Quantum Binary Polyhedral Groups And Their Actions On Quantum Planes

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Joint work with Kenneth Chan, Ellen Kirkman, and James Zhang

November 18, 2012

Goal

An investigation of noncommutative/ Hopf invariant theory...

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...quantizations of results in classical invariant theory

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Actions of finite subgroups of $SL_2(\mathbb{C})$

on

"planes" $\mathbb{C}[u, v]$

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Actions of quantum finite subgroups of $SL_2(\mathbb{C})$

on

"quantum planes": noncommutative $\mathbb{C}[u, v]$

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Let's recall some classical results. $\boxed{\text{Put } k = \mathbb{C}}$

Take G a finite subgroup of $GL_2(k)$ acting faithfully on $k[u, v]$.

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are classified up to conjugation.

types: $A_n \quad D_n \quad E_6 \quad E_7 \quad E_8$

"binary polyhedral groups" =: G_{BPG}

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[DuVal-McKay] Geometry of $k[u, v]^{G_{BPG}}$.

The "Kleinian" or "DuVal" singularities

$X = \mathrm{Spec}(k[u, v]^{G_{BPG}})$

are precisely the rational double points

and the resolution graph of X is Dynkin.

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$SL_q(2) \cdots\cdots\cdots\cdots \mathcal{O}_q(SL_2(k))$

G_q fin. subgrp $\cdots\cdots$ $\mathcal{O}_q(G)$ fin. Hopf quot.

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$\boxed{\text{AS regular algebras } R \text{ of gldim 2}}$

AS = Artin-Schelter
* R is graded with $R_0 = k$
* global dimension 2
* AS-Gorenstein
* polynomial growth

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Viewed as ‘noncommutative $k[u, v]$’ in Noncommutative Projective AG

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Classified up to isomorphism:

$$k_q[u,v] := k\langle u,v\rangle/(vu - quv), \;\; q \in k^{\times}$$
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H acts on R if R is a left H-*module algebra*: R is a left H-module and $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$ and $h \cdot 1_R = \epsilon(h)1_R$ for all $h \in H$, and for all $a, b \in R$

Let $H \neq k$ be a finite dimensional Hopf algebra acting on an AS regular algebra R of global dimension 2.

(H1) [notion of faithfulness]

.

(H2) H preserves the grading of R

(H3) [notion of H-action having 'determinant 1']
... as results involving G with $\det(G)$ =1 motivate our results.
See [DuVal-McKay] for instance.

Setting of Study

Let $H \neq k$ be a finite dimensional Hopf algebra acting on an AS regular algebra R of global dimension 2.

(H1) H acts on R *inner faithfully*:
there is not an induced action of H/I on R for any nonzero Hopf ideal I of H

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(H3) H-action of R have trivial "homological determinant".
here, $\text{hdet}_H R$: $H \to k$ and it is *trivial* if equal to the counit map ϵ

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Definition. A Hopf algebra H satisfying the conditions above is called a quantum binary polyhedral group, denoted by H_{QBPG}.

Main Result

Theorem. [CKWZ] The pairs (H_{QBPG}, R_{ASreg2}) are classified as follows.

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$\boxed{H \text{ noncom \& s.s.}}$

$(kG_{BPG}, k[u,v])$

G_{BPG} nonabelian

$(kD_{2n}, k_{-1}[u,v])$

$n \geq 3$

$(\mathcal{D}(G_{BPG})^{\circ}, k_{-1}[u,v])$

$\mathcal{D}(G_{BPG})$: Hopf deformation of nonabelian b.p.g. [BN]

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$(kG_{BPG}, k[u, v])$ G_{BPG} nonabelian	$(kC_2, \text{any } R)$ diagonal action
$(kD_{2n}, k_{-1}[u, v])$ $n \geq 3$	$(kC_2, k_{-1}[u, v])$ non-diagonal action
$(\mathcal{D}(G_{BPG})^\circ, k_{-1}[u, v])$ $\mathcal{D}(G_{BPG})$: Hopf deformation of nonabelian b.p.g. [BN]	$(kC_n, k_q[u, v])$ $n \geq 3$
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$\boxed{H \text{ nonsemisimple}}$

For q is a root of 1, $q^2 \neq 1$

$((T_{q,\alpha,n})^{\circ}, k_{q^{-1}}[u,v])$
$T_{q,\alpha,n}$: generalized Taft alg.

$(H, k_{q^{-1}}[u,v])$ ord(q) odd
$1 \to (kG_{BPG})^{\circ} \to H^{\circ} \to \overline{\mathcal{O}_q(SL_2)} \to 1$

$(H, k_{q^{-1}}[u,v])$ ord(q) even
$1 \to (kG_{PG})^{\circ} \to H^{\circ} \to \overline{\mathcal{O}_q(SL_2)} \to 1$

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Theorem. [CKWZ] The pairs (H_{QBPG}, R_{ASreg2}) are classified as follows.

$$R = k[u,v] \implies H = kG_{BPG}, \text{ no “new” } H$$

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For $R = k_{-1}[u, v]$

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$(kG_{BPG}, k[u, v])$ G_{BPG} nonabelian	$(kC_2, k_{-1}[u, v])$ diagonal action	For q is a root of 1, $q^2 \neq 1$
$(kD_{2n}, k_{-1}[u, v])$ $n \geq 3$	$(kC_2, k_{-1}[u, v])$ non-diagonal action	$((T_{q,\alpha,n})^\circ, k_{q^{-1}}[u, v])$ $T_{q,\alpha,n}$: generalized Taft alg.
$(\mathcal{D}(G_{BPG})^\circ, k_{-1}[u, v])$ $\mathcal{D}(G_{BPG})$: Hopf deformation of nonabelian b.p.g. [BN]	$(kC_n, k_{-1}[u, v])$ $n \geq 3$	$(H, k_{q^{-1}}[u, v])$ ord(q) odd $1 \to (kG_{BPG})^\circ \to H^\circ \to \overline{O_q(SL_2)} \to 1$
	$((kD_{2n})^\circ, k_{-1}[u, v])$ $n \geq 3$	$(H, k_{q^{-1}}[u, v])$ ord(q) even $1 \to (kG_{PG})^\circ \to H^\circ \to \overline{O_q(SL_2)} \to 1$

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For $R = k_q[u,v]$ with q a root of unity, $q^2 \neq 1$

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For $R = k_q[u, v]$ for q not a root of 1

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$(kD_{2n}, k_{-1}[u, v])$ $n \geq 3$	$(kC_2, k_{-1}[u, v])$ non-diagonal action	$((T_{q,\alpha,n})^{\circ}, k_{q^{-1}}[u, v])$ $T_{q,\alpha,n}$: generalized Taft alg.
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For $R = k_J[u, v]$

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Further Results

Given a pair ($H = H_{QBPG}$, $R = R_{ASreg2}$) in the main theorem, to say:

a finite dimensional Hopf algebra H acts inner faithfully and preserves the grading of an AS regular algebra R of gldim 2, with H-action having trivial homological determinant

we have the following results.

$$R^H = \{r \in R \mid h \cdot r = \epsilon(h)r \text{ for all } h \in H\}$$

[On the regularity of the invariant subring R^H, motivated by [STC]]

[On the Gorenstein condition for the invariant subring R^H, motivated by [Watanabe]]

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Proposition. [CKWZ] Let (H, R) be as above. The invariant subring R^H is AS-Gorenstein. (semisimple case by [KKZ])

Future Work

(1) Since R^H is Gorenstein and is not regular ...
Motivated by [DuVal-McKay] and others:

Study the geometry of 'noncommutative Gorenstein singularities' R^H for (H, R) in the main theorem, particularly with H semisimple.

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Study finite dimensional Hopf algebra actions on AS regular algebras of gldim 2 with *arbitrary* homological determinant.

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(2) Motivated by [STC] and others:

Study finite dimensional Hopf algebra actions on AS regular algebras of gldim 2 with *arbitrary* homological determinant.

(3) Since AS regular algebras of gldim 3 have been classified...

Study finite dim'l Hopf algebra actions on AS reg. algs of gldim 3.

... AS regular algebras of gldim > 3 have not been classified

References:

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[STC] = [Ben93, Theorem 7.2.1]

[Watanabe] = [Ben93, Theorem 4.6.2]

Thank you for listening!