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[k base field]

On Quantum Groups associated to a pair of preregular forms

joint w/ Alex Chirvasitu & Xingting Wang [arXiv. 1605.06428]

$\boxed{\text{Big Goal}}$ To understand universal quantum groups (UQGs) $\mathcal{O}_B$ ← w/decoration? that coact on finitely generated & finitely presented k-algebras $B = \frac{TV}{J}$.

Call a Hopf algebra $\mathcal{O}_B$ a UQG associated to B if $\mathcal{O}_B$ coacts on B (say from the right) and $\forall$ Hopf algebras H that coact on B, $\exists!$ Hopf alg map $\pi: \mathcal{O}_B \to H$ ∋.

$$\begin{tikzcd} B \arrow[r, "\mathcal{O}\text{-coaction}"] \arrow[dr, "H\text{-coaction}"'] & B \otimes \mathcal{O}_B \arrow[d, dashed, "id_B \otimes \pi"] \\ & B \otimes H \end{tikzcd}$$

commutes.

If the relations of B are governed by a form $\varphi: \underbrace{X}_{k\text{-vs}} \to k$ (ie B = superpotential algebra/ vacualgebra)

then can consider UQG $\mathcal{O}_\varphi$ that preserves this form.

That is, $\mathcal{O}_\varphi$ coacts universally on X ∋.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d] & k \arrow[d, "\text{trivial coaction}"] \\ X \otimes \mathcal{O}_\varphi \arrow[r, "\varphi \otimes id_{\mathcal{O}_\varphi}"] & k \otimes \mathcal{O}_\varphi \end{tikzcd}$$

commutes

If such UQGs exist, they're typically huge, and not necessarily finitely generated nor finitely presented ....

Towards a 'nicer' UQG....

Late 1980s: Manin constructed a UQG $\mathcal{O}_{A,B}$ associated to a pair of quadratic graded algs $A = \frac{TW}{I}$, $B = \frac{TV}{J}$ ∋. $\exists$ nondeg. pairing $\underset{\underset{W}{\shortparallel}}{A_1} \times \underset{\underset{V}{\shortparallel}}{B_1} \to k$

— generated in degree one —

Namely if $W = \bigoplus_i k y_i$ and $V = \bigoplus_i k x_i$, then $\mathcal{O}_{A,B}$ coacts on A and B simultaneously & universally as follows:

$$A \longrightarrow \mathcal{O}_{A,B} \otimes A \qquad \text{and} \qquad B \longrightarrow B \otimes \mathcal{O}_{A,B}$$

$$y_i \longmapsto \sum_j u_{ij} \otimes y_j \qquad\qquad x_i \longmapsto \sum_j x_j \otimes u_{ji}.$$

$\mathcal{O}_{A,B}$ always exists! "Hopf envelope"

Here, u_{ij} are generators of $\mathcal{O}_{A,B}$.

Manin's question to Artin, Schelter, Tate (AST) [paraphrased]

If A, B are
- homologically nice (• N Koszul • finite gl.dim • Gorenstein condition ....)
- ring-theoretically nice (• Noetherian • domain • poly'l growth $\Updownarrow$ finite GK dim)

, then is $\mathcal{O}_{A,B}$ as well?

(When do $\mathcal{O}_{A,B}$ and A, B resemble each other algebraically?)

* Fails for 'one-sided' UQGs: $\mathcal{O}_{k[x,y]}$ is not Noetherian, not commutative .....

AST's answer: yes! for $A = \dfrac{k\langle y_1,\dots,y_n\rangle}{(y_iy_j = q_{ij}y_jy_i)_{1\le i,j\le n}}$, $B = \dfrac{k\langle x_1,\dots,x_n\rangle}{(x_ix_j = q'_{ij}x_jx_i)_{1\le i,j\le n}}$ skew-poly'l algebras

[$(q_{ij}), (q'_{ij}) \in \mathrm{Mat}_n(k)$ multiplicatively anti-sym.]

for q_{ij}' = \lambda q_{ij} with i>j and \lambda \neq -1

Here, $\mathcal{O}_{A,B} \cong \mathcal{M}_{A,B}[D^{-1}]$ a central localization of an iterated Ore extn in n^2 variables

central elt D is sometimes known as the "quantum determinant"

Although, Manin's question was posed initially for A, B being 3-dim'l Sklyanin algs...

$\boxed{\text{Refined goal}}$ Address Manin's question for a wider class of homologically nice algebras: $\underbrace{\text{N-Koszul}}$ $\underbrace{\text{Artin-Schelter (AS) regular algs } B \text{ of dimension } d}$

guarantees a nice resolution of ${}_B k$ ...

We don't require finite GKdim

$\overset{\text{defn}}{\Longleftrightarrow}$ · B is connected $\mathbb{N}$-graded $B = k \oplus B_1 \oplus B_2 \oplus \cdots$

· gldim $B = d$

· $\mathrm{Ext}_B^i(k, B) \cong \delta_{i,d}\, B$ as left & right B-modules 'AS-Gorenstein'

That is: understand the universal quantum symmetry of Noncommutative Projective Spaces

Useful Fact [Dubois-Violette] N-Koszul AS regular algs are "superpotential algebras"

$$B = B(t, N) = TV/(\partial^{m-N} t) \qquad \text{Say } \dim V = n. \quad \text{Take } m \geq N \geq 2.$$

$t \in V^{\otimes m}$ $\underset{\text{superpotential}}{\phi\text{-twisted}}$ $= \phi$ cyclic elt for $\phi \in GL(V)$

$=$ sum of elts that satisfy $v_1 \otimes \cdots \otimes v_m = \phi v_m \otimes v_1 \otimes \cdots \otimes v_{m-1}$

$\partial^{m-N} t$ = space of $(m-N)$ fold left partial derivatives of t.

Note: t can be identified with an elt of $((V^*)^{\otimes m})^*$

or a form: $(V^*)^{\otimes m} \longrightarrow k$

t = ϕ-twisted superpotential $\Updownarrow$ this form is "preregular". [Hence the title].

Examples $\boxed{d=2}$ Zhang: All AS reg algs of dim 2 are N-homog. one-relator algs ($\Rightarrow$ N-Koszul)

$B = k\langle x_1, \ldots, x_n \rangle / (r)$

Here, $m = N = \deg r$ & r is the ϕ-twisted sup'l.

$\boxed{d=3}$ AS regular algs are only classified in the Noetherian ($\Leftrightarrow$ finite GK dim) case
[by Artin, Schelter, Tate, van den Bergh]

Most generic quadratic family: 3-dim'l Sklyanin algebras

$$S(a,b,c) = k\langle x,y,z\rangle \Big/ \left(\begin{array}{l} ayz+bzy+cx^2 \\ azx+bxz+cy^2 \\ axy+byx+cz^2 \end{array} \right) \begin{array}{l} \leftarrow \partial_x t = \text{apply } x^* \text{ on left to } t \\ \leftarrow \partial_y t = \\ \leftarrow \partial_z t - \text{ likewise.} \end{array}$$

$a,b,c \in k$ generic

Here, $m=3$, $N=2$, $t_{abc} = a(xyz+yzx+zxy)+b(xzy+zyx+yxz)+c(x^3+y^3+z^3)$.

Although, $S(a,b,c)$ evades Gröbner basis techniques,
we still think it's possible to study $\mathcal{O}_{S(a',b',c'),S(a,b,c)}$ ....

First thing's first: Presentation of $\mathcal{O}_{A,B}$. $\leftarrow$ again exists, apriori not nec. fin. gen., fin. presented

Question: If A,B are N-Koszul AS regular generated in degree 1, does $\mathcal{O}_{A,B}$ have a finite presentation?

Theorem: [CWW] Yes! If $A = \frac{TW}{I}$, $B = \frac{TV}{J}$ N-Koszul AS reg w/ $\dim V = \dim W = n$ (where $\exists$ nondeg pairing btw A_1 & B_1) $A_1 = W = V^*$, $B_1 = V$

then $\mathcal{O}_{A,B}$ has $\underbrace{n^2}+\underbrace{4}$ generators w/ finitely many relations.

n^2: the usual generators $(u_{ij})_{i,j=1,\dots,n}$ that define the coaction of $\mathcal{O}_{A,B}$ on A, B.

4: $D_A^{\pm1}, D_B^{\pm1}$ where D_A, D_B are the "quantum determinants" arising from coactions

/!\ $\uparrow$ not necessarily central elts (though central in many cases)

In particular, fix basis of V: $v_1, \dots, v_n$ & dual basis $\theta_1, \dots, \theta_n$ of $W = V^*$.

If $A = A(s, N)$ for $s \in (V^*)^{\otimes m}$ or $s: V^{\otimes m} \to k$

$B = B(t, N)$ for $t \in V^{\otimes m}$ or $t: (V^*)^{\otimes m} \to k$,

let $s_{i_1 \cdots i_m} := s(v_{i_1} \otimes \cdots \otimes v_{i_m}) \in k$.

$t_{i_1 \cdots i_m} := t(\theta_{i_1} \otimes \cdots \otimes \theta_{i_m})$

We get that $\mathcal{O}_{A,B}$ is generated by $\{u_{ij}\}_{i,j=1,\dots,n}$, $D_A^{\pm 1}$, $D_B^{\pm 1}$ with

relations:
$$\sum_{i_1, \dots, i_m = 1}^{n} s_{i_1 \cdots i_m} u_{i_1 j_1} \cdots u_{i_m j_m} = s_{j_1 \cdots j_m} D_A \quad (+) \quad \forall 1 \le j_1, \dots, j_m \le n$$

$2n^m$ relations of degree m

$$\sum_{i_1, \dots, i_m = 1}^{n} t_{i_1 \cdots i_m} u_{j_1 i_1} \cdots u_{j_m i_m} = t_{j_1 \cdots j_m} D_B^{-1} \quad (\ddagger) \quad \forall 1 \le j_1, \dots, j_m \le n$$

$$D_A D_A^{-1} = D_A^{-1} D_A = D_B D_B^{-1} = D_B^{-1} D_B = 1_{\mathcal{O}_{A,B}}.$$

Comult: $\Delta(u_{ij}) = \sum_{l=1}^{n} u_{il} \otimes u_{lj}$ $\qquad \Delta(D_A^{\pm 1}) = D_A^{\pm 1} \otimes D_A^{\pm 1}$ $\qquad \Delta(D_B^{\pm 1}) = D_B^{\pm 1} \otimes D_B^{\pm 1}$.

Counit: $\varepsilon(u_{ij}) = \delta_{ij}$ $\qquad \varepsilon(D_A^{\pm 1}) = \varepsilon(D_B^{\pm 1}) = 1$

antipode:
$$S(u_{ij}) = \sum_{i_1, \dots, i_{m-1}, j_1, \dots, j_{m-1} = 1}^{n} D_A^{-1} \tilde{s}_{i i_1 \cdots i_{m-1}} u_{j_1 i_1} \cdots u_{j_{m-1} i_{m-1}} s_{j_1 \cdots j_{m-1} j}$$

get n^2 relations of deg $m-1$

$$= \sum_{i_1, \dots, i_{m-1}, j_1, \dots, j_{m-1} = 1}^{n} t_{i i_1 \cdots i_{m-1}} u_{j_1 i_1} \cdots u_{j_{m-1} i_{m-1}} \tilde{t}_{j_1 \cdots j_{m-1} j} D_B \quad (\ddagger) \quad 1 \le i, j \le n$$

where $\tilde{s}, \tilde{t}$ are forms on $(V^*)^{\otimes m}$, $V^{\otimes m}$ resp so that

$$\sum_{l_1, \dots, l_{m-1} = 1}^{n} \tilde{s}_{i l_1 \cdots l_{m-1}} s_{l_1 \cdots l_{m-1} j} = \sum_{l_1, \dots, l_{m-1} = 1}^{n} t_{i l_1 \cdots l_{m-1}} \tilde{t}_{l_1 \cdots l_{m-1} j} = \delta_{ij}$$

These forms always exist. unique $\Leftrightarrow m = 2$

like the inverse of s, t

& $S(D_A^{\pm 1}) = D_A^{\mp 1}$, $S(D_B^{\pm 1}) = D_B^{\mp 1}$.

<u>Ex.</u> $A = k[x^*, y^*]$, $B = k[x,y]$: get $\mathcal{O}_{A,B} = \mathcal{O}(GL_2(k))$ commutative Hopf algebra.
where $D_A = D_B^{-1} = u_{11}u_{22} - u_{12}u_{21}$
(central)

relation $xy - yx$

$s_{ij}, \tilde{s}_{ij}$ defined similarly to:

$$\Rightarrow t_{11}=0,\ t_{12}=1 \qquad \Rightarrow \qquad \tilde{t}_{11}=0 \quad \tilde{t}_{12}=-1$$
$$t_{21}=-1,\ t_{22}=0 \qquad\qquad \tilde{t}_{21}=1 \quad \tilde{t}_{22}=0$$

(‡) yields the relation: $\underbrace{u_{11}u_{12}}_{(i_1,i_2)=(1,2)} - \underbrace{u_{12}u_{11}}_{(2,1)} = 0$ for $(j_1,j_2)=(1,1)$

$\underbrace{u_{11}u_{22}}_{(i_1,i_2)=(1,2)} - \underbrace{u_{12}u_{21}}_{(2,1)} = D_B^{-1}$ for $(j_1,j_2)=(1,2)$

⋮

get usual 8 relations of $\mathcal{O}(GL_2(k))$;

(‡) yields: $S(u_{11}) = (1)\, u_{22}\, (1) = u_{22}$.
$i=1, j=1 \Rightarrow i_1=2 \quad j_1=2$

$S(u_{12}) = (1)\, u_{12}\, (-1) = -u_{12}$
$i=1, j=2 \Rightarrow i_1=2 \quad j_1=1$

$\therefore\ S(u_{ij}) = \begin{pmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{pmatrix}$
as usual.

Also get for $\dim V = n$
$\mathcal{O}_{S(V^*),S(V)} = \mathcal{O}(GL_n(k))$
by similar computations

Don't like computations, you say?

<u>Prop [CWW]</u> $\mathcal{O}_{A,B}$ is realized as a <u>pushout</u> of the one-sided UQGs $\mathcal{O}_A$, $\mathcal{O}_B$.

Take $\mathcal{F}$ = free Hopf algebra that coacts on V universally

Get
$$\begin{tikzcd} & \mathcal{O}_A \\ \mathcal{F} \arrow[ur, bend left] \arrow[dr, bend right] & \\ & \mathcal{O}_B \end{tikzcd}$$
Hopf algebra maps

get induced $\mathcal{O}_A$-coaction on V

from $\mathcal{O}_B$-coaction on V

Have $\mathcal{O}_A \to \mathcal{O}_{A,B}$, $\mathcal{O}_B \to \mathcal{O}_{A,B}$ Hopf algebra maps so that
$$\begin{tikzcd} & \mathcal{O}_A \arrow[dr, bend left] & \\ \mathcal{F} \arrow[ur, bend left] \arrow[dr, bend right] & 2 & \mathcal{O}_{A,B} \\ & \mathcal{O}_B \arrow[ur, bend right] & \end{tikzcd}$$

& $\forall$ Hopf alg H that coact on A, B, get:
$$\begin{tikzcd} & \mathcal{O}_A \arrow[dr] \arrow[drr, bend left] & & \\ \mathcal{F} \arrow[ur, bend left] \arrow[dr, bend right] & & \mathcal{O}_{A,B} \arrow[r, dotted] & H \\ & \mathcal{O}_B \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

* Homological codeterminant of the $\mathcal{O}_{A,B}$-coaction on $\begin{cases} A \\ B \end{cases}$ is the det $\begin{cases} D_A \\ D_B \end{cases}$.

(as defined by Kirkman-Kuzmanovich-Zhang)

In fact,

Theorem [CWW] If $B = B(t, N)$ is N-Koszul AS-regular, then the homological codeterminant D of a Hopf algebra coaction on B arises as

$$\begin{cases} \rho(t) = t \otimes D^{-1} & \text{for right } H\text{-coaction} \\ \rho(t) = D \otimes t & \text{for left } H\text{-coaction} \end{cases}$$

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* have 'SL' versions / trivial homological codet ($D = 1_H$) / trivial quantum det versions for everything above.

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* Our construction recovers several other UQGs in the literature

* 2-sided $GL(n)$-like UQG of $\boxed{\text{AST}}$ for A, B skew polynomial rings

(& the special case of)

* 2-sided $GL(n)$-like UQG of $\boxed{\text{Takeuchi}}$ for $q_{ij} = q$, $q_{ij} = q'$ for $i > j$

* 1-sided $SL(2)$-like UQG of $\boxed{\text{Dubois-Violette \& Launer}}$ (or 2-sided) $\quad$ gldim A = gldim B = 2 [studied by Bichon, W-Wang]

(& the generalization of ...)

* 2-sided $GL(2)$-like UQG of $\boxed{\text{Mrozinski}}$ $\quad$ gldim A = gldim B = 2

* 1-sided $SL(n)$-like UQG of $\boxed{\text{Bichon \& Dubois-Violette}}$ preserving one preregular form.

* We also highlight genuinely new UQGs associated to N-Koszul algebras A, B

- 3-dim'l Sklyanin algs	$n=3$	$d=3$	$N=2$	$m=3$
- 4-dim'l Sklyanin algs	$n=4$	$d=4$	$N=2$	$m=4$
- Yang-Mills algebras	$n \geq 2$	$d=3$	$N=3$	$m=4$
	# of gens $= \dim V$	gldim	Koszulity #	$A = A(s,N)$, $B = B(t,N)$ for s,t in m-fold $\otimes$ of vspaces

Next up: Studying the ring-theoretic & hom'l properties & (co)representation-theoretic properties of $\mathcal{O}_{A,B}$

(work in progress w/ Chirvasitu & Wang —)