

Sept 2016 @ BIRS

On Quantum Groups associated to a pair of preregular forms

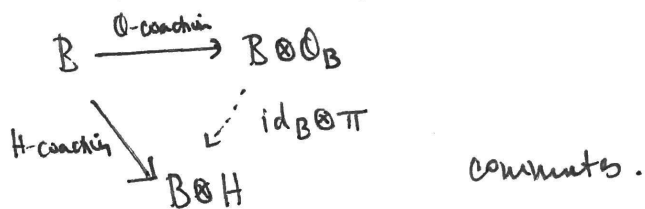
[k base field]

joint w/ Alex Chirvasitu & Xingting Wang [arXiv:1605.06428]

Big Goal

To understand universal quantum groups (UQGs) \mathcal{O}_B that coact on finitely generated & finitely presented k-algebras $B = \frac{T\langle V \rangle}{I}$.

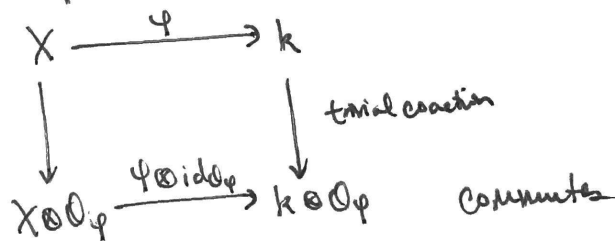
Call a Hopf algebra \mathcal{O}_B a UQG associated to B if \mathcal{O}_B coacts on B (say from the right) and \forall Hopf algebras H that coact on B, $\exists!$ Hopf alg map $\pi: \mathcal{O}_B \rightarrow H$.



If the relations of B are governed by a finite $\varphi: \underbrace{X}_{k\text{-vs}} \rightarrow k$ (ie B = superpotential algebra / vacualgebra)

then can consider UQG \mathcal{O}_φ that preserves this form.

That is, \mathcal{O}_φ coacts universally on X .



If such UQGs exist, they're typically huge, and not necessarily finitely generated nor finitely presented....

Towards a 'nicer' UQG....

Late 1980s: Manin constructed a UQG $\mathcal{O}_{A,B}$ associated to a pair of

quadratic graded algs $A = \frac{T\langle W \rangle}{I}$, $B = \frac{T\langle V \rangle}{J} \Rightarrow$ Frobenius pairing $A_1 \times B_1 \rightarrow k$
 — generated in degree one —

Namely if $W = \bigoplus_i ky_i$ and $V = \bigoplus_i kx_i$, then $\mathcal{O}_{A,B}$ coacts on A and B simultaneously & universally as follows:

$$A \longrightarrow \mathcal{O}_{A,B} \otimes A \quad \text{and} \quad B \longrightarrow B \otimes \mathcal{O}_{A,B}$$

$$y_i \longmapsto \sum_j u_{ij} \otimes y_j \quad \quad x_i \longmapsto \sum_j x_j \otimes u_{ji}$$

$\mathcal{O}_{A,B}$ always exists!
"Hopf-envelope"

Here, u_{ij} are generators of $\mathcal{O}_{A,B}$.

Manin's question to Artin, Schelter, Tate (AST) [paraphrased]

IF A, B are homologically nice (• N Koszul
• finite global dimension
• Gorenstein condition), then is $\mathcal{O}_{A,B}$ as well?
ring-theoretically nice (• Noetherian
• domain
• polyalgebraic
• finite GKdim)

(When do $\mathcal{O}_{A,B}$ and A, B resemble each other algebraically?)

* Fails for 'one-sided' UQGs: $\mathcal{O}_{k\langle x, y \rangle}$ is not Noetherian, not commutative

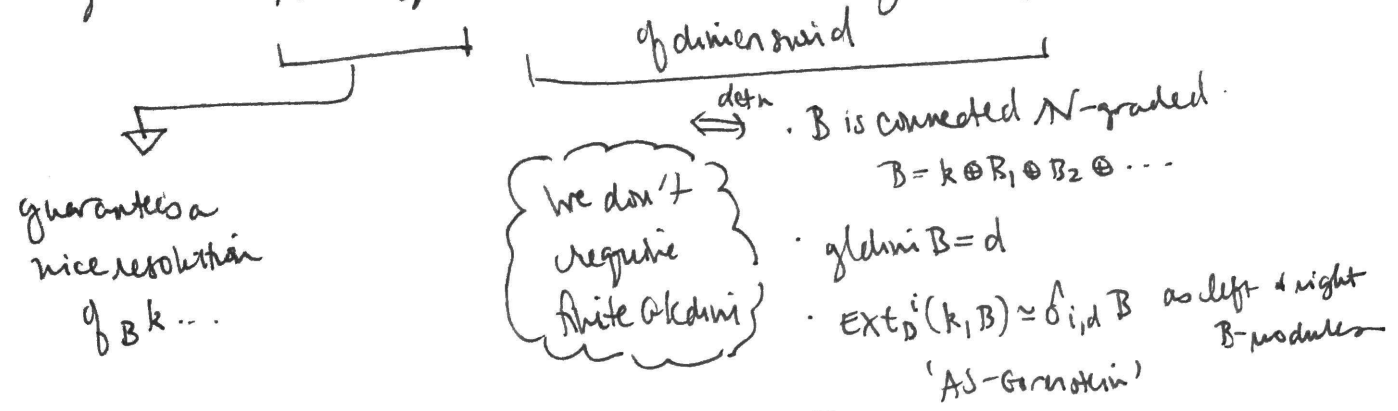
AST's answer: yes! for $A = k\langle y_1, \dots, y_n \rangle$, $B = k\langle x_1, \dots, x_n \rangle$ skew-polyalgebras
 $(q_{ij}, q_{ji}) \in \text{Mat}_n(k)$ multiplicatively anti-sym.
 $(y_i y_j = q_{ij} y_j y_i)_{1 \leq i, j \leq n}$ $(x_i x_j = q_{ji} x_j x_i)_{1 \leq i, j \leq n}$
 for $q_{ij} = \lambda q_{ji}$ with $i > j$ and $\lambda \neq -1$

Here, $\mathcal{O}_{A,B} \simeq M_{A,B}[D^{-1}]$ a central localization of an iterated Ore extn in n^2 variables

central elt D is sometimes known as the "quantum determinant"

Although, Manin's question was posed initially for A, B being 3-diml Sklyanin algs... -3-

Refined goal | Address Manin's question for a wider class of homologically nice algebras: N -Koszul Artin-Schelter (AS) regular algs B



That is: understand the universal quantum symmetry of Noncommutative Projective Spaces

Useful Fact [Dubois-Violette] N -Koszul AS regular algs are "superpotential algebras"

$B = B(t, N) = TV / (\partial^{m-N} t)$ Say $\dim V = n$. Take $m \geq N \geq 2$.

$t \in V^{\otimes m}$ ϕ -twisted Superpotential = ϕ cyclic elt for $\phi \in \text{GL}(V)$
 = sum of elts that satisfy $v_1 \otimes \dots \otimes v_m = \phi v_m \otimes v_1 \otimes \dots \otimes v_{m-1}$

$\partial^{m-N} t$ = space of $(m-N)$ fold left partial derivatives of t .

Note: t can be identified with an elt of $((V^*)^{\otimes m})^*$
 or a form: $(V^*)^{\otimes m} \longrightarrow k$

$t = \phi$ -twisted superpotential
 ↕
 this form is "preregular".

[Hence the title].

Examples
 $d=2$

Zhang: All AS reg algs of dim 2 are N -homog. one-relater algs ($\Rightarrow N$ -Koszul)

$B = \langle k\langle x_1, \dots, x_n \rangle / (r) \rangle$
 Here, $m=N=\text{deg } r$ & r is the ϕ -twisted sup'd.

d=3 AS regular algs are only classified in the Noetherian (\Leftrightarrow finite \mathcal{K} -dim) case
[by Artin, Schelter, Tate, vander Berg]

Most generic quadratic family: 3-dim'l Sklyanin algebras

$$S(a,b,c) = \mathcal{K}\langle x,y,z \rangle / \begin{cases} ayz + bzy + cx^2 \\ azx + bxz + cy^2 \\ axy + byx + cz^2 \end{cases}$$

$\xrightarrow{\partial_x t}$ = apply x^i on left to t
 $\xrightarrow{\partial_y t}$ = likewise
 $\xrightarrow{\partial_z t}$ = likewise

$a,b,c \in \mathcal{K}$
generic

Here, $m=3, N=2, t_{abc} = a(xyzy + yzxy + zxyx) + b(xzyz + zyxy + yxzx) + c(x^3 + y^3 + z^3)$.

Although, $S(a,b,c)$ evades Gröbner basis techniques,
we still think it's possible to study $\mathcal{O}_{S(a',b',c')}, \mathcal{O}_{S(a,b,c)}$

First thing's first:

Presentation of $\mathcal{O}_{A,B}$.

again exists, a priori not nec. fin. gen., fin. presented

Question: If A, B are N -Koszul AS regular, generated in degree 1,
does $\mathcal{O}_{A,B}$ have a finite presentation?

Theorem: [CWW] Yes! If $A = \frac{TW}{I}, B = \frac{TV}{J}$ N -Koszul AS reg w/ $\dim V = \dim W = n$
(where T non-deg pairing btw $A_1 + B_1$)

then $\mathcal{O}_{A,B}$ has $n^2 + 1$ generators w/ finitely many relations.

the usual generators $(u_{ij})_{i,j=1,\dots,n}$
that define the coaction of $\mathcal{O}_{A,B}$
on A, B .

$D_A^{\pm 1}, D_B^{\pm 1}$ where
 D_A, D_B are the "quantum determinants"
arising from coactions

! ! \nrightarrow not necessarily central elts
(though central in many cases)

In particular, fix basis of $V: v_1, \dots, v_n$ & dual basis $\theta_1, \dots, \theta_n$ of $W = V^*$.

If $A = A(s, N)$ for $s \in (V^*)^{\otimes m}$ or $s: V^{\otimes m} \rightarrow k$

$B = B(t, N)$ for $t \in V^{\otimes m}$ or $t: (V^*)^{\otimes m} \rightarrow k$,

let $\delta_{i_1 \dots i_m} := s(v_{i_1} \otimes \dots \otimes v_{i_m}) \in k$.

$t_{i_1 \dots i_m} := t(\theta_{i_1} \otimes \dots \otimes \theta_{i_m})$

We get that $\mathcal{O}_{A,B}$ is generated by $\{u_{ij}\}_{i,j=1,\dots,n}, D_A^{\pm 1}, D_B^{\pm 1}$ with

relations: $\sum_{i_1, \dots, i_m=1}^n \delta_{i_1 \dots i_m} u_{i_1 j_1} \dots u_{i_m j_m} = \delta_{j_1 \dots j_m} D_A$ (#) $\forall 1 \leq j_1, \dots, j_m \leq n$

$\sum_{i_1, \dots, i_m=1}^n t_{i_1 \dots i_m} u_{j_1 i_1} \dots u_{j_m i_m} = t_{j_1 \dots j_m} D_B^{-1}$ (#) $\forall 1 \leq j_1, \dots, j_m \leq n$

2m relations of degree m

$D_A D_A^{-1} = D_A^{-1} D_A = D_B D_B^{-1} = D_B^{-1} D_B = 1_{\mathcal{O}_{A,B}}$.

comult: $\Delta(u_{ij}) = \sum_{d=1}^n u_{id} \otimes u_{dj}$ $\Delta(D_A^{\pm 1}) = D_A^{\pm 1} \otimes D_A^{\pm 1}$ $\Delta(D_B^{\pm 1}) = D_B^{\pm 1} \otimes D_B^{\pm 1}$

comult: $\varepsilon(u_{ij}) = \delta_{ij}$ $\varepsilon(D_A^{\pm 1}) = \varepsilon(D_B^{\pm 1}) = 1$

antipode: $\mathcal{S}(u_{ij}) = \sum_{i_1, \dots, i_{m-1}, j_1, \dots, j_{m-1}=1}^n D_A^{-1} \tilde{s}_{i_1 \dots i_{m-1}} u_{j_1 i_1} \dots u_{j_{m-1} i_{m-1}} \delta_{j_1 \dots j_{m-1} j}$

$= \sum_{i_1, \dots, i_{m-1}, j_1, \dots, j_{m-1}=1}^n t_{i_1 \dots i_{m-1}} u_{j_1 i_1} \dots u_{j_{m-1} i_{m-1}} \tilde{t}_{j_1 \dots j_{m-1} j} D_B$ (#) $1 \leq i, j \leq n$

get n^2 relations of degree m-1

where \tilde{s}, \tilde{t} are forms on $(V^*)^{\otimes m}, V^{\otimes m}$ resp so that

$\sum_{d_1, \dots, d_{m-1}=1}^n \tilde{s}_{d_1 \dots d_{m-1}} \delta_{d_1 \dots d_{m-1} j} = \sum_{d_1, \dots, d_{m-1}=1}^n t_{d_1 \dots d_{m-1}} \tilde{t}_{d_1 \dots d_{m-1} j} = \delta_{ij}$

These forms always exist. unique $\Leftrightarrow m=2$

like the inverse of δ & t

$\neq \mathcal{S}(D_A^{\pm 1}) = D_A^{\mp 1}, \mathcal{S}(D_B^{\pm 1}) = D_B^{\mp 1}$

Ex. $A = k[x^*, y^*]$, $B = k[x, y]$ get $\mathcal{O}_{A,B} = \mathcal{O}(GL_2(k))$ commutative Hopf algebra. ⁻⁶⁻
 where $\mathcal{D}_A = \mathcal{D}_B^{-1} = u_{11}u_{22} - u_{12}u_{21}$ (central)
 relation $xy - yx$
 $\Rightarrow t_{11} = 0, t_{12} = 1 \Rightarrow \tilde{t}_{11} = 0, \tilde{t}_{12} = -1$
 $t_{21} = -1, t_{22} = 0 \Rightarrow \tilde{t}_{21} = 1, \tilde{t}_{22} = 0$
 $\delta_{ij}, \tilde{\delta}_{ij}$ defined similarly to

(\neq) yields the relations: $u_{11}u_{12} - u_{12}u_{11} = 0$ for $(j_1, j_2) = (1, 1)$
 $(i_1, i_2) = (1, 2)$ (2, 1)

$u_{11}u_{22} - u_{12}u_{21} = \mathcal{D}_B^{-1}$ for $(j_1, j_2) = (1, 2)$
 $(i_1, i_2) = (1, 2)$ (2, 1)

get usual 8 relations of $\mathcal{O}(GL_2(k))$

(\neq) yields: $S(u_{11}) = (\pm) u_{22} (\pm) = u_{22}$
 $i=1, j=1 \Rightarrow i_1=2, j_1=2$

$S(u_{12}) = (\pm) u_{12} (\pm) = -u_{12}$
 $i=1, j=2 \Rightarrow i_1=2, j_1=1$

$S(u_{ij}) = \begin{pmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{pmatrix}$
 as usual.

Also get for $\dim V = n$
 $\mathcal{O}(S(V)^*, S(V)) = \mathcal{O}(GL_n(k))$
 by similar computations

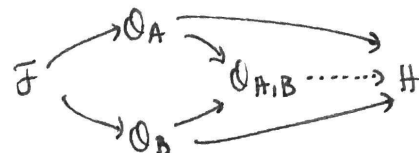
Don't like computations, you say?

Prop [CW] $\mathcal{O}_{A,B}$ is realized as a pushout of the one-sided UQOs $\mathcal{O}_A, \mathcal{O}_B$.

Take $\mathcal{F} =$ free Hopf algebra that coacts on V universally



\forall Hopf alg \mathcal{H} that coact on A, B , get:



* Homological codeterminant of the (A, B) -coaction on $\begin{cases} A \\ B \end{cases}$ is the $\det \begin{cases} D_A \\ D_B \end{cases}$
 (as defined by Kirichenko-Kuzmanovich-Zhang)

In fact,

Theorem [CW] If $B = B(t, N)$ is N -Koszul AS-regular, then
 the homological codeterminant D of a Hopf algebra coaction on B
 arises as $\begin{cases} \rho(t) = t \otimes D^{-1} & \text{for right H-coaction} \\ \rho(t) = D \otimes t & \text{for left H-coaction} \end{cases}$

* have 'SL' versions / trivial homological codet / trivial quantumdet versions
 ($D = 1, \#$)
 for everything above.

* Our construction recovers several other UQGs in the literature

* 2-sided $GL(n)$ -like UQG of AST for A, B skew polynomial rings
 (# the special case of)

* 2-sided $GL(n)$ -like UQG of Takeuchi for $q_{ij} = q$ $q'_{ij} = q'$ for $i > j$

* 1-sided $SL(2)$ -like UQG of Dubois-Violette & Launer $g \dim A = g \dim B = 2$
 (or 2-sided) [studied by Bichon, W-ways]

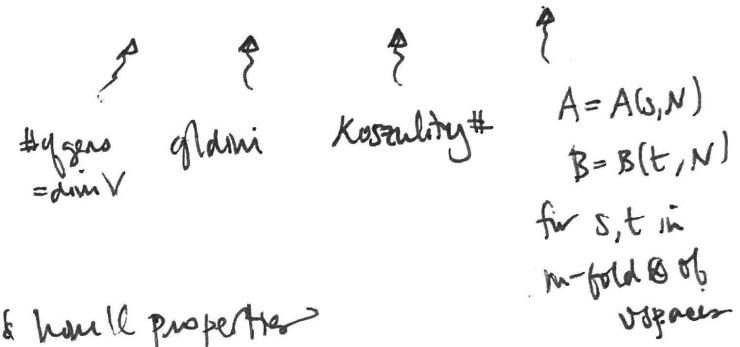
(# the generalization of ...)

* 2-sided $GL(2)$ -like UQG of Mrozinski $g \dim A = g \dim B = 2$

* 1-sided $SL(n)$ -like UQG of Bichon & Dubois-Violette preserving one preregular form.

* We also highlight genuinely new UQGs associated to N -Koszul algebras A, B

- 3-dim 12 Sklyanin algs $n=3$ $d=3$ $N=2$ $m=3$
- 4-dim 12 Sklyanin algs $n=4$ $d=4$ $N=2$ $m=4$
- Yangs-Mills algebras $n \geq 2$ $d=3$ $N=3$ $m=4$



Next up : Studying the algebraic & homological properties

* (a) representation-theoretic properties of $\mathcal{Q}_{A, B}$

(work in progress w/ Chirvaotiu & Wang -)