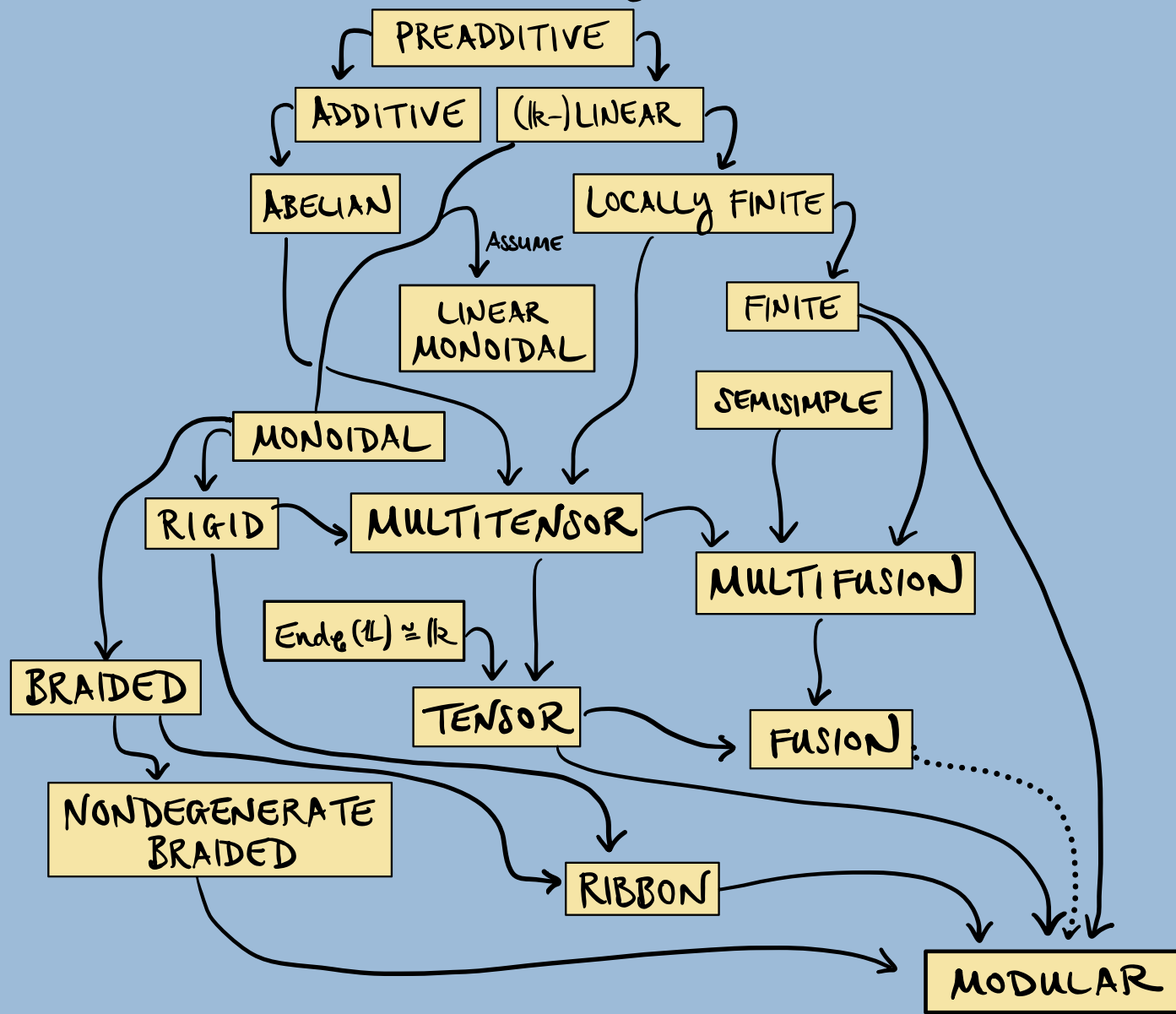


AN INTRODUCTION TO TENSOR CATEGORIES

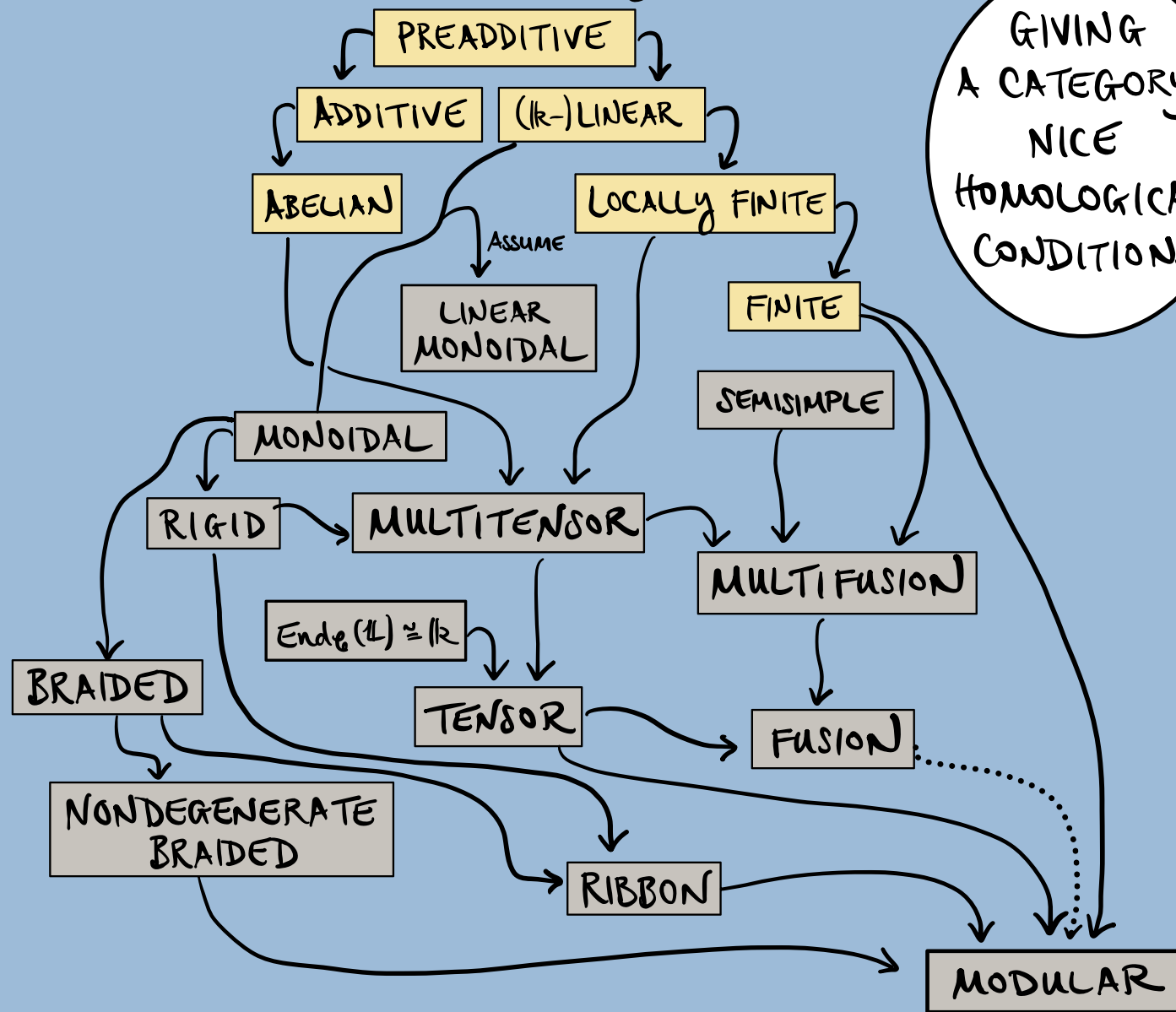
Hour 3 : REPRESENTATIONS IN & OF

CHELSEA WALTON
RICE UNIVERSITY
NOTLAW@RICE.EDU

LAST TIME: THE ROAD TO MODULARITY



LAST TIME: THE ROAD TO MODULARITY

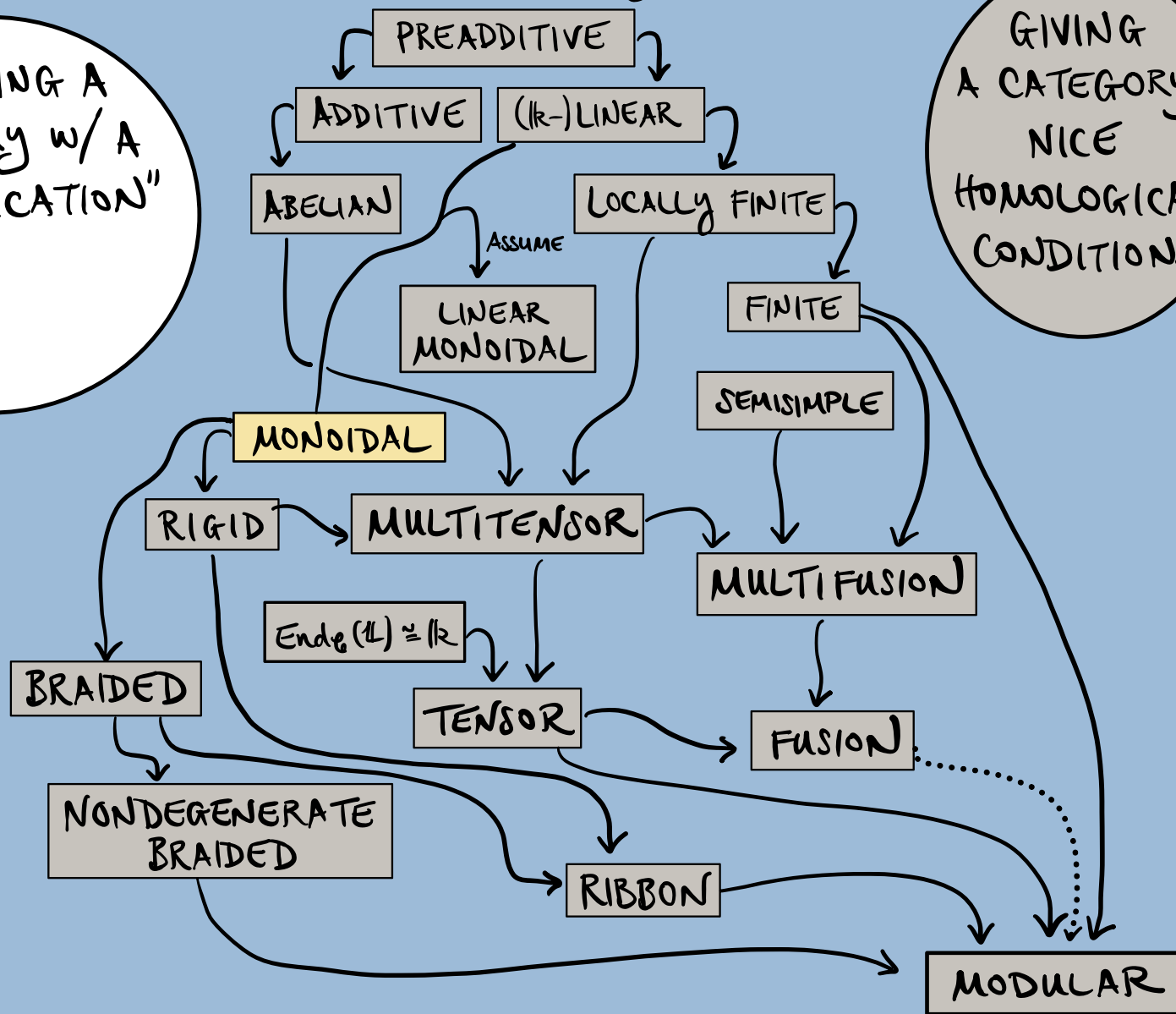


GIVING
A CATEGORY
NICE
HOMOLOGICAL
CONDITIONS

LAST TIME: THE ROAD TO MODULARITY

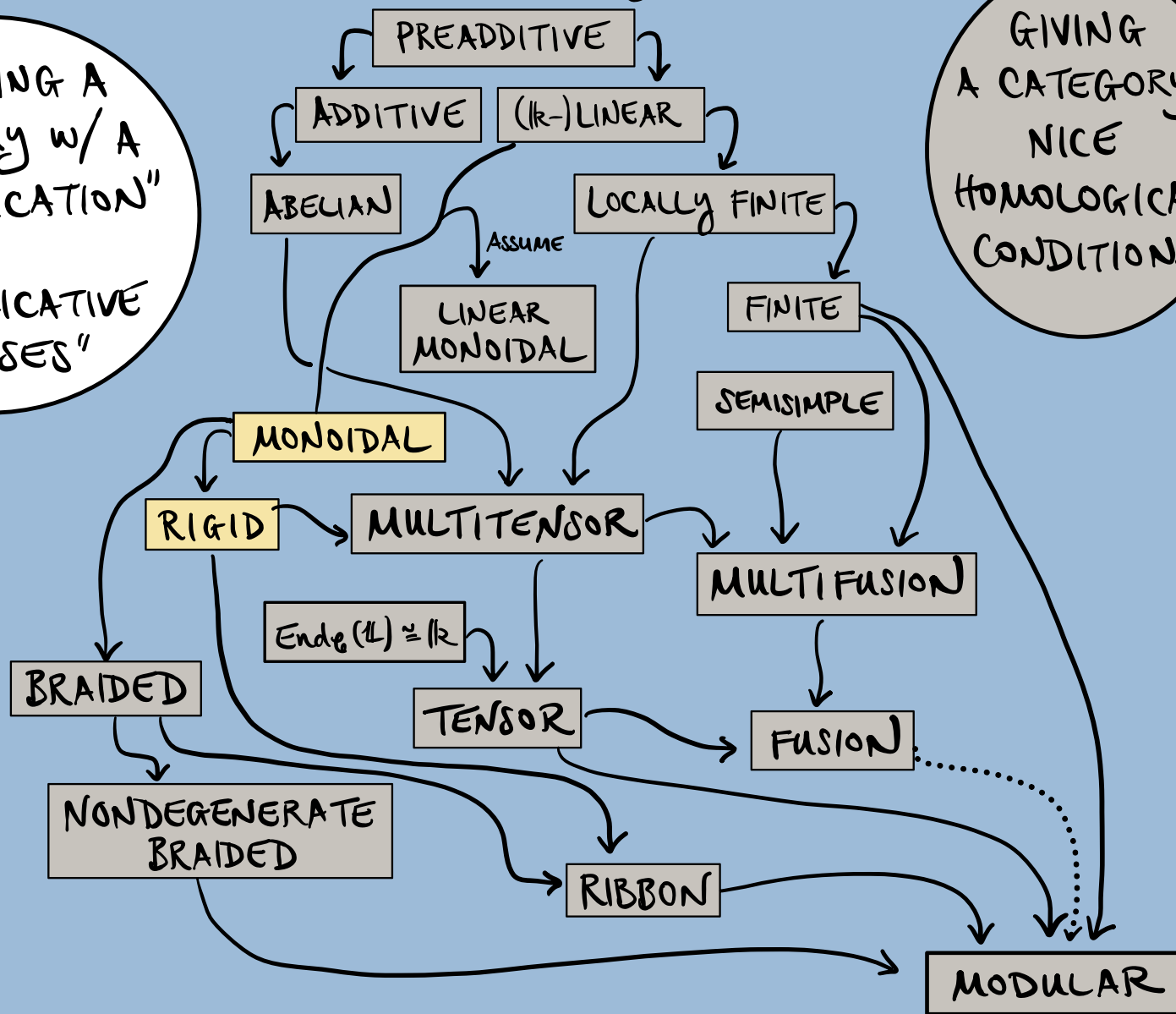
ENDOWING A CATEGORY w/ A "MULTIPLICATION"

GIVING A CATEGORY NICE HOMOLOGICAL CONDITIONS



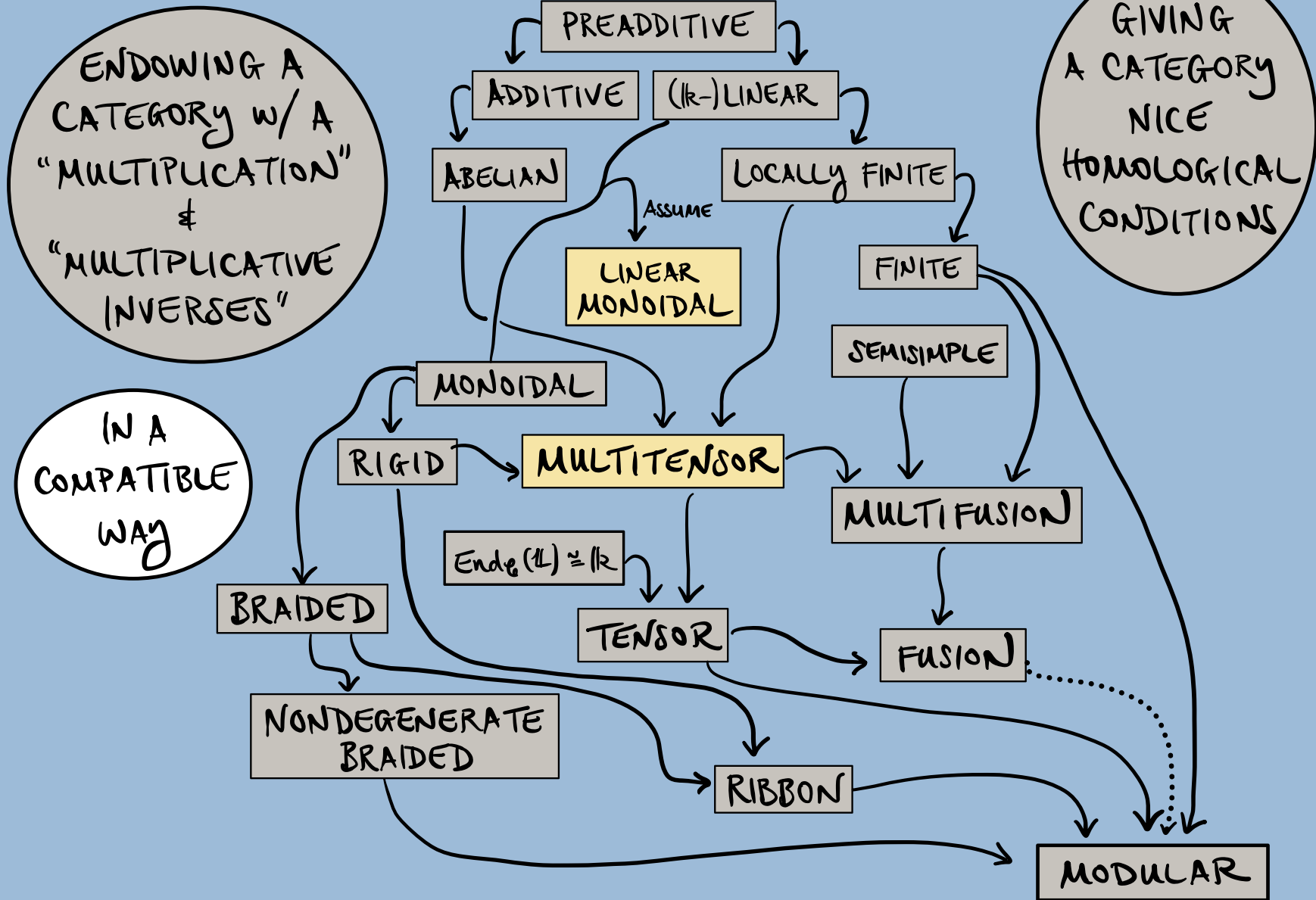
LAST TIME: THE ROAD TO MODULARITY

ENDOWING A CATEGORY w/ A "MULTIPLICATION" $\neq$ "MULTIPLICATIVE INVERSES"

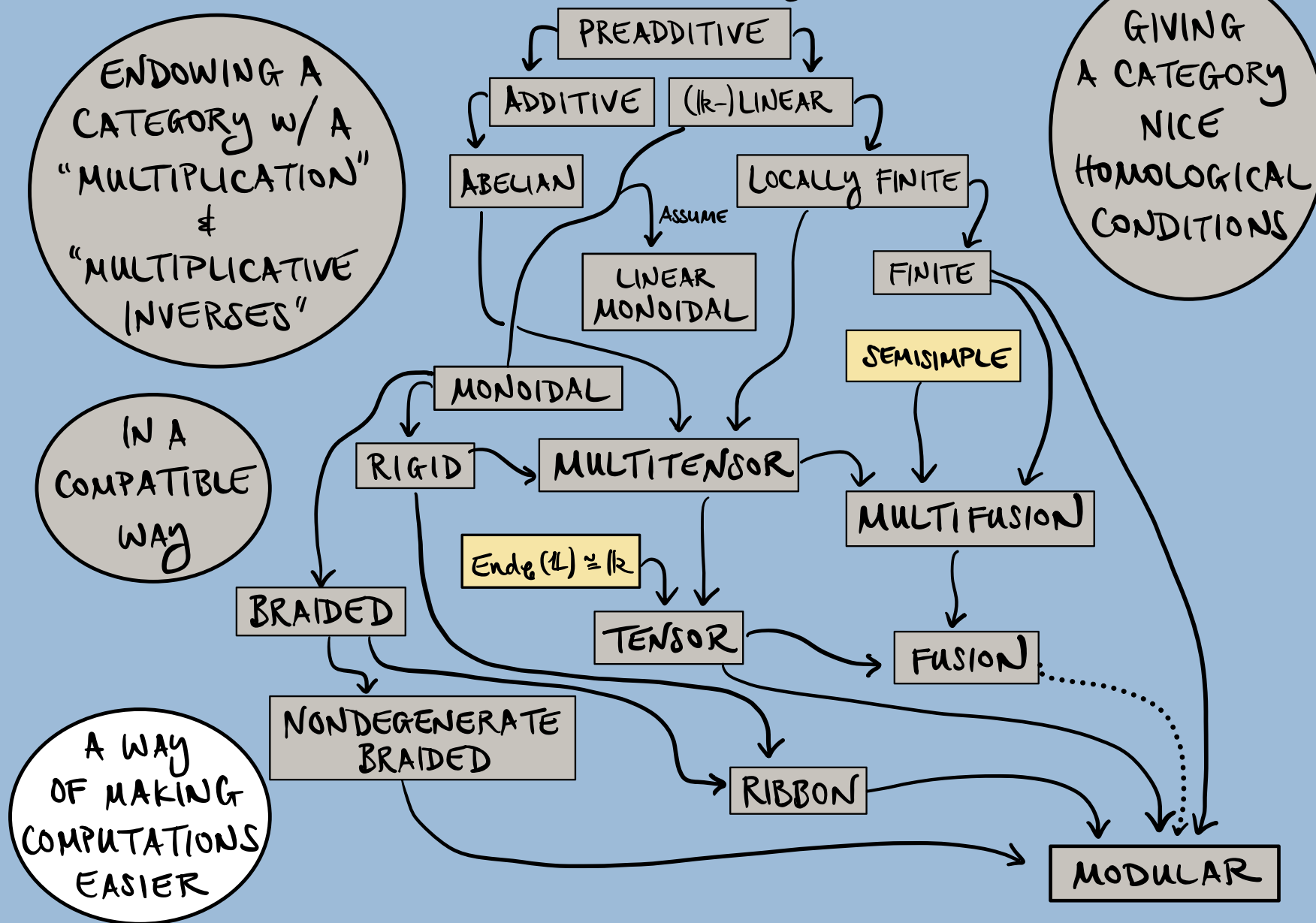


GIVING A CATEGORY NICE HOMOLOGICAL CONDITIONS

LAST TIME: THE ROAD TO MODULARITY



LAST TIME: THE ROAD TO MODULARITY



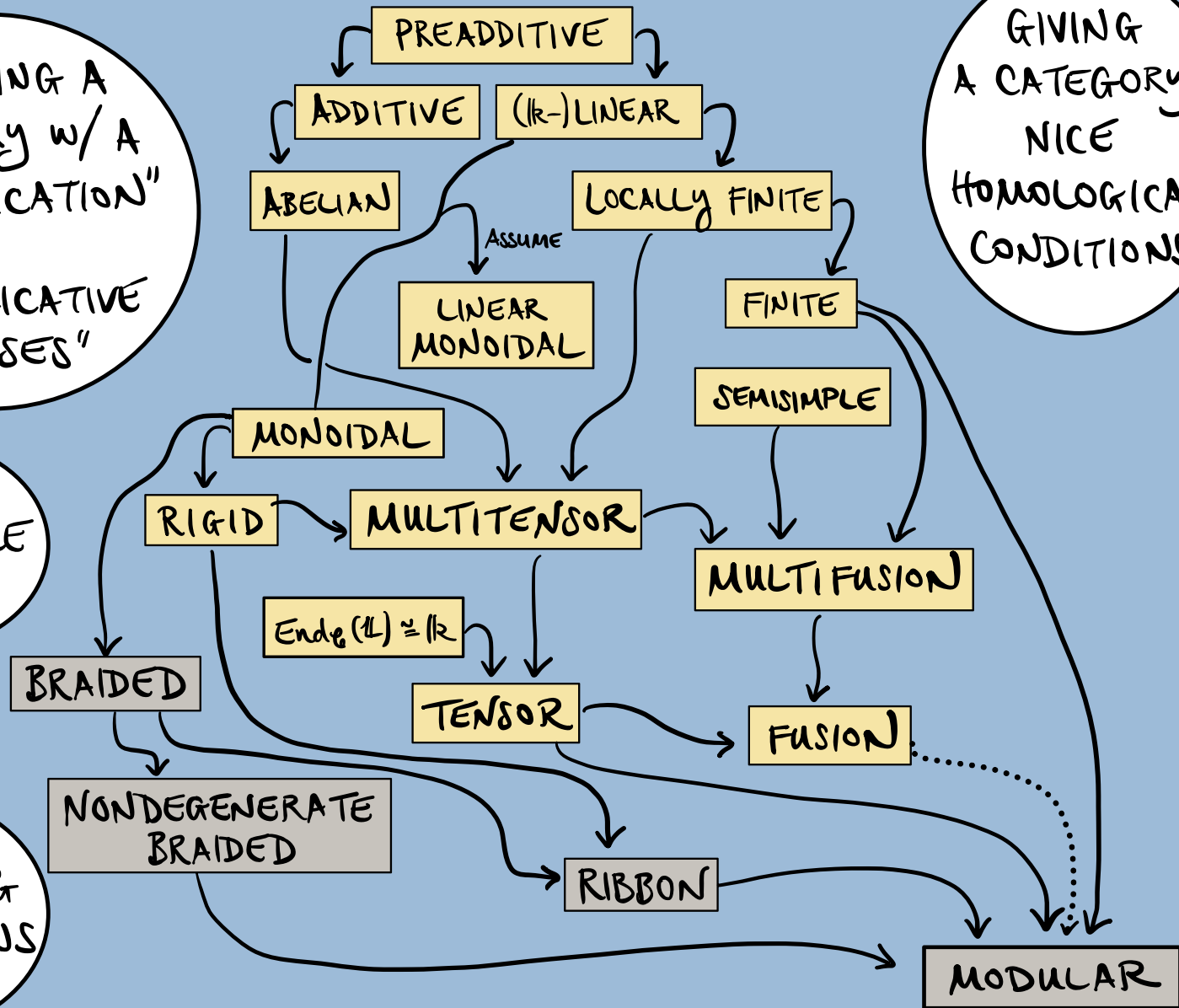
LAST TIME: THE ROAD TO MODULARITY

ENDOWING A CATEGORY w/ A "MULTIPLICATION" $\neq$ "MULTIPLICATIVE INVERSES"

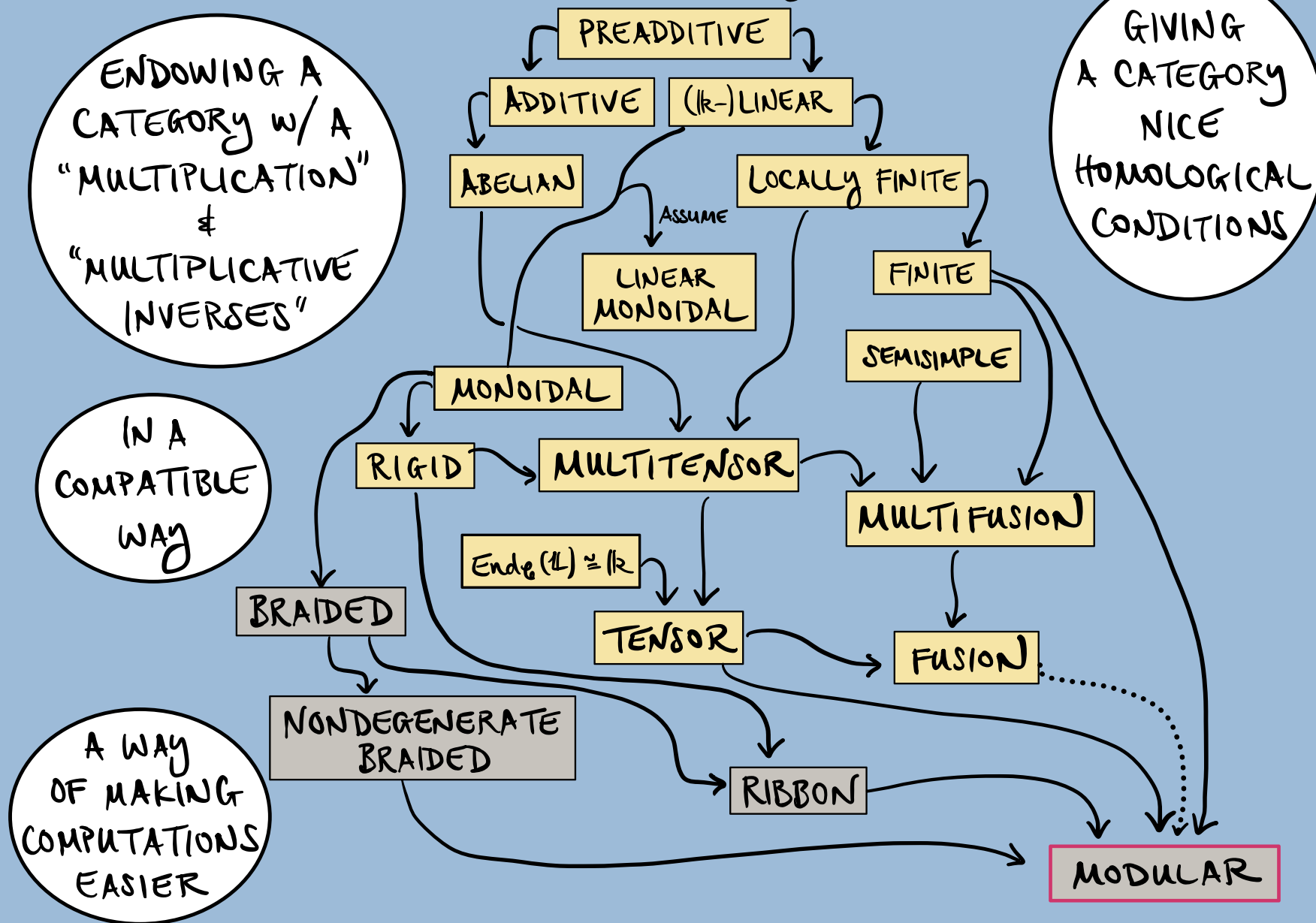
IN A COMPATIBLE WAY

A WAY OF MAKING COMPUTATIONS EASIER

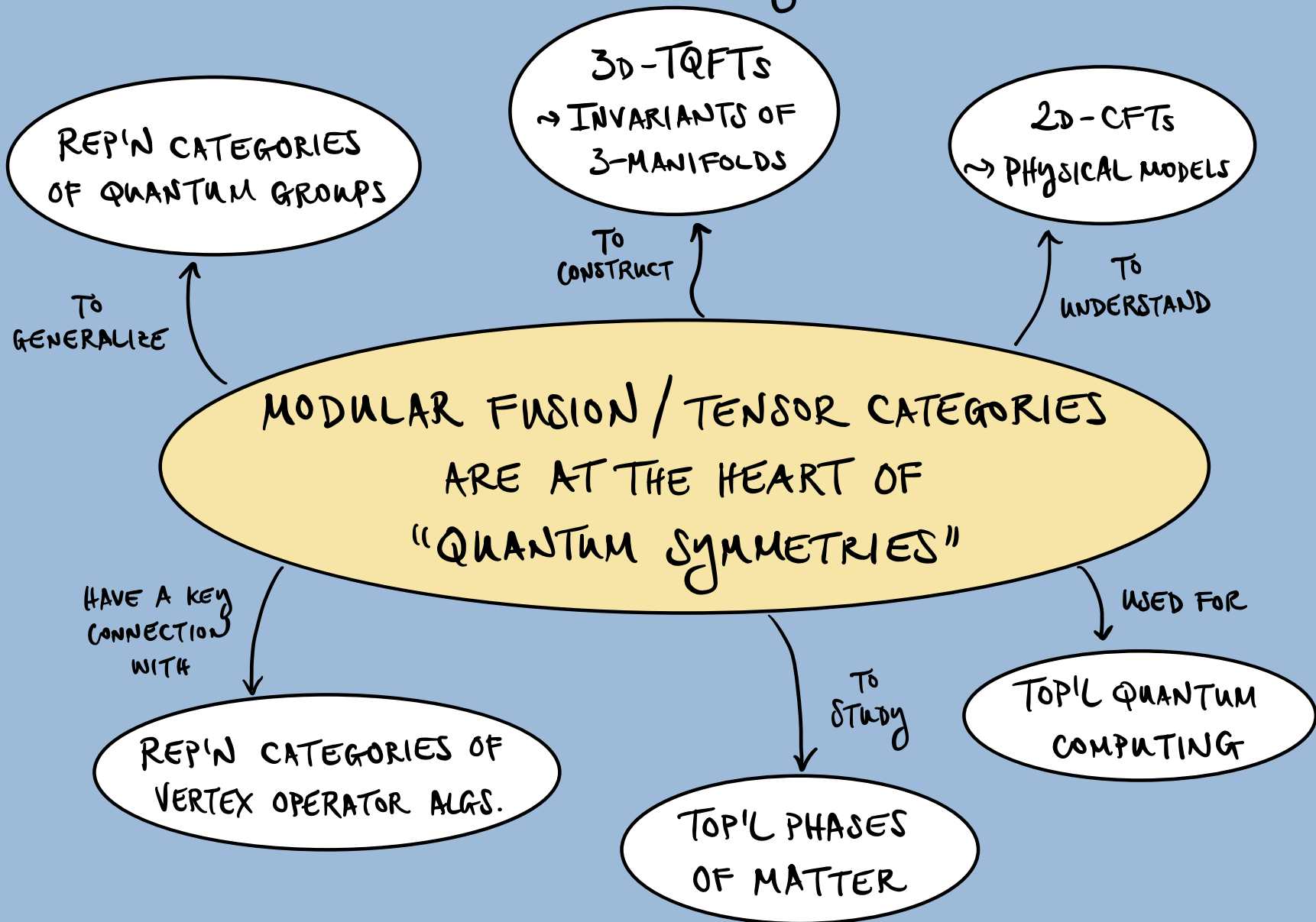
GIVING A CATEGORY NICE HOMOLOGICAL CONDITIONS



LAST TIME: THE ROAD TO MODULARITY



LAST TIME: THE ROAD TO MODULARITY



LAST TIME: THE ROAD TO MODULARITY

BRAIDED

MONOIDAL

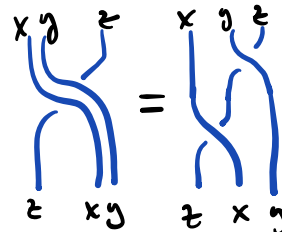
A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{I})^\#$ IS "BRAIDED" IF $\exists$ NAT'L ISOMORPHISM

$$C := \{c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x\}_{x,y \in \mathcal{C}}$$

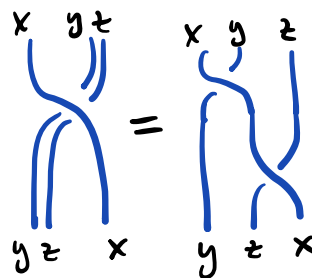


SUCH THAT

$$\begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{c_{x,y} \otimes \text{id}} & y \otimes x \otimes z \\ \text{id} \otimes c_{y,z} \searrow & \cong & \nearrow c_{x,z} \otimes \text{id} \\ & x \otimes z \otimes y & \end{array}$$



$$\begin{array}{ccc} x \otimes y \otimes z & \xrightarrow{c_{x,y} \otimes \text{id}} & y \otimes x \otimes z \\ c_{x,y} \otimes \text{id} \searrow & \cong & \nearrow \text{id} \otimes c_{x,z} \\ & y \otimes z \otimes x & \end{array}$$



VIA $\neq$ GRAPHICAL CALC.

$$x \otimes y \equiv \begin{array}{c} x \\ | \\ y \end{array}$$

$$f: x \rightarrow y \equiv \begin{array}{c} x \\ \boxed{f} \\ y \end{array}$$

$$c_{x,y} \equiv \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ y \quad x \end{array}$$

$$c_{y,x}^{-1} \equiv \begin{array}{c} x \quad y \\ \diagup \quad \diagdown \\ y \quad x \end{array}$$

$\neq$ ASSUMING "STRICT" $\equiv$ SUPPRESSING ASSOC, UNITALITY NAT'L ISOMS

LAST TIME: THE ROAD TO MODULARITY

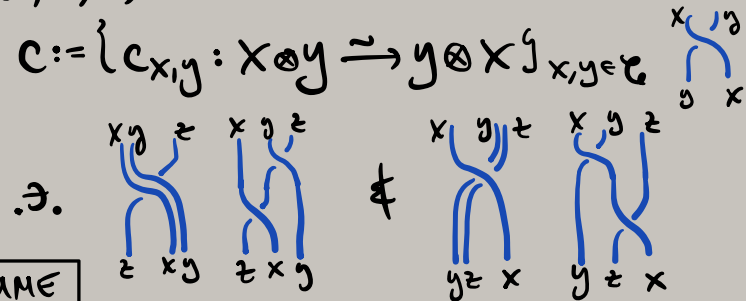
CONSIDER FULL SUBCATEGORY:
 "MÜGER CENTER" OF $(\mathcal{C}, \otimes, \mathbb{1}, c)$
 $\mathcal{Z}_2(\mathcal{C})$ WITH OBJECTS
 $\{X \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{X \otimes y} \forall y \in \mathcal{C}\}$

$(\mathcal{C}, \otimes, \mathbb{1}, c)$ IS "NONDEGENERATE"
 IF
 $\mathcal{Z}_2(\mathcal{C}) \simeq \text{Vec}$

$H_{\mathcal{C}}$ HAS A NONDEGENERATE
 HOPF PAIRING $\omega_{\mathcal{C}}: H_{\mathcal{C}} \otimes H_{\mathcal{C}} \rightarrow \mathbb{1}$

$\text{Hom}_{\mathcal{C}}(\mathbb{1}, H_{\mathcal{C}}) \rightarrow \text{Hom}_{\mathcal{C}}(H_{\mathcal{C}}, \mathbb{1})$
 $f \mapsto [\omega_{\mathcal{C}} \circ (f \otimes \text{id})]$
 IS BIJECTIVE

$(\mathcal{C}, \otimes, \mathbb{1})$ IS "BRAIDED" IF $\exists$ NAT'L ISOM:
 $c := \{c_{x,y}: X \otimes y \xrightarrow{\sim} y \otimes X\}_{x,y \in \mathcal{C}}$



ASSUME
TENSOR

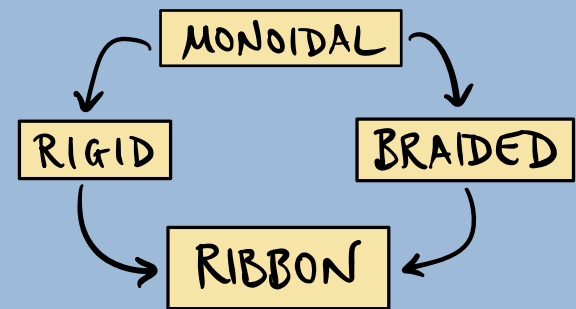
IF, FURTHER,
 $\mathcal{C}$ IS FINITE TENSOR

$H_{\mathcal{C}} := \int^{x \in \mathcal{C}} x^* \otimes x \in \text{HopfAlg}(\mathcal{C})$

IF, FURTHER,
 $\mathcal{C}$ IS FUSION

"S-MATRIX" OF $\mathcal{C}$
 IS INVERTIBLE

LAST TIME: THE ROAD TO MODULARITY



A BRAIDED, RIGID MONOIDAL CATEGORY

$$(\mathcal{C}, \otimes, \mathbb{1}, c, (-)^*, {}^*(-))$$

IS "RIBBON"

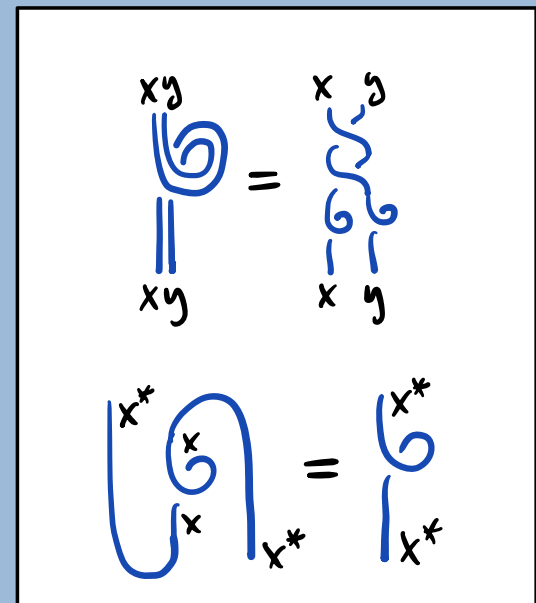
IF $\exists$ NATURAL ISOMORPHISM

$$\theta = \{\theta_x : X \xrightarrow{\sim} X^*\}_{x \in \mathcal{C}} \quad \begin{array}{c} x \\ \text{ } \\ x \end{array} \text{ (cup) } = \begin{array}{c} x \\ \text{ } \\ x \end{array} \text{ (cap) }$$

$\ni$.

$$\theta_{x \otimes y} = (\theta_x \otimes \theta_y) \circ c_{y,x} \circ c_{x,y}$$

$$\nexists (\theta_x)^* = \theta_{x^*} \quad \forall x, y \in \mathcal{C}$$



LAST TIME: THE ROAD TO MODULARITY

MODULAR TENSOR CAT. (MTC) $\equiv$ NONDEG. RIBBON FINITE TENSOR CAT.

MODULAR FUSION CAT. (MFC) $\equiv$ NONDEG. RIBBON FUSION CAT.

MODULAR HOPF ALGEBRA $H \equiv$ FIN. DIM'L (S.S.) FACTORIZABLE
 $\rightarrow H\text{-Mod} \equiv \text{MTC}$ RIBBON HOPF ALGEBRA

Ex. $u_q(\mathfrak{g})$

"SMALL QUANTUM GROUP"

$D(H)$

"DRINFELD DOUBLE"

OF A FIN. DIM'L (S.S.)

SPHERICAL HOPF ALGEBRA

E.G. $D(G)$

$\uparrow$
FINITE GROUP

CONSTRUCTIONS:

$\mathcal{C}(\mathfrak{g}, q)$

"VERLINDE
MODULAR CATS"

$\mathcal{Z}(\mathcal{C})$

"DRINFELD CENTERS OF
SPH. FINITE TENSOR CATS"

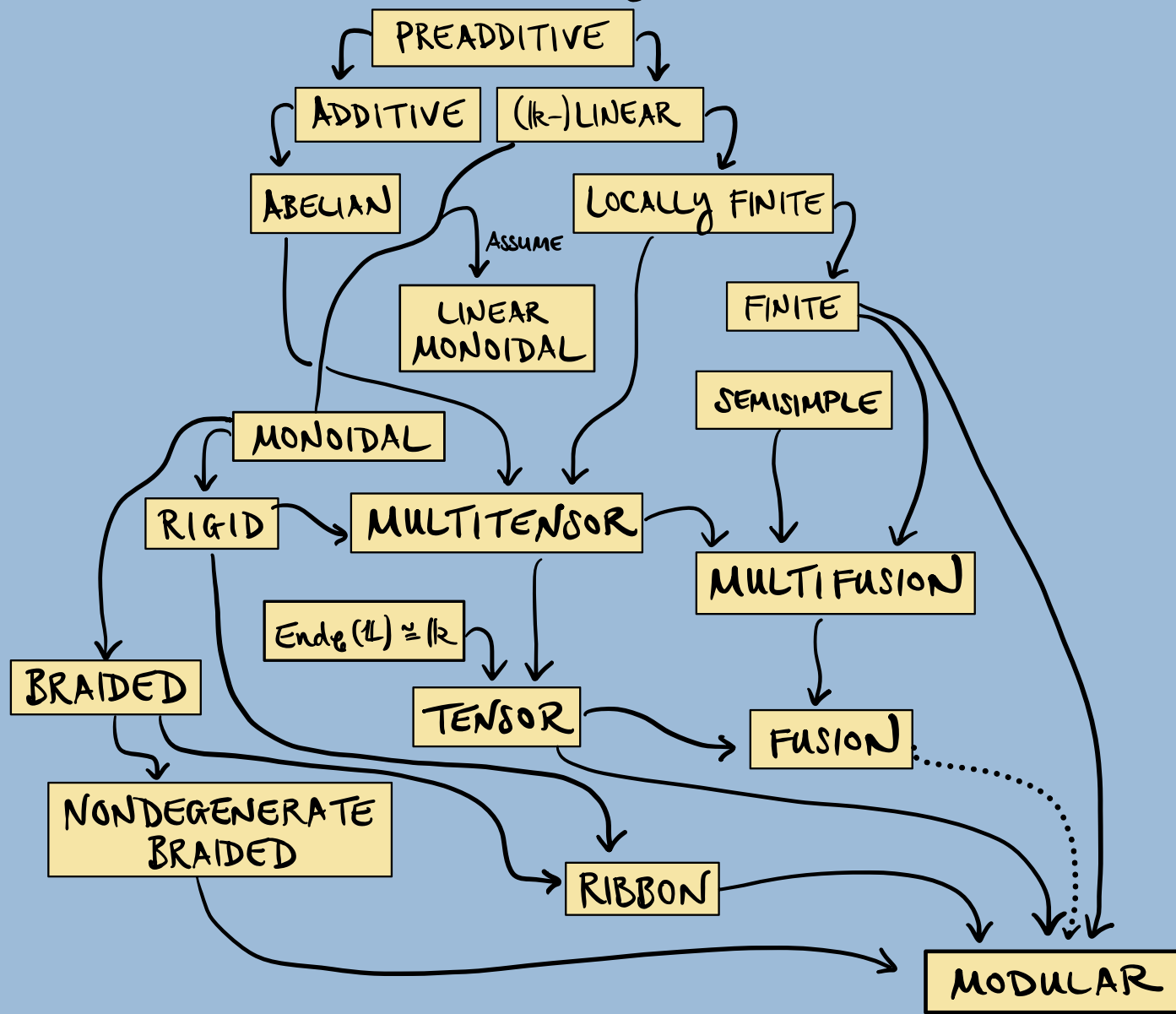
$\mathcal{Z}_g(\mathcal{C})$

"RELATIVE
CENTERS..."

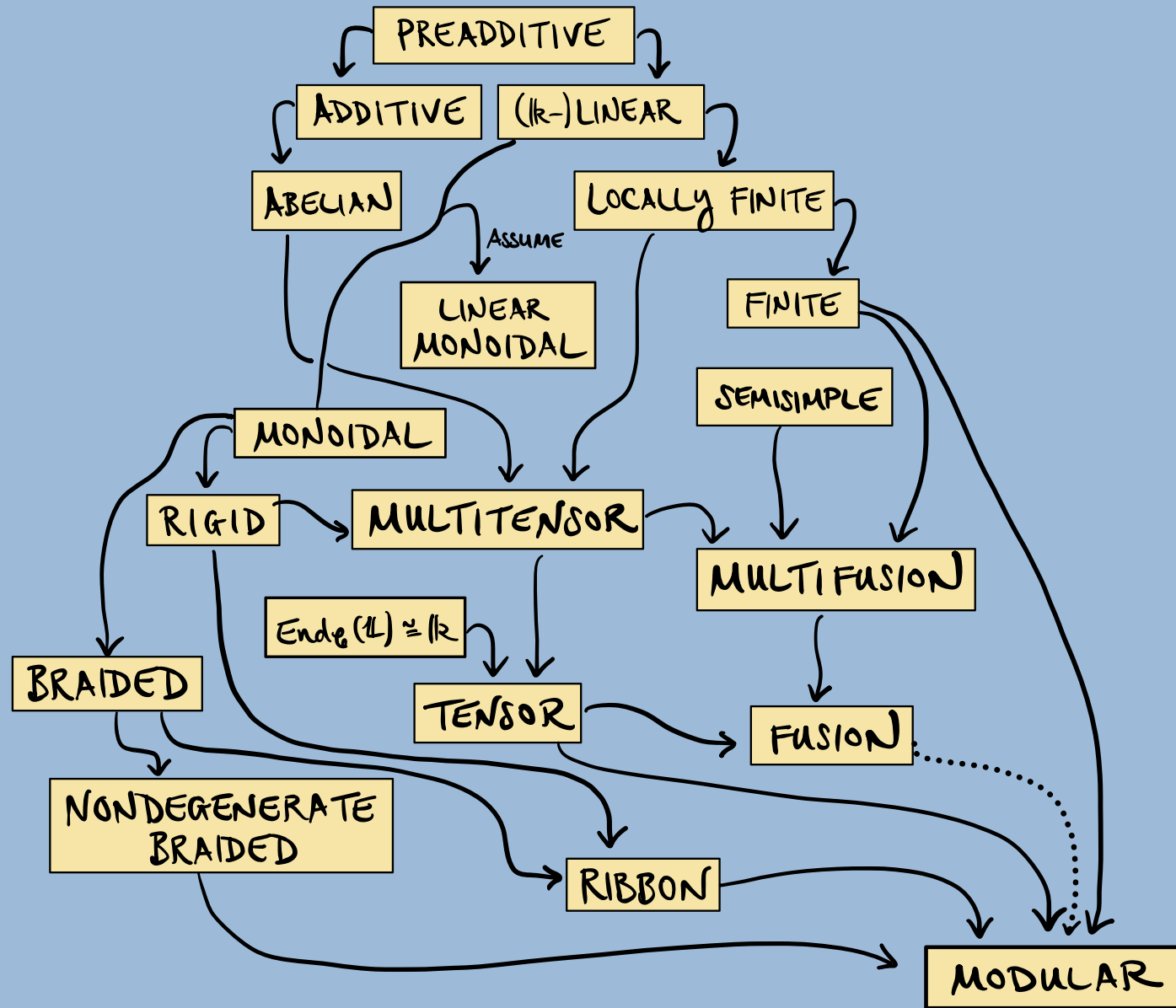
$\text{LocMod}_{\mathcal{C}}(A)$

"CATS OF
LOCAL MODULES..."

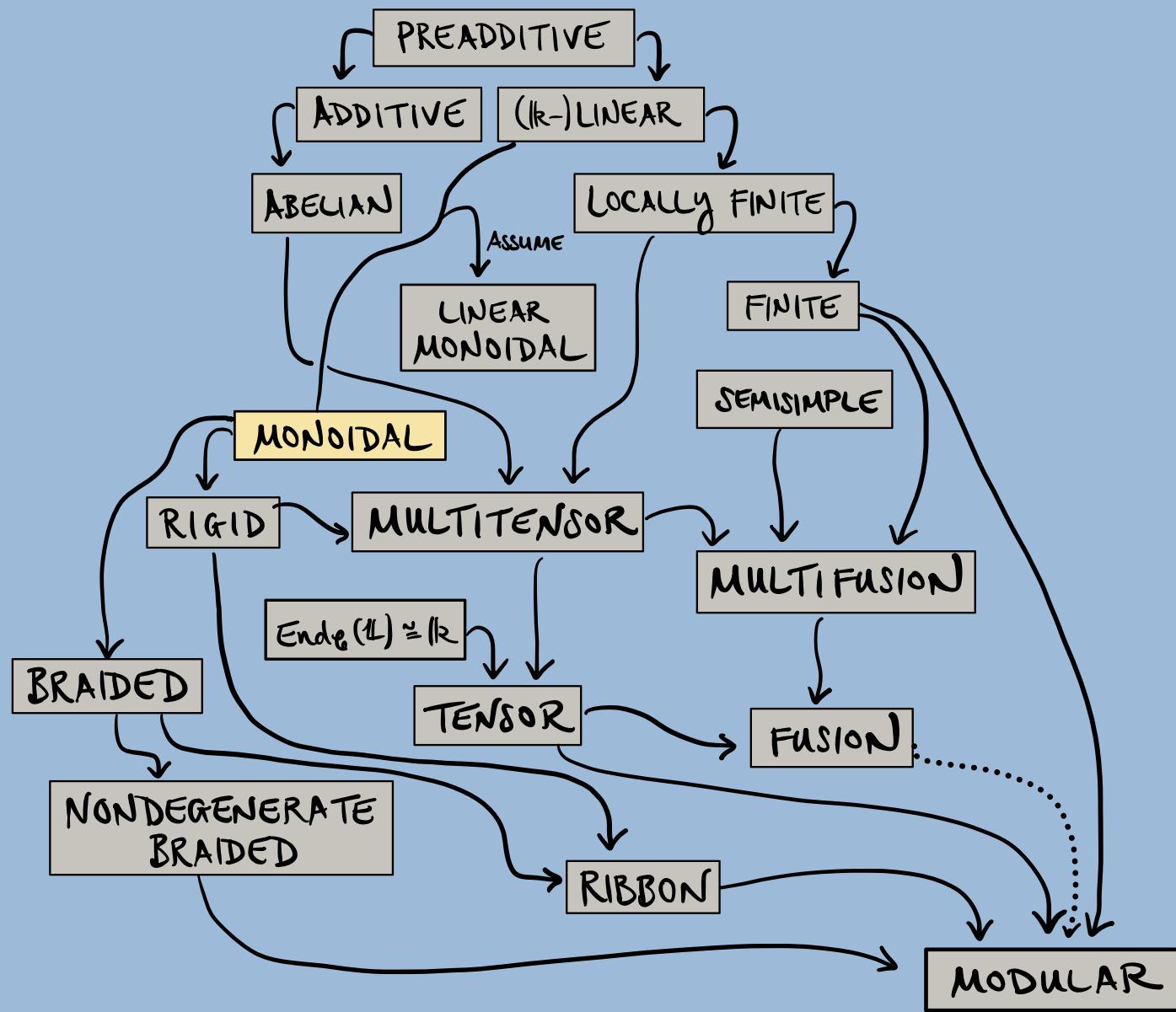
LAST TIME: THE ROAD TO MODULARITY



TODAY: REPRESENTATIONS IN & OF THESE



REPRESENTATIONS IN MONOIDAL CATEGORIES



REPRESENTATIONS IN MONOIDAL CATEGORIES

MONOIDAL

.

REPRESENTATIONS IN MONOIDAL CATEGORIES

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$

REPRESENTATIONS IN MONOIDAL CATEGORIES

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$
↑
CATEGORY

REPRESENTATIONS IN MONOIDAL CATEGORIES

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$

CATEGORY

BIFUNCTOR

$\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

REPRESENTATIONS IN MONOIDAL CATEGORIES

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$

CATEGORY

BIFUNCTOR

$\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

REPRESENTATIONS IN MONOIDAL CATEGORIES

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$

CATEGORY

BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

... BEHAVES LIKE
A MONOID

REPRESENTATIONS IN MONOIDAL CATEGORIES

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$

CATEGORY

BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

... BEHAVES LIKE
A MONOID

Ex.

$(\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

$\mathbb{K}$ -VECTOR SPACES

REPRESENTATIONS IN MONOIDAL CATEGORIES

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$

CATEGORY

BIFUNCTOR

$\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

... BEHAVES LIKE
A MONOID

Ex.

$(\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

$(\text{Rep}(G), \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

GROUP

REPRESENTATIONS IN MONOIDAL CATEGORIES

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$

CATEGORY

BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

... BEHAVES LIKE
A MONOID

Ex.

$(\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

$(\text{Rep}(G), \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

GROUP

$(\mathcal{B}\text{-Mod}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

BIALGEBRA

REPRESENTATIONS IN MONOIDAL CATEGORIES

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$

CATEGORY

BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

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GROUP

$(\mathcal{B}\text{-Mod}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

BIALGEBRA

$(\text{End}(\mathcal{A}), \circ, \text{Id}_{\mathcal{A}})$

CATEGORY

REPRESENTATIONS IN MONOIDAL CATEGORIES

AN "ALGEBRA" IN $(\mathcal{C}, \otimes, \mathbb{1})$ IS:

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$
↑
CATEGORY

BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

... BEHAVES LIKE
A MONOID

Ex.

$(\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

$(\text{Rep}(G), \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
↑
GROUP

$(\mathcal{B}\text{-Mod}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
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$\exists$.

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$
 CATEGORY

BIFUNCTOR
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DIST. OBJECT IN $\mathcal{C}$

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Ex.

$(\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

$(\text{Rep}(G), \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
 GROUP

$(B\text{-Mod}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
 BIALGEBRA

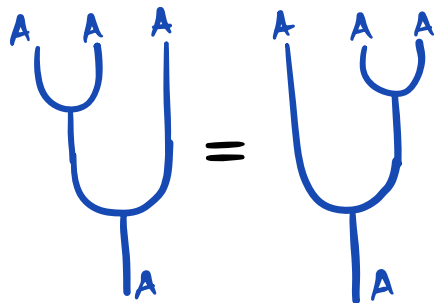
$(\text{End}(A), \circ, \text{Id}_A)$
 CATEGORY

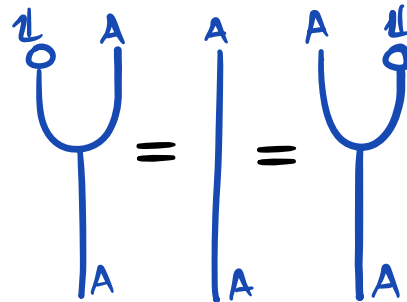
REPRESENTATIONS IN MONOIDAL CATEGORIES

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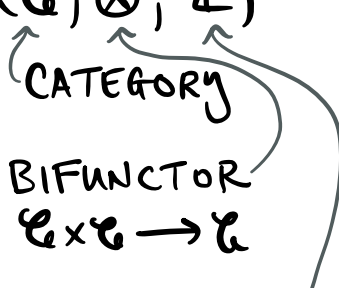
- AN OBJECT A IN $\mathcal{C}$,
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$\exists$. ASSOC. & UNITALITY AXIOMS HOLD:





MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$

 CATEGORY

BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

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 A MONOID

Ex.

$(\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

$(\text{Rep}(G), \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

 GROUP

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Ex.

$\mathbb{K}$ -ALGEBRAS

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$
↑
CATEGORY

BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

... BEHAVES LIKE
A MONOID

Ex.

$(\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

$(\text{Rep}(G), \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
↑
GROUP

$(B\text{-Mod}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
↑
BIALGEBRA

$(\text{End}(A), \circ, \text{Id}_A)$
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Ex.

$\mathbb{K}$ -ALGEBRAS

"G-MODULE ALGS"

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$
↑
CATEGORY

BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

... BEHAVES LIKE
A MONOID

Ex.

$(\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

$(\text{Rep}(G), \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
↑
GROUP

$(B\text{-Mod}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
↑
BIALGEBRA

$(\text{End}(A), \circ, \text{Id}_A)$
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Ex.

$\mathbb{K}$ -ALGEBRAS

" G -MODULE ALGS"

" B -MODULE ALGS"

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$
↑
CATEGORY

BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

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Ex.

$(\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

$(\text{Rep}(G), \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
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GROUP

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Ex.

$\mathbb{K}$ -ALGEBRAS

" G -MODULE ALGS"

" B -MODULE ALGS"

"MONADS"

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$
↑
CATEGORY

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 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

... BEHAVES LIKE
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Ex.

$(\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

$(\text{Rep}(G), \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
↑
 G GROUP

$(B\text{-Mod}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
↑
BIALGEBRA

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CATEGORY



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Ex.

$\mathbb{K}$ -ALGEBRAS

"G-MODULE ALGS"

"B-MODULE ALGS"

"MONADS"

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$
 $\uparrow$
 CATEGORY

BIFUNCTOR
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$(\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

$(\text{Rep}(G), \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
 $\uparrow$
 GROUP

$(B\text{-Mod}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
 $\uparrow$
 BIALGEBRA

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$\exists$. ASSOC. & UNITALITY AXIOMS HOLD.

Ex.

$\mathbb{K}$ -ALGEBRAS

"G-MODULE ALGS"

$\mathcal{B}$ = DUAL
OF $\mathbb{K}G$

"B-MODULE ALGS"

"MONADS"

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$
CATEGORY

BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

... BEHAVES LIKE
A MONOID

Ex.

$(\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

$(\text{Rep}(G), \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
GROUP

$(\mathcal{B}\text{-Mod}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
BIALGEBRA

$(\text{End}(\mathcal{A}), \circ, \text{Id}_{\mathcal{A}})$
CATEGORY

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Ex.

$\mathbb{K}$ -ALGEBRAS

G -GRADED ALGS

$B =$ DUAL
OF $\mathbb{K}G$

"B-MODULE ALGS"

"MONADS"

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$
CATEGORY

BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

... BEHAVES LIKE
A MONOID

Ex.

$(\text{Vec}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$

$(\text{Vec}_G \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
GROUP

$(B\text{-Mod}, \otimes_{\mathbb{K}}, \mathbb{1}_{\mathbb{K}})$
BIALGEBRA

$(\text{End}(A), \circ, \text{Id}_A)$
CATEGORY

REPRESENTATIONS IN MONOIDAL CATEGORIES

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$\exists$. ASSOC. & UNITALITY AXIOMS HOLD.

Ex.

$\mathbb{R}$ -ALGEBRAS

G -GRADED ALGS

"MONADS"

MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$
 $\uparrow$
 CATEGORY

BIFUNCTOR
 $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$

DIST. OBJECT IN $\mathcal{C}$

... BEHAVES LIKE
 A MONOID

Ex.

$(\text{Vec}, \otimes_{\mathbb{R}}, \mathbb{R})$

$(\text{Vec}_G \otimes_{\mathbb{R}}, \mathbb{R})$
 $\uparrow$
 GROUP

$(\text{End}(A), \circ, \text{Id}_A)$
 $\uparrow$
 CATEGORY

REPRESENTATIONS IN MONOIDAL CATEGORIES

AN "ALGEBRA" IN
 $(\mathcal{C}, \otimes, \mathbb{1})$ IS:

AN OBJECT A IN $\mathcal{C}$,

& MORPHISMS

$m: A \otimes A \rightarrow A$ IN $\mathcal{C}$,

$u: \mathbb{1} \rightarrow A$ IN $\mathcal{C}$,

$\exists$.

ASSOC. & UNITALITY
AXIOMS HOLD.

Ex.

$\mathbb{R}$ -ALGS IN Vec

G -GRADED ALGS IN Vec_G

"MONADS" IN $\text{End}(A)$

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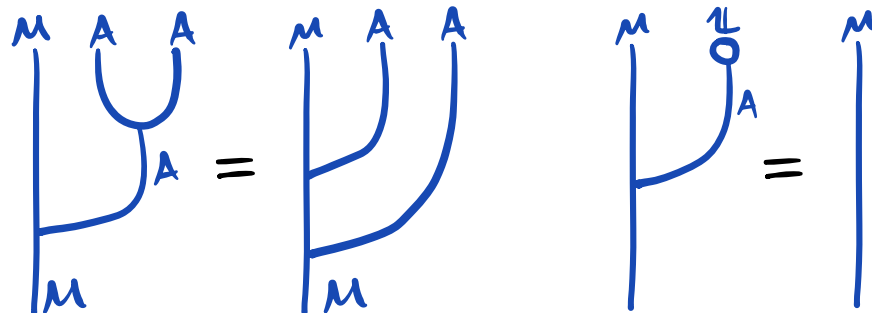
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$\rightarrow$ FORMS A CATEGORY $\boxed{\text{Mod-}A(\mathcal{C})}$

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Ex.

RIGHT A -MODULES/ $\mathbb{R}$

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RIGHT A -MODULES/ $\mathbb{R}$

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Ex.

RIGHT A -MODULES/ $\mathbb{R}$

RIGHT G -GRADED A -MODULES

"EILENBERG-MOORE CATEGS"

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G -GRADED ALGS IN Vec_G

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RIGHT A -MODULES/ $\mathbb{R}$

RIGHT G -GRADED A -MODULES

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Ex.

RIGHT A -MODULES/ $\mathbb{R}$

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ASSOC. &

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FORMS A
CATEGORY

$\text{Mod-}A(\mathcal{C})$

REPRESENTATIONS OF MONOIDAL CATEGORIES

GIVEN A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1})$
A "LEFT $\mathcal{C}$ -MODULE CATEGORY" IS:

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FORMS A
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EQUIPPED WITH NAT'L ISOMS
SAT. ASSOC. & UNITALITY AXIOMS

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FORMS A
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Ex.

- REG. MOD. CAT. ($\mathcal{M} := \mathcal{C}, \triangleright := \otimes$)

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FORMS A
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$\text{Mod-}A(\mathcal{C})$

REPRESENTATIONS OF MONOIDAL CATEGORIES

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Ex.

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$$(X, (M, \triangleleft_M)) \mapsto (X \otimes M, \triangleleft_{X \otimes M} := \text{id}_X \otimes \triangleleft_M)$$

GIVEN AN ALGEBRA
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FORMS A
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$\text{Mod-}A(\mathcal{C})$

REPRESENTATIONS OF MONOIDAL CATEGORIES

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FORMS A
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$\text{Mod-}A(\mathcal{C})$

REPRESENTATIONS OF FUSION CATEGORIES

GIVEN A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1})$

A "LEFT $\mathcal{C}$ -MODULE CATEGORY" IS:

- A CATEGORY $\mathcal{M}$
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EQUIPPED WITH NAT'L ISOMS
SAT. ASSOC. & UNITALITY AXIOMS

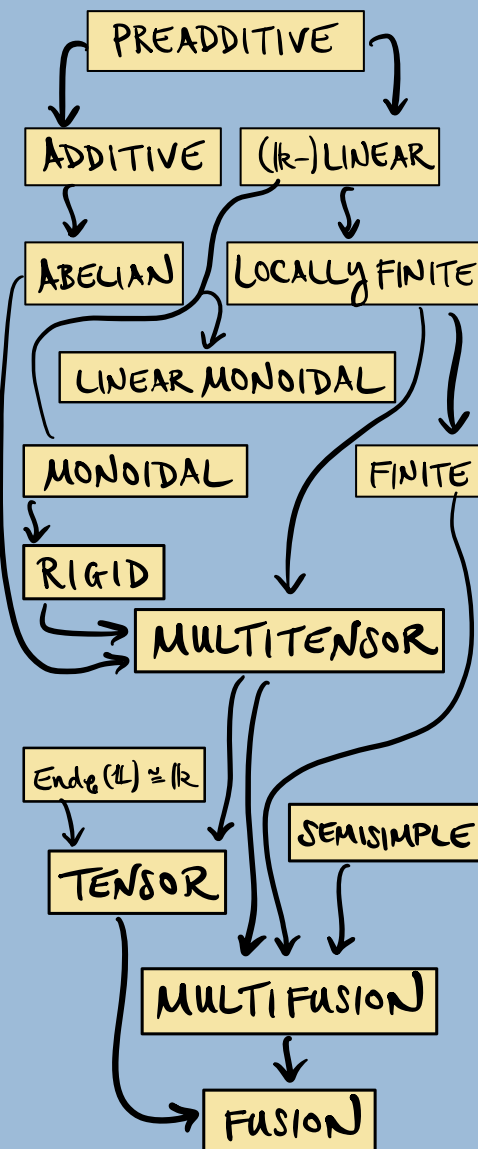
Ex.

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REPRESENTATIONS OF FUSION CATEGORIES

GIVEN $(\mathcal{C}, \otimes, \mathbb{1})$ AS IN $\rightarrow$

A "LEFT $\mathcal{C}$ -MODULE CATEGORY" IS:

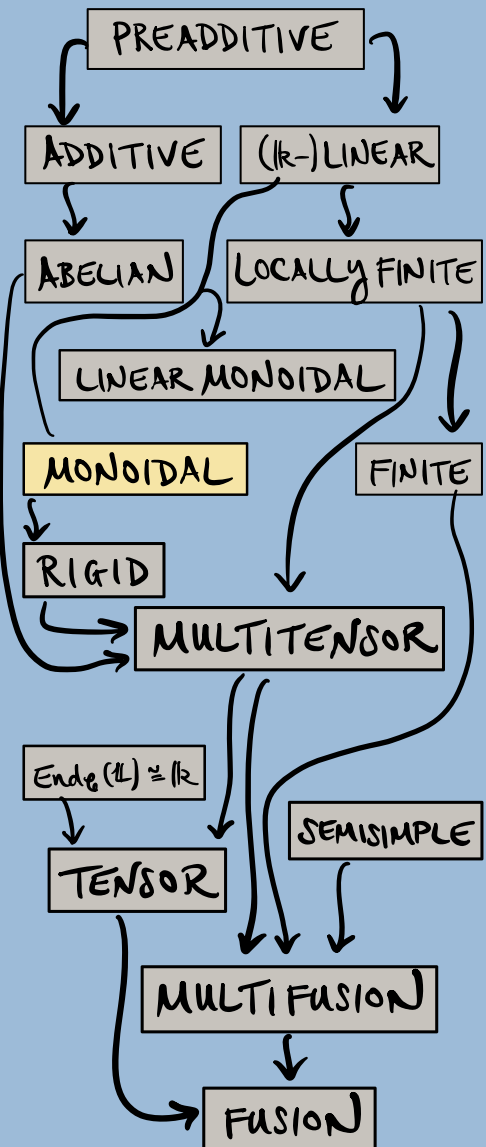
- A CATEGORY $\mathcal{M}$ THAT IS

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EQUIPPED W/ NAT'L ISOMS SAT. ASSOC. & UNIT. AXIOMS

EX. • REGULAR MODULE CATS.

- $\text{Mod-}\mathcal{A}(\mathcal{C})$



REPRESENTATIONS OF FUSION CATEGORIES

GIVEN $(\mathcal{C}, \otimes, \mathbb{1})$ AS IN $\rightarrow$

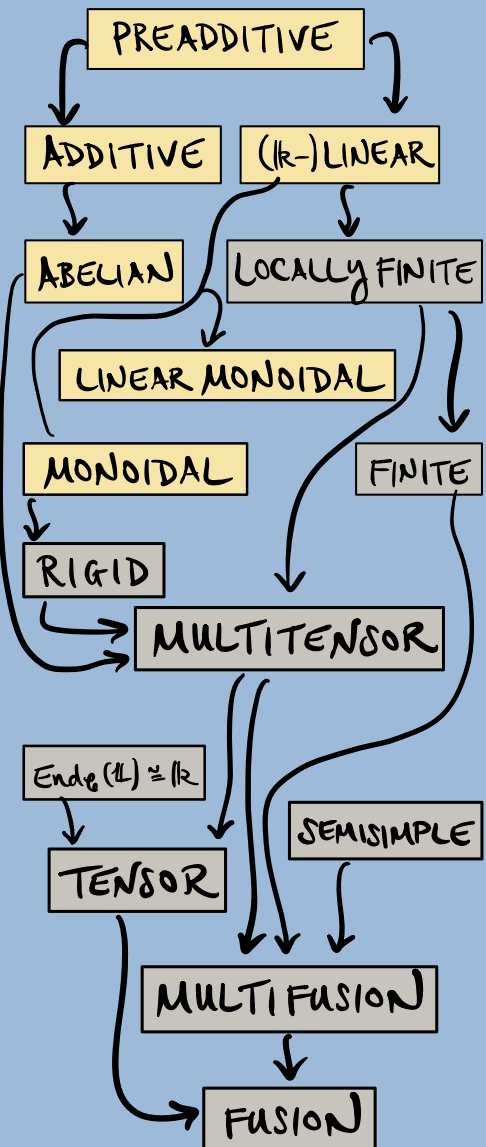
A "LEFT $\mathcal{C}$ -MODULE CATEGORY" IS:

- A CATEGORY $\mathcal{M}$ THAT IS
ABELIAN,
LINEAR
- A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M} \ni$
 $\triangleright$ IS LINEAR IN EACH SLOT

EQUIPPED W/ NAT'L ISOMS SAT. ASSOC. & UNIT. AXIOMS

EX. • REGULAR MODULE CATS.

- $\text{Mod-}\mathcal{A}(\mathcal{C})$ w/ $(-\otimes A) : \mathcal{C} \rightarrow \mathcal{C}$ LINEAR



REPRESENTATIONS OF FUSION CATEGORIES

GIVEN $(\mathcal{C}, \otimes, \mathbb{1})$ AS IN $\rightarrow$

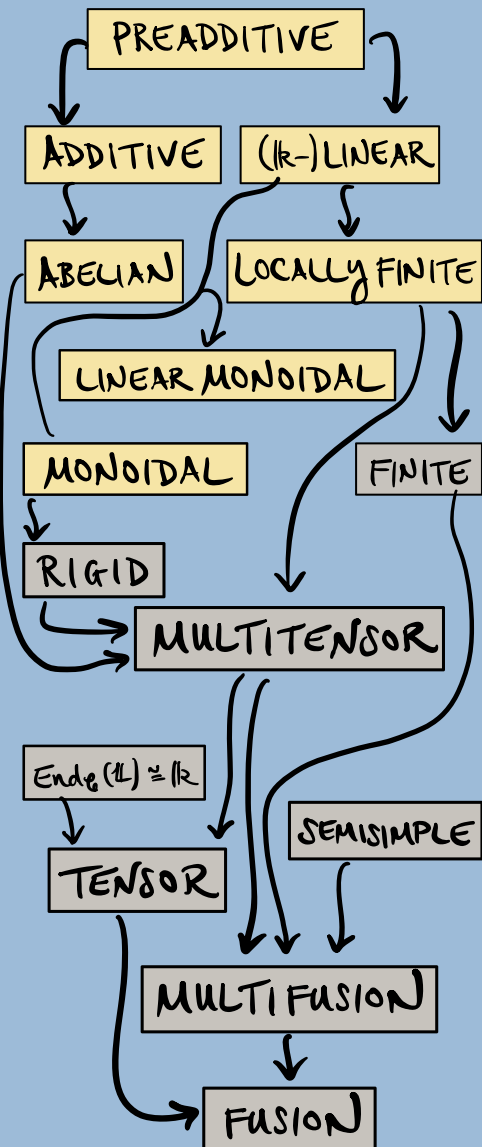
A "LEFT $\mathcal{C}$ -MODULE CATEGORY" IS:

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~~LINEAR~~, LOCALLY FINITE
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 $\triangleright$ IS LINEAR IN EACH SLOT

EQUIPPED W/ NAT'L ISOMS SAT. ASSOC. & UNIT. AXIOMS

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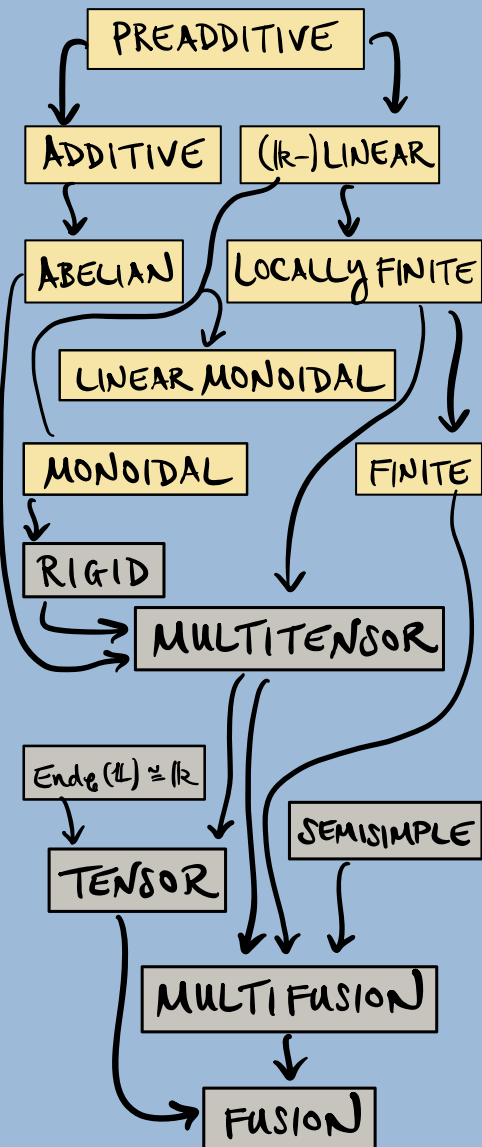
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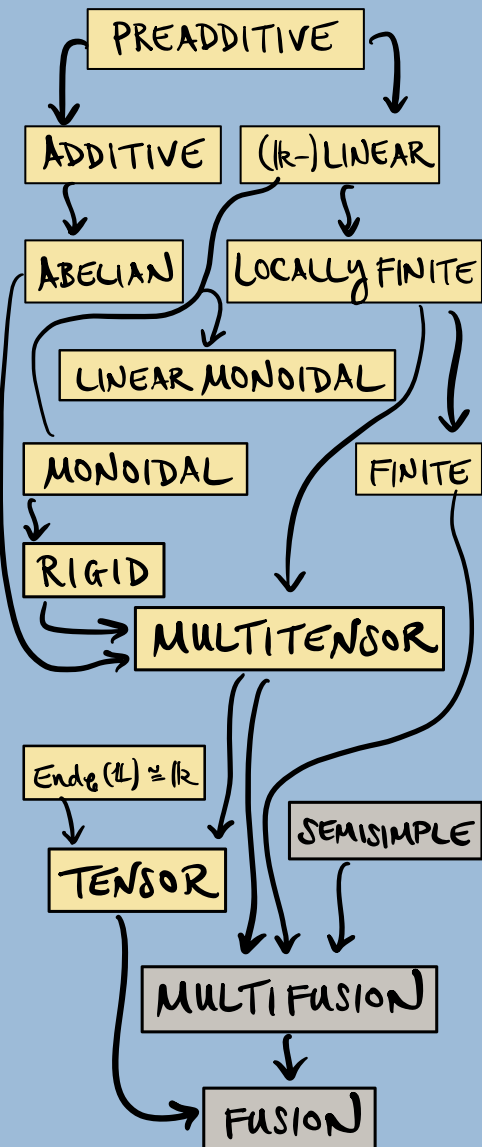
$\triangleright$ IS LINEAR IN EACH SLOT,

$(-\triangleright M) : \mathcal{C} \rightarrow \mathcal{M}$ IS EXACT $\forall M \in \mathcal{M}$

EQUIPPED W/ NAT'L ISOMS SAT. ASSOC. & UNIT. AXIOMS

EX. • REGULAR MODULE CATS.

- $\text{Mod-}\mathcal{A}(\mathcal{C})$ w/ $\begin{cases} (-\otimes A) : \mathcal{C} \rightarrow \mathcal{C} \text{ LINEAR} \\ \text{A "EXACT"} \end{cases}$



REPRESENTATIONS OF FUSION CATEGORIES

GIVEN $(\mathcal{C}, \otimes, \mathbb{1})$ AS IN $\rightarrow$

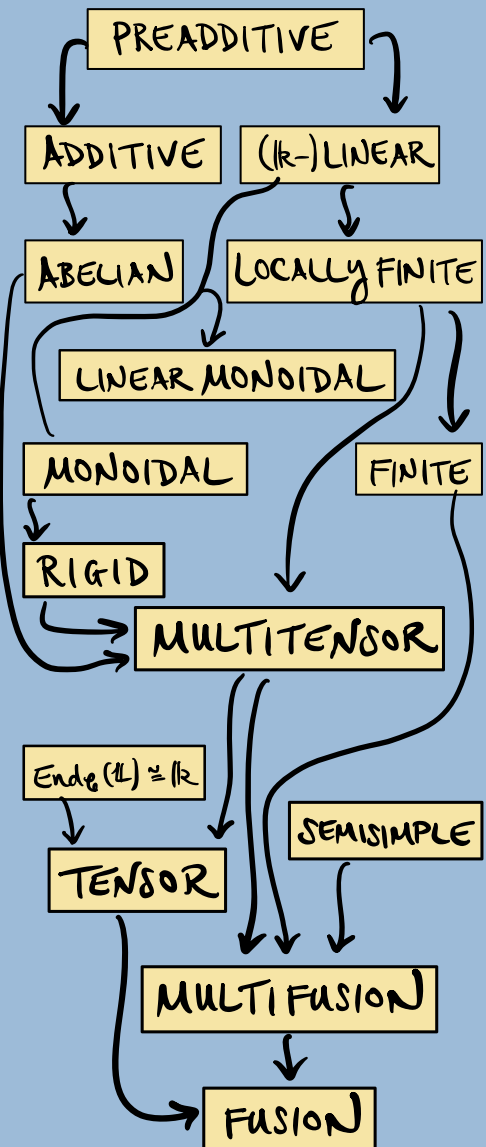
A "LEFT $\mathcal{C}$ -MODULE CATEGORY" IS:

- A CATEGORY $\mathcal{M}$ THAT IS
 - ABELIAN,
 - ~~LINEAR~~, ~~LOCALLY~~ FINITE,
 - SEMISIMPLE
- A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M} \ni$
 - $\triangleright$ IS LINEAR IN EACH SLOT,
 - $(-\triangleright M) : \mathcal{C} \rightarrow \mathcal{M}$ IS EXACT $\forall M \in \mathcal{M}$

EQUIPPED W/ NAT'L ISOMS SAT. ASSOC. & UNIT. AXIOMS

EX. • REGULAR MODULE CATS.

- $\text{Mod-}\mathcal{A}(\mathcal{C})$ w/ $\begin{cases} (-\otimes A) : \mathcal{C} \rightarrow \mathcal{C} \text{ LINEAR} \\ \text{A "EXACT", SEPARABLE} \end{cases}$



REPRESENTATIONS OF FUSION CATEGORIES

GIVEN A FUSION CAT. $\mathcal{C}$

A "LEFT $\mathcal{C}$ -MOD. CAT." IS:

- A CAT. $\mathcal{M}$ THAT IS

$$\left\{ \begin{array}{l} \text{ABELIAN,} \\ \text{FINITE,} \\ \text{SEMISIMPLE} \end{array} \right.$$

- A BIFUNCTOR

$$\triangleright : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M} \quad \exists.$$

$$\left\{ \begin{array}{l} \triangleright \text{ IS LINEAR IN EACH SLOT} \\ (-\triangleright M) \text{ IS EXACT } \forall M \in \mathcal{M} \end{array} \right.$$

EQUIPPED W/ NAT'L ISOMS

SAT. ASSOC. & UNIT. AXIOMS

EX. • REG. MODULE CATS.

- $\text{Mod-}\mathcal{A}(\mathcal{C})$ w/ $\left\{ \begin{array}{l} (-\otimes A) \text{ LINEAR} \\ A \text{ SEPARABLE} \end{array} \right.$

REPRESENTATIONS OF FUSION CATEGORIES

GIVEN A FUSION CAT. $\mathcal{C}$

A "LEFT $\mathcal{C}$ -MOD. CAT." IS:

- A CAT. $\mathcal{M}$ THAT IS

$\left\{ \begin{array}{l} \text{ABELIAN,} \\ \text{FINITE,} \\ \text{SEMISIMPLE} \end{array} \right.$

- A BIFUNCTOR

$$\triangleright : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M} \quad \exists.$$

$\left\{ \begin{array}{l} \triangleright \text{ IS LINEAR IN EACH SLOT} \\ (-\triangleright M) \text{ IS EXACT } \forall M \in \mathcal{M} \end{array} \right.$

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INTERNAL
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$\text{Mod-}\mathcal{A}(\mathcal{C})$

RIGHT $\mathcal{A}$ -MODS
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OVER ALG $\mathcal{A}$
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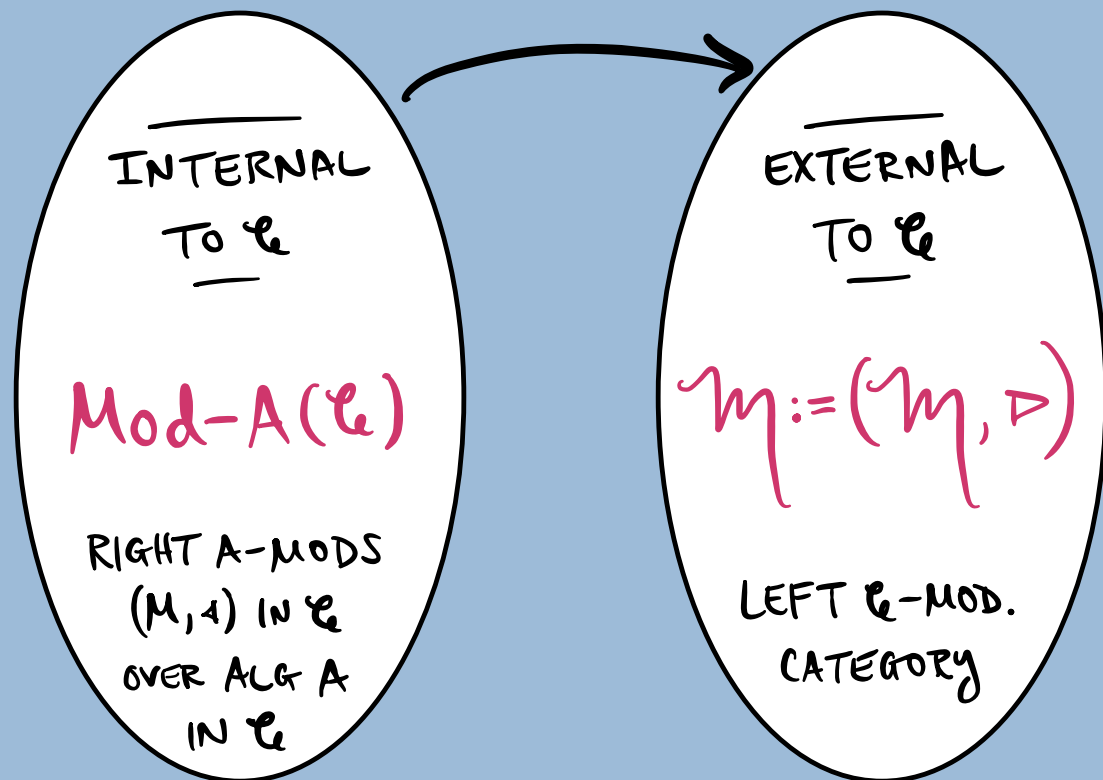
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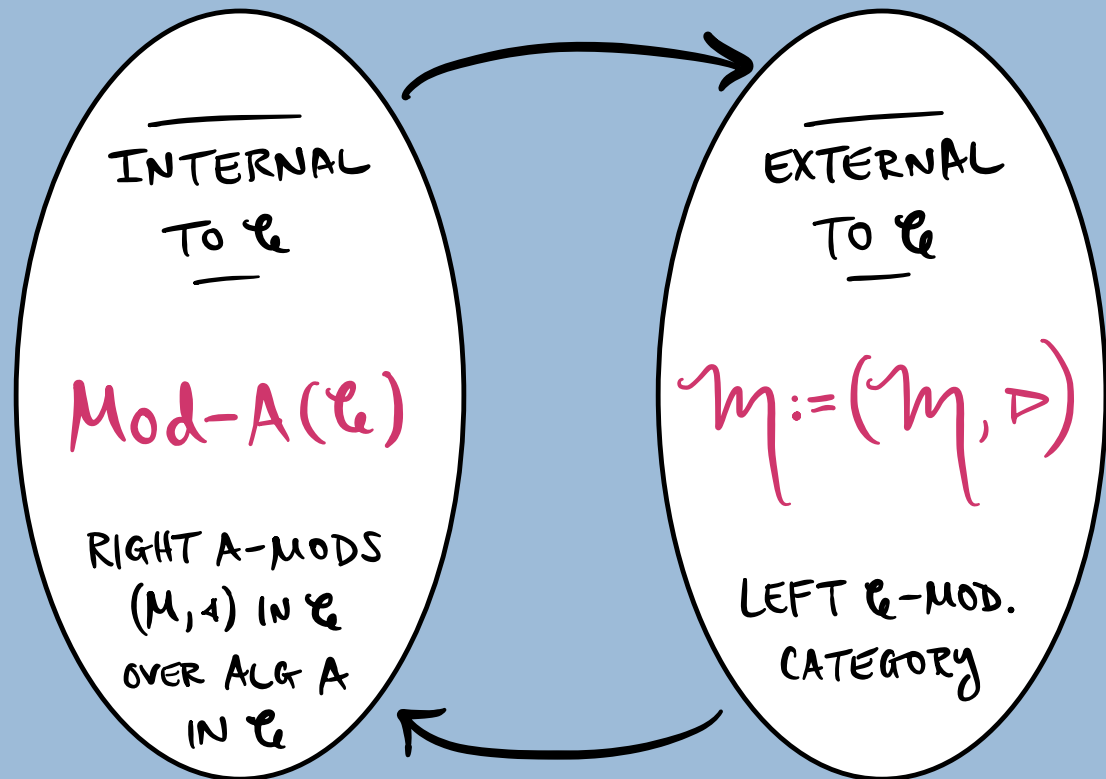
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NOW WE DISCUSS WHEN THIS OCCURS

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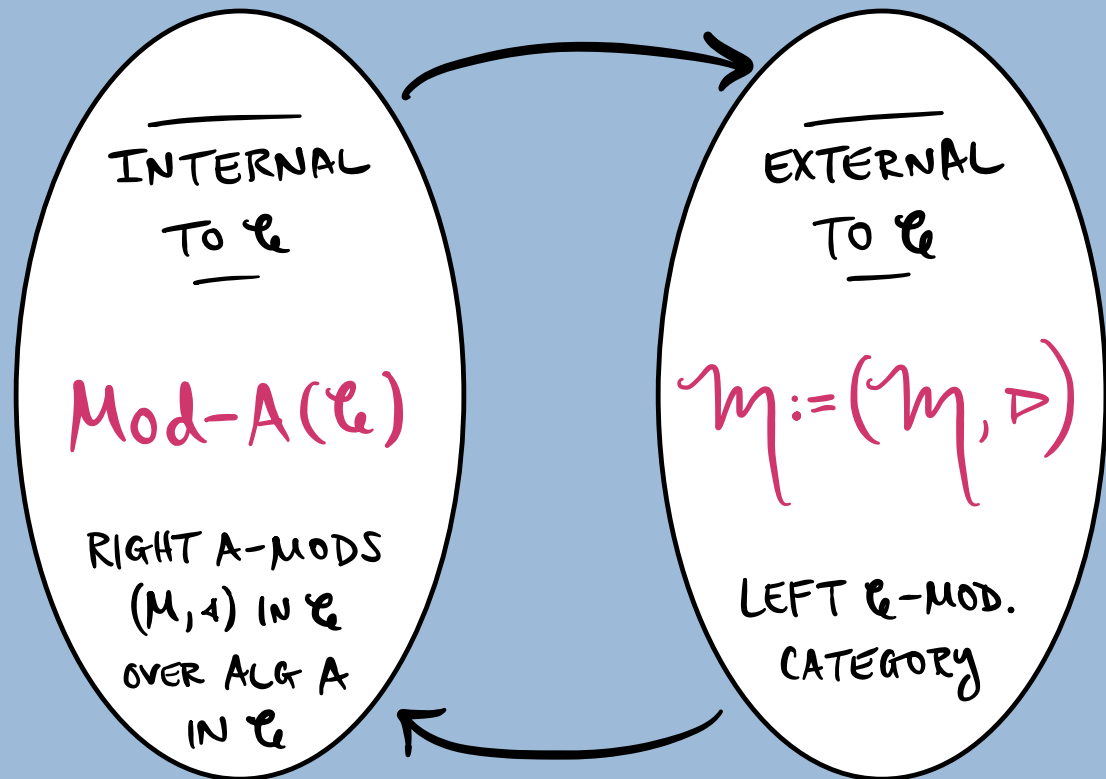
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NOW WE DISCUSS WHEN THIS OCCURS

... A WILL BE AN "INTERNAL END ALG."

INTERNAL END ALGEBRAS

GIVEN $(\mathcal{C}, \otimes, \mathbb{1})$,
"LEFT $\mathcal{C}$ -MOD. CAT.":

- A CATEGORY $\mathcal{M}$

- A BIFUNCTOR

$$\triangleright : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$$

EQUIPPED W/

NAT'L ISOMS

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CONSIDER $(- \triangleright M) : \mathcal{C} \rightarrow \mathcal{M}$

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EX. IF $\mathcal{C} = \text{FdVec}/_{\mathbb{K}}$ & $\mathcal{M} = \mathcal{C}_{\text{reg}}$

$$\text{Hom}_{\mathbb{K}}(X \otimes V, W) \cong \text{Hom}_{\mathbb{K}}(X, W \otimes V^*)$$

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$$m : V \otimes V^* \otimes V \otimes V^* \xrightarrow{\text{id} \otimes \text{ev} \otimes \text{id}} V \otimes V^*$$

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$$\leadsto \text{End}(V) \cong V \otimes V^* \cong \text{Mat}_n(\mathbb{K})$$

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FOR
 $\dim_{\mathbb{K}} V = n$

AS FIN DIM $\mathbb{K}$ -ALGS.

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OSTRIK'S THEOREM

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ RIGID

& $(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$

$\downarrow$

$(-\triangleright M): \mathcal{C} \rightarrow \mathcal{M}$

HAS RIGHT ADJOINT

$\underline{\text{Hom}}(M, -): \mathcal{M} \rightarrow \mathcal{C}$

$\forall M \in \mathcal{M}$

$\downarrow$

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THEOREM: TAKE $\mathcal{C}$ A MULTIFUSION CAT.

OSTRIK'S THEOREM

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THEOREM: TAKE $\mathcal{C}$ A MULTIFUSION CAT.

TAKE A LEFT $\mathcal{C}$ -MODULE CAT. $\mathcal{M}$.

IF $M \in \mathcal{M}$, THEN

$$\mathcal{M} \longrightarrow \text{Mod-}\underline{\text{End}}(M)(\mathcal{C})$$

$$N \longmapsto \underline{\text{Hom}}(M, N)$$

IS A $\mathcal{C}$ -MODULE FUNCTOR ...

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THEOREM: TAKE $\mathcal{C}$ A MULTIFUSION CAT.

TAKE A LEFT $\mathcal{C}$ -MODULE CAT. $\mathcal{M}^{\neq 0}$
 IF $M^{\neq 0} \in \mathcal{M}$, THEN

$$\mathcal{M} \xrightarrow{\sim} \text{Mod-}\underline{\text{End}}(M)(\mathcal{C})$$

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IS A $\mathcal{C}$ -MODULE EQUIVALENCE.

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IS A $\mathcal{C}$ -MODULE EQUIVALENCE.

Ex. $\mathcal{C} = \text{FdVec}/k$, $\mathcal{M} = \mathcal{C}_{\text{reg}} \cong \text{FdMod-}k$

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$$N \longmapsto \underline{\text{Hom}}(M, N)$$

IS A $\mathcal{C}$ -MODULE EQUIVALENCE.

Ex. $\mathcal{C} = \text{FdVec}/\mathbb{k}$, $\mathcal{M} = \mathcal{C}_{\text{reg}} \cong \text{FdMod-}\mathbb{k}$

FOR AN n -DIML $\mathbb{k}$ -VECTOR SPACE V

$$\rightarrow \text{FdMod-}\mathbb{k} \simeq \text{FdMod-}\underline{\text{End}}(V)$$

OSTRIK'S THEOREM

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ RIGID
 $\& (\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$



$(-\triangleright M): \mathcal{C} \rightarrow \mathcal{M}$
 HAS RIGHT ADJOINT
 $\underline{\text{Hom}}(M, -): \mathcal{M} \rightarrow \mathcal{C}$
 $\forall M \in \mathcal{M}$



$\underline{\text{End}}(M) := \underline{\text{Hom}}(M, M)$
 IS AN ALG. IN $\mathcal{C}$
 $\&$
 $\underline{\text{Hom}}(M, N)$ IS A
 RIGHT MODULE
 OVER $\underline{\text{End}}(M)$ IN $\mathcal{C}$

THEOREM: TAKE $\mathcal{C}$ A MULTIFUSION CAT.

TAKE A LEFT $\mathcal{C}$ -MODULE CAT. $\mathcal{M}^{\neq 0}$
 IF $M^{\neq 0} \in \mathcal{M}$, THEN

$$\mathcal{M} \xrightarrow{\sim} \text{Mod-}\underline{\text{End}}(M)(\mathcal{C})$$

$$N \longmapsto \underline{\text{Hom}}(M, N)$$

IS A $\mathcal{C}$ -MODULE EQUIVALENCE.

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FOR AN n -DIML $\mathbb{k}$ -VECTOR SPACE V

$$\rightarrow \text{FdMod-}\mathbb{k} \simeq \text{FdMod-Mat}_n(\mathbb{k})$$

OSTRIK'S THEOREM

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FOR AN n -DIML $\mathbb{k}$ -VECTOR SPACE V

$$\rightarrow \text{FdMod-}\mathbb{k} \simeq \text{FdMod-Mat}_n(\mathbb{k})$$

($\mathbb{k} \& \text{Mat}_n(\mathbb{k})$ ARE INDEED MORITA EQUIV.)

OSTRIK'S THEOREM

∃ MANY GENERALIZATIONS (SEE REFS)

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ RIGID
& $(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$

↓

$(-\triangleright M): \mathcal{C} \rightarrow \mathcal{M}$

HAS RIGHT ADJOINT

$\underline{\text{Hom}}(M, -): \mathcal{M} \rightarrow \mathcal{C}$

$\forall M \in \mathcal{M}$

↓

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IS A $\mathcal{C}$ -MODULE EQUIVALENCE.

Ex. $\mathcal{C} = \text{FdVec}/\mathbb{k}$, $\mathcal{M} = \mathcal{C}_{\text{reg}} \cong \text{FdMod-}\mathbb{k}$

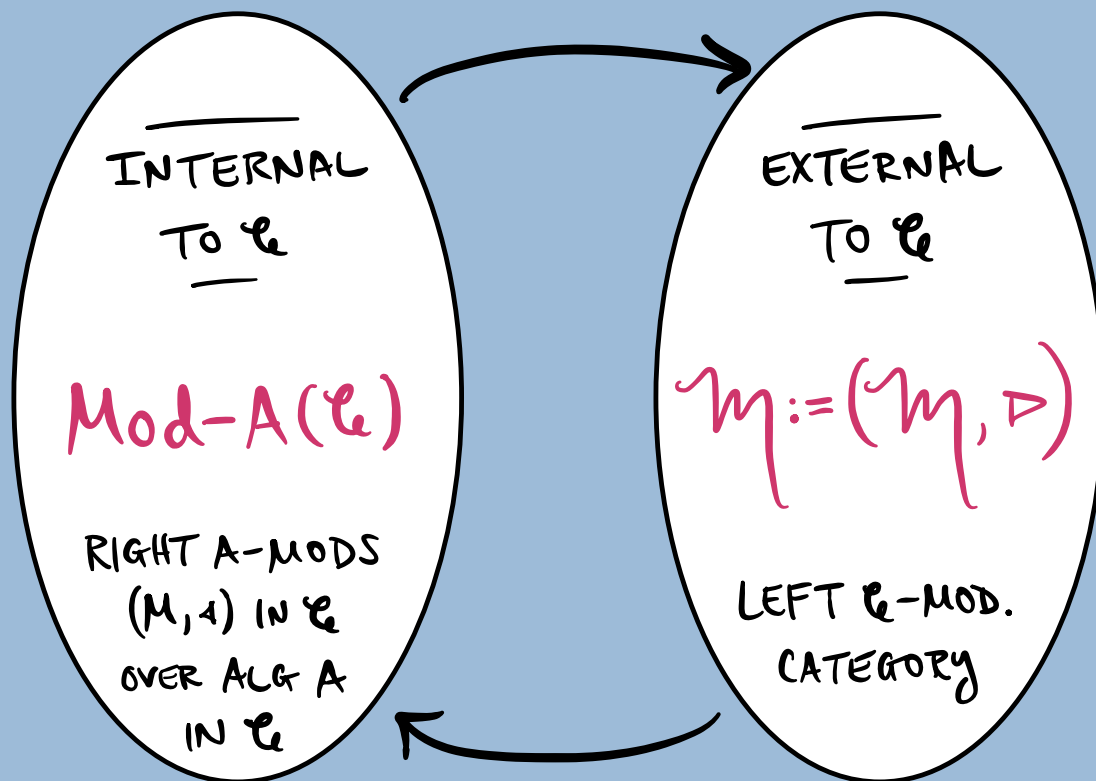
FOR AN n -DIML $\mathbb{k}$ -VECTOR SPACE V

$\rightarrow \text{FdMod-}\mathbb{k} \simeq \text{FdMod-Mat}_n(\mathbb{k})$

($\mathbb{k} \nsubseteq \text{Mat}_n(\mathbb{k})$ ARE INDEED MORITA EQUIV.)

REPRESENTATIONS OF FUSION CATEGORIES

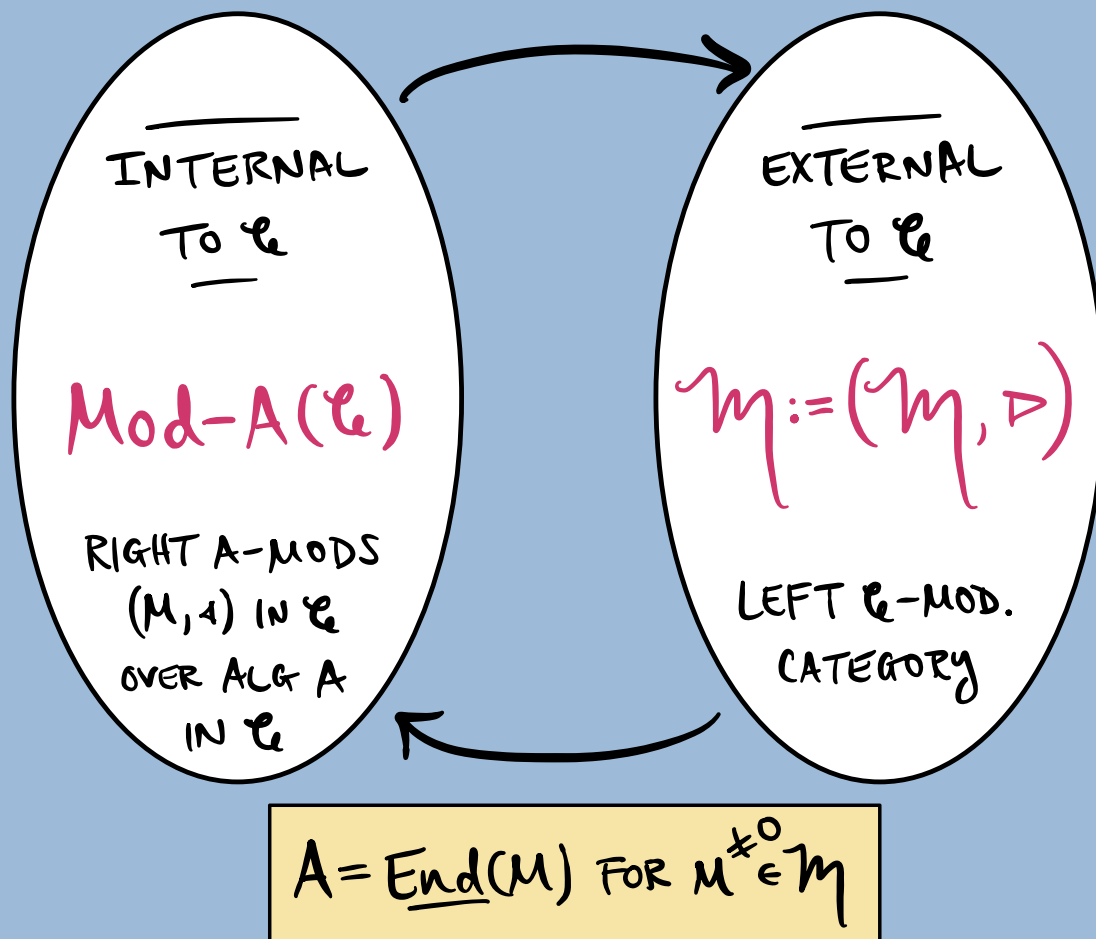
$$\text{Mod-}A(\mathcal{C}) \in \mathcal{C}\text{-Mod via } X \triangleright (M, \triangleleft) := (X \otimes M, \triangleleft = \text{id} \otimes \triangleleft)$$



$$A = \text{End}(M) \text{ FOR } M^{\neq 0} \in \mathcal{M}$$

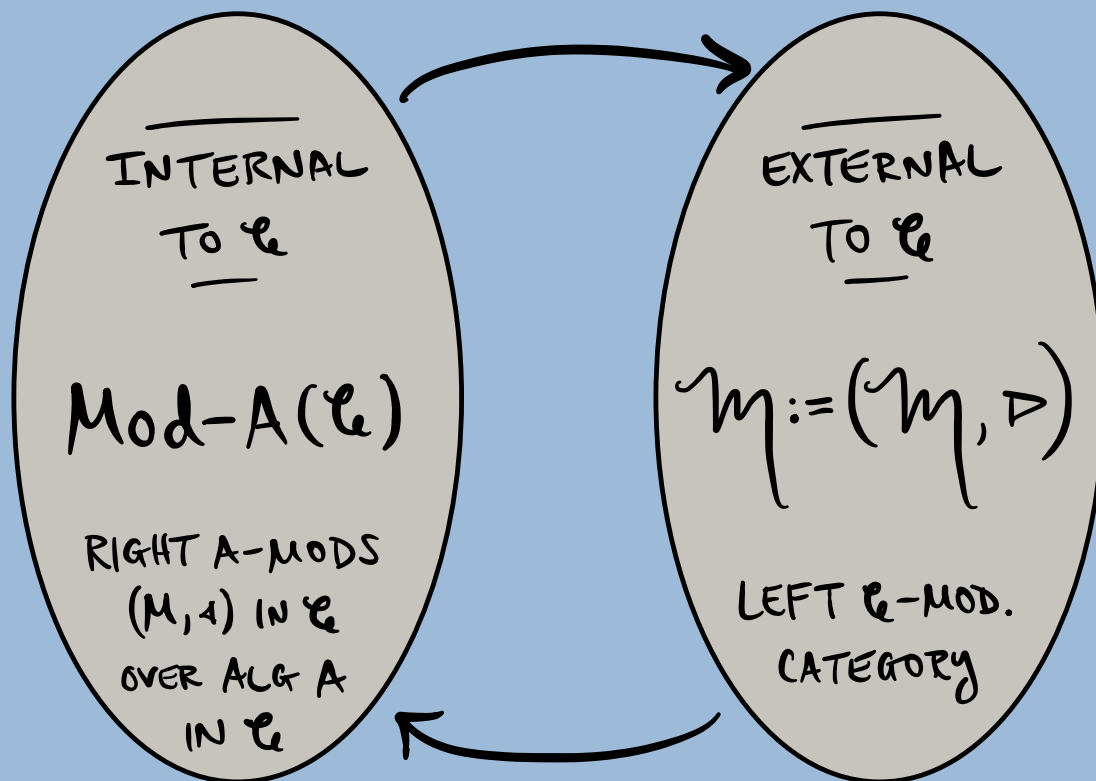
REPRESENTATIONS OF FINITE TENSOR CATEGORIES

$$\text{Mod-}A(\mathcal{C}) \in \mathcal{C}\text{-Mod via } X \triangleright (M, \triangleleft) := (X \otimes M, \triangleleft = \text{id} \otimes \triangleleft)$$



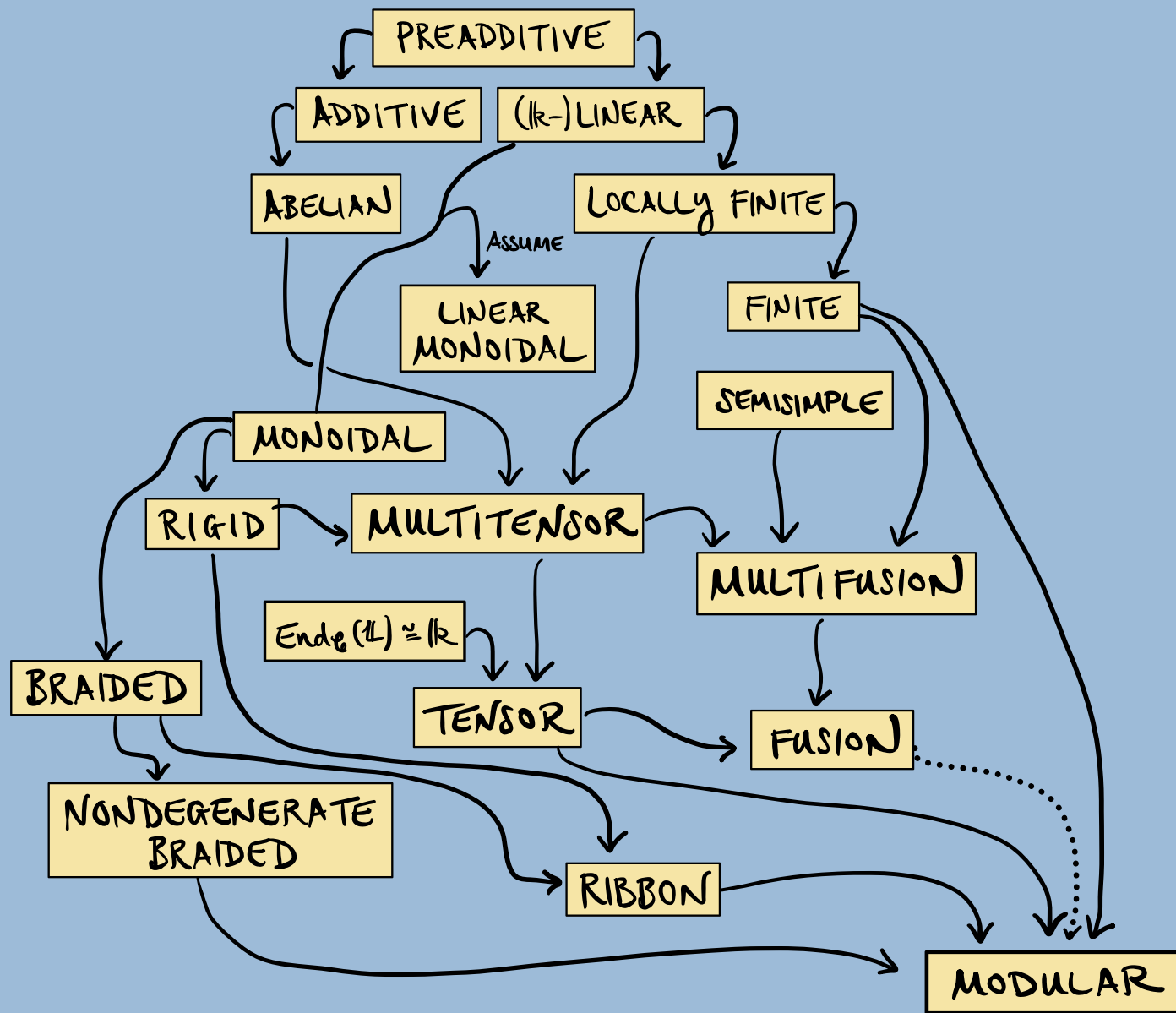
TOWARDS MODULARITY

$\text{Mod-}A(\mathcal{C}) \in \mathcal{C}\text{-Mod}$ via
 $X \triangleright (M, \triangleleft) := (X \otimes M, \triangleleft = \text{id} \otimes \triangleleft)$

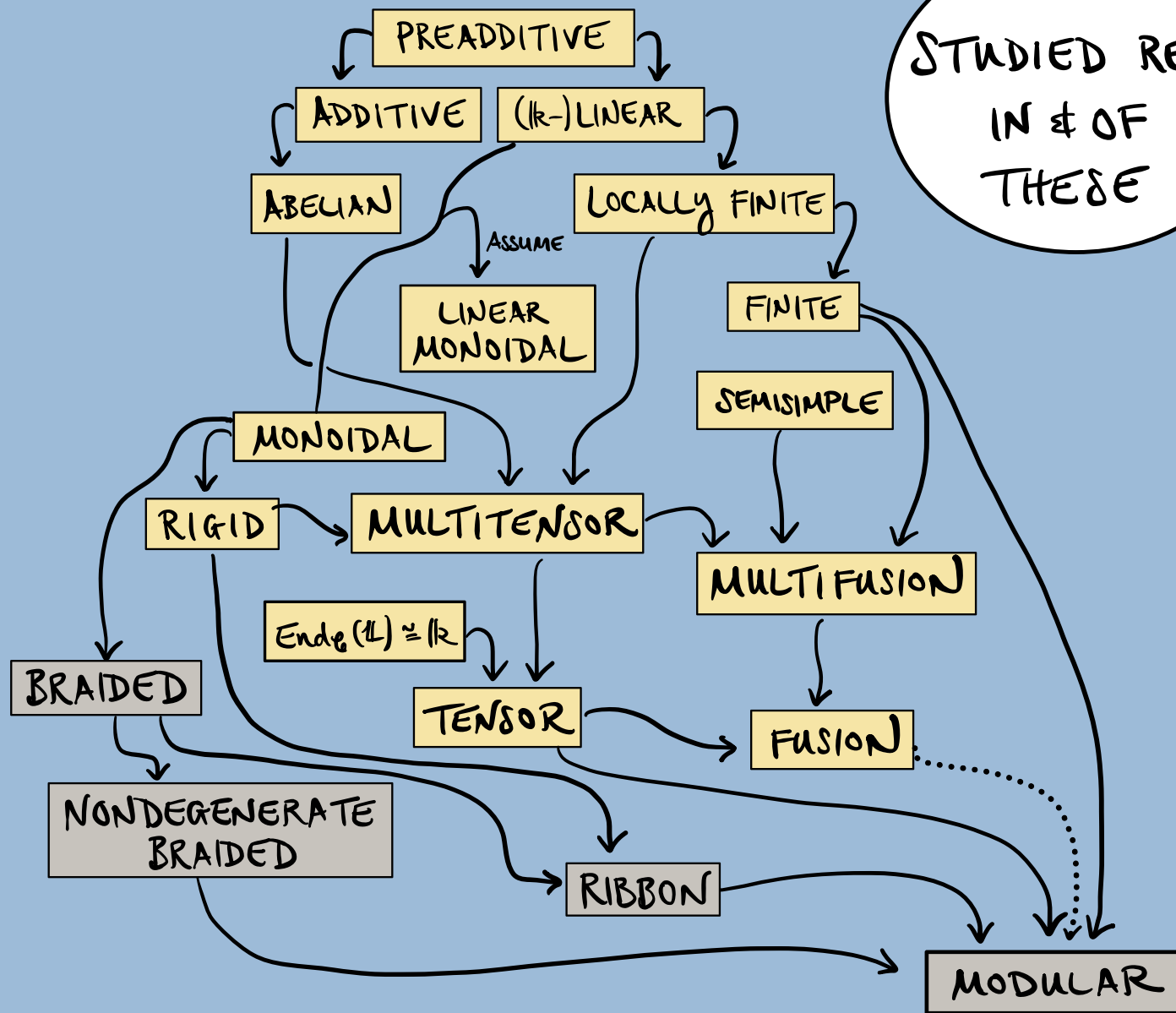


$A = \text{End}(M)$ FOR $M^{\neq 0} \in \mathcal{M}$

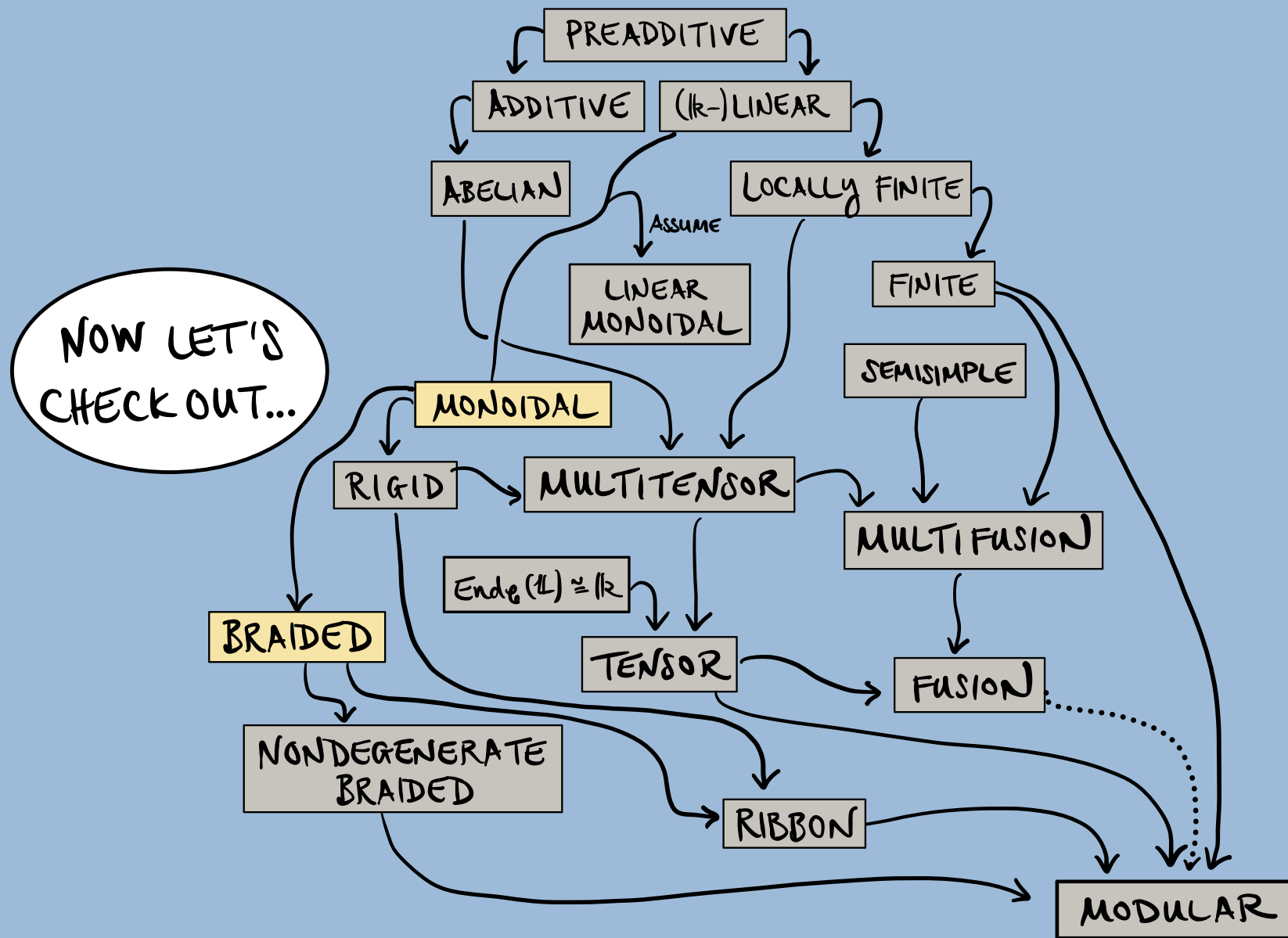
TOWARDS MODULARITY



TOWARDS MODULARITY



TOWARDS MODULARITY



LOCAL MODULES

MONOIDAL



BRAIDED

LOCAL MODULES

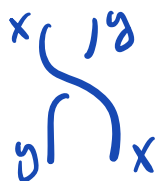
MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$

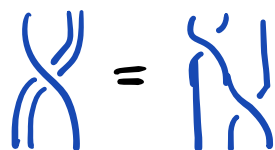
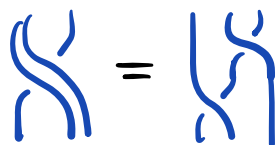
BRAIDED

$\exists$ NAT ISOM

$$X \otimes Y \xrightarrow{c_{X,Y}} Y \otimes X$$



SUCH THAT



LOCAL MODULES

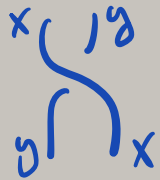
MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1})$

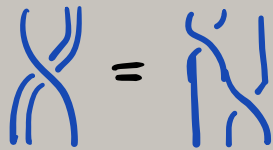
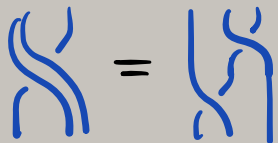
BRAIDED

$\exists$ NAT ISOM

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SUCH THAT



FIX BRAIDED MONOIDAL CATEGORY

$$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, c)$$

E.G. $H\text{-Mod}$

$\uparrow$ QUASI Δ HOPF ... QUANTUM GP ... $\mathbb{R}$ $\uparrow$ Vec

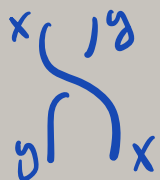
LOCAL MODULES

MONOIDAL

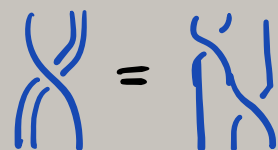
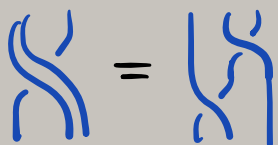
$(\mathcal{C}, \otimes, \mathbb{1})$

BRAIDED

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 $x \otimes y \xrightarrow{c_{x,y}} y \otimes x$



SUCH THAT



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TAKE A "COMMUTATIVE" ALGEBRA IN $\mathcal{C}$:

(A, m, u)

SUCH
THAT

$$m = m \circ c_{A,A}$$

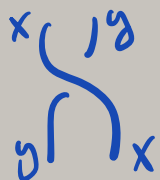
LOCAL MODULES

MONOIDAL

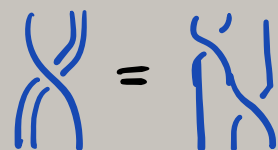
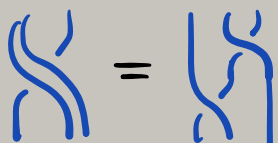
$(\mathcal{C}, \otimes, \mathbb{1})$

BRAIDED

∃ NAT ISOM
 $x \otimes y \xrightarrow{c_{x,y}} y \otimes x$



SUCH THAT



FIX BRAIDED MONOIDAL CATEGORY
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↑ QUASI Δ HOPF ... QUANTUM GP ... $\mathbb{R}$ ↑

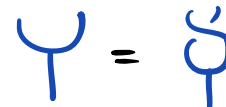
Vec

TAKE A "COMMUTATIVE" ALGEBRA IN $\mathcal{C}$:

(A, m, u)

SUCH
THAT

$M = M \circ c_{A,A}$



A "LOCAL" RIGHT A -MODULE IN $\mathcal{C}$

$= (M, \mu) \in \text{Mod-}A(\mathcal{C}) \dots$

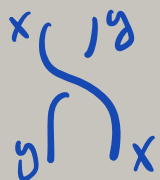
LOCAL MODULES

MONOIDAL

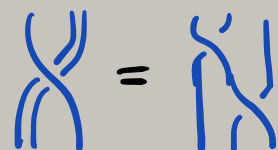
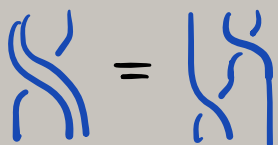
$(\mathcal{C}, \otimes, \mathbb{1})$

BRAIDED

$\exists$ NAT ISOM
 $x \otimes y \xrightarrow{c_{x,y}} y \otimes x$



SUCH THAT



FIX BRAIDED MONOIDAL CATEGORY
 $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, c)$

E.G. H -Mod

$\uparrow$ QUASI Δ HOPF ... QUANTUM GP ... $\mathbb{R}$ $\uparrow$ Vec

TAKE A "COMMUTATIVE" ALGEBRA IN $\mathcal{C}$:

(A, m, u)

SUCH
THAT

$m = m \circ c_{A,A}$
 $\Upsilon = \Upsilon$

A "LOCAL" RIGHT A -MODULE IN $\mathcal{C}$

$= (M, \triangleleft) \in \text{Mod-}A(\mathcal{C})$

SUCH THAT $\triangleleft = \triangleleft \circ c_{A,M} \circ c_{M,A}$

THAT $\Upsilon = \Upsilon$

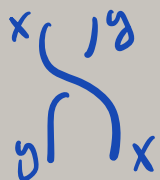
LOCAL MODULES

MONOIDAL

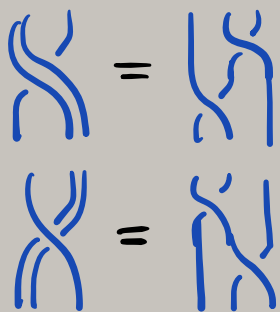
$(\mathcal{C}, \otimes, \mathbb{1})$

BRAIDED

FNAT ISOM
 $x \otimes y \xrightarrow{c_{x,y}} y \otimes x$



SUCH THAT



FIX BRAIDED MONOIDAL CATEGORY
 $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, c)$

E.G. $H\text{-Mod}$

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Vec

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SUCH
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 $\Upsilon = \Upsilon$

$\rightarrow$ CATEGORY:

$\text{LocMod}_{\mathcal{C}}(A)$

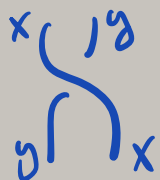
LOCAL MODULES

MONOIDAL

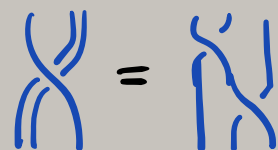
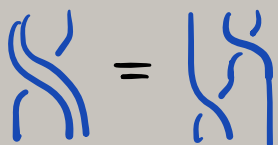
$(\mathcal{C}, \otimes, \mathbb{1})$

BRAIDED

$\exists$ NAT ISOM
 $x \otimes y \xrightarrow{c_{x,y}} y \otimes x$



SUCH THAT



FIX BRAIDED MONOIDAL CATEGORY
 $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, c)$

E.G. $H\text{-Mod}$

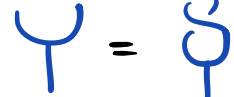
$\uparrow$ QUASI Δ HOPF ... QUANTUM GP ... $\mathbb{R} \uparrow$

Vec

TAKE A "COMMUTATIVE" ALGEBRA IN $\mathcal{C}$:

(A, m, u)


SUCH
THAT

$m = m \circ c_{A,A}$


A "LOCAL" RIGHT A -MODULE IN $\mathcal{C}$

$= (M, \triangleleft)$ $\in \text{Mod-}A(\mathcal{C})$


SUCH
THAT

$\triangleleft = \triangleleft \circ c_{A,M} \circ c_{M,A}$


$\triangleleft := \check{\triangleleft}$
 $\text{LocMod}_{\mathcal{C}}(A)$
 $\cong$
 $\text{Bimod}_{\mathcal{C}}(A)$
 $\otimes \text{CAT}$

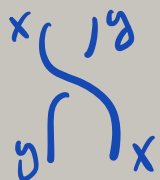
LOCAL MODULES

MONOIDAL

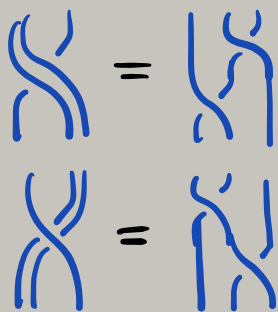
$(\mathcal{C}, \otimes, \mathbb{1})$

BRAIDED

$\exists \text{ NAT ISOM}$
 $x \otimes y \xrightarrow{c_{x,y}} y \otimes x$



SUCH THAT



FIX BRAIDED MONOIDAL CATEGORY
 $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, c)$

E.G. $H\text{-Mod}$

$\uparrow$ QUASI Δ HOPF ... QUANTUM GP ... $\mathbb{R} \uparrow$

Vec

TAKE A "COMMUTATIVE" ALGEBRA IN $\mathcal{C}$:

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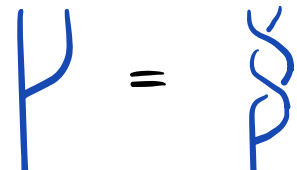

SUCH
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$= (M, \triangleleft)$ $\in \text{Mod-}A(\mathcal{C})$


SUCH
THAT

$\triangleleft = \triangleleft \circ c_{A,M} \circ c_{M,A}$


[PAREIGIS]

$\text{LocMod}_{\mathcal{C}}(A)$

BRAIDED $\otimes$

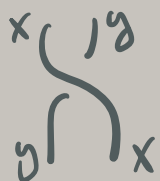
LOCAL MODULES

MONOIDAL

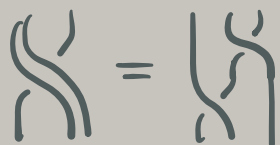
$(\mathcal{C}, \otimes, \mathbb{1})$

BRAIDED

FNAT ISOM
 $x \otimes y \xrightarrow{c_{x,y}} y \otimes x$



SUCH THAT



FIX BRAIDED MONOIDAL CATEGORY
 $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, c)$

Why Care?

Vec
 $\mathbb{R} \uparrow$

(A, m, u)
 $\Uparrow \uparrow$

SUCH
THAT

$m = m \circ c_{A,A}$
 $\Uparrow = \Uparrow$

A "LOCAL" RIGHT A-MODULE IN $\mathcal{C}$

$= (M, \triangleleft) \in \text{Mod-}A(\mathcal{C})$

SUCH
THAT

$\triangleleft = \triangleleft \circ c_{A,M} \circ c_{M,A}$
 $\Uparrow = \Uparrow$

[PAREIGIS]

$\text{LocMod}_{\mathcal{C}}(A)$

BRAIDED $\otimes$

LOCAL MODULES

$$(\mathcal{C}, \otimes, \mathbb{1}, c)$$

BRAIDED $\otimes$
CATEGORY

$$(A, m, u)$$

COMMUTATIVE
ALG IN $\mathcal{C}$

$$\Upsilon = \Upsilon'$$

"LOCAL" RIGHT
A-MOD IN $\mathcal{C}$
 $(M, \triangleleft) \rightarrow$

$$\mu = \mu'$$



$$\text{LocMod}_{\mathcal{C}}(A)$$

Why Care?

LOCAL MODULES

Why Care?

$$(\mathcal{C}, \otimes, \mathbb{1}, c)$$

BRAIDED $\otimes$
CATEGORY

$$(A, m, u)$$

COMMUTATIVE
ALG IN $\mathcal{C}$

$$\Upsilon = \Upsilon$$

"LOCAL" RIGHT
A-MOD IN $\mathcal{C}$
 $(M, \triangleleft) \rightarrow$

$$\mu = \mu$$



$$\text{LocMod}_{\mathcal{C}}(A)$$

$\checkmark$ VOA

LOCAL MODULES

Why Care?

$$(\mathcal{C}, \otimes, \mathbb{1}, c)$$

BRAIDED $\otimes$
CATEGORY

$$(A, m, u)$$

COMMUTATIVE
ALG IN $\mathcal{C}$

$$\Upsilon = \Upsilon$$

"LOCAL" RIGHT
A-MOD IN $\mathcal{C}$
 $(M, \triangleleft) \rightarrow$

$$\mu = \mu$$



$$\text{LocMod}_{\mathcal{C}}(A)$$

V VOA

[HUANG
-LEPOWSKY]

$$\mathcal{C} = \text{Rep}(V)$$

"VERTEX TENSOR CAT."
(BRAIDED $\otimes$)

LOCAL MODULES

Why Care?

$$(\mathcal{C}, \otimes, \mathbb{1}, c)$$

BRAIDED $\otimes$
CATEGORY

$$(A, m, u)$$

COMMUTATIVE
ALG IN $\mathcal{C}$

$$\Upsilon = \S$$

"LOCAL" RIGHT
A-MOD IN $\mathcal{C}$
 $(M, \triangleleft) \rightarrow$

$$\mu = \S$$



$$\text{LocMod}_{\mathcal{C}}(A)$$

$$V \text{ VOA}$$



$$V_{\text{ext}}$$

"EXTENDED VOA"
(COMM. ALG. IN $\mathcal{C}$)

[HUANG
-LEPOWSKY]

$$\mathcal{C} = \text{Rep}(V)$$

"VERTEX TENSOR CAT."
(BRAIDED $\otimes$)

LOCAL MODULES

Why Care?

$$(\mathcal{C}, \otimes, \mathbb{1}, c)$$

BRAIDED $\otimes$
CATEGORY

$$(A, m, u)$$

COMMUTATIVE
ALG IN $\mathcal{C}$

$$\Upsilon = \S$$

"LOCAL" RIGHT
A-MOD IN $\mathcal{C}$
(M, $\triangleleft$) $\rightarrow$

$$\mu = \S$$



$$\text{LocMod}_{\mathcal{C}}(A)$$

V VOA



V^{ext}

"EXTENDED VOA"
(COMM. ALG. IN $\mathcal{C}$)

[HUANG
- LEPOWSKY]

$\mathcal{C} = \text{Rep}(V)$
"VERTEX TENSOR CAT."
(BRAIDED $\otimes$)

[HUANG-KIRILLOV
- LEPOWSKY
1406.3420]



$\text{Rep}(V^{\text{ext}})$
 $\downarrow$
 $\text{LocMod}_{\mathcal{C}}(V^{\text{ext}})$

LOCAL MODULES

Why Care?

$$(\mathcal{C}, \otimes, \mathbb{1}, c)$$

BRAIDED $\otimes$
CATEGORY

$$(A, m, u)$$

COMMUTATIVE
ALG IN $\mathcal{C}$

$$\Upsilon = \S$$

"LOCAL" RIGHT
A-MOD IN $\mathcal{C}$
(M, $\triangleleft$) $\rightarrow$

$$\mu = \S$$



$$\text{LocMod}_{\mathcal{C}}(A)$$

V VOA



V^{ext}

"EXTENDED VOA"
(COMM. ALG. IN $\mathcal{C}$)

[HUANG
- LEPOWSKY]

$\mathcal{C} = \text{Rep}(V)$
"VERTEX TENSOR CAT."
(BRAIDED $\otimes$)

[HUANG-KIRILLOV
- LEPOWSKY
1406.3420]

$\text{Rep}(V^{\text{ext}})$

$\cong$

$\text{LocMod}_{\mathcal{C}}(V^{\text{ext}})$

"EXTENDED
CFT"

E.G.

[FUCHS-SCHWEIGERT,
0011160]

LOCAL MODULES

Why Care?

$$(\mathcal{C}, \otimes, \mathbb{1}, c)$$

BRAIDED $\otimes$
CATEGORY

$$(A, m, u)$$

COMMUTATIVE
ALG IN $\mathcal{C}$

$$\Upsilon = \S$$

"LOCAL" RIGHT
A-MOD IN $\mathcal{C}$
(M, $\triangleleft$) $\rightarrow$

$$\mu = \S$$



$$\text{LocMod}_{\mathcal{C}}(A)$$

$$V \text{ VOA}$$



$$V \text{ ext}$$

"EXTENDED VOA"
(COMM. ALG. IN $\mathcal{C}$)

[HUANG
-LEPOWSKY]

$\mathcal{C} = \text{Rep}(V)$
"VERTEX TENSOR CAT."
(BRAIDED $\otimes$)

[HUANG-KIRILLOV
-LEPOWSKY
1406.3420]

$$\text{Rep}(V \text{ ext})$$

SI

$$\text{LocMod}_{\mathcal{C}}(V \text{ ext})$$



"EXTENDED
CFT"

E.G.

[FUCHS-SCHWEIGERT,
0011160]

MODULARITY
IS USEFUL HERE

E.G. [GANNON, 2301.07287]

[ANGER-CREUTZIG-KANADE
-RUPERT, 1906.07212]

[CREUTZIG-KANADE-MCRAE,
1705.05017]

LOCAL MODULES

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

BRAIDED $\otimes$
CATEGORY

(A, m, u)

COMMUTATIVE
ALG IN $\mathcal{C}$

$$\Upsilon = \Upsilon'$$

"LOCAL" RIGHT
A-MOD IN $\mathcal{C}$
 $(M, \triangleleft) \rightarrow$

$$\mu = \mu'$$



$\text{LocMod}_{\mathcal{C}}(A)$

Why Care?

DAVYDOV'S EXAMPLES (J. ALG 2010)

LOCAL MODULES

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$$\text{LocMod}_{\mathcal{C}}(A)$$

Why Care?

DAVYDOV'S EXAMPLES (J. ALG 2010)

G = FINITE GROUP

$$\mathcal{C} =$$

$$\mathbb{Z}(\text{RG-FdMod})$$

DRINFELD CENTER

LOCAL MODULES

Why Care?

$$(\mathcal{C}, \otimes, \mathbb{1}, c)$$

BRAIDED $\otimes$
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$$\text{LocMod}_{\mathcal{C}}(A)$$

DAVYDOV'S EXAMPLES (J. ALG 2010)

$G = \text{FINITE GROUP}$

$$\mathcal{C} = \underbrace{D(G)\text{-FdMod}}_{\text{DRINFELD DOUBLE}} \stackrel{\text{BR.}\otimes}{\simeq} \mathbb{Z}(\underbrace{KG\text{-FdMod}}_{\text{DRINFELD CENTER}})$$

LOCAL MODULES

Why Care?

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$$\mathcal{C} = \underbrace{D(G)\text{-FdMod}}_{\text{DRINFELD DOUBLE}} \stackrel{\text{BR.}\otimes}{\simeq} \underbrace{\mathbb{Z}(kG\text{-FdMod})}_{\text{DRINFELD CENTER}}$$

$$\uparrow$$

$$= kG^* \otimes kG \text{ AS COALGEBRAS}$$

BUT MULTIPLICATION IS A
WEE BIT COMPLICATED

LOCAL MODULES

Why Care?

$$(\mathcal{C}, \otimes, \mathbb{1}, c)$$

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BUT MULTIPLICATION IS A
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R-MATRIX IS NICE:

$$R = \sum_{g \in G} \varepsilon_{kG} \otimes g \otimes g^* \otimes e$$

LOCAL MODULES

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$\rightarrow D(G)$ IS A MODULAR HOPF ALG.

LOCAL MODULES

Why Care?

$$(\mathcal{C}, \otimes, \mathbb{1}, c)$$

BRAIDED $\otimes$
CATEGORY

$$(A, m, u)$$

COMMUTATIVE
ALG IN $\mathcal{C}$

$$\Upsilon = \S$$

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$$\text{LocMod}_{\mathcal{C}}(A)$$

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FOR $H \leq G$:

$A = k(G/H)$ COMMUTATIVE ALG. IN $\mathcal{C}$

$\rightarrow D(G)$ IS A MODULAR HOPF ALG.

LOCAL MODULES

Why Care?

$$(\mathcal{C}, \otimes, \mathbb{1}, c)$$

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"LOCAL" RIGHT
A-MOD IN $\mathcal{C}$
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$$\text{LocMod}_{\mathcal{C}}(A)$$

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FOR $H \leq G$:

$A = k(G/H)$ COMMUTATIVE ALG. IN $\mathcal{C}$

$$\rightarrow \text{LocMod}_{\mathcal{C}}(A) \stackrel{\text{BR.}\otimes}{\simeq} D(H)\text{-FdMod}$$

LOCAL MODULES

Why Care?

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FOR $H \leq G$:

$A = k(G/H)$ COMMUTATIVE ALG. IN $\mathcal{C}$

$$\rightarrow \text{LocMod}_{\mathcal{C}}(A) \stackrel{\text{BR.}\otimes}{\simeq} D(H)\text{-FdMod}$$

Ex. $\text{LocMod}_{\mathcal{C}}(kG) \simeq \text{FdVec}$ FOR $H = \langle e \rangle$

LOCAL MODULES

Why Care?

$$(\mathcal{C}, \otimes, \mathbb{1}, c)$$

BRAIDED $\otimes$
CATEGORY

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$$\text{LocMod}_{\mathcal{C}}(A)$$

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FOR $H \leq G$:

$A = k(G/H)$ COMMUTATIVE ALG. IN $\mathcal{C}$

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Ex. $\text{LocMod}_{\mathcal{C}}(kG) \simeq \text{FdVec}$ FOR $H = \langle e \rangle$

$\text{LocMod}_{\mathcal{C}}(k) \simeq \mathcal{C}$ FOR $H = G$

LOCAL MODULES

Why Care?

$\mathcal{C}$ MTC

DAVYDOV'S EXAMPLES (J. ALG 2010)

G = FINITE GROUP

$$\mathcal{C} = \underbrace{D(G)\text{-FdMod}}_{\text{DRINFELD DOUBLE}} \stackrel{\text{BR.}\otimes}{\simeq} \underbrace{\mathbb{Z}(kG\text{-FdMod})}_{\text{DRINFELD CENTER}}$$

FOR $H \leq G$:

$A = k(G/H)$ COMMUTATIVE ALG. IN $\mathcal{C}$

$$\rightarrow \text{LocMod}_{\mathcal{C}}(A) \stackrel{\text{BR.}\otimes}{\simeq} D(H)\text{-FdMod}$$

Ex. $\text{LocMod}_{\mathcal{C}}(kG) \simeq \text{FdVec}$ FOR $H = \langle e \rangle$

$\text{LocMod}_{\mathcal{C}}(k) \simeq \mathcal{C}$ FOR $H = G$

LOCAL MODULES

Why Care?

$\mathbb{C}$ MTC

$\neq$

NICE
COM. ALG
A IN $\mathbb{C}$

DAVYDOV'S EXAMPLES (J. ALG 2010)

G = FINITE GROUP

$$\mathbb{C} = \underbrace{D(G)}_{\text{DRINFELD DOUBLE}} - \text{FdMod} \stackrel{\text{BR.} \otimes}{\simeq} \mathbb{Z}(\underbrace{1kG - \text{FdMod}}_{\text{DRINFELD CENTER}})$$

FOR $H \leq G$:

$$A = 1k(G/H) \text{ COMMUTATIVE ALG. IN } \mathbb{C}$$

$$\rightarrow \text{LocMod}_{\mathbb{C}}(A) \stackrel{\text{BR.} \otimes}{\simeq} D(H) - \text{FdMod}$$

$$\text{Ex. } \text{LocMod}_{\mathbb{C}}(1kG) \simeq \text{FdVec} \text{ FOR } H = \langle e \rangle$$

$$\text{LocMod}_{\mathbb{C}}(1k) \simeq \mathbb{C} \text{ FOR } H = G$$

LOCAL MODULES

Why Care?

$\mathbb{C}$ MTC

$\nLeftarrow$

NICE
COM. ALG
 A IN $\mathbb{C}$

$\downarrow$

LOCAL MOD.
CONSTRUCTION

$\downarrow$

MTC

DAVYDOV'S EXAMPLES (J. ALG 2010)

G = FINITE GROUP

$$\mathbb{C} = \underbrace{D(G)\text{-FdMod}}_{\text{DRINFELD DOUBLE}} \stackrel{\text{BR.}\otimes}{\simeq} \underbrace{\mathbb{Z}(kG\text{-FdMod})}_{\text{DRINFELD CENTER}}$$

FOR $H \leq G$:

$A = k(G/H)$ COMMUTATIVE ALG. IN $\mathbb{C}$

$$\rightarrow \text{LocMod}_{\mathbb{C}}(A) \stackrel{\text{BR.}\otimes}{\simeq} D(H)\text{-FdMod}$$

Ex. $\text{LocMod}_{\mathbb{C}}(kG) \simeq \text{FdVec}$ FOR $H = \langle e \rangle$

$\text{LocMod}_{\mathbb{C}}(k) \simeq \mathbb{C}$ FOR $H = G$

LOCAL MODULES

ROAD TO MODULARITY

$(\mathcal{C}, \otimes, \mathbb{1}, \text{DUALS}, c, \theta)$

MODULAR
TENSOR
CATEGORY

(A, m, u)

COM. ALG. IN $\mathcal{C}$

$$\Upsilon = \bar{\Upsilon}$$

"LOCAL" RIGHT

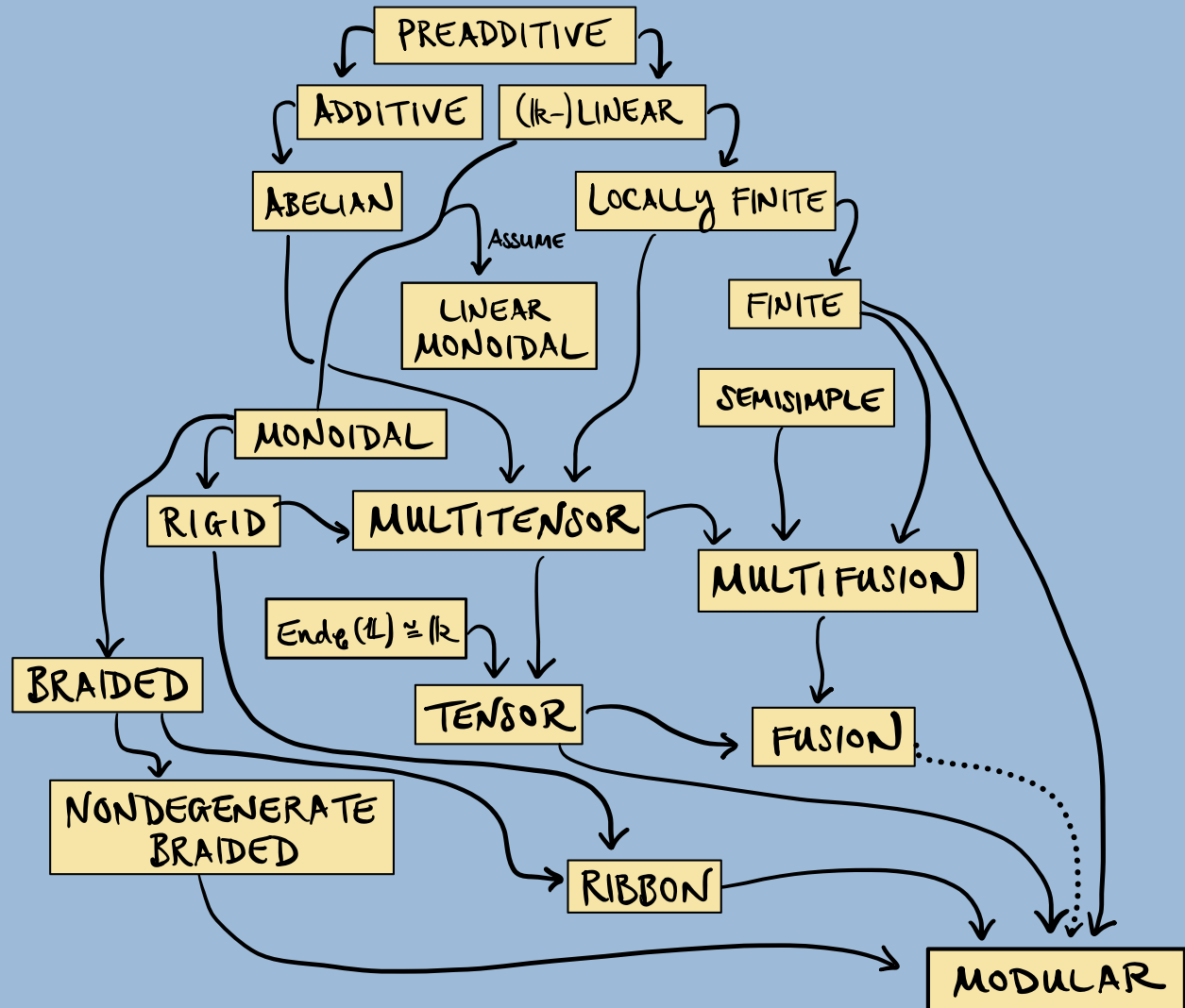
$A\text{-MOD}$ IN $\mathcal{C}$

$(M, \triangleleft) \rightarrow$

$$\mu = \bar{\mu}$$



$\text{LocMod}_{\mathcal{C}}(A)$



LOCAL MODULES

ROAD TO MODULARITY

$(\mathcal{C}, \otimes, \mathbb{1}, \text{DUALS}, c, \theta)$

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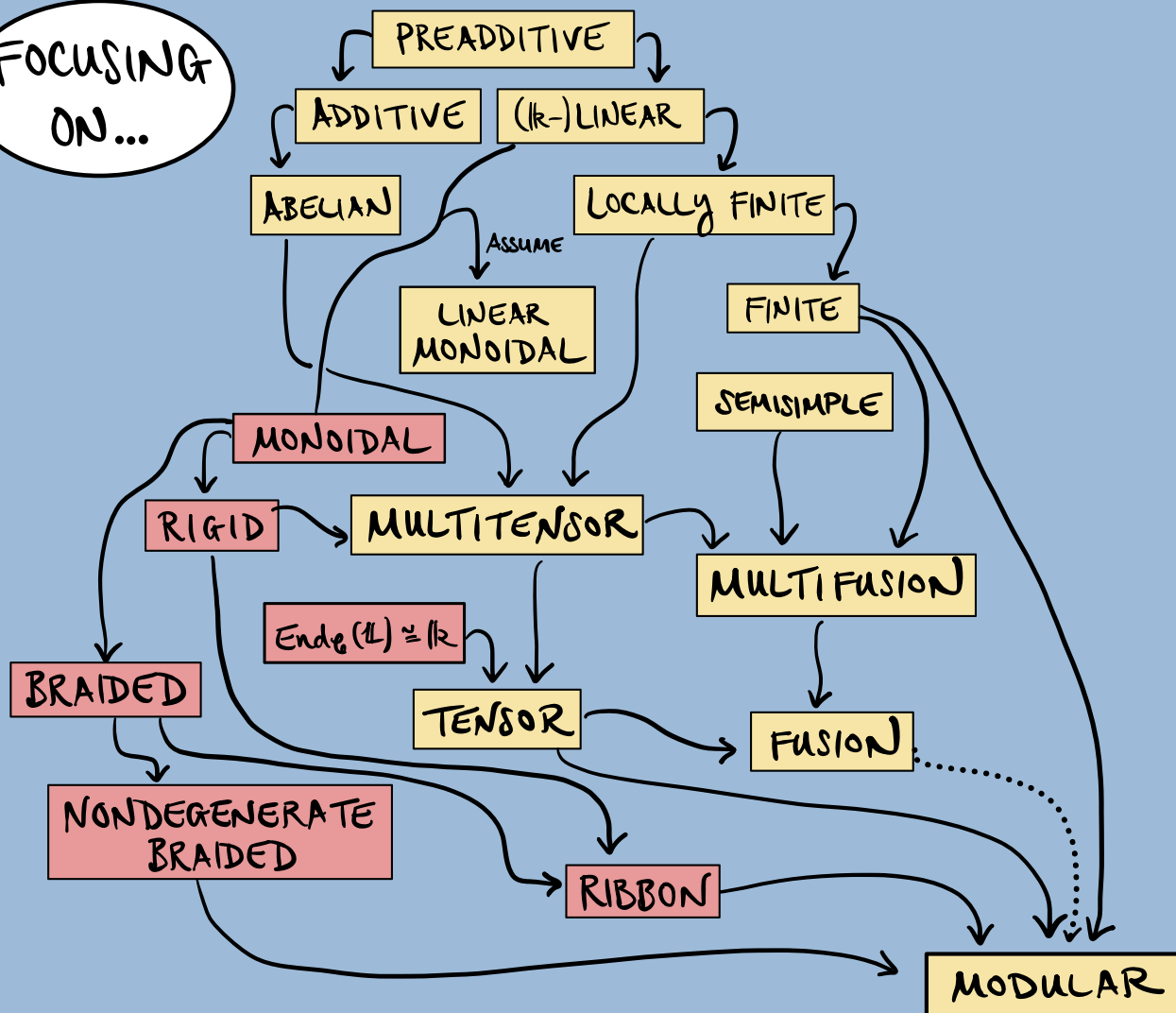
$A\text{-MOD}$ IN $\mathcal{C}$

$(M, \triangleleft) \rightarrow$

$$\mu = \sum p$$

$\text{LocMod}_{\mathcal{C}}(A)$

FOCUSING
ON...



LOCAL MODULES

ROAD TO MODULARITY

$(\mathcal{C}, \otimes, \mathbb{1}, \text{DUALS}, c, \theta)$

MODULAR
TENSOR
CATEGORY

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COM. ALG. IN $\mathcal{C}$

$$\Upsilon = \check{\Upsilon}$$

"LOCAL" RIGHT

A -MOD IN $\mathcal{C}$

$(M, \triangleleft) \rightarrow$

$$\mu = \sum_P \downarrow$$

$\text{LocMod}_{\mathcal{C}}(A)$

$\text{LocMod}_{\mathcal{C}}(A)$ IS :

BRAIDED
MONOIDAL

NONDEGENERATE

$$\text{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{K}$$

RIGID

RIBBON

WHEN A IS :

LOCAL MODULES

ROAD TO MODULARITY

$(\mathcal{C}, \otimes, \mathbb{1}, \text{DUALS}, c, \theta)$

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TENSOR
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WHEN A IS :

JUST
COMMUTATIVE
[PAREIGIS]

LOCAL MODULES

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[PAREIGIS]

—

LOCAL MODULES

ROAD TO MODULARITY

$(\mathcal{C}, \otimes, \mathbb{1}, \text{DUALS}, c, \theta)$

MODULAR
TENSOR
CATEGORY

(A, m, u)

COM. ALG. IN $\mathcal{C}$

$$\Upsilon = \check{\Upsilon}$$

"LOCAL" RIGHT
A-MOD IN $\mathcal{C}$

$(M, \triangleleft) \rightarrow$

$$\mu = \sum_P \downarrow$$

$\text{LocMod}_{\mathcal{C}}(A)$

$\text{LocMod}_{\mathcal{C}}(A)$ IS :

BRAIDED
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NONDEGENERATE

$$\text{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{1}_2$$

RIGID

RIBBON

WHEN A IS :

JUST
COMMUTATIVE
[PAREIGIS]

—

$$\text{LocMod}_{\mathcal{C}}(A) \boxtimes (\mathcal{C}, c^{-1})$$

LOCAL MODULES

ROAD TO MODULARITY

$(\mathcal{C}, \otimes, \mathbb{1}, \text{DUALS}, c, \theta)$

MODULAR
TENSOR
CATEGORY

(A, m, u)

COM. ALG. IN $\mathcal{C}$

$$\Upsilon = \tilde{\Upsilon}$$

"LOCAL" RIGHT

$A\text{-MOD}$ IN $\mathcal{C}$

$(M, \triangleleft) \rightarrow$

$$\mu = \sum \rho$$



$\text{LocMod}_{\mathcal{C}}(A)$

$\text{LocMod}_{\mathcal{C}}(A)$ IS :

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MONOIDAL

NONDEGENERATE

$$\text{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{1}_2$$

RIGID

RIBBON

WHEN A IS :

JUST
COMMUTATIVE
[PAREIGIS]

—

$$\text{LocMod}_{\mathcal{C}}(A) \boxtimes (\mathcal{C}, c^{-1})$$

SI

$$\text{LocMod}_{\mathcal{Z}(\mathcal{C})}(A)$$

SI [SCHAUENBURG]

$$\mathcal{Z}(\text{Mod-}A(\mathcal{C}))$$

LOCAL MODULES

ROAD TO MODULARITY

$(\mathcal{C}, \otimes, \mathbb{1}, \text{DUALS}, c, \theta)$

MODULAR
TENSOR
CATEGORY

(A, m, u)

COM. ALG. IN $\mathcal{C}$

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"LOCAL" RIGHT

$A\text{-MOD}$ IN $\mathcal{C}$

$(M, \triangleleft) \rightarrow$

$$\mu = \sum \bar{\mu}$$



$\text{LocMod}_{\mathcal{C}}(A)$

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MONOIDAL

NONDEGENERATE

$$\text{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{1}_2$$

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RIBBON

WHEN A IS:

JUST
COMMUTATIVE
[PAREIGIS]

—

$$\text{LocMod}_{\mathcal{C}}(A) \boxtimes (\mathcal{C}, c^{-1})$$

$\downarrow$

$$\text{LocMod}_{\mathbb{Z}(\mathcal{C})}(A)$$

$\downarrow$

$$\mathbb{Z}(\text{Mod-}A(\mathcal{C}))$$

NONDEG

LOCAL MODULES

ROAD TO MODULARITY

$(\mathcal{C}, \otimes, \mathbb{1}, \text{DUALS}, c, \theta)$

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TENSOR
CATEGORY

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A-MOD IN $\mathcal{C}$

$(M, \triangleleft) \rightarrow$

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$\text{LocMod}_{\mathcal{C}}(A)$

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$$\text{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{1}_2$$

RIGID

RIBBON

WHEN A IS:

JUST
COMMUTATIVE
[PAREIGIS]

$$\overbrace{\text{LocMod}_{\mathcal{C}}(A) \boxtimes (\mathcal{C}, c^{-1})}^{\text{NONDEG SI}}$$

$$\text{LocMod}_{\mathbb{Z}(\mathcal{C})}(A)$$

SI

$$\mathbb{Z}(\text{Mod-A}(\mathcal{C}))$$

NONDEG

LOCAL MODULES

ROAD TO MODULARITY

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$$\text{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{K}$$

RIGID

RIBBON

WHEN A IS:

JUST
COMMUTATIVE
[PAREIGIS]

—

"CONNECTED" (AKA "HAPLOID")

$$\dim_{\mathbb{K}} \text{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$$

LOCAL MODULES

ROAD TO MODULARITY

$(\mathcal{C}, \otimes, \mathbb{1}, \text{DUALS}, c, \theta)$

MODULAR
TENSOR
CATEGORY

(A, m, u)

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WHEN A IS:

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[PAREIGIS]

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"CONNECTED" (AKA "HAPLOID")

"EXACT"

LOCAL MODULES

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RIGID

RIBBON

WHEN A IS:

JUST
COMMUTATIVE
[PAREIGIS]

FOR ANY $\mathcal{M} \in \mathcal{C}\text{-Mod}$,
ANY $\mathcal{C}\text{-MOD. FUNCTOR}$
 $\text{Mod-A}(\mathcal{C}) \rightarrow \mathcal{M}$
IS EXACT

"EXACT"

LOCAL MODULES

ROAD TO MODULARITY

$(\mathcal{C}, \otimes, \mathbb{1}, \text{DUALS}, c, \theta)$

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[PAREIGIS]

—

"CONNECTED" (AKA "HAPLOID")

"EXACT"

"SPECIAL FROBENIUS"

LOCAL MODULES

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MONOIDAL

NONDEGENERATE

$$\text{End}_{\mathcal{C}}(\mathbb{1}) \cong \mathbb{K}$$

RIGID

RIBBON

WHEN A IS:

JUST
COMMUTATIVE
[PABEUGIS]

$(A, m, u, \Delta, \varepsilon)$

$$\Upsilon \quad \Upsilon \quad \Upsilon \quad \Upsilon$$

SUCH THAT

$$\Upsilon = \Upsilon = \Upsilon$$

$$\Upsilon = | \quad \Upsilon = \vdots$$

"SPECIAL FROBENIUS"

LOCAL MODULES

ROAD TO MODULARITY

$(\mathcal{C}, \otimes, \mathbb{1}, \text{DUALS}, c, \theta)$

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TENSOR
CATEGORY

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$\text{LocMod}_{\mathcal{C}}(A)$ IS :

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RIGID

RIBBON

WHEN A IS :

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[PAREIGIS]

—

"CONNECTED" (AKA "HAPLOID")

"EXACT"

"SPECIAL FROBENIUS"

LOCAL MODULES

ROAD TO MODULARITY

$A = k(G/H)$ IN $D(G)\text{-FdMod}$
TICKS THESE BOXES

(A, m, u)
COM. ALG. IN $\mathbb{C}$
 $\Upsilon = \tilde{\Upsilon}$

"LOCAL" RIGHT
A-MOD IN $\mathbb{C}$
 $(M, \triangleleft) \rightarrow$

$\mu = \sum p$
 $\downarrow$

$\text{LocMod}_{\mathbb{C}}(A)$

NONDEGENERATE

$\text{End}_{\mathbb{C}}(\mathbb{L}) \cong k$

RIGID

RIBBON

WHEN A IS

JUST
COMMUTATIVE
[PAREIGIS]

—

"CONNECTED"

"EXACT"

"SPECIAL FROBENIUS"

LOCAL MODULES

ROAD TO MODULARITY

$A = k(G/H)$ IN $D(G)\text{-FdMod}$
TICKS THESE BOXES

(A, M, u)
COM. ALG. IN $\mathbb{Q}$
()

SO DO MANY GENERALIZATIONS
[LAUGWITZ-W, 2023]

"LOCAL" RIGHT
 $A\text{-MOD}$ IN $\mathbb{Q}$
 $(M, \triangleleft) \rightarrow$

$$\mu = \sum_P$$

↓

$\text{LocMod}_{\mathbb{Q}}(A)$

ALGEBRAIC

$$\text{End}_{\mathbb{Q}}(U) \cong \mathbb{R}$$

RIGID

RIBBON

WHEN A IS ...

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COMMUTATIVE
[PAREIGIS]

—

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COM. ALG. IN $\mathbb{K}$
()

ALGEBRAIC

SO DO MANY GENERALIZATIONS
[LAUGWITZ-W, 2023]

"LOCAL" RIGHT
 $A\text{-MOD}$ IN $\mathbb{K}$
 $(M, \triangleleft) \rightarrow$

$\text{End}_{\mathbb{K}}(U) \cong \mathbb{K}$

SO ARE KEY VOA CONSTRUCTIONS
[sy, 2408.06314] [CMSy, 2409.14618]

$\text{LocMod}_{\mathbb{K}}(A)$

RIBBON

REFERENCES

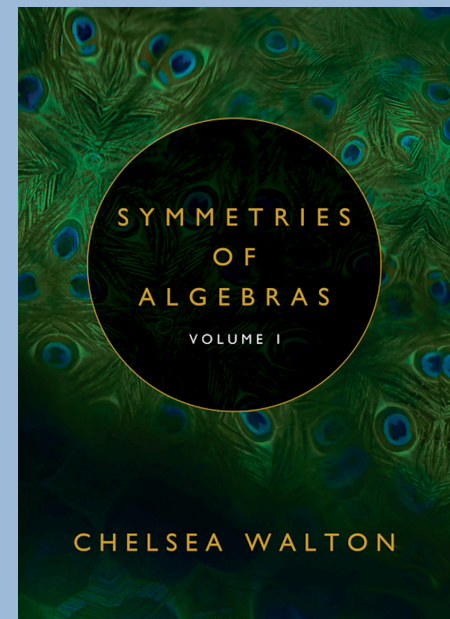
ALGEBRAS & MODULES IN MONOIDAL CATEGORIES —
PAREIGIS' "NONADDITIVE RING AND MODULE THEORY, I, II, III" (1977-78)

MODULE CATEGORIES —

- EGNO'S "TENSOR CATEGORIES" (2015)
- OSTRIK'S "MODULE CATEGORIES, WEAK HOPF ALGS. AND MODULAR INVARIANTS" (2003)
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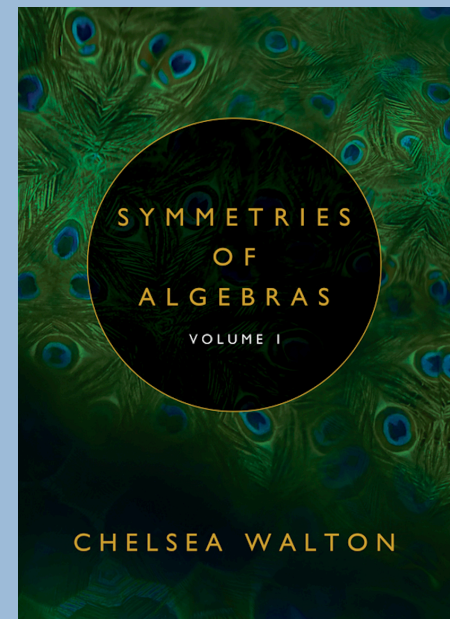
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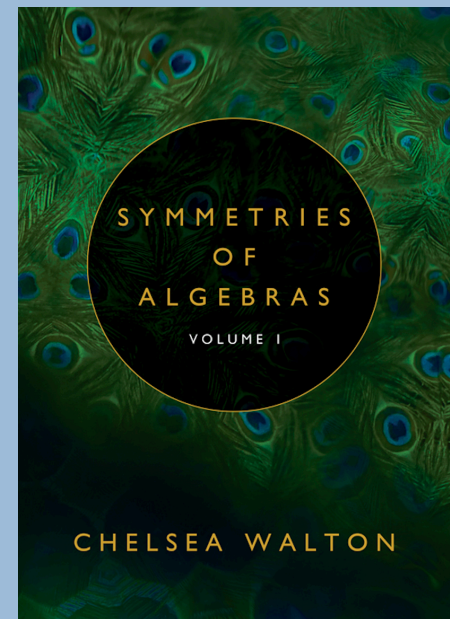
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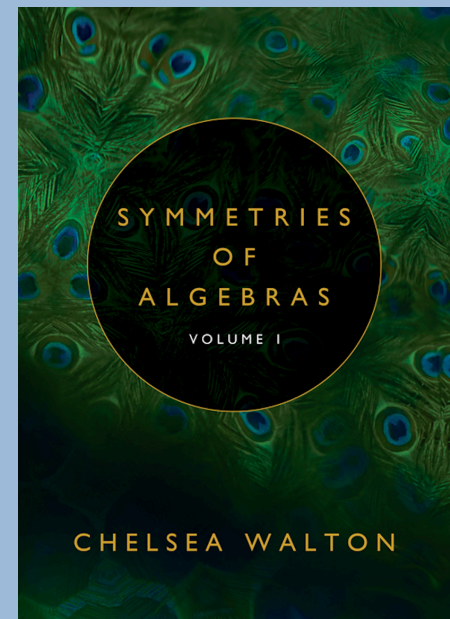
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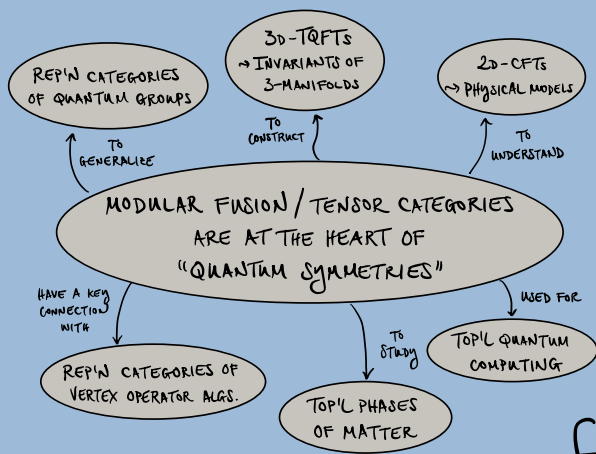
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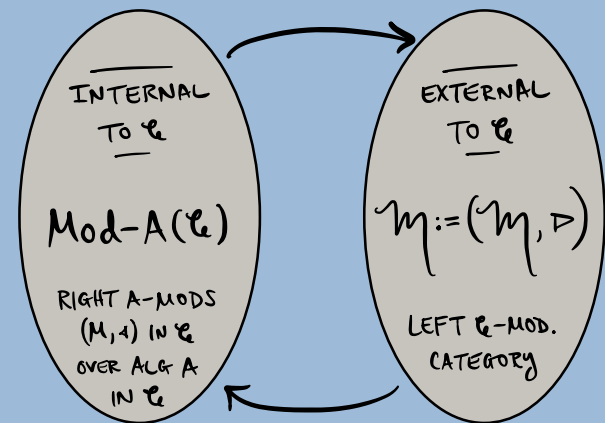
RECONSTRUCTION THEOREM:

"CAT. OF PAIRS" $\simeq$ "CAT. OF STRUCTURES/ $\mathbb{R}$ "

$\{(\mathcal{C}, \mathcal{U})\} \simeq \text{Struc}$

$(\mathcal{C}, \mathcal{U}) \longmapsto (\text{Co})\text{End}_{\mathcal{C}}(\mathcal{U})$

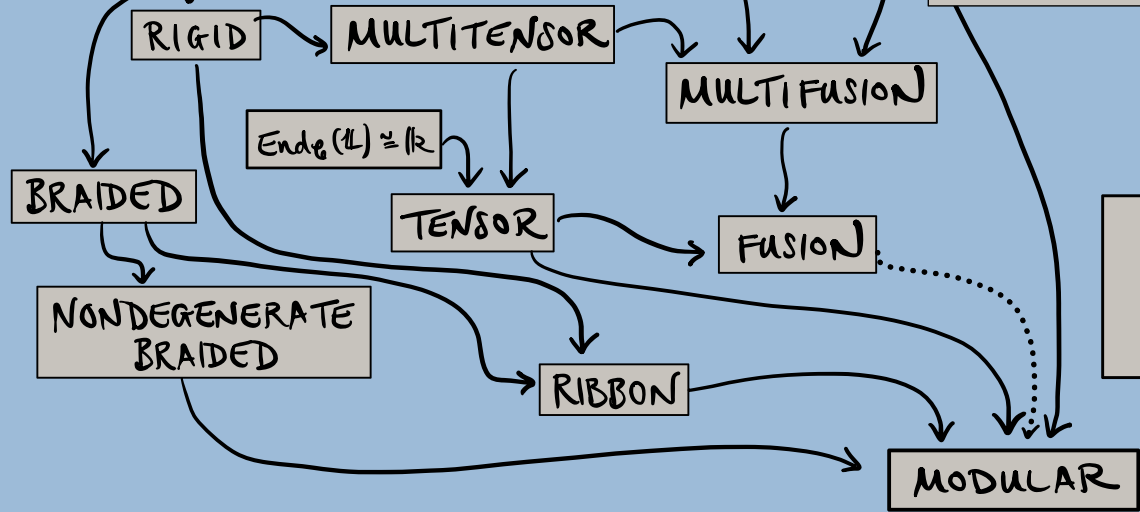
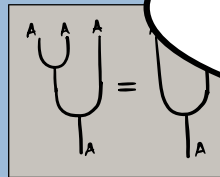
$((\text{Co})\text{Mod}/\mathcal{S}, \text{Frg}) \longleftarrow \mathcal{S}$



PREADDITIVE

ADDITIVE $(\mathbb{R})$ -LINEAR

Thanks for listening!



Rep(V)

"TENSOR CAT." (BRAIDED $\otimes$)

EXTENDED VOA" (COMM. ALG. IN $\mathcal{C}$) $\rightarrow$ Rep(Vext) s.t. Loc Mod $_{\mathcal{C}}(Vext)$

