[U. Waterloo
- Algebra seminar]

November 9, 2018 On the quadratic dual of the Fomin-Kirillov algebras
joint with James Zhang, arXiv: 1806.09263.

/$\Bbbk$ field.

I'm interested in all things related to noncommutative algebra
- representations
- symmetries
- and, of course, ring-theoretic & homological properties

In particular, I like to study noncom algebras A that are
- $\mathbb{N}$-graded: $A = \bigoplus_{i\geq 0} \underbrace{A_i}_{\Bbbk\text{-vs}}$, $A_i \cdot A_j \subseteq A_{i+j}$
- connected: $A_0 = \Bbbk$ (nice if $\dim A_i < \infty\ \forall i$)
- generated in degree 1 & quadratic: $A = T(A_1)/(R)$, $R \subseteq A_1 \otimes A_1$

... which includes commutative poly'l algebras
& noncom. graded algebras that behave like them.

So when I stumbled across the following noncom. algebras,
which had many open questions attached to them, I was intrigued —

Def'n For $n \geq 2$, the Fomin-Kirillov algebra $\boxed{E_n}$ is an associative $\Bbbk$-alg generated by $\{x_{ij}\}_{1\leq i<j\leq n}$ of degree 1, subject to relations:

$$\left|\begin{array}{ll}
x_{ij}^2 = 0 & \forall i<j \\
x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij} = 0 & \forall i<j<k \\
x_{jk}x_{ij} - x_{ik}x_{jk} - x_{ij}x_{ik} = 0 & \forall i<j<k \\
x_{ij}x_{kl} - x_{kl}x_{ij} = 0 & \forall \{i,j\}\cap\{k,l\} = \emptyset,\ i<j,\ k<l
\end{array}\right.$$

introduced by Sergey Fomin & Anatol Kirillov in 1999 to study
the ordinary & quantum cohomology of flag manifolds Fl_n.

- In particular, there's a nice collection of elts of $\mathcal{E}_n$, "Dunkl elements" that form a commutative subalgebra, say $\mathcal{F}_n$, of $\mathcal{E}_n$.

Theorem [FK, §7] $\mathcal{F}_n$ is canonically $\cong$ to cohomology algebra of Fl_n

- Soon after, Postnikov resolved a conjecture in [FK, §13] that the comm. subalgebra of $\mathcal{E}_n^q$ (quantized $\mathcal{E}_n$) is $\cong$ to quantum cohom. algebra of Fl_n.
- We'll come back to these commutative algebras at the end of the talk, if time permits. (Have more questions than answers pertaining to this)

Since then, the FK algebras have appeared in several fields: alg. combinatorics, number theory, noncommutative geometry, Hopf algebras, & more.

There are also several unresolved questions about their fundamental structure; the main one being:

$$\boxed{\text{Q: Is } \mathcal{E}_n \text{ finite dimensional (as a } \mathbb{k}\text{-vs)?}}$$

A: $\dim_{\mathbb{k}} \mathcal{E}_n = \begin{cases} 2 & n=2 \\ 12 & n=3 \\ 576 & n=4 \\ 8{,}294{,}400 & n=5 \\ ??? & n\geq 6. \end{cases}$

↖ Naturally I was drawn to this question & tried to solve it! ... and so far -- failed. But at least this led to interesting directions ...

At this point I'll present the main results of our work & will then discuss the "what" and "why care" after.

Recall that for a quadratic algebra $A = T(V)/(R)$, V $\Bbbk$-vs, $\dim V < \infty$, $R \subseteq V \otimes V$ its <u>quadratic dual</u> is a $\Bbbk$-algebra $A^! = T(V^*)/(R^\perp)$, $R^\perp$ = orthogonal complement of R $= \{f \in V^* \otimes V^* \mid f(r) = 0 \ \forall r \in R\}$

(also known as the <u>Koszul dual</u>)

Ex. $S(V)^! = \Lambda(V)$, V $\Bbbk$-vs

Ex. Let $y_{ij} := x_{ij}^*$

$$\mathcal{E}_2 = \Bbbk[x_{12}]/(x_{12}^2) \rightsquigarrow \mathcal{E}_2^! = \Bbbk[y_{12}]$$

$$\mathcal{E}_3 = \frac{\Bbbk\langle x_{12}, x_{13}, x_{23}\rangle}{\left(\begin{array}{ccc} x_{12}^2 & x_{13}^2 & x_{23}^2 \\ \multicolumn{3}{c}{x_{12}x_{23} - x_{23}x_{13} - x_{13}x_{12}} \\ \multicolumn{3}{c}{x_{23}x_{12} - x_{13}x_{23} - x_{12}x_{13}} \end{array}\right)} \rightsquigarrow \mathcal{E}_3^! = \frac{\Bbbk\langle y_{12}, y_{13}, y_{23}\rangle}{\begin{pmatrix} y_{12}y_{23} + y_{23}y_{13} & y_{12}y_{23} + y_{13}y_{12} \\ y_{23}y_{12} + y_{13}y_{23} & y_{23}y_{12} + y_{12}y_{13} \end{pmatrix}}$$

<u>Lemma</u>: By taking $y_{ji} = -y_{ij}$ for $i<j$, we get that in general:

$$\mathcal{E}_n^! = \Bbbk\langle y_{ij}\rangle_{1 \le i < j \le n} \Big/ \left(\begin{array}{ll} y_{ij}y_{jk} + y_{jk}y_{ki} & \forall i,j,k \text{ distinct} \\ y_{ij}y_{kl} + y_{kl}y_{ij} & \forall \{i,j\} \cap \{k,l\} = \emptyset, \ \substack{i<j \\ k<l} \end{array}\right)$$

Main Theorem [WZ] The algebras $\mathcal{E}_n^!$ satisfy the following conditions:

<u>Ring-theoretic</u>

$$\left.\begin{array}{l} \text{① Noetherian} \\ \text{② module finite / center} \\ \text{③ Gelfand Kirillov dimension } \lfloor \tfrac{n}{2} \rfloor \end{array}\right] \forall n \geq 2$$

④ not prime ($\Rightarrow$ not a domain) $\big]$ $\forall n \geq 3$

<u>Homological</u>

⑤ AS-, Aus-regular $\Leftrightarrow$ $n=2$

⑥ AS-, Aus-Gorenstein $\Leftrightarrow$ $n=2,3$

⑦ AS-CM, CM $\Leftrightarrow$ $n=2,3$

⑧ depth ≤ 1 $\forall n \geq 2$.

AS = Artin-Schelter, Aus = Auslander
CM = Cohen-Macaulay

Why care about quadratic duals?

In the nice case, when $A = T(V)/(R)$ (connected $\mathbb{N}$-graded quadratic) is **Koszul** [= the trivial A-module $\Bbbk = A/\bigoplus_{i\geq 1} A_i$ has a linear resolution by free A-modules]

we get that $A^! \cong \mathrm{Ext}_A^*(\Bbbk, \Bbbk) =: E(A)$

$\rightsquigarrow$ $A^!$ carries a lot of cohomological information about A & viceversa because $(A^!)^! \cong A$.

But what makes E_n so difficult to study, in this context, is the fact that E_n is not Koszul $\forall n \geq 3$ [Roos].

Still, $E_n^!$ is useful cohomologically —

Fact For A connected $\mathbb{N}$-graded quadratic, not nec. Koszul, get that $A^! \cong \bigoplus_{i\geq 0} \mathrm{Ext}_A^{i,i}(\Bbbk, \Bbbk)$, the "diagonal" subalg of $E(A)$ generated in deg 1.

Loose Fact: "The homological growth (e.g. gldim) of $E(A)$, and of $A^!$, is finite $\Leftrightarrow$ the $\Bbbk$-vs dimension of A is finite ...

... in the A_∞ setting"

see "Koszul equivalences in A_∞-setting" by Lu, Palmieri, Wu, Zhang (2008) for more details

Speaking of growth, let's discuss ring-theoretic properties of noncom. (graded) algs —

① Noetherian condition = ACC on left & on right ideals &

② $E_n^!$ is a module/$Z(E_n^!)$ of finite rank

buy us a lot of leverage ...

④ A is <u>prime</u> $\iff \forall a^{\neq 0}, b^{\neq 0} \in A$ get $aAb \neq 0$.

weaker than domain condition, & still desirable

Eg. matrix rings $\mathrm{Mat}_d(\mathbb{k})$ are not domains, but are prime.

$E_n^!$ is <u>not</u> prime for $n \geq 3$: $\underbrace{y_{jk}}_{(\text{generator}) \neq 0} (\underbrace{y_{ij}^2 - y_{ik}^2}_{\neq 0 \ (\text{can show})}) = 0$ $\forall$ distinct i,j,k

$(\exists a^{\neq 0}, b^{\neq 0} \Rightarrow aAb = 0)$

<u>Hierarchy</u>: Domain $\Rightarrow$ Prime $\Rightarrow$ Semiprime (= contains no nilpotent ideals)

X X ??

<u>Question</u> Is $E_n^!$ semiprime $\forall n \geq 2$??

③ <u>GKdimension</u> of a connected, $\mathbb{N}$-graded, locally-finite algebra A

$(A = \bigoplus_{i \geq 0} A_i, \ \dim A_i < \infty \ \forall i, \ A_0 = \mathbb{k})$

is defined by:

$$\mathrm{GKdim}(A) = \limsup_{n \to \infty} \frac{\log\left(\sum_{i=0}^{n} \dim A_i\right)}{\log(n)}$$

a very useful growth measure ...

- $\mathrm{GKdim}(A) = 0 \iff \dim A < \infty$
- $\mathrm{GKdim}\, \mathbb{k}[x_1, \ldots, x_n] = n$
- Say A is of <u>poly'l growth</u> if $\mathrm{GKdim}(A) \in \mathbb{Z}_+$
- A commutative $\Rightarrow \mathrm{GKdim}(A) = \mathrm{Krull\,dim}(A)$.

On homological properties of noncom. graded algebras –

This mimics the hierarchy of nice homological properties of commutative local algebras $(R, \mathfrak{m})$. Assume R Noeth., $\text{Kdim}(R) = d < \infty$.

$$\begin{tikzcd}
\boxed{\text{regular}} \arrow[r, Rightarrow] & \boxed{\text{Gorenstein}} \arrow[r, Rightarrow] & \boxed{\text{Cohen-Macaulay}}
\end{tikzcd}$$

regular: $\text{gldim}\, R = d < \infty$

Gorenstein: $\text{injdim}\, R = d < \infty$

$$\Updownarrow$$

$$\text{Ext}^i_R(\Bbbk, R) \cong \begin{cases} \Bbbk, & i = d \\ 0, & i \neq d \end{cases}$$

Cohen-Macaulay: depth = max{ length of reg-seq. in $\mathfrak{m}$ } $(\leq d)$

$$\text{CM} \overset{\text{def.}}{\Longleftrightarrow} \text{depth}(R) = d.$$

Have analogous hierarchy for noncom. con. graded algebras $A = \bigoplus_{i \geq 0} A_i$

Take $\mathfrak{m} = \bigoplus_{i \geq 1} A_i$ (augmentation ideal). Assume: A Noeth, $\text{GKdim} A = d < \infty$

$$\begin{tikzcd}
\boxed{\text{AS-regular}} \arrow[r, Rightarrow] & \boxed{\text{AS-Gorenstein}} \arrow[rr, Rightarrow] & & \boxed{\text{AS-CM}} \arrow[dl, Leftrightarrow] \arrow[d, Leftrightarrow, "\text{if } A \text{ mod fin over } Z(A)"] \\
 & & \boxed{\begin{array}{c}\text{depth}(A) \\ = \text{GKdim}(A)\end{array}} & \boxed{\begin{array}{c} j_A(M) + GK(M) \geq d \\ \forall M \in A\text{-mod, f.g.}\end{array}} \\
\boxed{\text{Ausl.-regular}} \arrow[uu, Rightarrow] \arrow[r, Rightarrow] & \boxed{\text{Ausl.-Gorenstein}} \arrow[uu, Rightarrow] & & \boxed{\text{CM}} \arrow[u, Rightarrow] \arrow[u, dotted, bend right, "\text{(have equality)}"']
\end{tikzcd}$$

AS-regular: $\text{gldim} A = d < \infty$ & $\text{Ext}^i_A(\Bbbk, A) = \begin{cases} \Bbbk & i = d \\ 0 & i \neq d \end{cases}$ as left & right A-mods

AS-Gorenstein: $\text{injdim}\, A = d < \infty$ & same Ext-cond.

AS-CM: "local coh. is concentrated in one deg" $H^i_{\mathfrak{m}}(A) = 0$ unless $i = d$.

Ausl.-regular: $\text{gldim}(A) = d < \infty$ & technical Ext cond.

{$\forall q \geq 0$, get $j_A(N) \geq q$ ... for all right/left A-submod. M of $\text{Ext}^q_A(N, A)$} → technical Ext cond.

Ausl.-Gorenstein: $\text{inj dim}(A) = d < \infty$ & same Ext. cond.

Here, for $N \in A$-mod:

- grade $j_A(N) = \inf\{i \mid \text{Ext}^i_A(N, A) \neq 0\}$
- depth $\text{depth}(N) = \inf\{i \mid \text{Ext}^i_A(\Bbbk, N) \neq 0\}$

In summary, we understand a lot about $E_n^!$
& this may help provide insight into the structure of E_n.

In the proof of the Main Theorem,
there were two important commutative subalgs of $E_n^!$ that were used:

$\mathcal{C}_n$ = subalgebra of $E_n^!$ generated by $\{y_{ij}^2 =: a_{ij}\}_{1\le i<j\le n}$

$$\mathcal{D}_n = \Bbbk\langle a_{ij}\rangle_{1\le i<j\le n} \Big/ (a_{ij}a_{jk} - a_{ij}a_{ik})\ \forall i,j,k \text{ distinct}.$$

With Pavel Etingof, we showed that

- $\mathcal{C}_n \cong \mathcal{D}_n$
- computed its Hilbert series (combinatorial formula)
- showed that it's reduced & thus semiprime

$\quad\quad\quad\quad\quad\quad\quad\quad\uparrow$ no nonzero nilp. elts

Question Is $Z(E_n^!) \cong$ this com. subalg?

More crucially –

Question: What is the relationship between
the com subalg $\mathcal{C}_n$ of $E_n^!$
& the (quantum) cohomology alg. of the flag mfld?
(which pertains to Fomin-Kirillov's original work)