

November 9, 2018    On the quadratic dual of the Fomin-Kirillov algebras  
 joint with James Zhang, arXiv: 1806.09263.

$\mathbb{k}$  field.

I'm interested in all things related to noncommutative algebras

- representations
- symmetries
- and, of course, ring theoretic & homological properties

In particular, I like to study noncomm algebras  $A$  that are

- $N$ -graded:  $A = \bigoplus_{i \geq 0} A_i$ ,  $A_i \cdot A_j \subseteq A_{i+j}$
- connected:  $A_0 = \mathbb{k}$      $\mathbb{k}$ -vs (nice if  $\dim A_i < \infty \forall i$ )
- generated in degree 1 & quadratic:  $A = T(A_1) / (R)$ ,  $R \subseteq A_1 \otimes A_1$   
 ... which includes commutative poly  $\mathbb{k}$  algebras  
 & noncomm. graded algebras that behave like them.

So when I stumbled across the following noncomm algebras, which had many open questions attached to them, I was intrigued —

Defn For  $n \geq 2$ , the Fomin-Kirillov algebra  $[E_n]$  is an associative  $\mathbb{k}$ -alg generated by  $\{x_{ij} \mid 1 \leq i < j \leq n\}$  of degree 1, subject to relations:

$x_{ij}^2 = 0$	$\forall i < j$
$x_{ij} x_{jk} - x_{jk} x_{ik} - x_{ik} x_{ij} = 0$	$\forall i < j < k$
$x_{jk} x_{ij} - x_{ik} x_{jk} - x_{ij} x_{ik} = 0$	$\forall i < j < k$
$x_{ij} x_{kl} - x_{kl} x_{ij} = 0$	$\forall \{i, j\} \cap \{k, l\} = \emptyset, i < j, k < l$

introduced by Sergey Fomin & Anatol Kirillov in 1999 to study the ordinary & quantum cohomology of flag manifolds  $Fl_n$ .

- In particular, there's a nice collection of alts of  $E_n$ , "Dunkl elements" that form a commutative subalgebra, say  $F_n$ , of  $E_n$ .

Theorem [FK, §7]  $F_n$  is canonically  $\cong$  to cohomology algebra of  $FL_n$

- Soon after, Postnikov resolved a conjecture in [FK, §13] that the com-subalgebra of  $E_n^q$  (quantized  $E_n$ ) is  $\cong$  to quantum cohom. algebra of  $FL_n$ .

- We'll come back to these commutative algebras at the end of the talk, if time permits. (Have more questions than answers pertaining to this)

Since then, the FK algebras have appeared in several fields: alg. combinatorics, number theory, noncommutative geometry, Hopf algebras, & more.

There are also several unresolved questions about their fundamental structure; the main one being:

Q: Is  $E_n$  finite dimensional (as a  $k$ -vs)?

A: $\dim_k E_n =$ {	2	$n=2$
	12	$n=3$
	576	$n=4$
	8,294,400	$n=5$
	???	$n \geq 6$ .

Naturally I was drawn to this question & tried to solve it!  
... and so far -- failed.

But at least this led to interesting directions ...

At this point I'll present the main results of our work & will then discuss the "what" and "why care" after.

Recall that for a quadratic algebra  $A = T(V)/(R)$ ,  $V$   $k$ -vs,  $\dim V < \infty$ ,  $R \subseteq V \otimes V$   
 its quadratic dual is a  $k$ -algebra  $A' = T(V^*)/(R^\perp)$ ,  $R^\perp = \text{orthogonal complement of } R$   
 (also known as the Koszul dual)  
 $= \{f \in V^* \otimes V^* \mid f(r) = 0 \forall r \in R\}$

ex.  $S(V)' = \Lambda(V)$ ,  $V$   $k$ -vs

ex. let  $y_{ij} := x_{ij}^*$

$$E_2 = k[x_{12}] / (x_{12}^2) \rightsquigarrow E_2' = k[y_{12}]$$

$$E_3 = k\langle x_{12}, x_{13}, x_{23} \rangle / \begin{pmatrix} x_{12}^2 & x_{13}^2 & x_{23}^2 \\ x_{12}x_{23} - x_{23}x_{13} - x_{13}x_{12} \\ x_{23}x_{12} - x_{13}x_{23} - x_{12}x_{13} \end{pmatrix} \rightsquigarrow E_3' = k\langle y_{12}, y_{13}, y_{23} \rangle / \begin{pmatrix} y_{12}y_{23} + y_{23}y_{13} & y_{12}y_{23} + y_{13}y_{12} \\ y_{23}y_{12} + y_{13}y_{23} & y_{23}y_{12} + y_{12}y_{13} \end{pmatrix}$$

Lemma: By taking  $y_{ji} = -y_{ij}$  for  $i < j$ , we get that in general:

$$E_n' = k\langle y_{ij} \mid 1 \leq i < j \leq n \rangle / \begin{pmatrix} y_{ij}y_{jk} + y_{jk}y_{ik} & \forall i, j, k \text{ distinct} \\ y_{ij}y_{kl} + y_{kl}y_{ij} & \forall \{i, j\} \cap \{k, l\} = \emptyset, \begin{matrix} i < j \\ k < l \end{matrix} \end{pmatrix}$$

**Main Theorem [Wi2]** The algebras  $E_n'$  satisfy the following conditions:

Ring-theoretic

- ① Noetherian
  - ② module finite/center
  - ③ Gelfand-Kirillov dimension  $\lfloor \frac{n}{2} \rfloor$
  - ④ not prime ( $\Rightarrow$  not a domain)
- ]  $\forall n \geq 2$   
]  $\forall n \geq 3$

Homological

- ⑤ AS-, Aus-regular  $\Leftrightarrow n=2$
- ⑥ AS-, Aus-Gorenstein  $\Leftrightarrow n=2, 3$
- ⑦ AS-CM, CM  $\Leftrightarrow n=2, 3$
- ⑧ depth  $\leq 1$   $\forall n \geq 2$ .

AS = Artin-Schelter, Aus = Auslander  
 CM = Cohen-Macaulay

Why care about quadratic duals?

In the nice case, when  $A = \mathbb{T}(V)/(R)$  (connected  $N$ -graded quadratic) is Koszul [= the trivial  $A$ -module  $k = A / \bigoplus_{i \geq 1} A_i$  has a linear resolution by free  $A$ -modules] we get that  $A^! \cong \text{Ext}_A^*(k, k) =: E(A)$ .

$\rightsquigarrow$   $A^!$  carries a lot of cohomological information about  $A$  & vice versa because  $(A^!)^! \cong A$ .

But what makes  $E_n$  so difficult to study, in this context, is the fact that  $E_n$  is not Koszul  $\forall n \geq 3$  [Roos].

Still,  $E_n^!$  is useful cohomologically -

Fact For  $A$  connected  $N$ -graded quadratic, not nec. Koszul, get that  $A^! \cong \bigoplus_{i \geq 0} \text{Ext}_A^{i,i}(k, k)$ , the "diagonal" subalg of  $E(A)$  generated in deg 1.

Loose Fact: "The homological growth (e.g. gldim) of  $E(A)$ , and of  $A^!$ , is finite

see "Koszul equivalences in  $A_{\infty}$ -setting" by Lu, Palmieri, Wu, Zhang (2008) for more details

$\Leftrightarrow$  the  $k$ - $k$  dimension of  $A$  is finite ...  
"in the  $A_{\infty}$  setting"

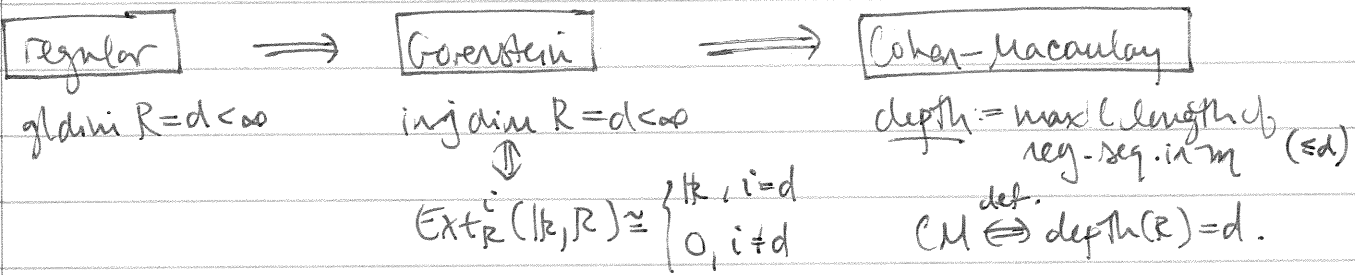
Speaking of growth, let's discuss ring-theoretic properties of noncom. (graded) algs -

- ① Noetherian condition = ACC on left & on right ideals  $\left\{ \begin{array}{l} \leftarrow \text{buy us a lot} \\ \leftarrow \text{of leverage...} \end{array} \right.$
- ②  $E_n^!$  is a module /  $\mathcal{Z}(E_n^!)$  of finite rank

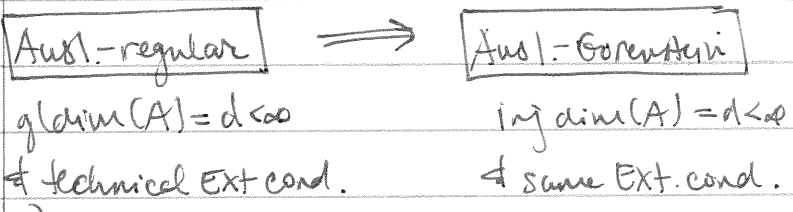
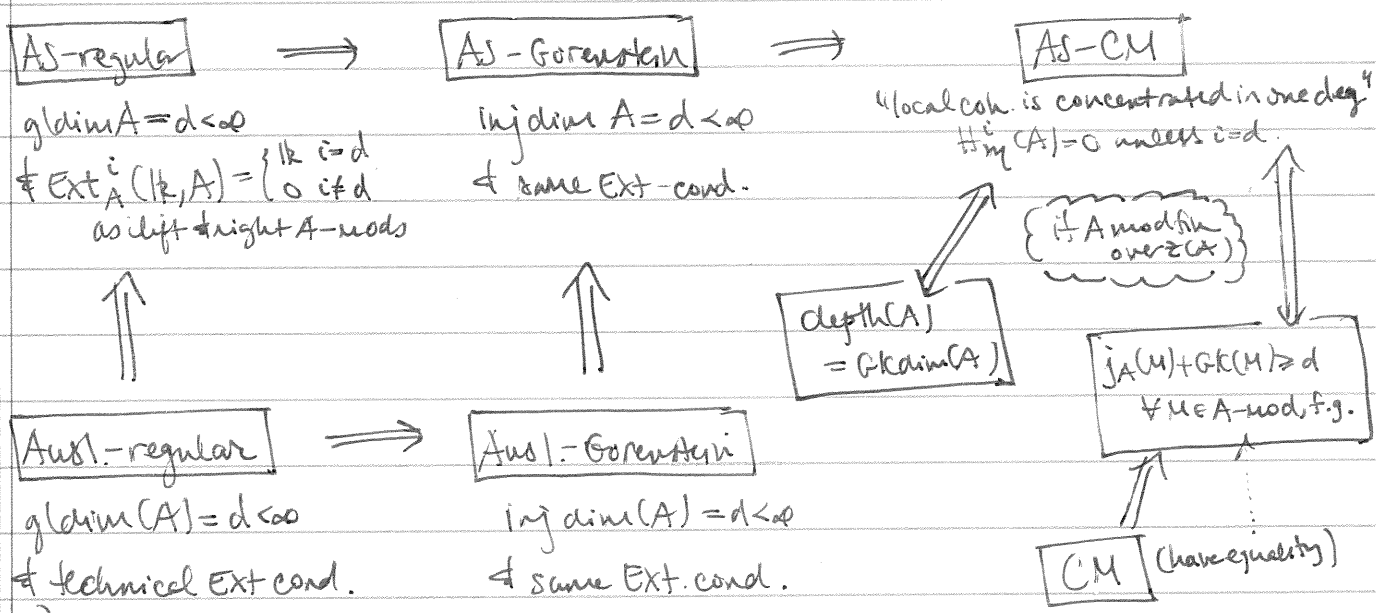


On homological properties of noncom. graded algebras

This mimics the hierarchy of nice homological properties of commutative local algebras  $(R, \mathfrak{m})$ . Assume  $R$  Noeth.,  $\text{Kdim}(R) = d < \infty$ .



Have analogous hierarchy for noncom. com. graded algebras  $A = \bigoplus_{i \geq 0} A_i$   
 Take  $\mathfrak{m} = \bigoplus_{i \geq 1} A_i$  (augmentation ideal). Assume:  $A$  Noeth.,  $\text{gldim } A = d < \infty$



$\forall q \geq 0,$   
 get  $j_A(N) \geq ? \dots$   
 for all right/left  
 $A$ -submod.  $M$  of  
 $\text{Ext}_A^q(N/A)$

Here, for  $N \in A\text{-mod}$ :

- grade  $j_A(N) = \inf \{ i \mid \text{Ext}_A^i(N, A) \neq 0 \}$
- depth  $\text{depth}(N) = \inf \{ i \mid \text{Ext}_A^i(k, N) \neq 0 \}$

In summary, we understand a lot about  $En!$

& this may help provide insight into the structure of  $En$ .

In the proof of the Main Theorem,

there were two important commutative subalgs of  $En!$  that were used:

$\mathcal{C}en =$  subalgebra of  $En!$  generated by  $\{y_{ij}^2 =: a_{ij}^2 \mid 1 \leq i < j \leq n\}$

$\mathcal{S}en = \mathbb{K}\langle a_{ij} \mid 1 \leq i < j \leq n \rangle / (a_{ij}a_{jk} - a_{jka_{ik}})_{i,j,k \text{ distinct}}$

With Pavel Ettinger, we showed that

- $\mathcal{C}en \cong \mathcal{S}en$
- computed its Hilbert series (combinatorial formula)
- showed that it's reduced & thus semiprime  
     ↑  
     no nonzero nilp. elts

Question Is  $Z(En!)$   $\cong$  this com. subalg?

More crucially -

Question: What is the relationship between  
the com subalg  $\mathcal{C}en$  of  $En!$

& the (quantum) cohomology alg. of the flag mfd?  
(which pertains to Fomin-Kirillov's original work)