GAP XV - PSU
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Poisson Geometry of PI 3-dim'l Sklyanin algebras

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Goal: Study the rep thy of certain noncom. algebras that are module-finite over their center

... with Poisson geometry / $k = \bar{k}$, $\operatorname{ch} k = 0$

Dichotomy - study the rep of noncom algebras A

module-infinite over $Z(A)$
$\Downarrow$
typically the ∞-dim'l irred. reps are the most interesting
$\Downarrow$
writing them down precisely is nearly impossible as there's a zoo of them. To get at them, use some pretty sophisticated tools as discussed in Pavel's, Travis' talks

module-finite over $Z(A)$
$\Downarrow$
irred reps have dim bounded by an invariant of A, called the PI degree. In fact, most irreducible reps of A have dimension equal to the PI degree A.

Let's be more precise. A review of standard PI theory -

* If A is module-finite / $Z(A)$, then A satisfies a polyl identity PI
($\exists$ monic multilinear polyl $f \in \mathbb{Z}\langle t\rangle \ni f(a_1, \dots, a_n) = 0 \ \forall a_i \in A$)

$\rightsquigarrow$ PIdeg $(A) \overset{\text{def}}{=} \frac{1}{2}$ (min'l deg of such f) when A is prime ($aAb \neq 0$, $\forall a, b \neq 0$)

"measure of a ring's noncommutativity"

<u>Examples of prime PI algs</u>

$\underline{A}$	$\underline{Z(A)}$	$\underline{\text{PI deg } A}$
$k[x]$	$k[x]$	$1.$
$\mathrm{Mat}_n(k)$	k	n
$k\langle x,y\rangle/(xy+yx)$	$k[x^2,y^2]$	$2.$

* quantized env. algs $U_\varepsilon(\mathfrak{g})$, at a root of unity
* quantized fcn algs $\mathcal{O}_\varepsilon(G)$, " " " " "

Tacking on mild hypothesis on A, can study rep thy A via the affine geometry $Z(A)$.

Take A = prime, PI ($\Leftarrow$ mod. finite$/Z(A)$), fin. gen., Noetherian
<u>Theorem</u>: irreps of A are fin-dim'l & $\dim(\text{irrep of } A) \leq \text{PI deg } A \overset{!!}{=} n$

[loads of names
Artin,
Brown,
Goodearl
DeConcini
Procesi
small
?]

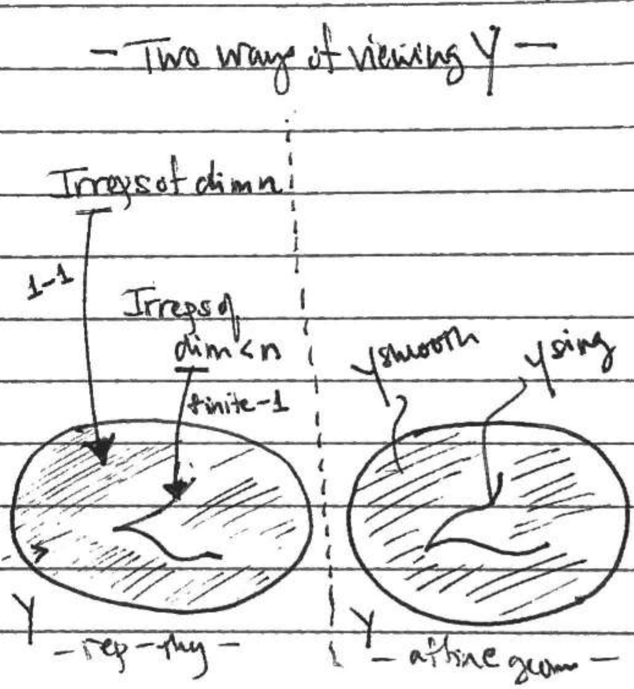
Have map: $\underset{\text{(equiv. classes)}}{\mathrm{Irred}(A)}$

Ψ well defined, surjective

$$\begin{array}{c} \psi: A \twoheadrightarrow \mathrm{Mat}_d(k) \\ \downarrow \text{"central char"} \\ (\ker\psi)\cap Z =: m_\psi \end{array}$$

$\mathrm{maxSpec}(Z) =: Y$

— Two ways of viewing Y —



$\mathcal{A}_A$ = <u>Azumaya locus</u>, open & dense in Y

{convenient when pictures coincide!}

* $\mathcal{X}_A \subseteq Y^{smooth}$; though useful, it's tough to show $=$

* $=$ shown for many algebras $U_\varepsilon(\mathfrak{g})$, $\mathcal{O}_\varepsilon(G)$
  & others that have a nice PBW basis

Here we work with an algebra that's not so "easy" to control algebraically: 3-dim'l Sklyanin algebras S
(There is recent work of Iyudu-Shkarin on PBW basis of Sklyanin algebras, but combinatorially it's complicated)

$$S = S(a,b,c) = k\langle \underset{\deg 1}{x,y,z} \rangle \Big/ \begin{pmatrix} axy + byx + cz^2 \\ ayz + bzy + cx^2 \\ azx + axz + cy^2 \end{pmatrix}$$

$[a:b:c] \in \mathbb{P}^2_k$ omitting some points

* arose in Artin-Schelter's classification of noncom. graded alg's that behave (homologically & ring-theor'ly) like $k[x,y,z]$: "AS regular algebras of dim 3"
    * gldim 3, Gorenstein condition.
    * prime, Noetherian
    * Hilbert series $\frac{1}{(1-t)^3}$

[Artin-Tate -van den Bergh]

* difficult to understand algebraically, so projective algebro-geometric data was assigned to (& actually can be used to recover) $S = S(a,b,c)$

• E_{abc} elliptic curve $\mathbb{V}\left((abc)(v_0^3+v_1^3+v_2^3) - (a^3+b^3+c^3)v_0v_1v_2\right) \subseteq \mathbb{P}^2$

• $\sigma_{abc} \in \mathrm{Aut}(E)$ given by translation a point $[a:b:c]$ when $0 = [1:-1:0]$ (origin)

* higher dim'l analogues of these "elliptic algebras" were discussed in Travis' talk (even glaim)

How to use (E,σ) to study S?

* Get $S \twoheadrightarrow B(E, \overbrace{\mathcal{O}_{\mathbb{P}^2}(1)|_E}^{\mathcal{L}}, \sigma)$
geometric ring: "twisted hom. coordinate ring"
$\bigoplus_{i \geq 0} H^0(E, \mathcal{L}_{\sigma^i})$
twisted sheaf

with kernel $= \langle g \rangle$
reg. central homog. degree 3 elt.

$\rightarrow$ nice properties of B (Noetherian, domain) established with geometric techniques lift up to S.

* S and B are module finite over their centers
$\Leftrightarrow |\sigma| < \infty$
In this case, PIdeg S = PIdeg $B = |\sigma| =: n$.

Fix $|\sigma| =: n < \infty$. Goal Study rep thy of S via Poisson geometry of $Z(S)$.

[Walton] * If $(3,n)=1$, then every non-trivial irrep has dim n
* If $3 | n$, then every non-trivial irrep has $\frac{n}{3} \leq \dim \leq n$

Conjecture [WWY] $\dim = n/3$ or n (Address by Delaet 1707.03813)

[Smith] * $Z(S) = k[\underset{\deg n}{z_1, z_2, z_3}, \underset{\deg 3}{g}] / \underset{\deg 3n}{(F)}$ $Y = \mathrm{maxSpec}(Z) = \mathbb{V}(F) \subseteq \mathbb{A}^4$

Technology leading up to conjecture & more —
— We give S the structure of a Poisson Z-order

[Brown -Gordon]

* A = algebra module finite over $Z(A) =: Z$
  A = Poisson Z-order if $\exists$ k-linear map $\partial : Z \to \mathrm{Der}_k(A/Z)$ [space of k-deriv of A fixing Z]
  that induces on Z the structure of a Poisson alg.
  via $\{z, z'\} =: \partial_z(z')$ $\forall z, z' \in Z$.

* How to build these? Specialization
  Take a formal def $A_\hbar$ of A $\ni$ have $\theta : A_\hbar \xrightarrow{\text{can'l proj}} A \cong A_\hbar/(\hbar)$
  * take $\iota : Z \hookrightarrow A_\hbar$ $\ni$ $\theta\iota = \mathrm{id}_Z$

(*) [* define $\partial : Z_1 \longrightarrow \mathrm{Der}_k(A/Z_1)$ (well-def.)
  $z_1 \longmapsto \partial_z : z' \longmapsto \theta\left(\frac{1}{\hbar^N}[\iota(z), \tilde{z}']\right)$
  where $\tilde{z}' \in \theta^{-1}(z')$ (N is always ≥ 1)

[B-G only considered P.O. of level 1st we generalized for our purpose]

Then (A, Z, ∂) is a Poisson order of level N

Theorem 1 [WWY] S = 3-dim'l Skly. alg mod-fin/$Z(S) =: Z$
① S admits the structure of a Poisson Z-order for which induced $\{,\}$ on Z is non-trivial
② $\{,\}$ on Z given by $\{z_i, z_{i+1}\} = \partial_{z_{i+2}} F$, indices mod 3 w/ g in the Poisson center.
③ $\{,\}$ on Z is $\Sigma' = \mathbb{Z}_3 \times k^\times$ -equivariant
permutes a certain basis of generating space, dilation

Pf of ① was the hardest part: Trick ⓐ Showed $\exists$ max value N for which S is a Pois Z-ord. of lev. N & ⓑ Showed that if induced bracket $\{,\}$ on Z of level N is trivial, then we can construct a special^n $\ni$ induced $\{,\}$ on Z has level $N+1$. #

What about rep thy of S?

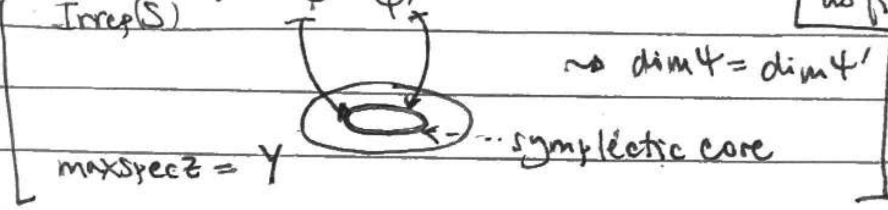
[Brown-Gordon] * Can partition $Y = \text{maxSpec}\, Z(S)$ via its symplectic cores [coarsening of symplectic leaf partition via closures]

equiv. classes of $m \in Y$ are determined:

$$m \sim m' \Leftrightarrow \mathcal{P}(m) = \mathcal{P}(m')$$

$\mathcal{P}(m)$: the unique max Poisson ideal $\subseteq m$

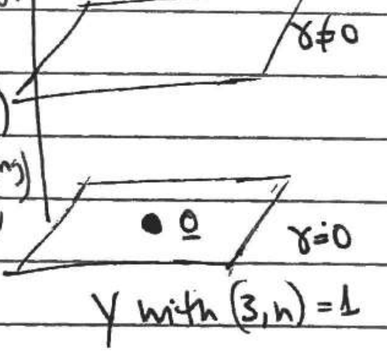
* Theorem [B-G] m, m' are in same symplectic core $\Rightarrow S/mS \simeq S/m'S$ as finite-dim'l alg's



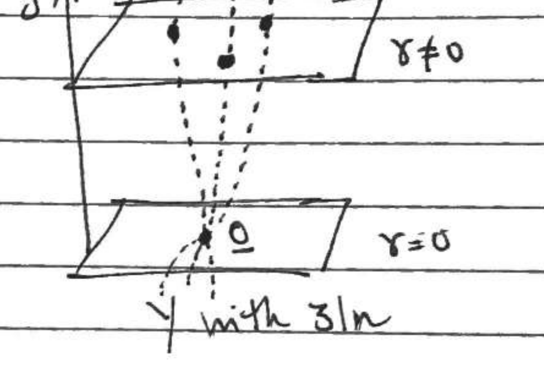
Theorem 2 [WWY] Take $\Sigma_1 = \mathbb{Z}_3 \times k^\times$ as in Thm 1.

① Can slice up Y into 2-dim'l Poisson subvarieties $Y_\gamma = Y \cap \mathbb{V}(g - \gamma)$ with singularities ($\bullet$) pictured as follows.

do'ly many (unlike Farnell's talk), get finitely many on each slice: Symp cores $= \coprod Y_\gamma^{sing} \cup (Y_\gamma \setminus Y_\gamma^{sing})$, slice Y_γ



Y with $(3, n) = 1$



Y with $3 | n$

② Get decomp. of Y in terms of Σ_1-orbits of symp. cores:

$$Y = \begin{cases} \{\underline{0}\} \amalg (Y_0 \setminus \{\underline{0}\}) \amalg (Y \setminus Y_0), & (3, n) = 1 \\ \underset{(\dagger)}{\{\underline{0}\}} \amalg \underset{(\ddagger)}{(Y_0 \setminus \{\underline{0}\})} \amalg \underset{(\ddagger\!\!\!\!\ddagger)}{(C_1 \cup C_2 \cup C_3 \setminus \{\underline{0}\})} \amalg \underset{(\ddagger\!\!\!\!\ddagger\!\!\!\!\ddagger)}{(Y \setminus (C_1 \cup C_2 \cup C_3))}, & 3 | n \end{cases}$$

③ Get Azumaya locus of S, $\mathcal{A}_S$ = smooth locus of Y, Y^{smooth}

Pf of (3) was the coolest part:

$(3,n)=1 \to$ done by [W]

$3|n$ case:

via density argument —

$$X := (\neq) \cup (\not\equiv) \subseteq \mathcal{A}_S \subseteq Y^{\text{smooth}}$$

Also have that

$$\underset{\substack{\| \\ Y \setminus Y^{\text{smooth}}}}{Y^{\text{sing}}} = C_1 \cup C_2 \cup C_3 = \underset{\substack{\| \\ Y \setminus X}}{(+) \cup (\not\equiv)}$$

$$\therefore Y^{\text{smooth}} = X \subseteq \mathcal{A}_S \subseteq Y^{\text{smooth}} \quad ///.$$

Some comments

* Have explicit examples in paper for $\mathcal{S}$ with $|\sigma|=2, 6$.

* Works in preparation in analyzing other elliptic algebras that are module-finite over their center including PI 4-dim'l Sklyanin algebras

* Generic quantum algebras are typically module infinite over their (small) center yet they are often deformations of a nice com. (Poisson) alg & people exploit this... (recall various earlier talks) even the theme of the conf "quant'n speak to this"

On the other hand, non-generic algebras A those that are module-finite over their center are just as important

Takeaway pt: the Poisson alg here is the center of A if A admits the structure of a Poisson order (Brown-Gordon) there are rep'n theoretic benefits by using this technology.