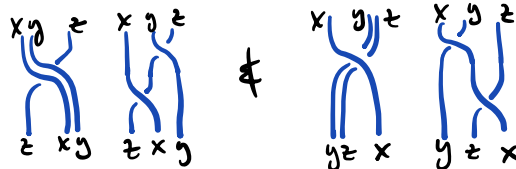
BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH
THAT



BRAIDED

MONOIDAL



REPRESENTATIONS OF
BRAID GROUPS

ARTIN BRAID GROUPS OF TYPE A, FOR $n \geq 2$:

$$B_n := \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ FOR } i=1, \dots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{array} \right\rangle$$

TAKE AN OBJECT X IN $(\mathcal{C}, \otimes, \mathbb{1}, c)$.

$\rightarrow$ FORM GROUP OF SYMMETRIES OF $\text{Aut}_{\mathcal{C}}(X^{\otimes n})$

$\rightarrow$ REPRN $\pi: B_n \longrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$
 $\equiv$
 GROUP HOMOM

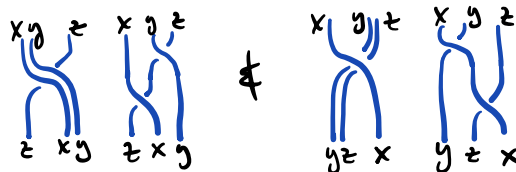
BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH
THAT



BRAIDED

MONOIDAL



REP'NS OF
BRAID GROUPS

ARTIN BRAID GROUPS OF TYPE A, FOR $n \geq 2$:

$$B_n := \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ FOR } i=1, \dots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{array} \right\rangle$$

TAKE AN OBJECT X IN $(\mathcal{C}, \otimes, \mathbb{1}, c)$.

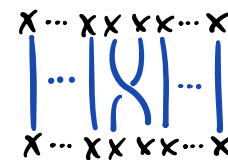
$\rightarrow$ FORM GROUP OF SYMMETRIES OF $\text{Aut}_{\mathcal{C}}(X^{\otimes n})$

$\rightarrow$ REP'N

Group Homom

$$\pi : B_n \longrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

$$\sigma_i \longmapsto \text{id}_{X^{\otimes i-1}} \otimes c_{X,X} \otimes \text{id}_{X^{\otimes n-i-1}}$$



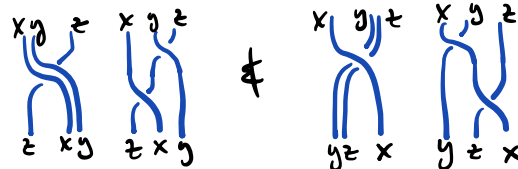
BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH THAT



BRAIDED

MONOIDAL



REPRESENTATIONS OF BRAID GROUPS

ARTIN BRAID GROUPS OF TYPE A, FOR $n \geq 2$:

$$B_n := \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ FOR } i=1, \dots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{array} \right\rangle$$

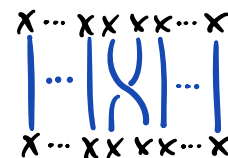
TAKE AN OBJECT X IN $(\mathcal{C}, \otimes, \mathbb{1}, c)$.

$\rightarrow$ FORM GROUP OF SYMMETRIES OF $\text{Aut}_{\mathcal{C}}(X^{\otimes n})$

$\rightarrow$ REPRN
|||
GROUP HOMOM

$$\pi : B_n \longrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

$$\sigma_i \longmapsto \text{id}_{X^{\otimes i-1}} \otimes c_{X,X} \otimes \text{id}_{X^{\otimes n-i-1}}$$



EX.

$$\pi(\sigma_1 \sigma_2 \sigma_1) = \text{[Braid Diagram]} = \dots$$

$$\text{[Braid Diagram]} = \pi(\sigma_2 \sigma_1 \sigma_2)$$

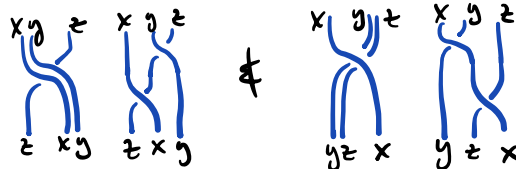
BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH THAT



BRAIDED

MONOIDAL



REPRESENTATIONS OF BRAID GROUPS

ARTIN BRAID GROUPS OF TYPE A, FOR $n \geq 2$:

$$B_n := \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ FOR } i=1, \dots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{array} \right\rangle$$

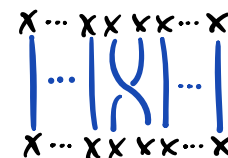
TAKE AN OBJECT X IN $(\mathcal{C}, \otimes, \mathbb{1}, c)$.

$\rightarrow$ FORM GROUP OF SYMMETRIES OF $\text{Aut}_{\mathcal{C}}(X^{\otimes n})$

$\rightarrow$ REPR
|||
GROUP HOMOM

$$\pi : B_n \longrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

$$\sigma_i \longmapsto \text{id}_{X^{\otimes i-1}} \otimes c_{X,X} \otimes \text{id}_{X^{\otimes n-i-1}}$$



EX.

$$\pi(\sigma_1 \sigma_2 \sigma_1) =$$



BRAID
AXIOM
=



= ...



BRAID
AXIOM
=



$$= \pi(\sigma_2 \sigma_1 \sigma_2)$$

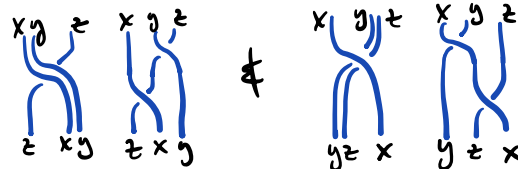
BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH THAT



BRAIDED

MONOIDAL



REPRESENTATIONS OF BRAID GROUPS

ARTIN BRAID GROUPS OF TYPE A, FOR $n \geq 2$:

$$B_n := \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ FOR } i=1, \dots, n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ FOR } |i-j| \geq 2 \end{array} \right\rangle$$

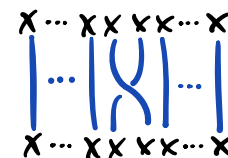
TAKE AN OBJECT X IN $(\mathcal{C}, \otimes, \mathbb{1}, c)$.

$\rightarrow$ FORM GROUP OF SYMMETRIES OF $\text{Aut}_{\mathcal{C}}(X^{\otimes n})$

$\rightarrow$ REPR
|||
GROUP HOMOM

$$\pi : B_n \longrightarrow \text{Aut}_{\mathcal{C}}(X^{\otimes n})$$

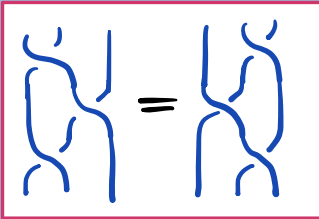
$$\sigma_i \longmapsto \text{id}_{X^{\otimes i-1}} \otimes c_{X,X} \otimes \text{id}_{X^{\otimes n-i-1}}$$



EX.

$$\pi(\sigma_1 \sigma_2 \sigma_1) = \text{Braid Axiom} = \text{Naturality} = \text{Braid Axiom} = \pi(\sigma_2 \sigma_1 \sigma_2)$$

BRAIDINGS

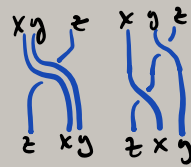


$(\mathcal{C}, \otimes, \mathbb{I})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

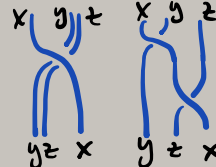
$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH
THAT



$\neq$



BRAIDED

MONOIDAL

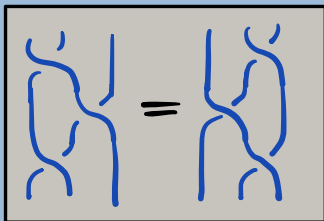


SOLUTIONS TO
THE YBE

|||

"R-MATRICES"

BRAIDINGS

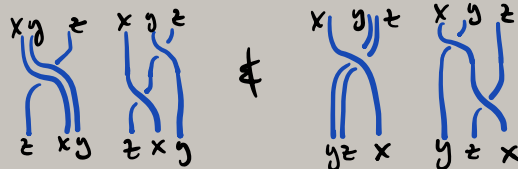


$(\mathcal{C}, \otimes, \mathbb{I})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH
THAT



BRAIDED

MONOIDAL



SOLUTIONS TO
THE YBE

|||

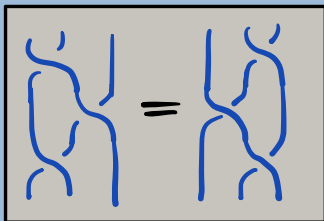
"R-MATRICES"

A BIALGEBRA/HOPF ALGEBRA H IS
"QUASITRIANGULAR" IF $\exists$ INVERTIBLE ELT

$$R := R' \otimes R'' \in H \otimes H$$

SUCH
THAT

BRAIDINGS

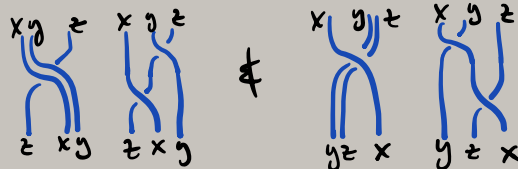


$(\mathcal{C}, \otimes, \mathbb{I})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH
THAT



BRAIDED

MONOIDAL



SOLUTIONS TO
THE YBE

|||

"R-MATRICES"

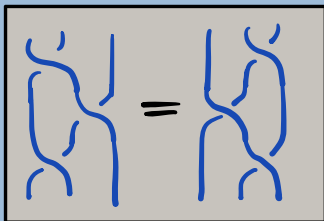
A BIALGEBRA/HOPF ALGEBRA H IS
"QUASITRIANGULAR" IF $\exists$ INVERTIBLE ELT

$$R := R' \otimes R'' \in H \otimes H$$

SUCH
THAT

$$\left\{ \begin{array}{l} R \Delta(h) R^{-1} = \Delta^{\text{cop}}(h) \quad \forall h \in H \end{array} \right.$$

BRAIDINGS

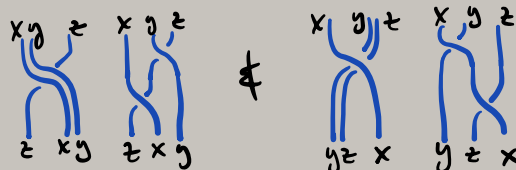


$(\mathcal{C}, \otimes, \mathbb{I})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x \mid x, y \in \mathcal{C}\}$$



SUCH
THAT



BRAIDED

MONOIDAL



SOLUTIONS TO
THE YBE

|||

"R-MATRICES"

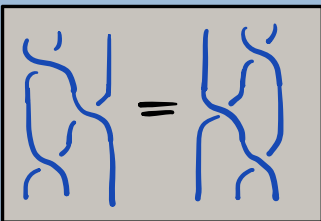
A BIALGEBRA/HOPF ALGEBRA H IS
"QUASITRIANGULAR" IF $\exists$ INVERTIBLE ELT

$$R := R' \otimes R'' \in H \otimes H$$

$$\text{SUCH THAT } \begin{cases} \Delta(R') \otimes R'' = R' \otimes \tilde{R}' \otimes R'' \tilde{R}'' \\ R' \otimes \Delta(R'') = R' \tilde{R}' \otimes \tilde{R}'' \otimes R'' \\ R \Delta(h) R^{-1} = \Delta^{\text{cop}}(h) \quad \forall h \in H \end{cases}$$

$$\tilde{R} = \text{copy of } R$$

BRAIDINGS

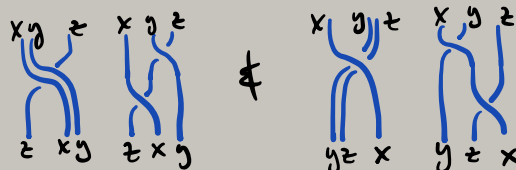


$(\mathcal{C}, \otimes, \mathbb{I})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x \mid x, y \in \mathcal{C}\}$$



SUCH
THAT



BRAIDED

MONOIDAL



SOLUTIONS TO
THE YBE

|||

"R-MATRICES"

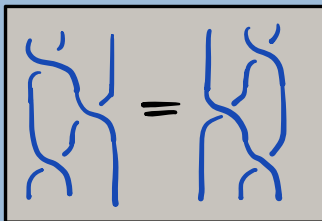
A BIALGEBRA/HOPF ALGEBRA H IS
"QUASITRIANGULAR" IF $\exists$ INVERTIBLE ELT

$$R := R' \otimes R'' \in H \otimes H$$

$$\text{SUCH THAT } \begin{cases} \Delta(R') \otimes R'' = R' \otimes \tilde{R}' \otimes R'' \tilde{R}'' \\ R' \otimes \Delta(R'') = R' \tilde{R}' \otimes \tilde{R}'' \otimes R'' \\ R \Delta(h) R^{-1} = \Delta^{\text{cop}}(h) \quad \forall h \in H \end{cases}$$

$$\tilde{R} = \text{copy of } R$$

BRAIDINGS

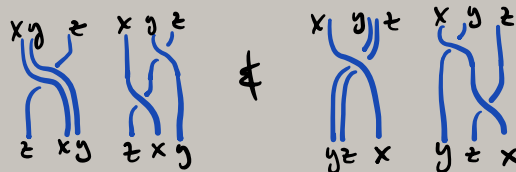


$(\mathcal{C}, \otimes, \mathbb{I})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH
THAT



BRAIDED

MONOIDAL



SOLUTIONS TO
THE YBE

III

"R-MATRICES"

A BIALGEBRA/HOPF ALGEBRA H IS
"QUASITRIANGULAR" IF $\exists$ INVERTIBLE ELT

$$R := R' \otimes R'' \in H \otimes H$$

$$\text{SUCH THAT } \begin{cases} \Delta(R') \otimes R'' = R' \otimes \tilde{R}' \otimes R'' \tilde{R}'' \\ R' \otimes \Delta(R'') = R' \tilde{R}' \otimes \tilde{R}'' \otimes R'' \\ R \Delta(h) R^{-1} = \Delta^{\text{cop}}(h) \quad \forall h \in H \end{cases}$$

$$\tilde{R} = \text{copy of } R$$

$H\text{-Mod}$ IS BRAIDED WITH

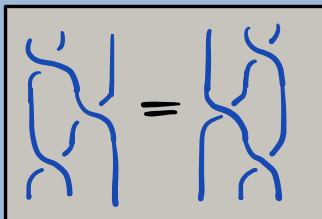
$$c_{x,y}(x \otimes y) = \dots$$

FOR $X, Y \in H\text{-Mod}$, $x \in X$, $y \in Y$



...

BRAIDINGS

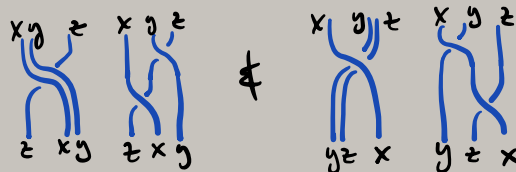


$(\mathcal{C}, \otimes, \mathbb{I})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH
THAT



BRAIDED

MONOIDAL



SOLUTIONS TO
THE YBE

III

"R-MATRICES"

A BIALGEBRA/HOPF ALGEBRA H IS
"QUASITRIANGULAR" IF $\exists$ INVERTIBLE ELT

$$R := R' \otimes R'' \in H \otimes H$$

SUCH
THAT

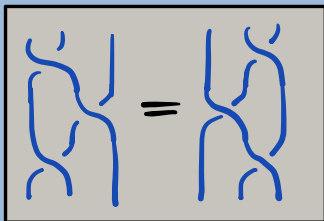
$$\begin{cases} \Delta(R') \otimes R'' = R' \otimes \tilde{R}' \otimes R'' \tilde{R}'' \\ R' \otimes \Delta(R'') = R' \tilde{R}' \otimes \tilde{R}'' \otimes R'' \\ R \Delta(h) R^{-1} = \Delta^{\text{cop}}(h) \quad \forall h \in H \\ \tilde{R} = \text{copy of } R \end{cases}$$

$H\text{-Mod}$ IS BRAIDED WITH
 $c_{x,y}(x \otimes y) = (R'' \triangleright y) \otimes (R' \triangleright x)$
 FOR $x, y \in H\text{-Mod}$, $x \in X$, $y \in Y$



$R = R' \otimes R'' \in H \otimes H$ IS
AN R-MATRIX

BRAIDINGS

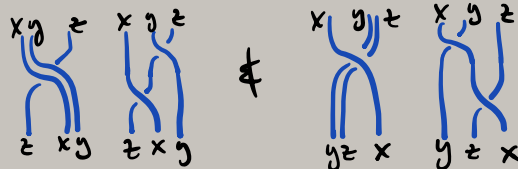


$(\mathcal{C}, \otimes, \mathbb{I})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH
THAT



BRAIDED

MONOIDAL



SOLUTIONS TO
THE YBE

|||

"R-MATRICES"

A BIALGEBRA/HOPF ALGEBRA H IS
"QUASITRIANGULAR" IF $\exists$ INVERTIBLE ELT

$$R := R' \otimes R'' \in H \otimes H$$

$$\text{SUCH THAT } \begin{cases} \Delta(R') \otimes R'' = R' \otimes \tilde{R}' \otimes R'' \tilde{R}'' \\ R' \otimes \Delta(R'') = R' \tilde{R}' \otimes \tilde{R}'' \otimes R'' \\ R \Delta(h) R^{-1} = \Delta^{\text{cop}}(h) \quad \forall h \in H \end{cases}$$

$$\tilde{R} = \text{copy of } R$$

$H\text{-Mod}$ IS BRAIDED WITH
 $c_{x,y}(x \otimes y) = (R'' \triangleright y) \otimes (R' \triangleright x)$
FOR $x, y \in H\text{-Mod}$, $x \in X$, $y \in Y$



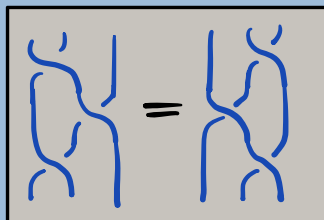
$R = R' \otimes R'' \in H \otimes H$ IS
AN R-MATRIX

EXAMPLES:

H COCOMMUTATIVE,

$$R = 1 \otimes 1$$

BRAIDINGS

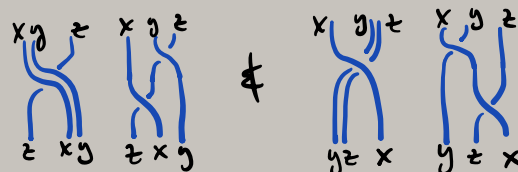


$(\mathcal{C}, \otimes, \mathbb{I})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH THAT



BRAIDED

MONOIDAL



SOLUTIONS TO THE YBE

|||

"R-MATRICES"

A BIALGEBRA/HOPF ALGEBRA H IS "QUASITRIANGULAR" IF $\exists$ INVERTIBLE ELT

$$R := R' \otimes R'' \in H \otimes H$$

$$\text{SUCH THAT } \begin{cases} \Delta(R') \otimes R'' = R' \otimes \tilde{R}' \otimes R'' \tilde{R}'' \\ R' \otimes \Delta(R'') = R' \tilde{R}' \otimes \tilde{R}'' \otimes R'' \\ R \Delta(h) R^{-1} = \Delta^{\text{cop}}(h) \quad \forall h \in H \end{cases}$$

$$\tilde{R} = \text{copy of } R$$

$H\text{-Mod}$ IS BRAIDED WITH
 $c_{x,y}(x \otimes y) = (R'' \triangleright y) \otimes (R' \triangleright x)$
 FOR $x, y \in H\text{-Mod}$, $x \in X$, $y \in Y$



$R = R' \otimes R'' \in H \otimes H$ IS AN R-MATRIX

EXAMPLES:

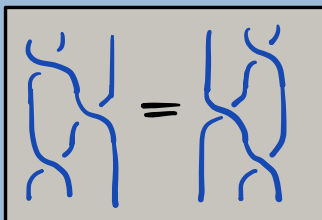
H COCOMMUTATIVE,
 $R = 1 \otimes 1$

$\mathbb{K}G$

FOR $u \in Z(G)$ w/ $u^2 = e$

$$R_u = \frac{1}{2}(e \otimes e + u \otimes e + e \otimes u - u \otimes u)$$

BRAIDINGS

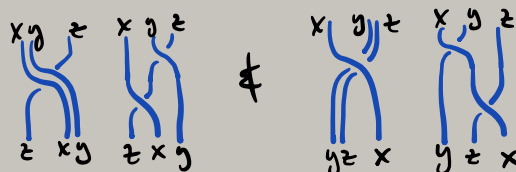


$(\mathcal{C}, \otimes, \mathbb{I})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



SUCH THAT



BRAIDED

MONOIDAL



SOLUTIONS TO THE YBE

|||

"R-MATRICES"

A BIALGEBRA/HOPF ALGEBRA H IS "QUASITRIANGULAR" IF $\exists$ INVERTIBLE ELT

$$R := R' \otimes R'' \in H \otimes H$$

$$\text{SUCH THAT } \begin{cases} \Delta(R') \otimes R'' = R' \otimes \tilde{R}' \otimes R'' \tilde{R}'' \\ R' \otimes \Delta(R'') = R' \tilde{R}' \otimes \tilde{R}'' \otimes R'' \\ R \Delta(h) R^{-1} = \Delta^{\text{cop}}(h) \quad \forall h \in H \end{cases}$$

$$\tilde{R} = \text{copy of } R$$

$H\text{-Mod}$ IS BRAIDED WITH
 $c_{x,y}(x \otimes y) = (R'' \triangleright y) \otimes (R' \triangleright x)$
 FOR $x, y \in H\text{-Mod}$, $x \in X$, $y \in Y$



$R = R' \otimes R'' \in H \otimes H$ IS AN R-MATRIX

EXAMPLES:

H COCOMMUTATIVE,
 $R = 1 \otimes 1$

$\mathbb{K}G$

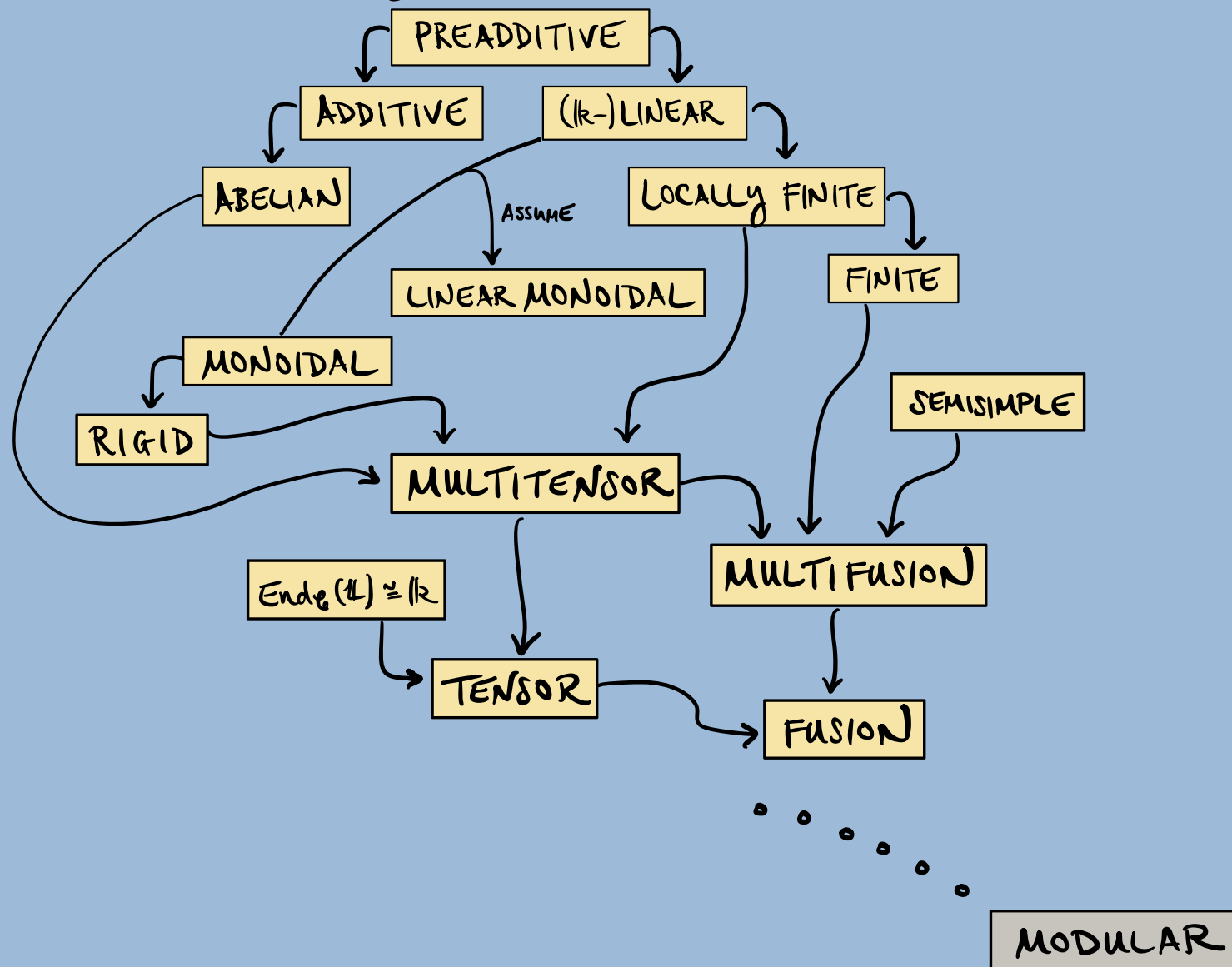
FOR $u \in Z(G)$ w/ $u^2 = e$

$$R_u = \frac{1}{2}(e \otimes e + u \otimes e + e \otimes u - u \otimes u)$$

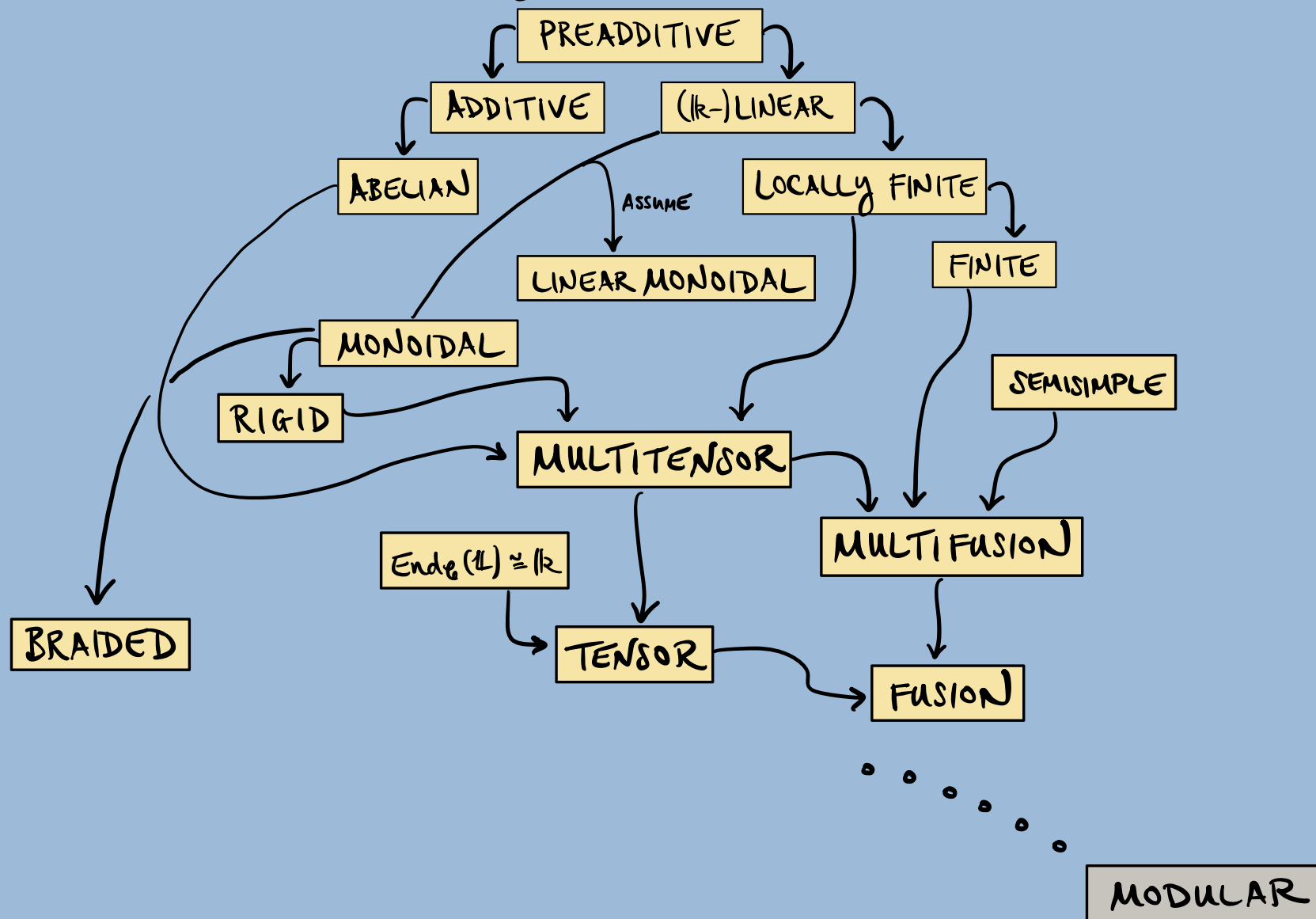
SMALL Q.GROUP

$$u_q(\mathfrak{sl}_2)$$

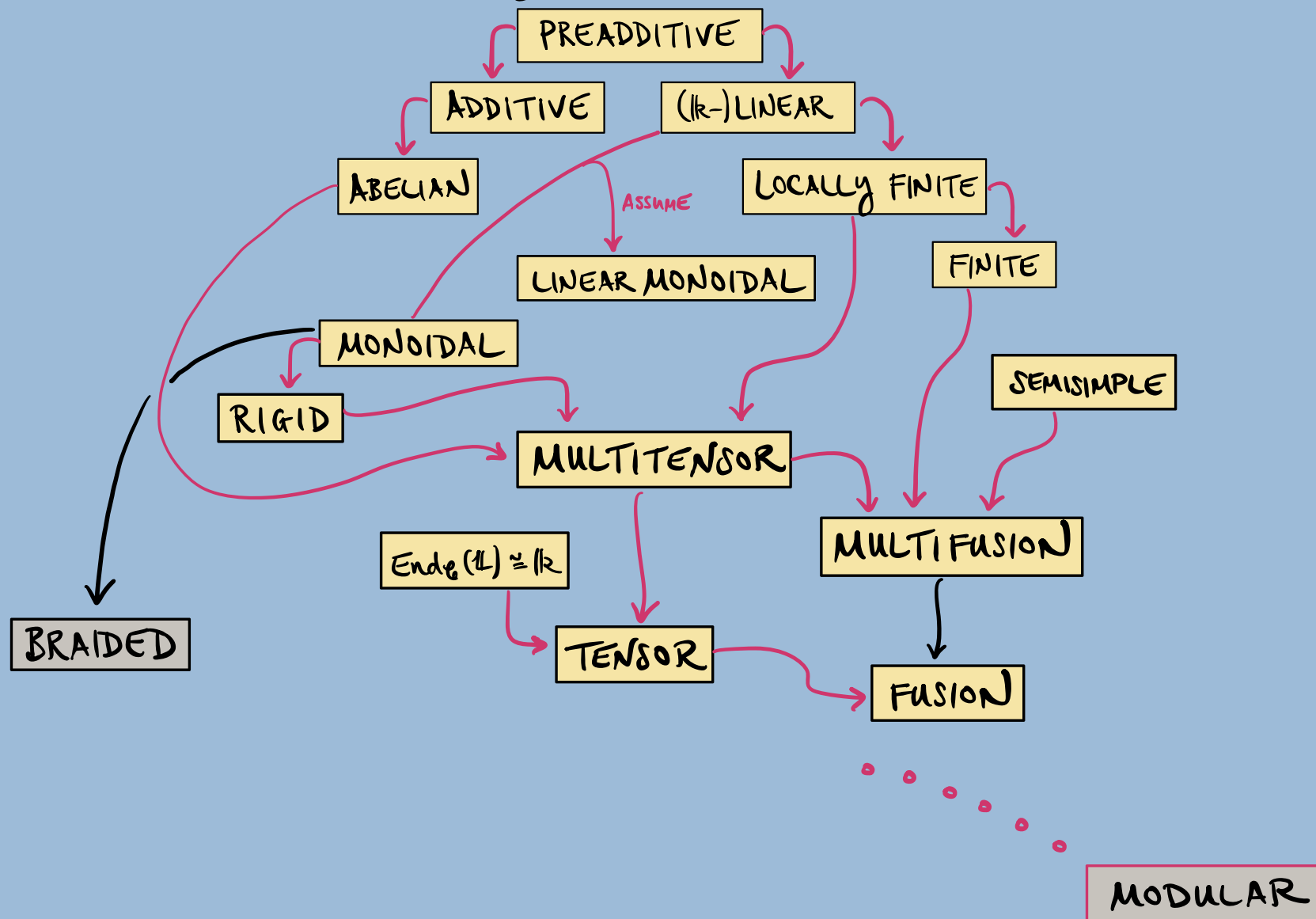
THE ROAD TO MODULARITY



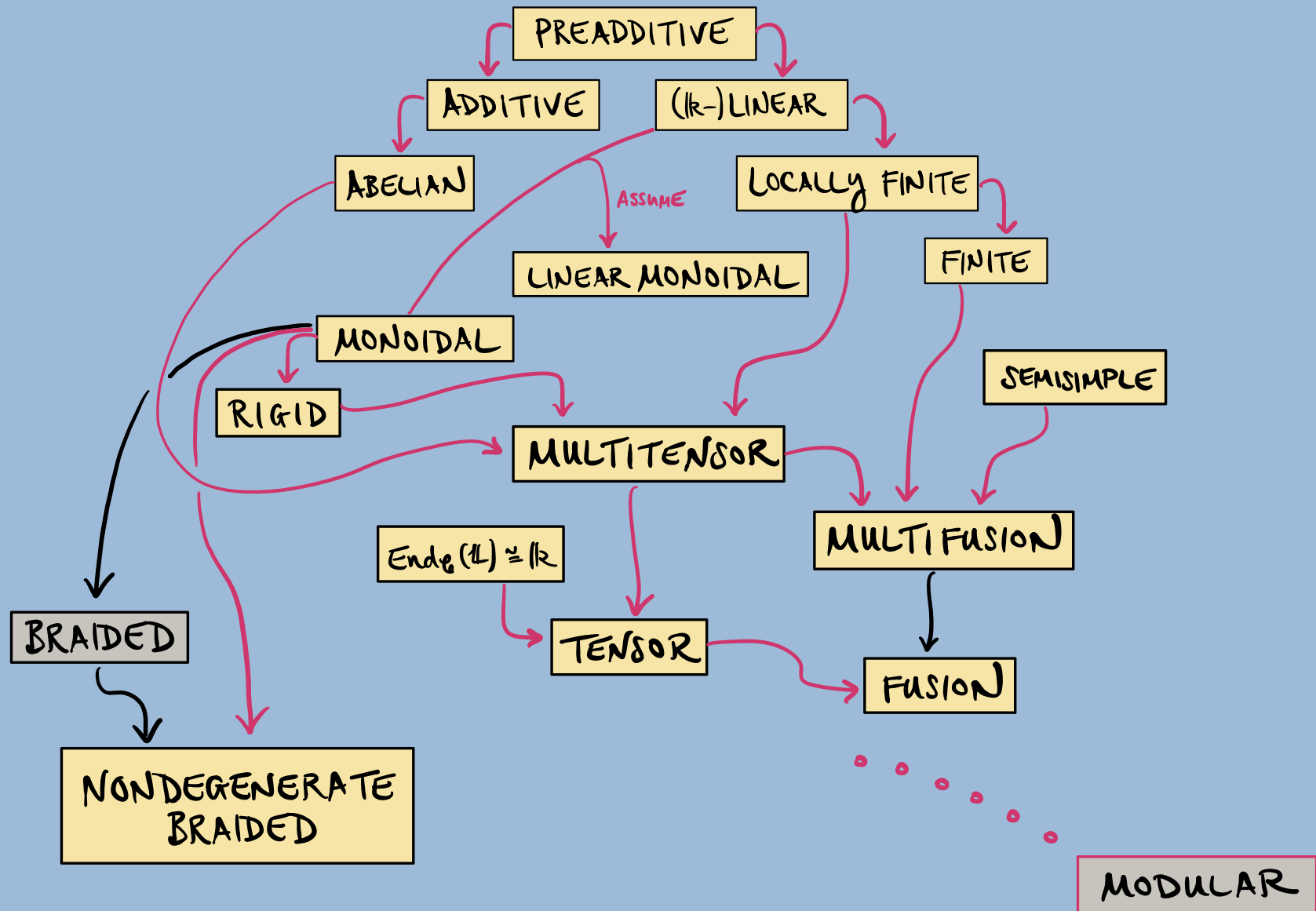
THE ROAD TO MODULARITY



THE ROAD TO MODULARITY



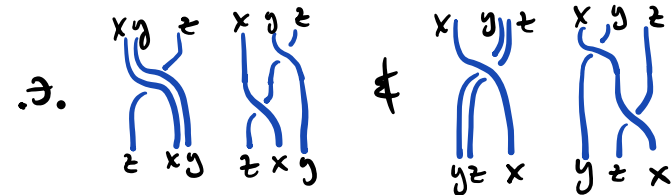
THE ROAD TO MODULARITY



NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{L})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

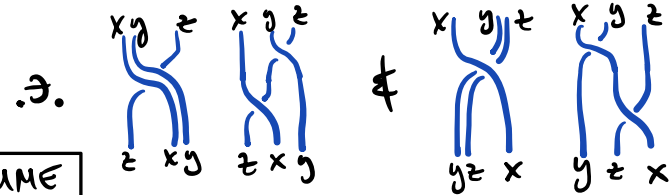
$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{L})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$



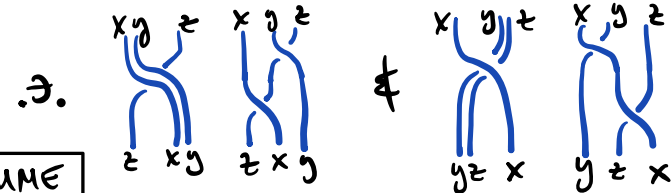
ASSUME
TENSOR

NONDEGENERATE BRAIDINGS

CONSIDER FULL SUBCATEGORY:
 "MÜGER CENTER" OF $(\mathcal{C}, \otimes, \mathbb{1}, c)$
 $\mathcal{Z}_2(\mathcal{C})$ WITH OBJECTS
 $\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{X \otimes y} \forall y \in \mathcal{C}\}$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y} : X \otimes y \xrightarrow{\sim} y \otimes X\}_{x,y \in \mathcal{C}}$$



ASSUME
TENSOR

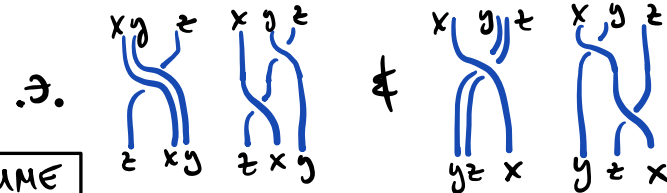
NONDEGENERATE BRAIDINGS

CONSIDER FULL SUBCATEGORY:
 "MÜGER CENTER" OF $(\mathcal{C}, \otimes, \mathbb{1}, c)$
 $\mathcal{Z}_2(\mathcal{C})$ WITH OBJECTS
 $\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{X \otimes y} \forall y \in \mathcal{C}\}$

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
 IF
 $\mathcal{Z}_2(\mathcal{C}) \simeq \text{Vec}$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y} : X \otimes y \xrightarrow{\sim} y \otimes X\}_{x,y \in \mathcal{C}}$$



ASSUME
TENSOR

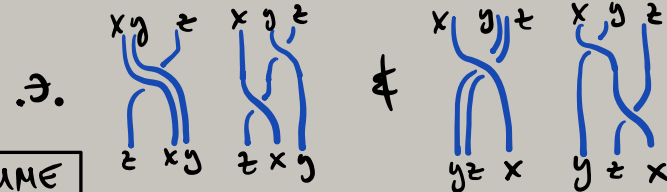
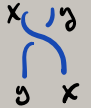
NONDEGENERATE BRAIDINGS

CONSIDER FULL SUBCATEGORY:
 "MÜGER CENTER" OF $(\mathcal{C}, \otimes, \mathbb{1}, c)$
 $\mathcal{Z}_2(\mathcal{C})$ WITH OBJECTS
 $\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{X \otimes y} \forall y \in \mathcal{C}\}$

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
 IF
 $\mathcal{Z}_2(\mathcal{C}) \simeq \text{Vec}$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y} : X \otimes y \xrightarrow{\sim} y \otimes X\}_{x,y \in \mathcal{C}}$$



ASSUME
TENSOR

IF, FURTHER,
 $\mathcal{C}$ IS FINITE TENSOR

$$H_{\mathcal{C}} := \int^{x \in \mathcal{C}} x^* \otimes x \in \text{HopfAlg}(\mathcal{C})$$

NONDEGENERATE BRAIDINGS

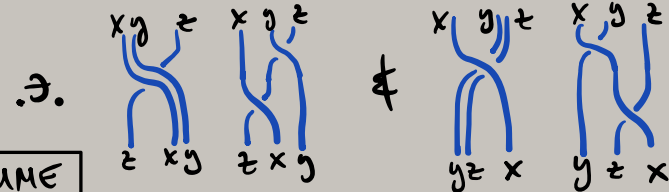
CONSIDER FULL SUBCATEGORY:
 "MÜGER CENTER" OF $(\mathcal{C}, \otimes, \mathbb{1}, c)$
 $\mathcal{Z}_2(\mathcal{C})$ WITH OBJECTS
 $\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{X \otimes y} \forall y \in \mathcal{C}\}$

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
 IF
 $\mathcal{Z}_2(\mathcal{C}) \simeq \text{Vec}$

$H_{\mathcal{C}}$ HAS A NONDEGENERATE
 HOPF PAIRING $\omega_{\mathcal{C}}: H_{\mathcal{C}} \otimes H_{\mathcal{C}} \rightarrow \mathbb{1}$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y}: X \otimes y \xrightarrow{\sim} y \otimes X\}_{x,y \in \mathcal{C}}$$



ASSUME
TENSOR

IF, FURTHER,
 $\mathcal{C}$ IS FINITE TENSOR

$$H_{\mathcal{C}} := \int^{x \in \mathcal{C}} x^* \otimes x \in \text{HopfAlg}(\mathcal{C})$$

NONDEGENERATE BRAIDINGS

CONSIDER FULL SUBCATEGORY:
 "MÜGER CENTER" OF $(\mathcal{C}, \otimes, \mathbb{1}, c)$
 $\mathcal{Z}_2(\mathcal{C})$ WITH OBJECTS
 $\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{X \otimes y} \forall y \in \mathcal{C}\}$

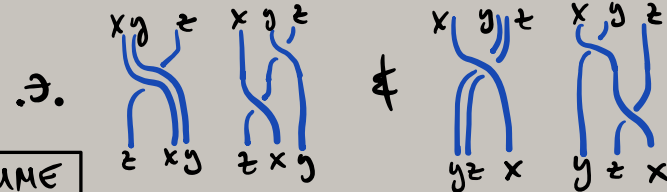
$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
 IF
 $\mathcal{Z}_2(\mathcal{C}) \simeq \text{Vec}$

$H_{\mathcal{C}}$ HAS A NONDEGENERATE
 HOPF PAIRING $\omega_{\mathcal{C}}: H_{\mathcal{C}} \otimes H_{\mathcal{C}} \rightarrow \mathbb{1}$

$\text{Hom}_{\mathcal{C}}(\mathbb{1}, H_{\mathcal{C}}) \rightarrow \text{Hom}_{\mathcal{C}}(H_{\mathcal{C}}, \mathbb{1})$
 $f \mapsto [\omega_{\mathcal{C}} \circ (f \otimes \text{id})]$
 IS BIJECTIVE

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y}: X \otimes y \xrightarrow{\sim} y \otimes X\}_{x,y \in \mathcal{C}}$$



ASSUME
TENSOR

IF, FURTHER,
 $\mathcal{C}$ IS FINITE TENSOR

$$H_{\mathcal{C}} := \int^{x \in \mathcal{C}} x^* \otimes x \in \text{HopfAlg}(\mathcal{C})$$

NONDEGENERATE BRAIDINGS

CONSIDER FULL SUBCATEGORY:
 "MÜGER CENTER" OF $(\mathcal{C}, \otimes, \mathbb{1}, c)$
 $\mathcal{Z}_2(\mathcal{C})$ WITH OBJECTS
 $\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{X \otimes y} \forall y \in \mathcal{C}\}$

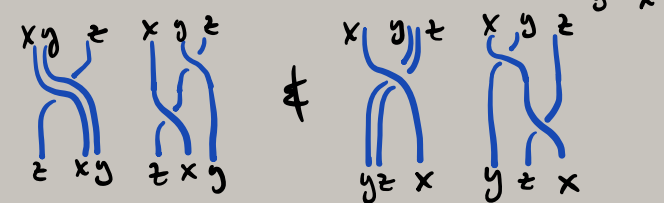
$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
 IF
 $\mathcal{Z}_2(\mathcal{C}) \simeq \text{Vec}$

$H_{\mathcal{C}}$ HAS A NONDEGENERATE
 HOPF PAIRING $\omega_{\mathcal{C}}: H_{\mathcal{C}} \otimes H_{\mathcal{C}} \rightarrow \mathbb{1}$

$\text{Hom}_{\mathcal{C}}(\mathbb{1}, H_{\mathcal{C}}) \rightarrow \text{Hom}_{\mathcal{C}}(H_{\mathcal{C}}, \mathbb{1})$
 $f \mapsto [\omega_{\mathcal{C}} \circ (f \otimes \text{id})]$
 IS BIJECTIVE

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y}: X \otimes y \xrightarrow{\sim} y \otimes X\}_{x,y \in \mathcal{C}}$$

$\exists$ 

ASSUME
TENSOR

IF, FURTHER,
 $\mathcal{C}$ IS FINITE TENSOR

$$H_{\mathcal{C}} := \int^{x \in \mathcal{C}} x^* \otimes x \in \text{HopfAlg}(\mathcal{C})$$

IF, FURTHER,
 $\mathcal{C}$ IS FUSION

"S-MATRIX" OF $\mathcal{C}$
 IS INVERTIBLE

NONDEGENERATE BRAIDINGS

CONSIDER FULL SUBCATEGORY:
 "MÜGER CENTER" OF $(\mathcal{C}, \otimes, \mathbb{1}, c)$
 $\mathcal{Z}_2(\mathcal{C})$ WITH OBJECTS
 $\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{X \otimes y} \forall y \in \mathcal{C}\}$

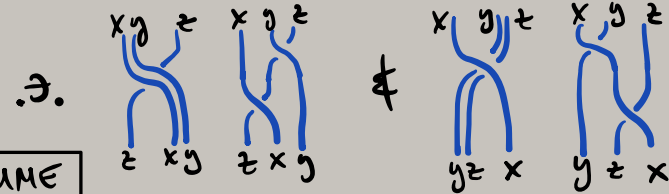
$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
 IF
 $\mathcal{Z}_2(\mathcal{C}) \simeq \text{Vec}$

$H_{\mathcal{C}}$ HAS A NONDEGENERATE
 HOPF PAIRING $\omega_{\mathcal{C}}: H_{\mathcal{C}} \otimes H_{\mathcal{C}} \rightarrow \mathbb{1}$

$\text{Hom}_{\mathcal{C}}(\mathbb{1}, H_{\mathcal{C}}) \rightarrow \text{Hom}_{\mathcal{C}}(H_{\mathcal{C}}, \mathbb{1})$
 $f \mapsto [\omega_{\mathcal{C}} \circ (f \otimes \text{id})]$
 IS BIJECTIVE

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:

$$c := \{c_{x,y}: X \otimes y \xrightarrow{\sim} y \otimes X\}_{x,y \in \mathcal{C}}$$



ASSUME
TENSOR

IF, FURTHER,
 $\mathcal{C}$ IS FINITE TENSOR

$$H_{\mathcal{C}} := \int^{x \in \mathcal{C}} x^* \otimes x \in \text{HopfAlg}(\mathcal{C})$$

[SHIMIZU: 1602.06534]

IF, FURTHER,
 $\mathcal{C}$ IS FUSION

"S-MATRIX" OF $\mathcal{C}$
 IS INVERTIBLE

NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
IF $Z_2(\mathcal{C}) \simeq \text{Vec}$

$$\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y} \forall y \in \mathcal{C}\}$$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM

$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$

SATISFYING BRAID AXIOMS

ASSUME
TENSOR

NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
IF $Z_2(\mathcal{C}) \simeq \text{Vec}$

$$\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y} \forall y \in \mathcal{C}\}$$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM

$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$

SATISFYING BRAID AXIOMS

ASSUME
TENSOR

TAKE A QUASITRIANGULAR HOPF ALGEBRA $(H, R := R' \otimes R'' \in H \otimes H)$

NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
IF $Z_2(\mathcal{C}) \simeq \text{Vec}$

$$\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y} \forall y \in \mathcal{C}\}$$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM
 $c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$
SATISFYING BRAID AXIOMS

ASSUME
TENSOR

TAKE A QUASITRIANGULAR HOPF ALGEBRA $(H, R := R' \otimes R'' \in H \otimes H)$
 $\rightarrow$ FORM "MONODROMY MATRIX" OF H : $Q := R'' \tilde{R}' \otimes R' \tilde{R}''$

NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
IF $Z_2(\mathcal{C}) \simeq \text{Vec}$

$$\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y} \forall y \in \mathcal{C}\}$$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM

$$c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$

SATISFYING BRAID AXIOMS

ASSUME
TENSOR

TAKE A QUASITRIANGULAR HOPF ALGEBRA $(H, R := R' \otimes R'' \in H \otimes H)$

$\rightarrow$ FORM "MONODROMY MATRIX" OF H : $Q := R'' \tilde{R}' \otimes R' \tilde{R}''$

(H, R) IS "FACTORIZABLE"

$$\text{IF } Q = \sum_d h_d \otimes l_d$$

FOR $\{h_d\}, \{l_d\}$ BASES OF H

NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
IF $\mathbb{Z}_2(\mathcal{C}) \simeq \text{Vec}$

$$\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y} \forall y \in \mathcal{C}\}$$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists \text{ NAT'L ISOM}$
 $c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$
SATISFYING BRAID AXIOMS

ASSUME
TENSOR

TAKE A QUASITRIANGULAR HOPF ALGEBRA $(H, R := R' \otimes R'' \in H \otimes H)$
 $\rightarrow$ FORM "MONODROMY MATRIX" OF H : $Q := R'' \tilde{R}' \otimes R' \tilde{R}''$

(H, R) IS "FACTORIZABLE"

$$\text{IF } Q = \sum_d h_d \otimes l_d$$

FOR $\{h_d\}, \{l_d\}$ BASES OF H

(H, R) IS "TRIANGULAR"

$$\text{IF } Q = 1_H \otimes 1_H$$

NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
IF $Z_2(\mathcal{C}) \simeq \text{Vec}$

$$\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y} \forall y \in \mathcal{C}\}$$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists \text{ NAT'L ISOM}$
 $c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$
SATISFYING BRAID AXIOMS

ASSUME
TENSOR

TAKE A QUASITRIANGULAR HOPF ALGEBRA $(H, R := R' \otimes R'' \in H \otimes H)$
 $\rightarrow$ FORM "MONODROMY MATRIX" OF H : $Q := R'' \tilde{R}' \otimes R' \tilde{R}''$

(H, R) IS "FACTORIZABLE"
IF $Q = \sum_d h_d \otimes l_d$
FOR $\{h_d\}, \{l_d\}$ BASES OF H

FOR
 $\mathcal{C} = H\text{-Mod}$

(H, R) IS "TRIANGULAR"
IF $Q = 1_H \otimes 1_H$

NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
IF $\mathbb{Z}_2(\mathcal{C}) \simeq \text{Vec}$

$$\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y} \forall y \in \mathcal{C}\}$$

TAKE A QUASITRIANGULAR HOPF ALGEBRA $(H, R := R' \otimes R'' \in H \otimes H)$
→ FORM "MONODROMY MATRIX" OF H : $Q := R'' \tilde{R}' \otimes R' \tilde{R}''$

(H, R) IS "FACTORIZABLE"
IF $Q = \sum_d h_d \otimes l_d$
FOR $\{h_d\}, \{l_d\}$ BASES OF H

FOR
 $\mathcal{C} = H\text{-Mod}$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists \text{ NAT'L ISOM}$
 $c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$
SATISFYING BRAID AXIOMS

"SYMMETRIC" $\iff \mathbb{Z}_2(\mathcal{C}) \simeq \mathcal{C}$

(H, R) IS "TRIANGULAR"
IF $Q = 1_H \otimes 1_H$

NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
IF $\mathbb{Z}_2(\mathcal{C}) \simeq \text{Vec}$

$$\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y} \forall y \in \mathcal{C}\}$$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists \text{ NAT'L ISOM}$
 $c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$
SATISFYING BRAID AXIOMS

"SYMMETRIC" $\iff \mathbb{Z}_2(\mathcal{C}) \simeq \mathcal{C}$

TAKE A QUASITRIANGULAR HOPF ALGEBRA $(H, R := R' \otimes R'' \in H \otimes H)$
 $\rightarrow$ FORM "MONODROMY MATRIX" OF H : $Q := R'' \tilde{R}' \otimes R' \tilde{R}$

(H, R) IS "FACTORIZABLE"
IF $Q = \sum_d h_d \otimes l_d$
FOR $\{h_d\}, \{l_d\}$ BASES OF H

FOR
 $\mathcal{C} = H\text{-Mod}$

(H, R) IS "TRIANGULAR"
IF $Q = 1_H \otimes 1_H$

EXAMPLES:

H COCOMMUTATIVE,
 $R = 1 \otimes 1$

$\mathbb{K}G$
FOR $u \in \mathbb{Z}(G)$ w/ $u^2 = e$
 $R_u = \frac{1}{2}(e \otimes e + u \otimes e + e \otimes u - u \otimes u)$

SMALL Q.GROUP
 $u_q(\mathfrak{sl}_2)$

NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
IF $\mathbb{Z}_2(\mathcal{C}) \cong \text{Vec}$

$$\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y} \forall y \in \mathcal{C}\}$$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists \text{ NAT'L ISOM}$
 $c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$
SATISFYING BRAID AXIOMS

"SYMMETRIC" $\iff \mathbb{Z}_2(\mathcal{C}) \cong \mathcal{C}$

TAKE A QUASITRIANGULAR HOPF ALGEBRA $(H, R := R' \otimes R'' \in H \otimes H)$
 $\rightarrow$ FORM "MONODROMY MATRIX" OF H : $Q := R'' \tilde{R}' \otimes R' \tilde{R}''$

(H, R) IS "FACTORIZABLE"
IF $Q = \sum_d h_d \otimes l_d$
FOR $\{h_d\}, \{l_d\}$ BASES OF H

FOR
 $\mathcal{C} = H\text{-Mod}$

(H, R) IS "TRIANGULAR"
IF $Q = 1_H \otimes 1_H$

EXAMPLES:

H COCOMMUTATIVE,
 $R = 1 \otimes 1$
 $\equiv \text{TRIANGULAR} \equiv$

$\mathbb{K}G$
FOR $u \in \mathbb{Z}(G)$ w/ $u^2 = e$
 $R_u = \frac{1}{2}(e \otimes e + u \otimes e + e \otimes u - u \otimes u)$

SMALL Q.GROUP
 $u_q(\mathfrak{sl}_2)$

NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
IF $\mathbb{Z}_2(\mathcal{C}) \cong \text{Vec}$

$$\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y} \forall y \in \mathcal{C}\}$$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists \text{ NAT'L ISOM}$
 $c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$
SATISFYING BRAID AXIOMS

"SYMMETRIC" $\iff \mathbb{Z}_2(\mathcal{C}) \cong \mathcal{C}$

TAKE A QUASITRIANGULAR HOPF ALGEBRA $(H, R := R' \otimes R'' \in H \otimes H)$
 $\rightarrow$ FORM "MONODROMY MATRIX" OF H : $Q := R'' \tilde{R}' \otimes R' \tilde{R}$

(H, R) IS "FACTORIZABLE"
IF $Q = \sum_d h_d \otimes l_d$
FOR $\{h_d\}, \{l_d\}$ BASES OF H

FOR
 $\mathcal{C} = H\text{-Mod}$

(H, R) IS "TRIANGULAR"
IF $Q = 1_H \otimes 1_H$

EXAMPLES:

H COCOMMUTATIVE,
 $R = 1 \otimes 1$
 $\equiv \text{TRIANGULAR} \equiv$

$\mathbb{K}G$
FOR $u \in \mathbb{Z}(G)$ w/ $u^2 = e$
 $R_u = \frac{1}{2}(e \otimes e + u \otimes e + e \otimes u - u \otimes u)$,
 $\equiv \text{TRIANGULAR} \equiv$

SMALL Q.GROUP
 $u_q(\mathfrak{sl}_2)$

NONDEGENERATE BRAIDINGS

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
IF $\mathbb{Z}_2(\mathcal{C}) \simeq \text{Vec}$

$$\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y} \forall y \in \mathcal{C}\}$$

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists \text{ NAT'L ISOM}$
 $c := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$
SATISFYING BRAID AXIOMS

"SYMMETRIC" $\iff \mathbb{Z}_2(\mathcal{C}) \simeq \mathcal{C}$

TAKE A QUASITRIANGULAR HOPF ALGEBRA $(H, R := R' \otimes R'' \in H \otimes H)$
 $\rightarrow$ FORM "MONODROMY MATRIX" OF H : $Q := R'' \tilde{R}' \otimes R' \tilde{R}$

(H, R) IS "FACTORIZABLE"
IF $Q = \sum_d h_d \otimes l_d$
FOR $\{h_d\}, \{l_d\}$ BASES OF H

FOR
 $\mathcal{C} = H\text{-Mod}$

(H, R) IS "TRIANGULAR"
IF $Q = 1_H \otimes 1_H$

EXAMPLES:

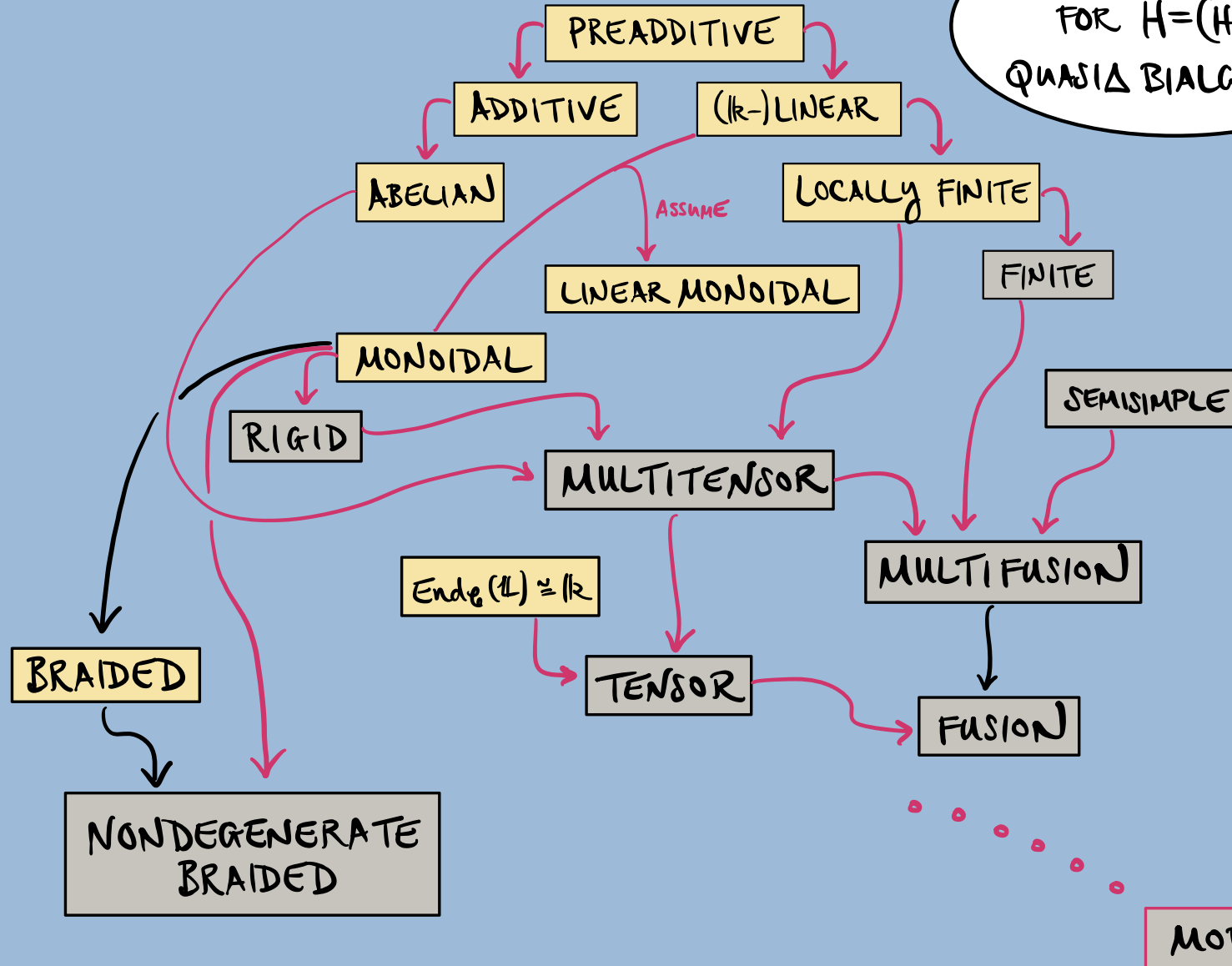
H COCOMMUTATIVE,
 $R = 1 \otimes 1$
 $\equiv \text{TRIANGULAR} \equiv$

$\mathbb{K}G$
FOR $u \in \mathbb{Z}(G)$ w/ $u^2 = e$
 $R_u = \frac{1}{2}(e \otimes e + u \otimes e + e \otimes u - u \otimes u)$
 $\equiv \text{TRIANGULAR} \equiv$

SMALL Q.GROUP
 $u_q(\mathfrak{sl}_2)$
 $\equiv \text{FACTORIZABLE} \equiv$

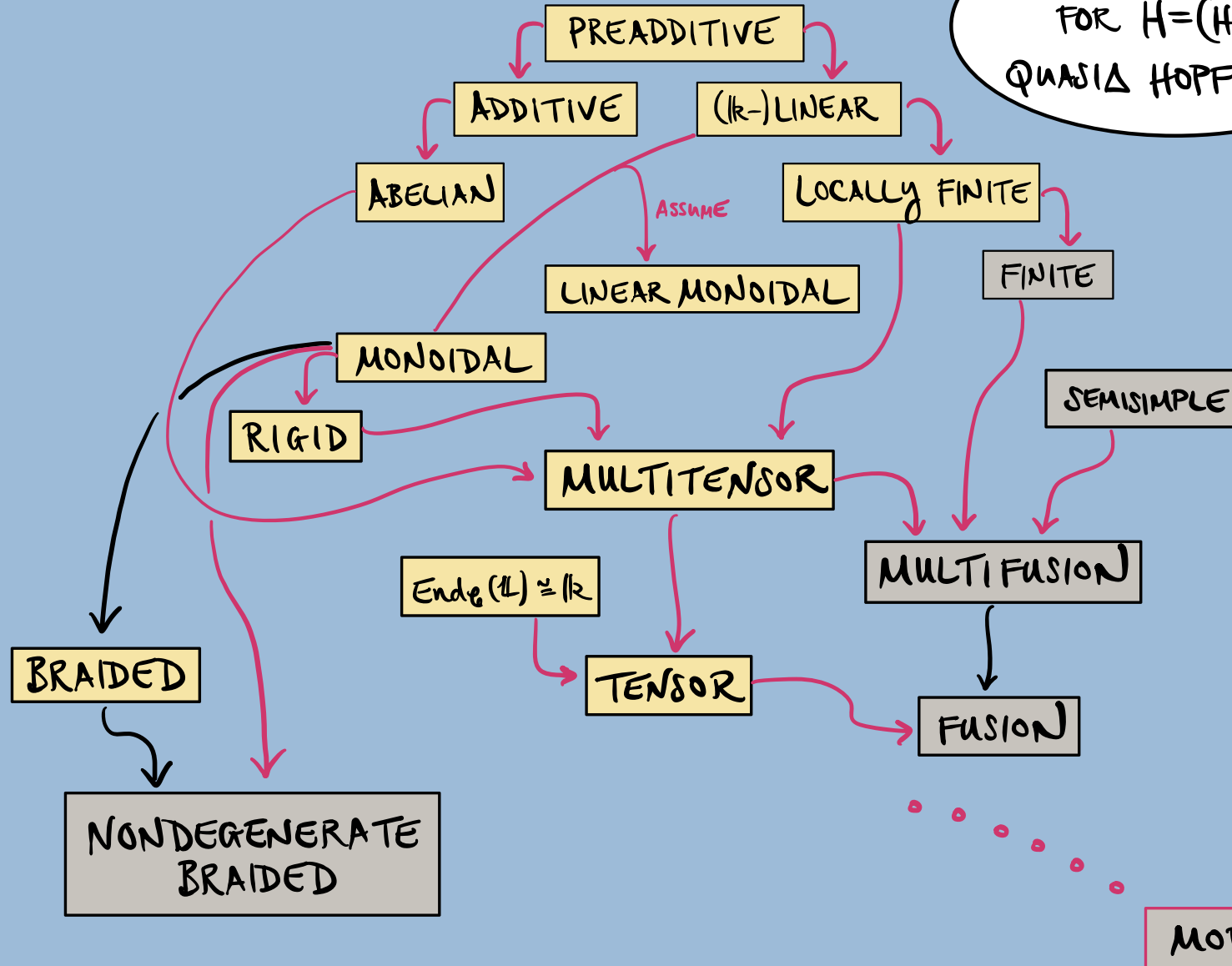
THE ROAD TO MODULARITY

EX. $H\text{-FdMod}$
FOR $H=(H,R)$
QUASIA Δ BIALGEBRA



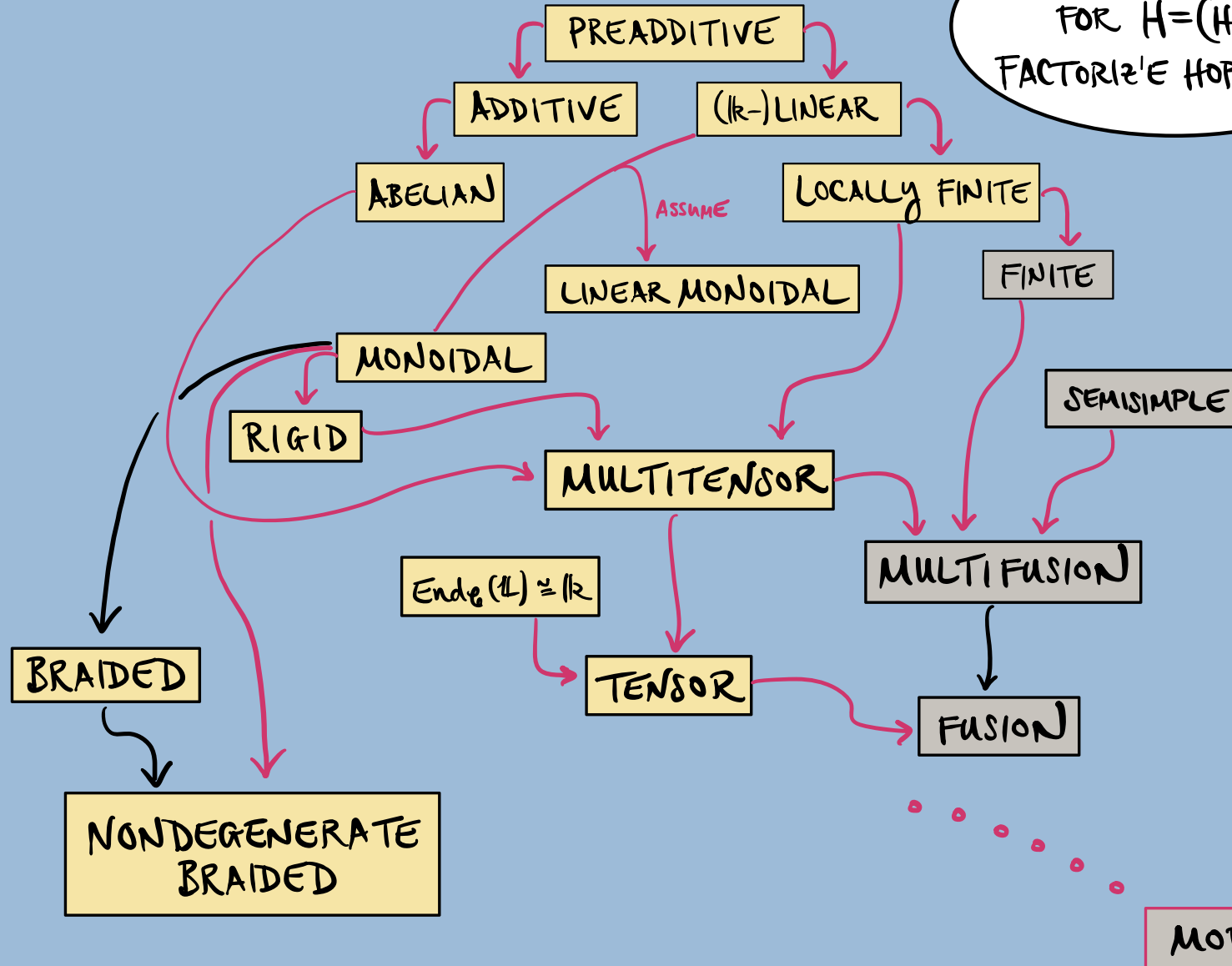
THE ROAD TO MODULARITY

EX. $H\text{-FdMod}$
FOR $H=(H,R)$
QUASIA HOPF ALG.



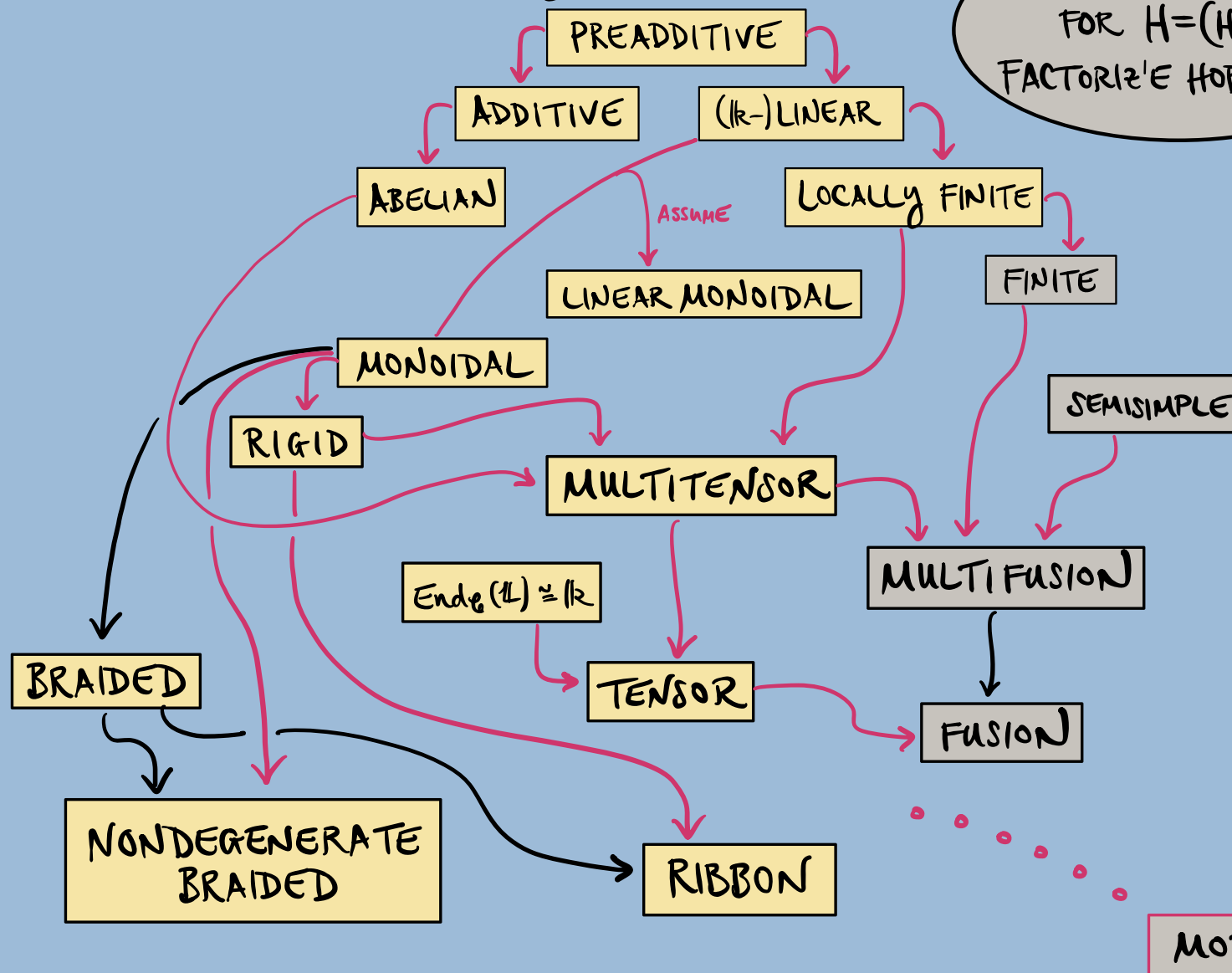
THE ROAD TO MODULARITY

EX. $H\text{-FdMod}$
FOR $H=(H,R)$
FACTORIZ'E HOPF ALG.

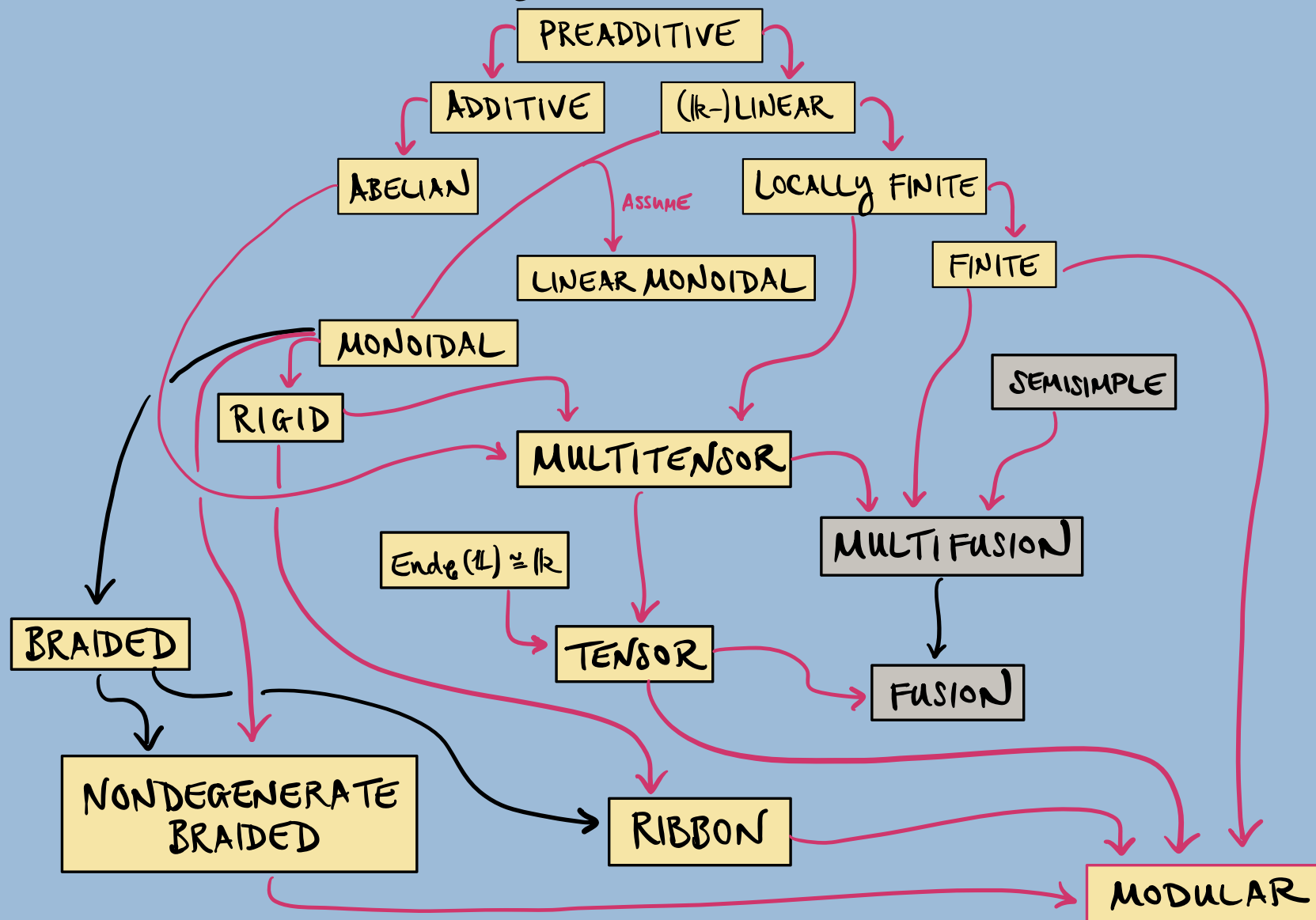


THE ROAD TO MODULARITY

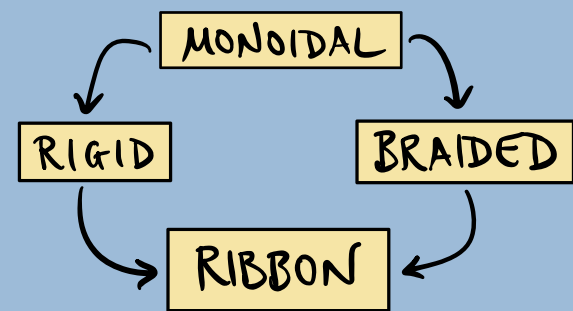
EX. $H\text{-FdMod}$
FOR $H=(H,R)$
FACTORIZ'E HOPF ALG.



THE ROAD TO MODULARITY



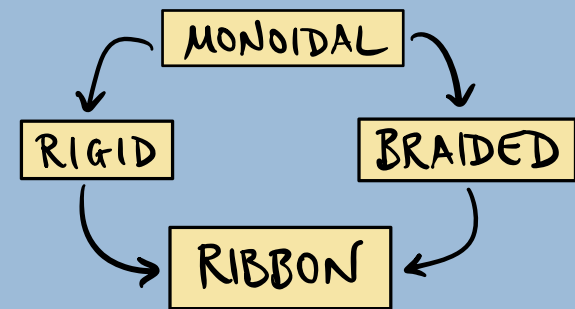
RIBBON CATEGORIES



RIBBON CATEGORIES

A BRAIDED, RIGID MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c, (-)^*, {}^*(-))$

IS "RIBBON"



RIBBON CATEGORIES

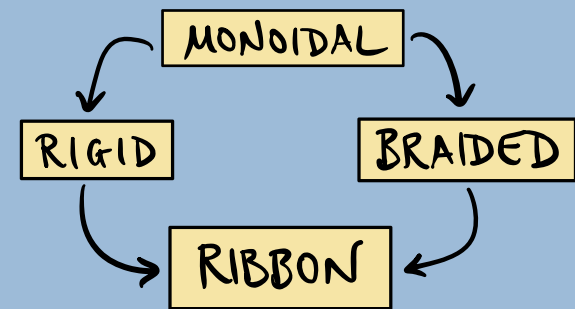
A BRAIDED, RIGID MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c, (-)^*, {}^*(-))$

IS "RIBBON" IF $\exists$ NATURAL ISOMORPHISM

$$\theta = \{\theta_x : X \xrightarrow{\sim} X^*\}_{x \in \mathcal{C}} \quad \ni.$$

$$\theta_{x \otimes y} = (\theta_x \otimes \theta_y) \circ c_{y,x} \circ c_{x,y} \quad \& \quad (\theta_x)^* = \theta_{x^*}$$

$$\forall x, y \in \mathcal{C}$$



RIBBON CATEGORIES

A BRAIDED, RIGID MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c, (-)^*, {}^*(-))$

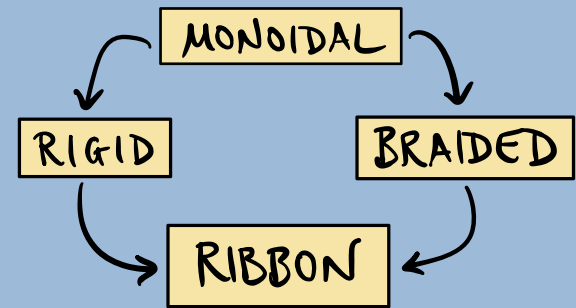
IS "RIBBON" IF $\exists$ NATURAL ISOMORPHISM

$$\theta = \{\theta_x : X \xrightarrow{\sim} X\}_{x \in \mathcal{C}} \quad \text{with diagram} \quad \begin{array}{c} x \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x \end{array} \Rightarrow$$

$$\theta_{x \otimes y} = (\theta_x \otimes \theta_y) \circ c_{y,x} \circ c_{x,y} \quad \& \quad (\theta_x)^* = \theta_{x^*}$$

$$\begin{array}{c} xy \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ xy \end{array} = \begin{array}{c} x \quad y \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x \quad y \end{array}$$

$$\begin{array}{c} x^* \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x^* \end{array} = \begin{array}{c} x^* \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x^* \end{array} \quad \forall x, y \in \mathcal{C}$$



RIBBON CATEGORIES

A BRAIDED, RIGID MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c, (-)^*, {}^*(-))$

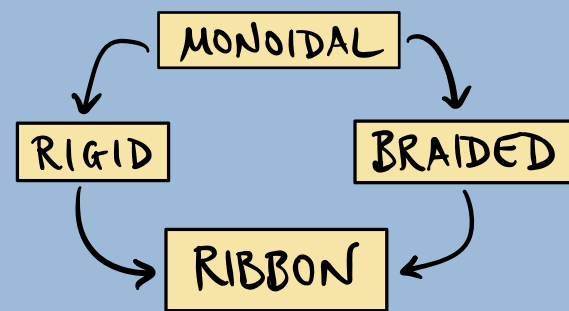
IS "RIBBON" IF $\exists$ NATURAL ISOMORPHISM

$$\theta = \{\theta_x : X \xrightarrow{\sim} X^*\}_{x \in \mathcal{C}} \quad \begin{array}{c} x \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x \end{array} \Rightarrow$$

$$\theta_{x \otimes y} = (\theta_x \otimes \theta_y) \circ c_{y/x} \circ c_{x/y} \quad \& \quad (\theta_x)^* = \theta_{x^*}$$

$$\begin{array}{c} x \ y \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x \ y \end{array} = \begin{array}{c} x \ y \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x \ y \end{array}$$

$$\begin{array}{c} x^* \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x^* \end{array} = \begin{array}{c} x^* \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x^* \end{array} \quad \forall x, y \in \mathcal{C}$$



[SHUM, 1994]

CATEGORY OF FRAMED,
ORIENTED TANGLES

SI

FREE RIBBON CATEG.
ON ONE GENERATOR

RIBBON CATEGORIES

A BRAIDED, RIGID MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c, (-)^*, {}^*(-))$

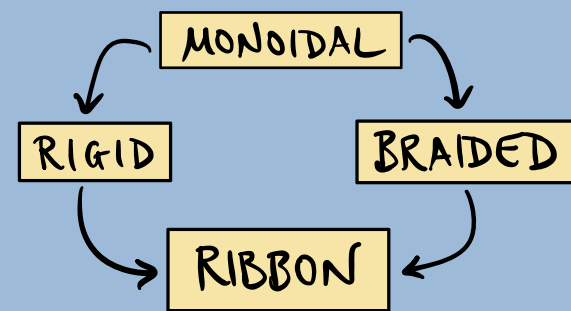
IS "RIBBON" IF $\exists$ NATURAL ISOMORPHISM

$$\theta = \{\theta_x : X \xrightarrow{\sim} X^*\}_{x \in \mathcal{C}} \quad \begin{array}{c} x \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x \end{array} \Rightarrow$$

$$\theta_{x \otimes y} = (\theta_x \otimes \theta_y) \circ c_{y,x} \circ c_{x,y} \quad \& \quad (\theta_x)^* = \theta_{x^*}$$

$$\begin{array}{c} x \ y \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x \ y \end{array} = \begin{array}{c} x \ y \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x \ y \end{array}$$

$$\begin{array}{c} x^* \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x^* \end{array} = \begin{array}{c} x^* \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x^* \end{array} \quad \forall x, y \in \mathcal{C}$$



[SHUM, 1994]

CATEGORY OF FRAMED,
ORIENTED TANGLES

SI

FREE RIBBON CATEG.
ON ONE GENERATOR

A QUASITRIANGULAR HOPF ALGEBRA (H, R) IS "RIBBON"

IF $\exists$ INVERTIBLE ELT $v \in Z(H)$ SATISFYING

$$\Delta(v) = (v \otimes v) Q \quad \& \quad S(v) = v$$

(Q = MONODROMY MATRIX)

RIBBON CATEGORIES

A BRAIDED, RIGID MONOIDAL CATEGORY
 $(\mathcal{C}, \otimes, \mathbb{1}, c, (-)^*, {}^*(-))$

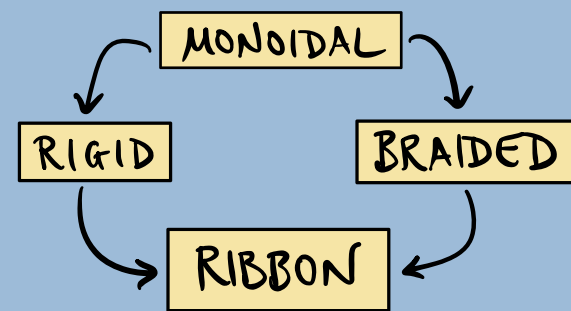
IS "RIBBON" IF $\exists$ NATURAL ISOMORPHISM

$$\theta = \{\theta_x : X \xrightarrow{\sim} X^*\}_{x \in \mathcal{C}} \quad \begin{array}{c} x \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x \end{array} \Rightarrow$$

$$\theta_{x \otimes y} = (\theta_x \otimes \theta_y) \circ c_{y,x} \circ c_{x,y} \quad \& \quad (\theta_x)^* = \theta_{x^*}$$

$$\begin{array}{c} x \ y \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x \ y \end{array} = \begin{array}{c} x \ y \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x \ y \end{array}$$

$$\begin{array}{c} x^* \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x^* \end{array} = \begin{array}{c} x^* \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ x^* \end{array} \quad \forall x, y \in \mathcal{C}$$



[SHUM, 1994]

CATEGORY OF FRAMED,
ORIENTED TANGLES

SI

FREE RIBBON CATEG.
ON ONE GENERATOR

A QUASITRIANGULAR HOPF ALGEBRA (H, R) IS "RIBBON"

IF $\exists$ INVERTIBLE ELT $v \in Z(H)$ SATISFYING

$$\Delta(v) = (v \otimes v) Q \quad \& \quad S(v) = v$$

(Q = MONODROMY MATRIX)

$\leadsto$ H-Mod RIBBON

MODULAR CATEGORIES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR CATEGORIES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR CATEGORIES



REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR CATEGORIES

→ REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE

→ $H\text{-Mod} \equiv \text{MTC}$

RIBBON HOPF ALGEBRA

MODULAR CATEGORIES

→ REPRESENTATIONS OF $SL_2(\mathbb{Z})$ VIA "S & T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE

→ $H\text{-Mod} \equiv \text{MTC}$

RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

MODULAR CATEGORIES

→ REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE

→ $H\text{-Mod} \equiv \text{MTC}$

RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

MODULAR CATEGORIES

→ REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE

→ $H\text{-Mod} \equiv \text{MTC}$

RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

E.G. $D(G)$

↑
FINITE GROUP

MODULAR CATEGORIES

→ REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE

→ $H\text{-Mod} \equiv \text{MTC}$

RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

E.G. $D(G)$

↑
FINITE GROUP

CONSTRUCTIONS:

MODULAR CATEGORIES

→ REPRS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE

→ $H\text{-Mod} \equiv \text{MTC}$

RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

E.G. $D(G)$

↑
FINITE GROUP

CONSTRUCTIONS:

$\mathcal{C}(\mathfrak{g}, q)$

"VERLINDE
MODULAR CATS"

[J:1810.09055]

& REFS WITHIN

MODULAR CATEGORIES

→ REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE

→ $H\text{-Mod} \equiv \text{MTC}$

RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

E.G. $D(G)$

↑
FINITE GROUP

CONSTRUCTIONS:

$\mathcal{C}(\mathfrak{g}, q)$

$\text{Rep}(u_q(\mathfrak{g}))_{ss}$

"VERLINDE
MODULAR CATS"

[J:1810.09055]

& REFS WITHIN

MODULAR CATEGORIES

→ REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE

→ $H\text{-Mod} \equiv \text{MTC}$

RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

E.G. $D(G)$

↑
FINITE GROUP

CONSTRUCTIONS:

$\mathcal{C}(\mathfrak{g}, q)$

"VERLINDE
MODULAR CATS"

[S: 1810.09055]

$\text{Rep}(u_q(\mathfrak{g}))_{ss}$

$\mathbb{Z}(\mathcal{C})$

"DRINFELD CENTERS OF
SPH. FINITE TENSOR CATS"

[LW: 2010.11872] [LW: 2202.08644] [S: 1707.09691] & REFS WITHIN

MODULAR CATEGORIES

→ REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE
→ $H\text{-Mod} \equiv$ MTC RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

E.G. $D(G)$

↑
FINITE GROUP

CONSTRUCTIONS:

$\mathcal{C}(\mathfrak{g}, q)$

"VERLINDE
MODULAR CATS"

[S: 1810.09055]

$\text{Rep}(u_q(\mathfrak{g}))_{ss}$

$\mathbb{Z}(\mathcal{C})$

"DRINFELD CENTERS OF
SPH. FINITE TENSOR CATS"

[LW: 2010.11872]

Ex. $\mathbb{Z}(H\text{-Mod}) \stackrel{BR \otimes}{\cong} D(H)\text{-Mod}$

[LW: 2202.08644]

[S: 1707.09691]

& REFS WITHIN

MODULAR CATEGORIES

→ REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE

→ $H\text{-Mod} \equiv$ MTC

RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

E.G. $D(G)$

↑
FINITE GROUP

CONSTRUCTIONS:

Ex. $Z(H\text{-Mod}) \stackrel{BR \otimes}{\cong} D(H)\text{-Mod}$

$\mathcal{C}(\mathfrak{g}, q)$

"VERLINDE
MODULAR CATS"

[S: 1810.09055]

$\text{Rep}(u_q(\mathfrak{g}))_{ss}$

$Z(\mathcal{C})$

"DRINFELD CENTERS OF
SPH. FINITE TENSOR CATS"

[LW: 2010.11872]

$Z_\otimes(\mathcal{C})$

"RELATIVE
CENTERS..."

[LW: 2202.08644] [S: 1707.09691]

& REFS WITHIN

MODULAR CATEGORIES

→ REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE

→ $H\text{-Mod} \equiv \text{MTC}$

RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

E.G. $D(G)$

↑
FINITE GROUP

CONSTRUCTIONS:

Ex. $Z(H\text{-Mod}) \stackrel{\text{BR}\otimes}{\cong} D(H)\text{-Mod}$

$\mathcal{C}(\mathfrak{g}, q)$

"VERLINDE
MODULAR CATS"

[S: 1810.09055]

$\text{Rep}(u_q(\mathfrak{g}))_{\text{ss}}$

$Z(\mathcal{C})$

"DRINFELD CENTERS OF
SPH. FINITE TENSOR CATS"

[LW: 2010.11872]

$Z_\delta(\mathcal{C})$

"RELATIVE
CENTERS..."

[LW: 2202.08644]

$\text{LocMod}_{\mathcal{C}}(A)$

"CATS OF
LOCAL MODULES..."

[S: 1707.09691] & REFS WITHIN

MODULAR CATEGORIES

→ REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE

→ $H\text{-Mod} \equiv$ MTC

RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

E.G. $D(G)$

↑
FINITE GROUP

CONSTRUCTIONS:

$\mathcal{C}(\mathfrak{g}, q)$

"VERLINDE
MODULAR CATS"

[S: 1810.09055]

$\text{Rep}(u_q(\mathfrak{g}))_{ss}$

$\mathcal{Z}(\mathcal{C})$

"DRINFELD CENTERS OF
SPH. FINITE TENSOR CATS"

[LW: 2010.11872]

Ex. $\mathcal{Z}(H\text{-Mod}) \stackrel{BR \otimes}{\cong} D(H)\text{-Mod}$

$\mathcal{Z}_\delta(\mathcal{C})$

"RELATIVE
CENTERS..."

[S: 1707.09691]

$A \mathcal{Y}_v^v = A \mathcal{Y}_v^{uv}$

$\text{LocMod}_\delta(A)$

"CATS OF
LOCAL MODULES..."

& REFS WITHIN

MODULAR CATEGORIES

→ REPS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE

→ $H\text{-Mod} \equiv \text{MTC}$

RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

E.G. $D(G)$

↑
FINITE GROUP

CONSTRUCTIONS:

$\mathcal{C}(\mathfrak{g}, q)$

"VERLINDE
MODULAR CATS"

[S: 1810.09055]

$\mathcal{Z}(\mathcal{C})$

"DRINFELD CENTERS OF
SPH. FINITE TENSOR CATS"

[LW: 2010.11872]

$\mathcal{Z}_\delta(\mathcal{C})$

"RELATIVE
CENTERS..."

[S: 1707.09691]

$\text{LocMod}_{\mathcal{C}}(A)$

"CATS OF
LOCAL MODULES..."

& REFS WITHIN

MODULAR CATEGORIES

→ REPRS OF $SL_2(\mathbb{Z})$ VIA "S&T" MATRICES

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE

$\rightarrow H\text{-Mod} \equiv \text{MTC}$

RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

E.G. $D(G)$

$\uparrow$
FINITE GROUP

CONSTRUCTIONS:

$\equiv$ MORE ON THIS IN HOUR 3 $\equiv$

$\mathcal{C}(\mathfrak{g}, q)$

"VERLINDE
MODULAR CATS"

[S: 1810.09055]

$\mathcal{Z}(\mathcal{C})$

"DRINFELD CENTERS OF
SPH. FINITE TENSOR CATS"

[LW: 2010.11872]

$\mathcal{Z}_\delta(\mathcal{C})$

"RELATIVE
CENTERS..."

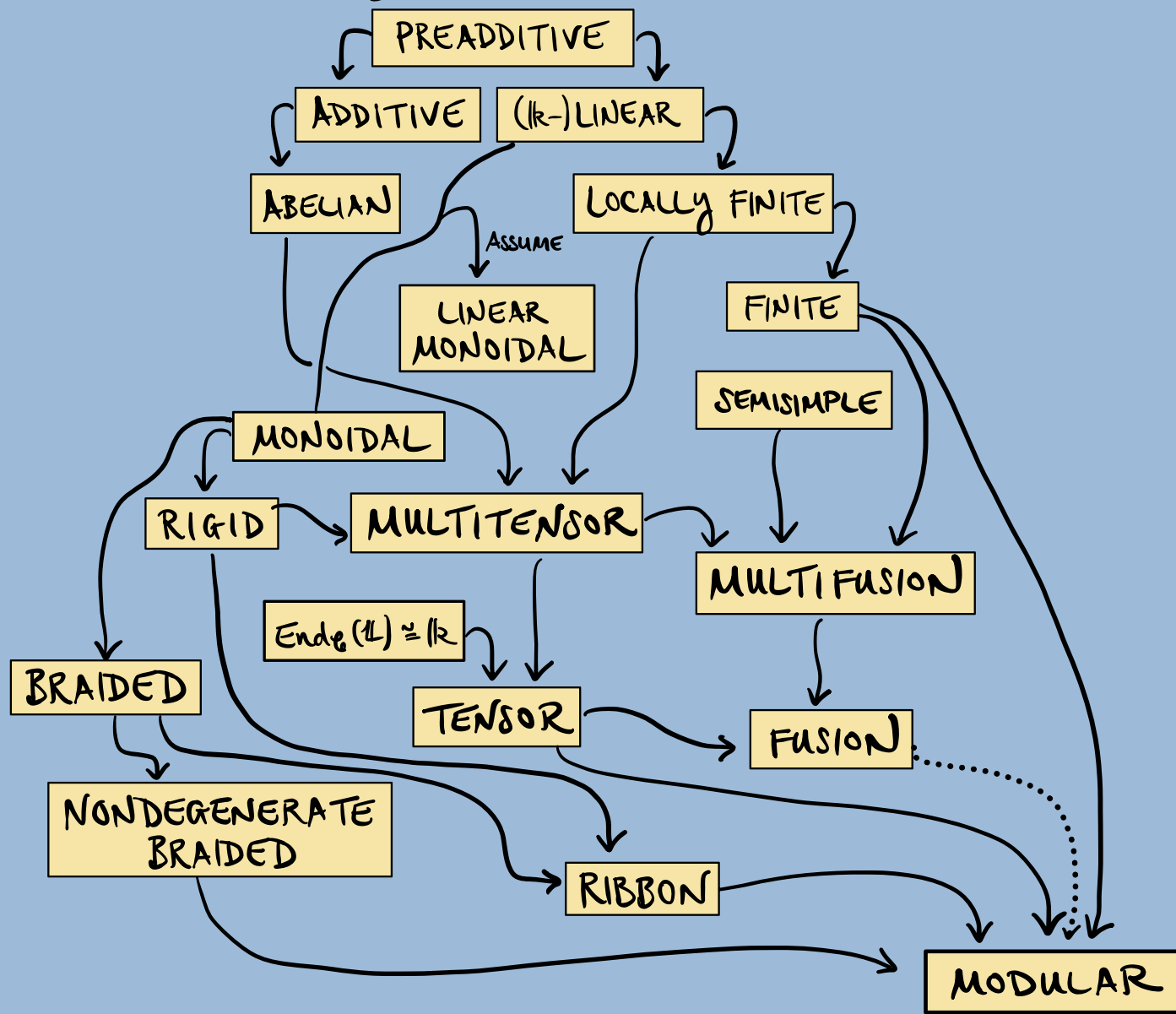
[LW: 2202.08644]

$\text{LocMod}_{\mathcal{C}}(A)$

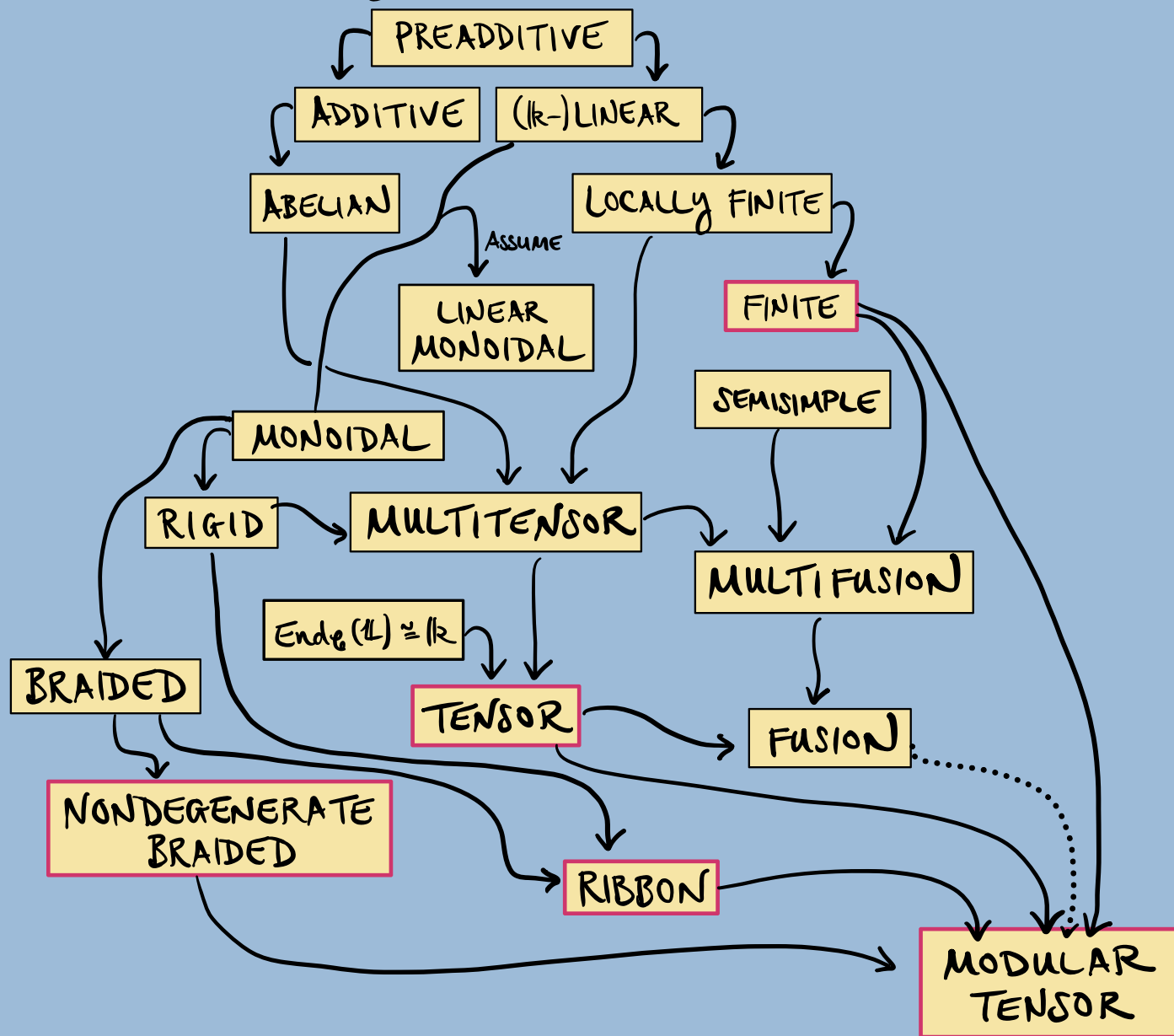
"CATS OF
LOCAL MODULES..."

[S: 1707.09691] & REFS WITHIN

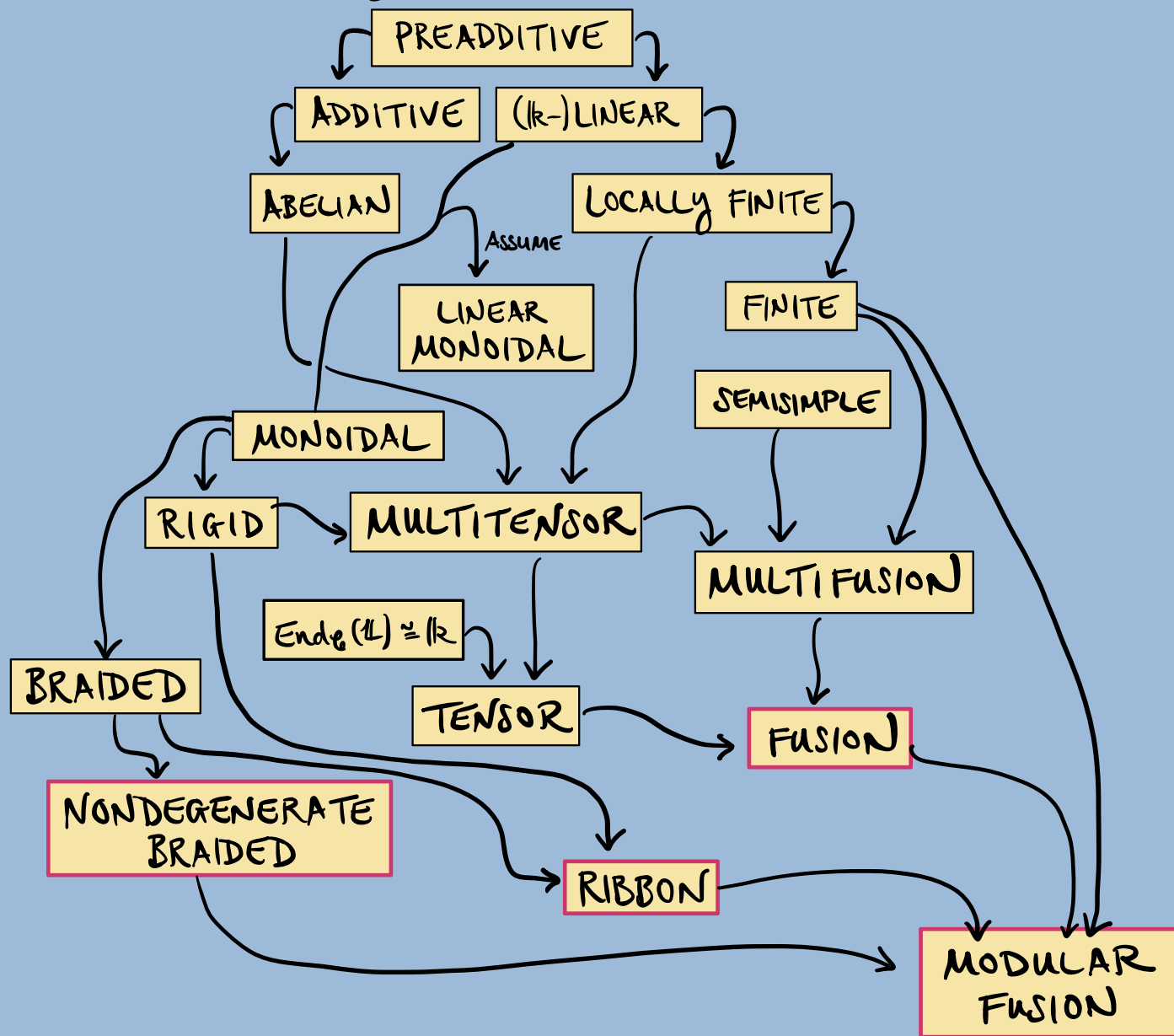
THE ROAD TO MODULARITY



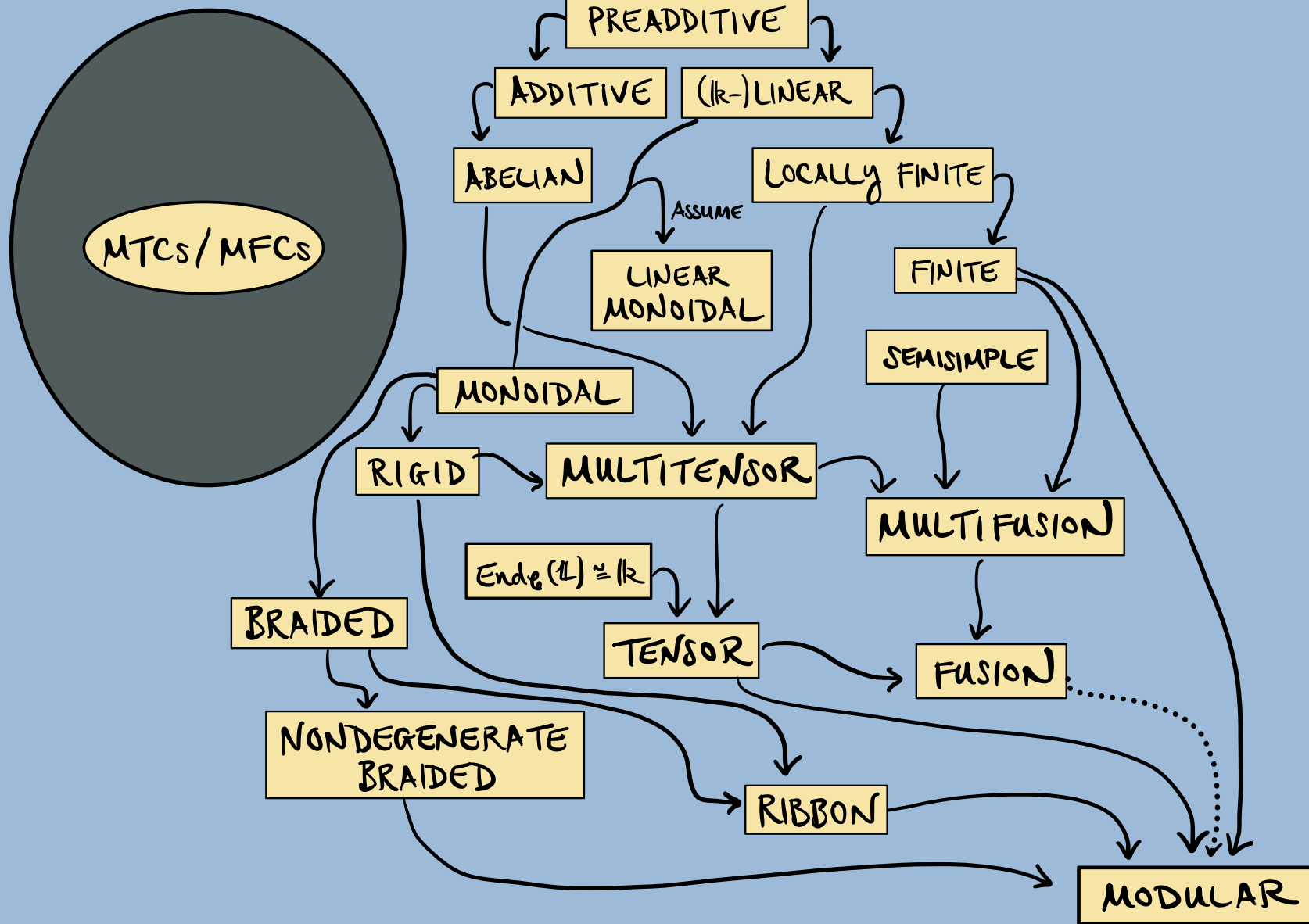
THE ROAD TO MODULARITY



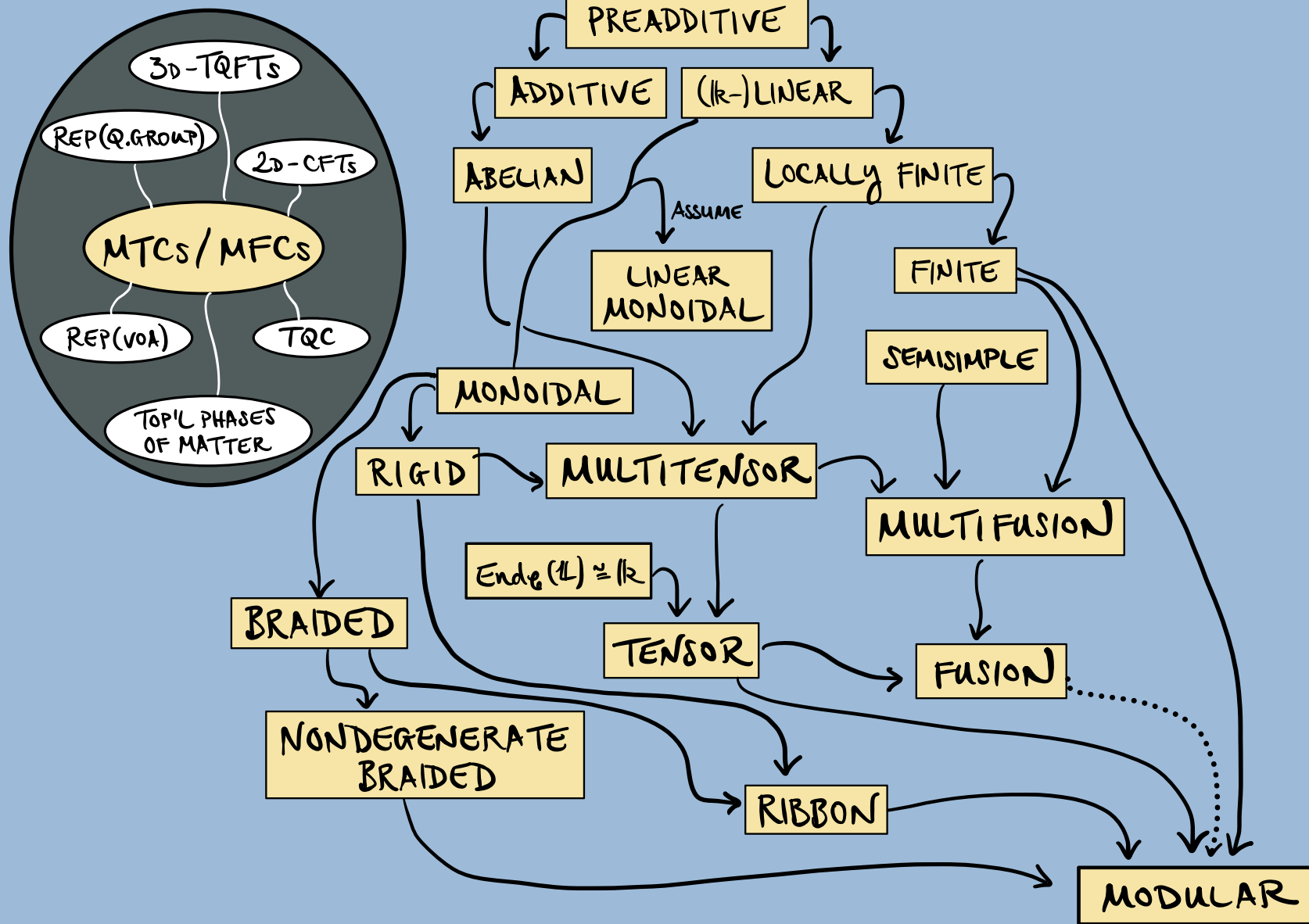
THE ROAD TO MODULARITY



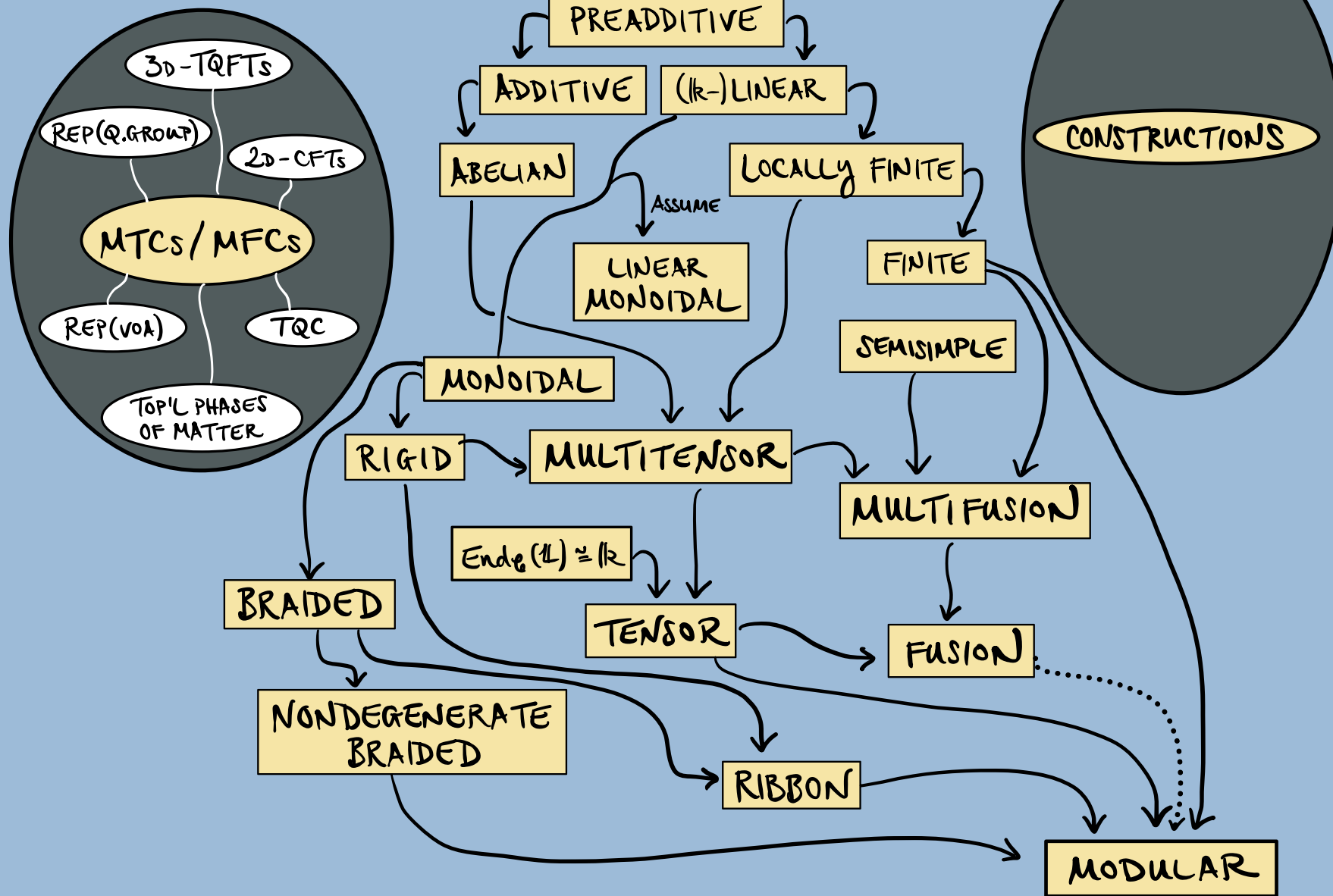
THE ROAD TO MODULARITY



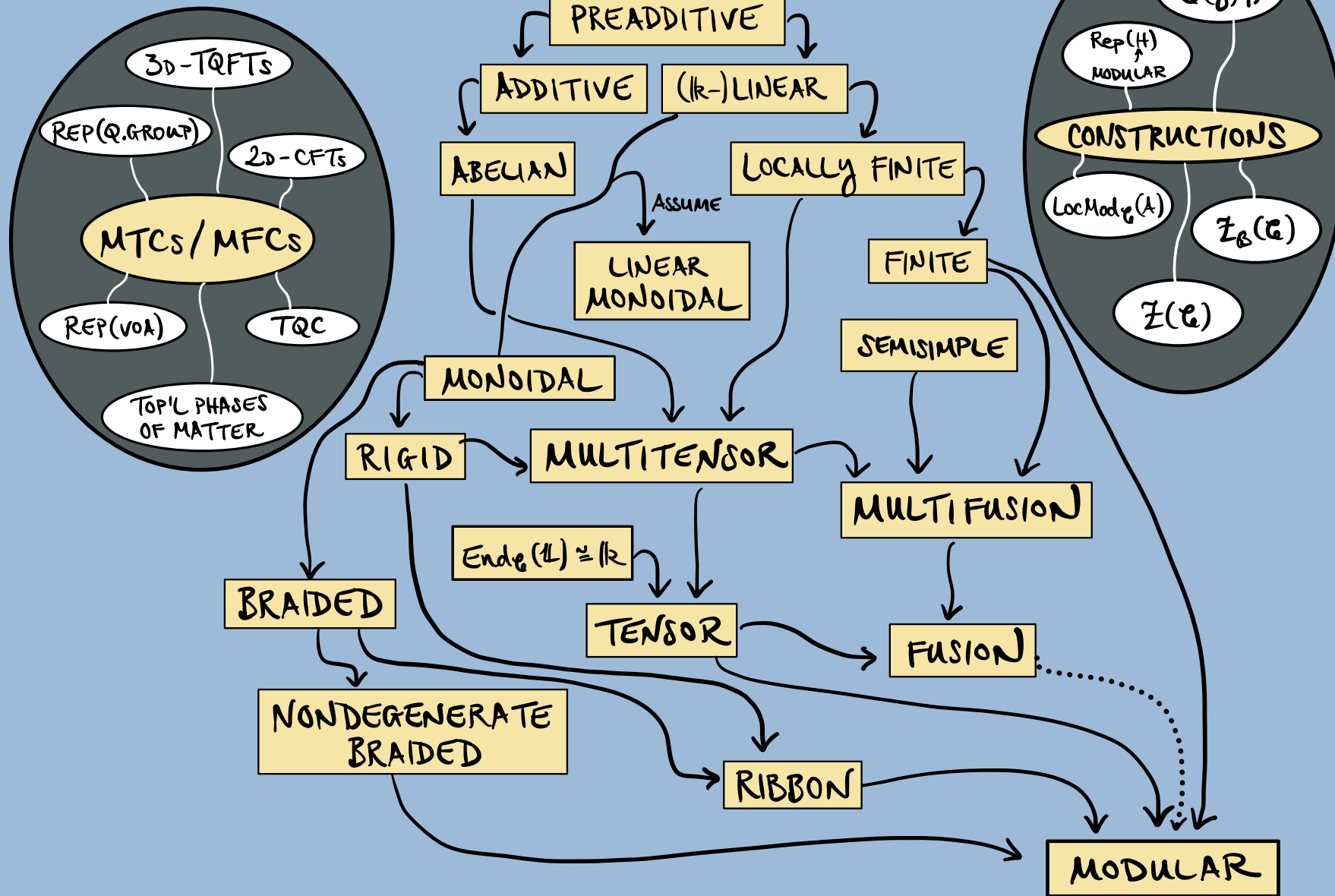
THE ROAD TO MODULARITY



THE ROAD TO MODULARITY



THE ROAD TO MODULARITY



REFERENCES

BRAIDED & MODULAR CATEGORIES —

- BAKALOV-KIRILLOV'S "LECTURES ON TENSOR CATEGORIES AND MODULAR FUNCTORS" (2001)
- KERLER-LYUBASHENKO'S "NON-SEMISIMPLE TQFTS FOR 3-MANIFOLDS WITH CORNERS" (2001)
- EGNO'S "TENSOR CATEGORIES" (2015)
- TURAEV-VIRELIZIER'S "MONOIDAL CATEGORIES AND TOPOLOGICAL FIELD THEORY" (2017)

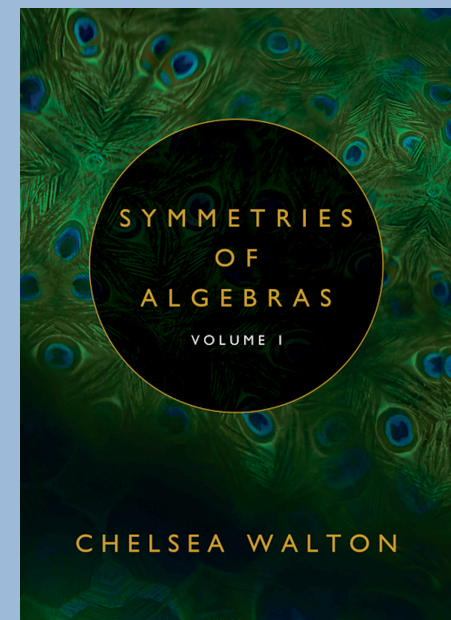
NONDEGENERATE CATEGORIES —

SHIMIZU'S "NON-DEGENERACY CONDITIONS FOR BRAIDED FINITE TENSOR CATEGORIES" (2019)

RECONSTRUCTION THEORY —

SCHAUENBURG'S "TANNAKA DUALITY FOR ARBITRARY HOPF ALGEBRAS" (1992)

Thanks for
listening!



PLEASE CONTACT ME IF
INTERESTED IN **VOL.2**
NOTLAW@RICE.EDU