August 17, 2016
[BIRS - CMO]

Quantum Symmetry (survey)
(via Hopf algebra (co)actions)

Take k = alg closed

Interested in symmetries of $\underbrace{k\text{-algebras}}$ $A = (\underset{k\text{-vs}}{A}, \underset{\substack{\text{multip}\\ A\otimes A\to A}}{m_A}, \underset{\substack{\text{unit}\\ k\to A}}{u_A})$

$\Downarrow$ function algebras $\mathcal{O}(X)$ of geometric objects X

$\Downarrow$ algebras in general....

Motivation for looking at (co)actions of Hopf algebras

Classical Geometry $\xleftarrow{\text{contravariant}}$ Commutative Algebra

$Aff/_k$ (category of affine varieties) | $ComAlg/_k$

$G \times X \longrightarrow X$ group action on an affine variety

morphism in Aff_k

$\Rightarrow$ G linear algebraic group (group + affine variety)

$\Rightarrow$ $G = (G, \underset{\text{multip}}{m}, \underset{\text{unit elt}}{e}, \underset{\text{inversion}}{i})$

morphisms in Aff_k

Satisfies group axioms

ⓐ $m(\sigma, e) = m(e, \sigma) = \sigma$

ⓑ $m(\sigma, i(\sigma)) = m(i(\sigma), \sigma) = e$

ⓒ associativity

$\forall \sigma \in G$

$\mathcal{O}(X) \longrightarrow \mathcal{O}(X) \otimes \mathcal{O}(G)$

com. alg. equipped with

$\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ comult.

$\varepsilon : \mathcal{O}(G) \to k$ counit

$f \mapsto f(e)$

$S : \mathcal{O}(G) \to \mathcal{O}(G)$ antipode

Satisfying Hopf algebra axioms

ⓐ $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id$

ⓑ antipode axiom

ⓒ coassociativity

$\therefore$ $\mathcal{O}(G)$ is a com. Hopf alg

<u>Noncom Geometry</u> ⟵ - - - - - - ⟶ <u>Noncom Alg</u>

quantum { variety ↻ groups action

noncom { algs, Hopf algebras } coaction

(Since Hopf algebras have a dual structure so that coaction of H correspond to actions of its dual H° ...) ← also a Hopf algebra

$\boxed{\text{Goal}}$ Study (co)actions of Hopf algebras H on (classes of) k-algebras A.

<u>Defn</u> • Say H <u>(co)acts</u> on A if A is an algebra in the $\otimes$ category H-(co)mod.

[A is an H-(co)module and m_A, u_A are H-(co)module maps]

Helpful to have a notion of faithfulness...

• Say H (co)acts on A <u>inner-faithfully</u> if $\nexists$ proper factor Hopf algebra (Hopf subalgebra) that (co)acts on A.

• A admits $\boxed{\text{No Quantum Symmetry}}$ -OR- admits $\boxed{\text{Genuine Quantum Symmetry}}$

If all Hopf actions on A factor through action of <u>cocom.</u> Hopf algebra → ($\tau\Delta = \Delta$, eg. kG, $U(\mathfrak{g})$)

[$H \curvearrowright A$ innerfaithfully $\Rightarrow$ H cocom]

(back in classical setting)

otherwise.

That is, $\exists$ an innerfaithful action of a noncocom H on A

or $\exists$ an innerfaithful coaction of a noncom Hopf alg on A

H cocom / k char $k = 0$
$\Rightarrow H \cong U(\mathfrak{g}) \rtimes kG$
due to Cartier-Gabriel-Kostant

$x \in \mathfrak{g}$: $\Delta(x) = 1 \otimes x + x \otimes 1$

$g \in G$: $\Delta(g) = g \otimes g$

• Interested in <u>all</u> Quantum Symmetries?

Call a Hopf algebra $\mathcal{O}_A$ a $\boxed{\text{universal quantum group}}$ coacting on A if $\forall$ coactions of H on A $\exists!$ Hopf map $\pi: \mathcal{O}_A \to H$ so that

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A \otimes \mathcal{O}_A \arrow[d, dashed, "\mathrm{id} \otimes \pi"] \\ & A \otimes H \end{tikzcd}$$

($\mathcal{O}_A$ coacts on A innerfaithfully)

The 3 problems: No Quantum Symmetry, Genuine Quantum Symmetry, Universal Quantum Symmetry are difficult ...

... Helpful to impose hypothesis

On H: finite dim'l? (as a k-vs)
- semisimple? (as a k-alg) (Involutory? $S^2 = Id_H$)
- pointed? (all simple H-comodules are 1-dim'l)
  - Cartan type?
  - super type?
  - ⋮

On A:
- ring-theoretic
  - non/commutative?
  - PI? (module finite/center)?
  - domain?
  - growth condition (GK-dimension)?
  - ⋮
- homological
  - global dimension?
  - Gorenstein condition?
  - ⋮

On (co)action
- (inner faithfulness)
- preservation of filtration/grading of A if A is filtered/graded
- homological (co)determinant?
  - ↳ trivial? (e.g. $G \leq SL_2(k)$ acting on $k[u,v]$)

Different techniques are needed to attack the 3 problems above for different choices of hypotheses.

Now Results & Questions ...
- algebraic setting
- analytic setting

No Quantum Symmetry in algebraic setting
$\Downarrow$
No Quantum Symmetry in analytic setting

No Quantum Symmetry Results

Results in blue

* Question/conjecture in Red

$k=\bar{k}$ char k?	Hypotheses on H	On A (X = smooth irred affine variety)	On H-action on A	Reference (Preprint #)
0	semisimple	com. domain $=\mathcal{O}(X)$	—	Etingof-W 1801.4161
$p>0$	semisimple & cos.	com. domain	—	" — "
* $p>0$	cosemisimple, f.d. not nec semisimple	com. domain	—	posed in 1301.4161
0	semisimple	com. fin-gen w/ no homog deg 2 relations	—	Etingof-Goswami-Mandal-W 1507.08486
0	semisimple	$A_n(k[x_1,\dots,x_s])$ Weyl alg.	—	Cuadra-Etingof-W 1409.1644
$p>>0$	semisimple & cos	$A_n(k[x_1,\dots,x_s])$	—	" — "
$p\geq 0$	semisimple & cos	division alg D	$(\deg D, (\dim H)!)=1$	" — "
0	finite dim'l	$A_n(k)$	—	Cuadra-Etingof-W 1509.01165
0	finite dim'l	ring of diff'l ops $D(X)$	—	" — "
0	semisimple or finite dim'l (†)	{ formal deformation of com. domain A_0; filtered deformation of A_0	(†) { Poisson center of $\mathcal{O}(A_0)$ is trivial & filtration of A preserved	Etingof-W 1602.00532
0	semisimple or finite dim'l (†)	{ filtered deformation that is PI modulo $p>>0$; generic skew poly'l rings, Sklyanin algs & THCRs	(†) under a nondeg. condition	Etingof-W 1605.00560

In the analytic setting

INCOMPLETE LIST

$k=\mathbb{C}$	On $A=\mathcal{O}(X)$ (or X) C^*-algebra	On H-coaction on A	Reference (see References therein too!)
$H=\mathcal{O}_X$ (universal) quantum group C^*-Hopf algebra	X = n points $n\leq 3$	See References for details	Wang 9807091 ★
	X = n-cycle $n\neq 4$		Banica 0304025
" — " preserving isometries	$X=S^n$ spheres		Bhowmick-Goswami 0707.2648
	$X=\mathbb{T}^n$ n-tori		Bhowmick 0803.4434
	X = smooth cpt connected Riemannian mfld		Goswami-Joardar 1309.1294
	X = cpt connected mfld w/ neg curvature		Chirvasitu 1503.07984

Genuine Quantum Symmetry Results & Applications (Quantum Invariant Thy & Deformations)

$k=\bar{k}$ char $k=0$

On H	On A	On H-action on A	Reference	Application(s)
finite dim'l nonsemisimple pointed of finite Cartan type (w/ braiding matrix governed by root of 1 of 'high order')	Commutative domain (boils down to fields)	(inner faithful & k-linear)	Etingof-W 1403.4673 1511.09320	Quasiclassical analogue: classify faithful Poisson actions of Poisson alg groups G on irred alg varieties X w/ 0 Pois. ideals (eg. classify $X=G/G'$ Pois. homog spaces)
semisimple	PI domain $\Uparrow$ module finite over center	inner faithful	posed in Etingof 1301.4161 **Question** Is PIdeg $H^* \leq (\text{PIdeg } A)^2$? * Bound is sharp * A com $\Rightarrow$ PI deg $A=1$ $\Rightarrow$ PIdeg H^* $\Rightarrow$ H com $\Rightarrow$ $H \simeq kG$ (No Q. Sym) * True in Hopf-Galois case [Cuadra-Etingof 1508.01251]	
finite dim'l pointed: Taft algs $T_S(n)$, $u_q(sl_2)$, $D(T_S(n))$ double	path alg kQ, Q finite, loopless, no parallel arrows	(not nec. inner faithful) preserves path length filtration	Kinser-W 1410.7696	
semisimple	— " — & more	(not nec. inner faithful) preserves path length grading	Kinser-W & others? (in progress)	
finite dim'l (ex. kG, $G \leq GL_2(k)$ finite)	Artin-Schelter reg. of dim 2: gldim 2, GKdim $<\infty$ (polyl growth), AS Gorenstein (graded alg) (ex. $A=k[x,y]$)	inner faithful, preserves grading, trivial homological determinant (ex. $G \leq SL_2(k)$ finite, kG acts linearly & faithfully on $k[x,y]$)	Chan-Kirkman-W-Zhang 1303.7203	when H semisimple: Quantum McKay correspondence I. CKWZ 1607.06977 II. CKWZ in preparation

Universal Quantum Symmetry

INCOMPLETE LIST

Object(s) $(X, X', ..)$ being coacted upon / preserved	Condition on coaction(s)	Def'n of universal quantum group (UQG)	Presentation of UQG	Properties of UQG
Pair of quadratic algebras A, A' (e.g. Koszul) ⇝ Can be extended to N-homog. A, A' (e.g. N-Koszul)	Coaction on A, A' is "simultaneous"	Manin 1988 ★ book: Q. groups & NC Geom.	— (not nec. fin. gen. or fin. presented)	—
A, A' skew-poly'l rings (which are Noeth. Artin-Schelter regular)	" — "	" — "	Artin-Schelter-Tate 1991 Takeuchi (special case $q_{ij} = q$) 1990	AST: shares ring-theoretic properties w/ A, A'
A, A' not nec. Noetherian AS-regular of dim 2 (⟹ Koszul) classified by J. Zhang 1998	" — "	" — " triv. homl'l codet → arbitrary homl'l codet →	Dubois-Violette & Launer 1990 (though this was not their intention) Mrozinski 1201.3494	Bichon 0411114 1204.0687 W-Wang 1503.09185
A, A' not nec. Noetherian, N-Koszul AS regular, dim ≥ 2 Noetherian dim = 3 classified by Artin-Schelter 1987 otherwise open	" — "	" — "	Chirvasitu-W-Wang 1605.06428	Chirvasitu-W-Wang (in progress)

In the analytic setting

INCOMPLETE LIST

$A = \mathcal{O}(X)$ C^*-algebra			
X = finite set of points	see References for details	Wang (1998)	see References for details
X = finite graph		Bichon (2003)	
X = finite dim'l Hilbert space		Van Daele-Wang (1996)	
X = {finite, cpt} metric space		Banica (2005) Goswami 1205.6099	
X = Riemannian mfld		Bhowmick-Goswami (2009)	