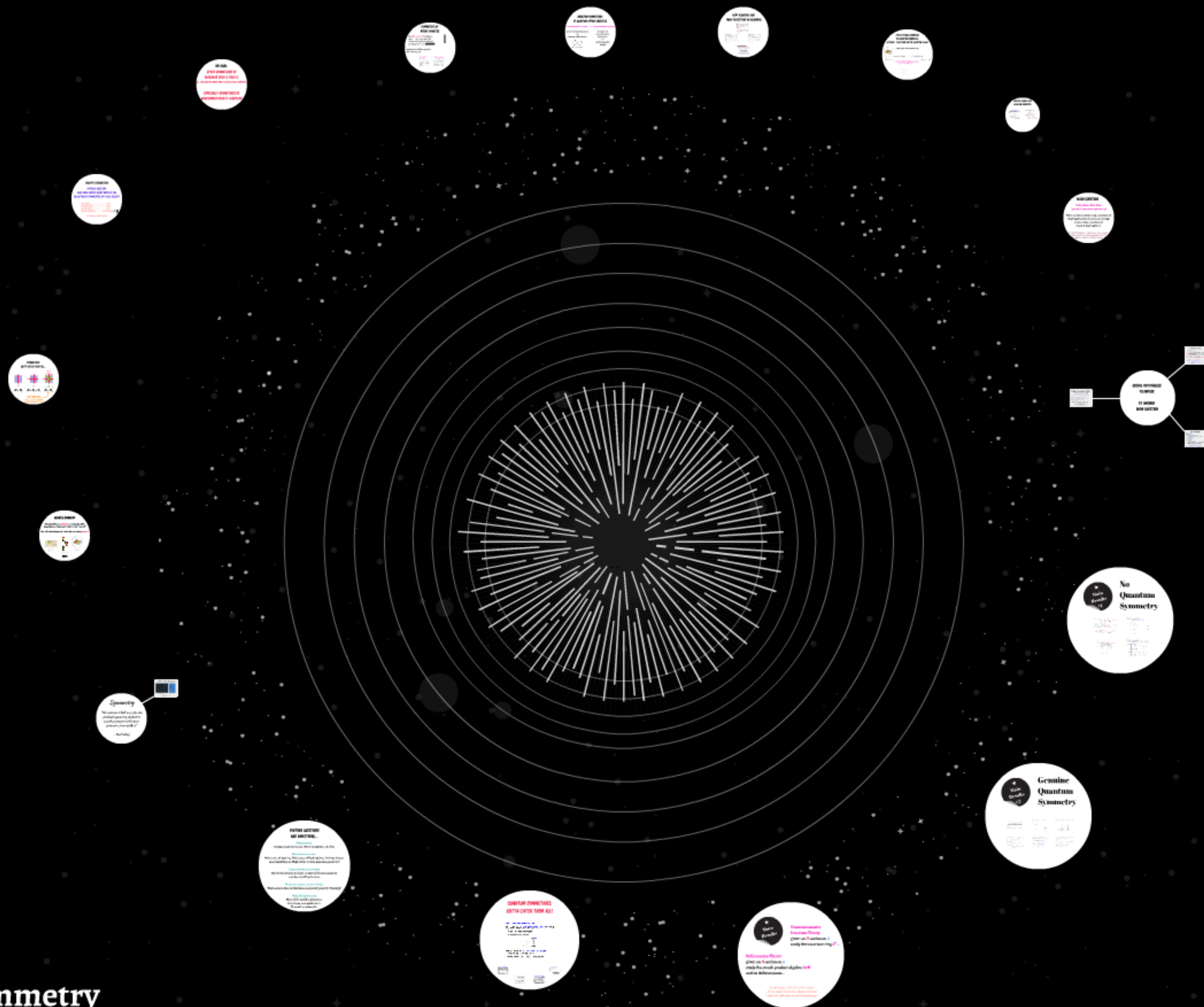

Quantum Symmetry

Chelsea Walton, Temple University

Miami U. Ohio, Annual Math Conference
September 2017

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Symmetry

"The universe is built on a plan the profound symmetry of which is somehow present in the inner structure of our intellect."

– Paul Valery

"...symmetry is often a constituent of beauty..."

-Winston Churchill

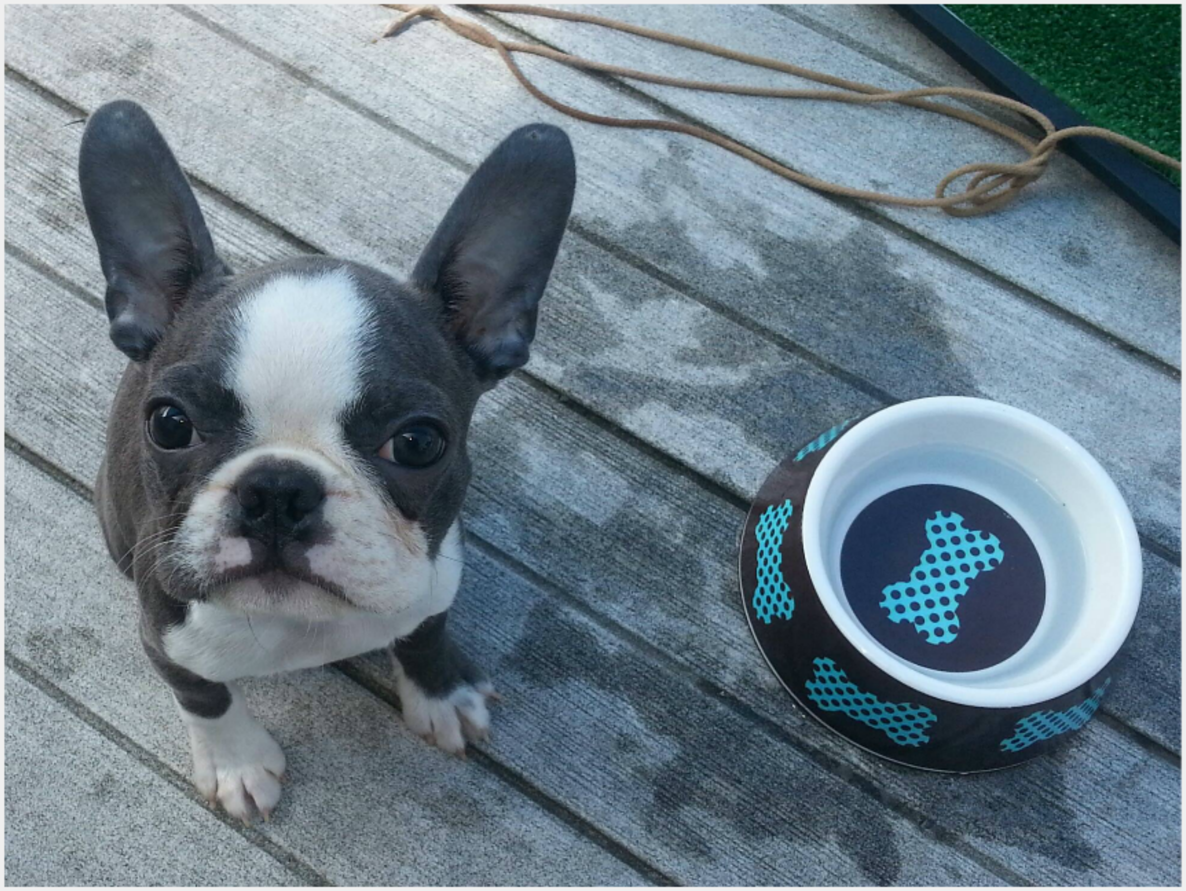


Fig: Mr. Mischief Maker

Fig: Dr. Boom-Boom

WHO IS MORE BEAUTIFUL?

GROUPS & SYMMETRY

Given an object *X*, a symmetry of *X* is an invertible property-preserving transformation from *X* to itself.

The collection of symmetries of an object *X* forms a group *G*.

Fig: Butterfly

$$G = \mathbb{Z}_2$$

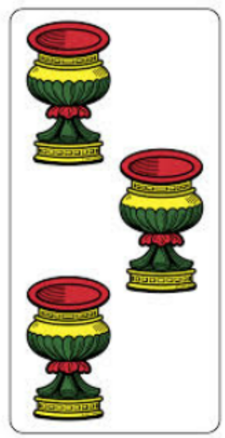


Fig: Configuration of 3 cups

$$G = S_3$$

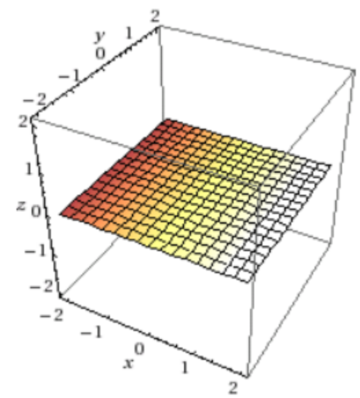
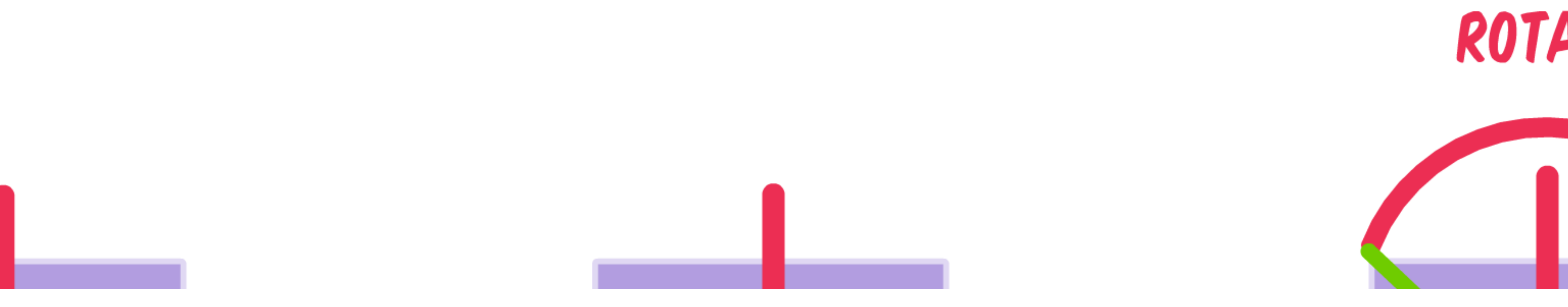
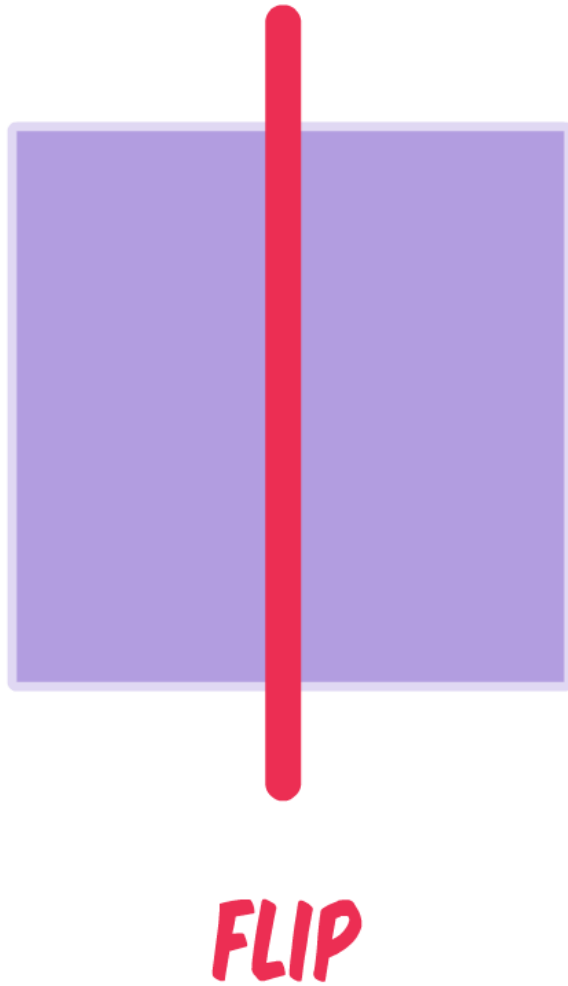


Fig: Real 2-space

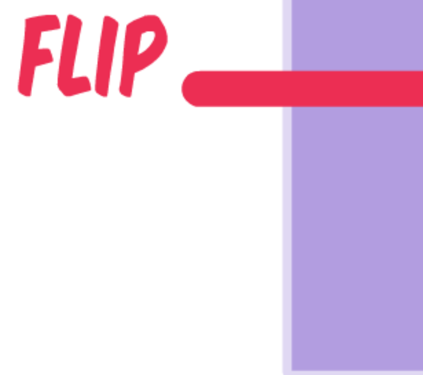
$$G = GL_2(\mathbb{R})$$

SYMMETRIES:
GOTTA CATCH THEM ALL...

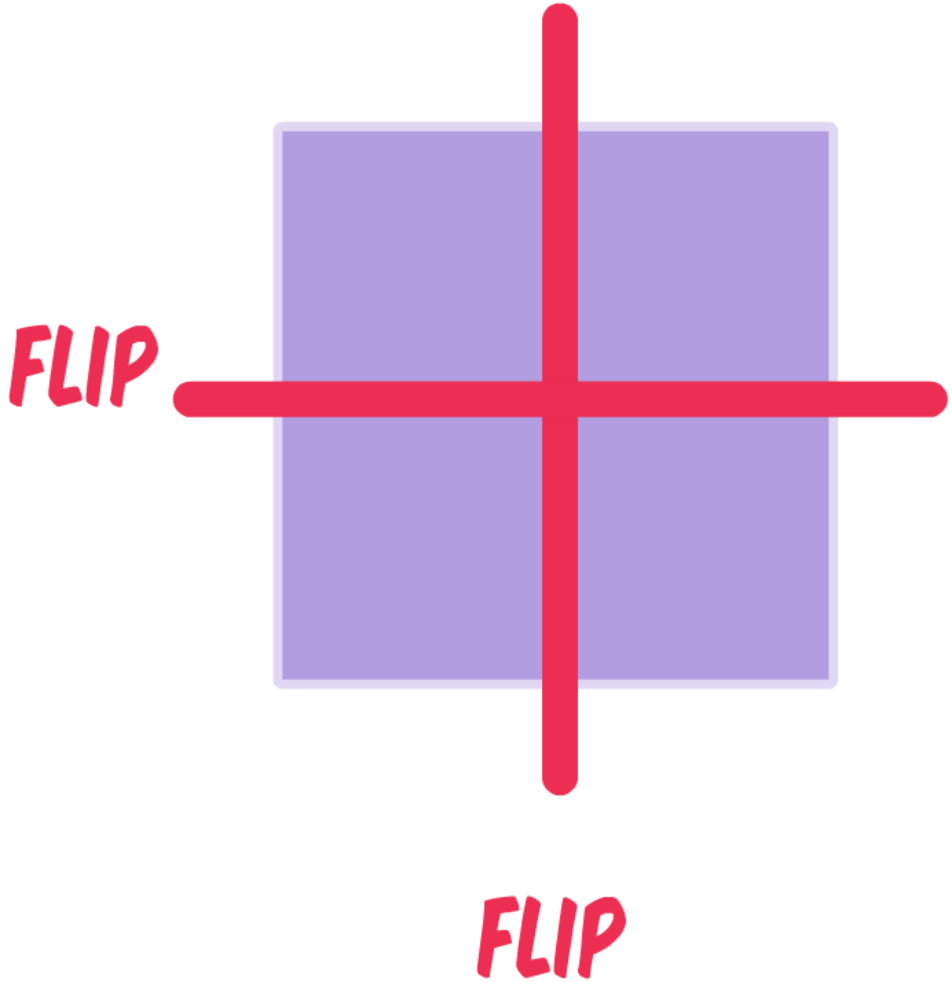


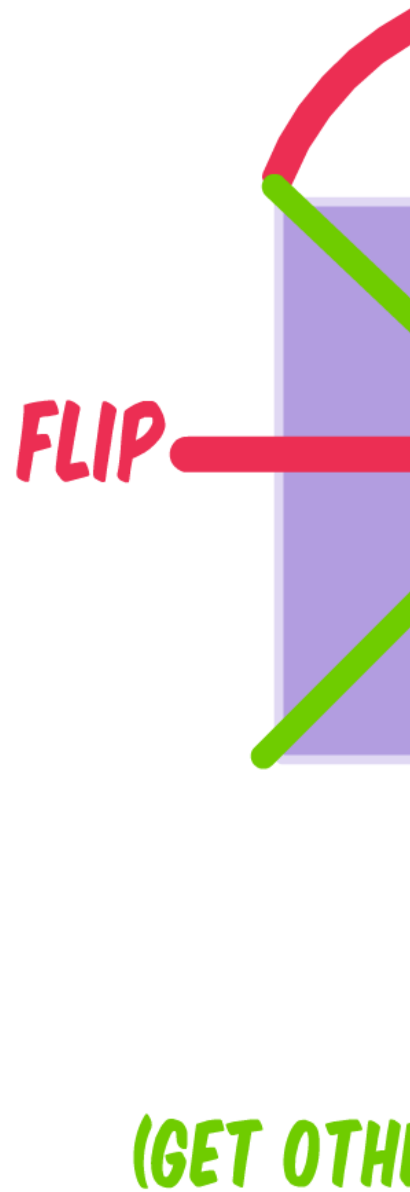


$$G = \mathbb{Z}_2$$



$$G =$$

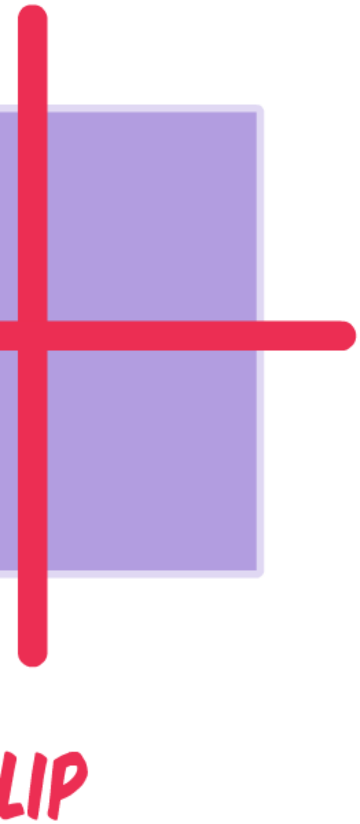


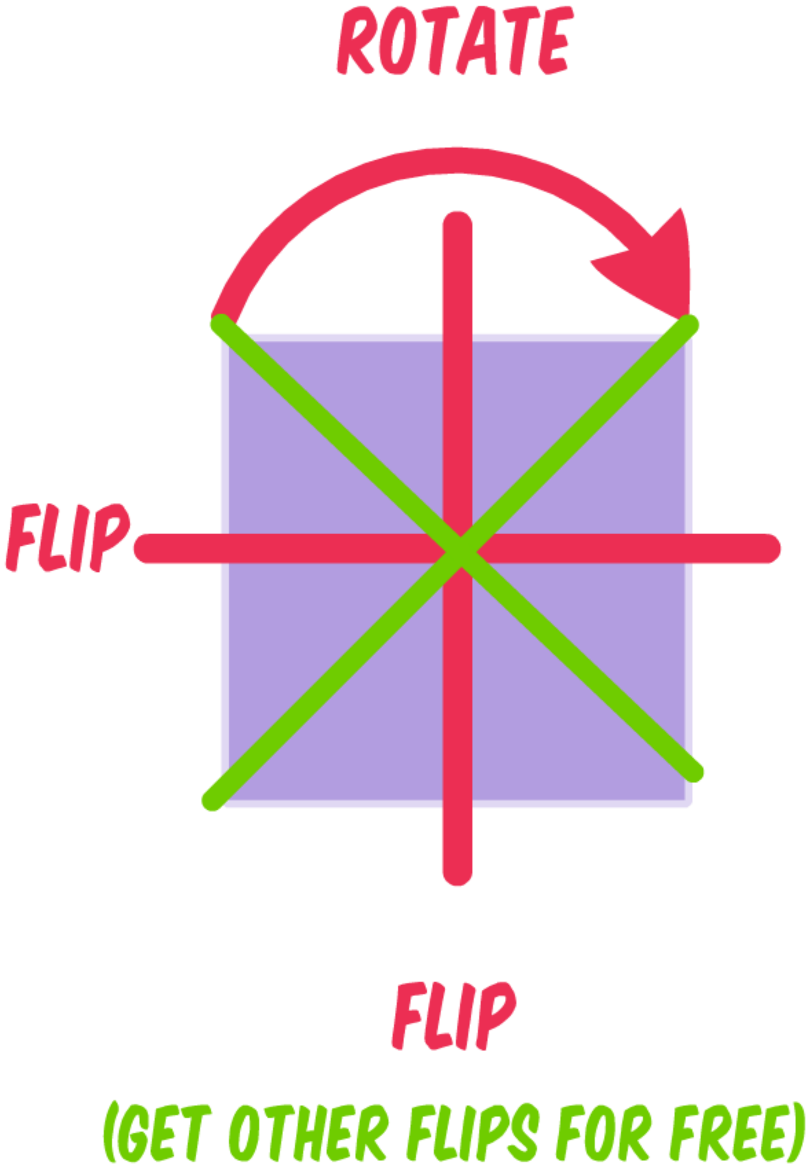


$$\mathbb{Z}_2$$

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$G$$

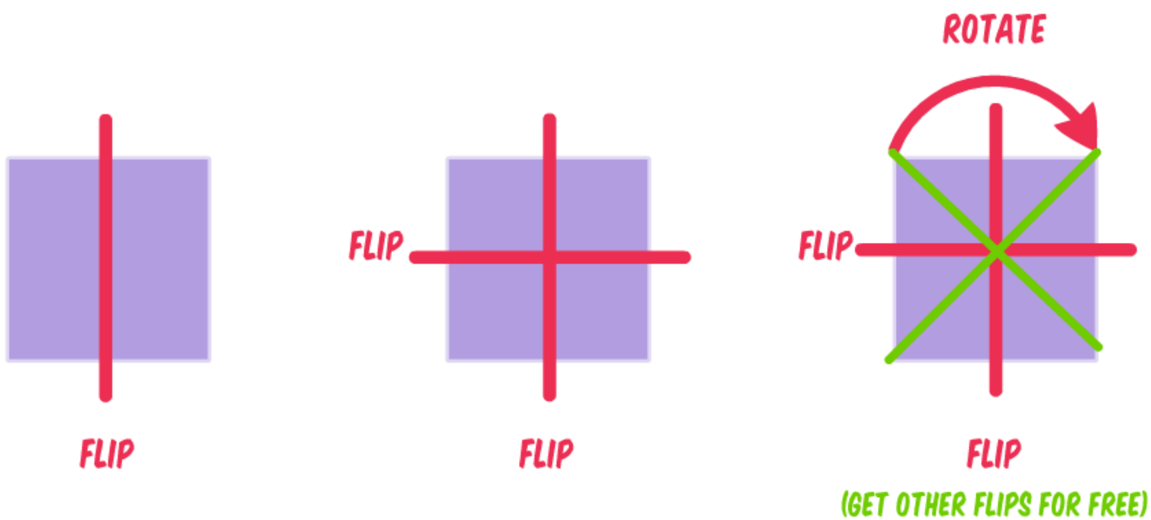




$\mathbb{Z}_2 \times \mathbb{Z}_2$

$G = D_8$

SYMMETRIES:
GOTTA CATCH THEM ALL...



$G = \mathbb{Z}_2$ $\quad$ $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\quad$ $G = D_8$

GOT THEM ALL!

(WELL, IF YOU DON'T DO ANYTHING WEIRD,
LIKE MOVE THE SQUARE OFF OF THE PLANE)

GROUPS & SYMMETRIES

NATURAL QUESTION:
DOES EACH (FINITE) GROUP ARISE AS THE COLLECTION OF SYMMETRIES OF A NICE OBJECT?

CYCLIC GROUPS..CHECK
SYMMETRIC GROUPS..................................CHECK
DIHEDRAL GROUPS......................................CHECK
THE QUATERNION GROUP...................YEP, CHECK. SEE:  

(STILL AN OPEN QUESTION THOUGH)

The Quaternion Group as a Symmetry Group

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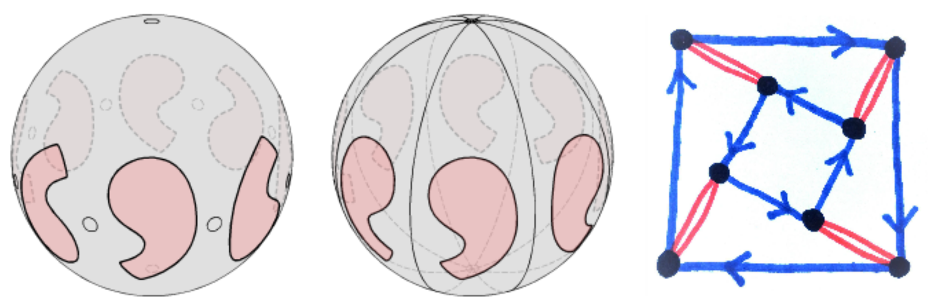
Abstract

We briefly review the distinction between abstract groups and symmetry groups of objects, and discuss the question of which groups have appeared as the symmetry groups of physical objects. To our knowledge, the quaternion group (a beautiful group with eight elements) has not appeared in this fashion. We describe the quaternion group, both formally and intuitively, and give our strategy for representing the quaternion group as the symmetry group of a physical sculpture.

1 Introduction

A *symmetry* of an object is a geometric transformation which leaves the object unchanged. So, for example, an object with 3-fold rotational symmetry has three symmetries: rotation by 120°, rotation by 240°, and the trivial symmetry, where we do nothing. The symmetries of an object naturally form a group under composition. Care must be taken to differentiate between the symmetry group of an object, consisting of geometric transformations that leave the object unchanged, and the abstract group, which only contains information about how the elements of the group interact with each other under composition.



(a) An object with symmetry group isomorphic to D_4.

(b) Another object with symmetry group isomorphic to D_4.

(c) A Cayley graph for D_4. The edges with arrows correspond to rotations, the other edges correspond to reflections.

Figure 1: Symmetric designs on the sphere.

As an example, consider the objects pictured in Figure 1. The object shown in Figure 1a has no planes of mirror symmetry, but has ten *gyration points* (points of rotational symmetry, marked with a small circle). The symmetry group of this object consists of rotations about these gyration points by multiples of either 90° or 180° (depending on the kind of gyration point). The object shown in Figure 1b has four planes of

the six neighbouring cubical cells, thro
copies of itself following the same mat

The obvious (perhaps even canonical) choice for such a design is, of course, a monkey. See Figure 6. With appropriate posing of the monkey, its bilateral symmetry can be broken, and including the head and tail it has six limbs, one for each face of the cube. The monkey's left foot stands on the head of a neighbour, the left hand grabs a neighbour's right foot, and the right hand grabs a neighbour's tail. By symmetry, everything that goes around comes around – so the other three neighbours of this monkey are standing on this monkey's head, grabbing its right foot, and grabbing its tail.

The monkey was designed in a Euclidean cube. It was then run through eight different transformations in order to move eight copies of it to the appropriate positions in S^3 and then back to $\mathbb{R}^3$ by stereographic projection. The first step of all of these transformations is to project the Euclidean cube into a curved cube in S^3. This is done in exactly the same w
hypersphere S^3. To be precise, we thi
the point $(1,x,y,z)$ on one of the cells c

Now that the design is on S^3, we (
eight transformations, and stereograph
from – we put the north pole at a verte
are as far from infinity as possible. Thi
each other as possible. Very small fea
entire sculpture up, but only so far as t

The resulting sculpture is shown i
at all: every monkey is different if we
the appropriate isometries of the 3-sph
identical. From this vantage point the t
two larger, outer monkeys and two sma
hand-foot and hand-tail connections.

Each monkey sits inside of a cel
dimensional faces of the hypercube.

metry Group

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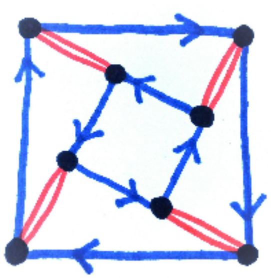

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sphere.

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e gyration points by multiples of either
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the six neighbouring cubical cells, through the six square faces of the cube. The design must connect onto copies of itself following the same matching rules as the monkey blocks of Section 3.

The obvious (perhaps even canonical) choice for such a design is, of course, a monkey. See Figure 6. With appropriate posing of the monkey, its bilateral symmetry can be broken, and including the head and tail it has six limbs, one for each face of the cube. The monkey's left foot stands on the head of a neighbour, the left hand grabs a neighbour's right foot, and the right hand grabs a neighbour's tail. By symmetry, everything that goes around comes around – so the other three neighbours of this monkey are standing on this monkey's head, grabbing its right foot, and grabbing its tail.

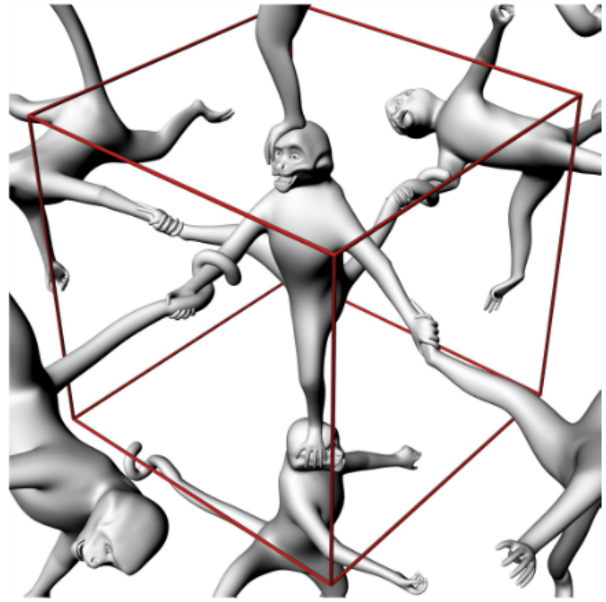


Figure 6: The monkey in a Euclidean cube, prior to mapping into S^3, with six neighbours.

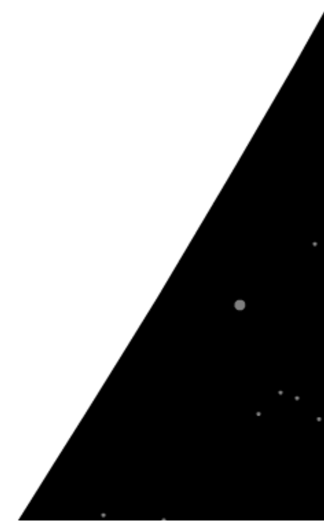
The monkey was designed in a Euclidean cube. It was then run through eight different transformations in order to move eight copies of it to the appropriate positions in S^3 and then back to $\mathbb{R}^3$ by stereographic projection. The first step of all of these transformations is to project the Euclidean cube into a curved cube in S^3. This is done in exactly the same way that a cubical cell of the hypercube is radially projected onto the hypersphere S^3. To be precise, we think of a point (x,y,z) in the Euclidean cube $[-1,1]^3$ as actually being the point $(1,x,y,z)$ on one of the cells of the Euclidean hypercube $[-1,1]^4$, and map it to S^3 by

$$(x,y,z) \mapsto \frac{(1,x,y,z)}{\sqrt{1+x^2+y^2+z^2}}.$$

Now that the design is on S^3, we (right) multiply it by $1,i,j,k,-1,-i,-j$ and $-k$ respectively for the eight transformations, and stereographically project each back to $\mathbb{R}^3$. There is a choice of where to project from – we put the north pole at a vertex of the hypercube, so that in the projection the copies of the monkey are as far from infinity as possible. This makes the resulting features of the eight monkeys as near in size to each other as possible. Very small features may be too fragile to 3D print – to avoid this we can scale the entire sculpture up, but only so far as the largest features fit within the printer and our budget.

The resulting sculpture is shown in Figure 7. Note that the sculpture has no "ordinary" symmetries at all: every monkey is different if we only consider isometries of 3-dimensional space. However, under the appropriate isometries of the 3-sphere (as seen through the lens of stereographic projection) they are all identical. From this vantage point the three pairs of axes of rotation have equal billing: each circle consists of two larger, outer monkeys and two smaller, inner monkeys. The three pairs of axes go through the head-foot, hand-foot and hand-tail connections.

Each monkey sits inside of a cell of the hypercube and connects to its neighbours through the 2-dimensional faces of the hypercube. Therefore, taken together they form the edges and vertices of the



MY GOAL:

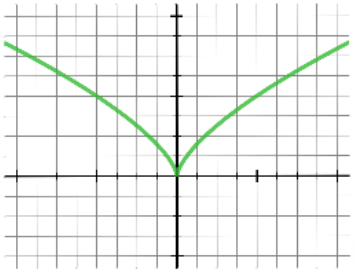
STUDY SYMMETRIES OF ALGEBRAS OVER A FIELD K,

(I.E. THAT HAVE AN UNDERLYING K-VECTOR SPACE STRUCTURE)

ESPECIALLY SYMMETRIES OF NONCOMMUTATIVE K-ALGEBRAS

SYMMETRIES OF AFFINE VARIETIES

An affine variety *X* in affine *n*-space $\mathbb{A}^n$ over a ground field $\Bbbk$ is the vanishing set of a (finite) set of polynomials in $\Bbbk[x_1, \ldots, x_n]$.



$\mathbb{V}(x^2 - y^3) \subset \mathbb{A}^2$

Symmetries of affine varieties also form a group.

$$G = \mathbb{Z}_2 = \langle x \mapsto -x \rangle$$

Classical Geometry <-----> **Commutative Algebra**

Category of Affine Varieties $\mathbf{Aff}_{\Bbbk}$ - - - contravariant functor - - - >

Action of a group G on an affine variety X...

$$G \times X \to X$$

... is a morphism in $\mathbf{Aff}_{\Bbbk}$.

$\therefore$ G is a group & an affine variety, hence a linear algebraic group

$$\therefore G = (G, m, e, \iota)$$

$m : G \times G \to G$, multiplication map (morphism in $\mathbf{Aff}_{\Bbbk}$)
$e \in G$, identity element (object in $\mathbf{Aff}_{\Bbbk}$)
$\iota : G \to G$, inversion map (morphism in $\mathbf{Aff}_{\Bbbk}$)

satisfying group axioms:

(a) $m(\sigma, e) = m(e, \sigma) = \sigma$
(b) $m(\sigma, \iota(\sigma)) = m(\iota(\sigma), \sigma) = e$
(c) $m(\sigma, m(\tau, \gamma)) = m(m(\sigma, \tau), \gamma)$ (associativity)

for all $\sigma, \tau, \gamma \in G$.

- - - $X \mapsto \mathcal{O}(X)$ - - - > Category of Commutative Algebras $\mathbf{ComAlg}_{\Bbbk}$

Coaction of the coordinate algebra $\mathcal{O}(G)$ on $\mathcal{O}(X)$...

$$\mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathcal{O}(G)$$

... is a morphism in $\mathbf{ComAlg}_{\Bbbk}$.

$\therefore$ $\mathcal{O}(G)$ is a commutative algebra
with multiplication $m : \mathcal{O}(G) \otimes \mathcal{O}(G) \to \mathcal{O}(G)$ and unit $u : \Bbbk \to \mathcal{O}(G)$
equipped with structure $\mathcal{O}(G) = (\mathcal{O}(G), \Delta, \epsilon, S)$

$\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$, comultiplication
$\epsilon : \mathcal{O}(G) \to \Bbbk$, $f \mapsto f(e)$, counit
$S : \mathcal{O}(G) \to \mathcal{O}(G)$, antipode

morphisms in $\mathbf{ComAlg}_{\Bbbk}$ satisfying Hopf algebra axioms:

(a) $(\mathrm{id} \otimes \epsilon)\Delta = (\epsilon \otimes \mathrm{id})\Delta = \mathrm{id}$ (counit axiom)
(b) $m(S \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes S)\Delta = u\epsilon$ (antipode axiom)
(c) $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$. (coassociativity)

Category of Affine Varieties $\mathbf{Aff}_{\Bbbk}$

contravariant functor $X \mapsto \mathcal{O}(X)$...
- >

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(c) $m(\sigma, m(\tau, \gamma)) = m(m(\sigma, \tau), \gamma)$ [associativity]

for all $\sigma, \tau, \gamma \in G$.

Classical Geometry <-----> Commutative Algebra

Category of Affine Varieties $\mathbf{Aff}_{\Bbbk}$ — contravariant functor $X \mapsto \mathcal{O}(X)$... — - - - > Category of Commutative Algebras $\mathbf{ComAlg}_{\Bbbk}$

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$$- - - - \overset{...X \mapsto \mathcal{O}(X)}{- - - - - - - -} >$$ Category of Commutative Algebras $\mathbf{ComAlg}_{\Bbbk}$

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QUANTUM SYMMETRIES OF QUANTUM AFFINE VARIETIES

Noncommutative Geometry <---> Noncommutative Algebra

Action of a Quantum Group on a Quantum Affine Variety

Fig: Quantum Variety

Coaction of a noncommutative Hopf algebra on a noncommutative algebra

HOPF ALGEBRAS AND THEIR (CO)ACTIONS ON ALGEBRAS

Hopf algebras

A Hopf algebra $H = (H, m_H, u_H, \Delta, \epsilon, S)$ over a field $\Bbbk$ is an associative algebra (H, m_H, u_H), a coassociative coalgebra (H, Δ, ε), with antipode map S, satisfying compatibility conditions.

Take $\tau : H \otimes H \to H \otimes H$, with $h \otimes \ell \mapsto \ell \otimes h$.
H is commutative if (H, m_H, u_H) is commutative: i.e., $m_H \circ \tau = m_H$.
H is cocommutative if (H, Δ, ε) is cocommutative: i.e., $\tau \circ \Delta = \Delta$.

Classical Examples:

- group algebra $\Bbbk G$: we have for $g \in G$
  m ✓ u ✓ $\Delta(g) = g \otimes g$ $\epsilon(g) = 1_{\Bbbk}$ $S(g) = g^{-1}$.
- universal enveloping algebra of a Lie algebra $U(\mathfrak{g})$: for $x \in \mathfrak{g}$
  m ✓ u ✓ $\Delta(x) = 1_H \otimes x + x \otimes 1_H$ $\epsilon(x) = 0_{\Bbbk}$ $S(x) = -x$.
- $\Bbbk G$ and $U(\mathfrak{g})$ are cocommutative.
- $\Bbbk G$ (resp., $U(\mathfrak{g})$) are commutative $\iff$ G (resp., $\mathfrak{g}$) is abelian
- $\mathcal{O}(G)$ is commutative.

Hopf actions on algebras

We say that a Hopf algebra $H = (H, m_H, u_H, \Delta, \epsilon, S)$ over $\Bbbk$ acts on an algebra $A = (A, m_A, u_A)$ over $\Bbbk$ if

A is an H-module algebra:

A is an H-module, and m_A and u_A of A are H-morphisms.

We need boxed equations to hold below for any $a, b \in A$ and $h \in H$ with $\Delta(h) = \sum h_1 \otimes h_2$ (Sweedler notation):

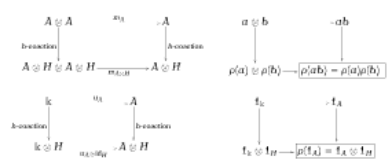
Hopf coactions on algebras

We say that a Hopf algebra $H = (H, m_H, u_H, \Delta, \epsilon, S)$ over $\Bbbk$ coacts on an algebra $A = (A, m_A, u_A)$ over $\Bbbk$ if

A is an H-comodule algebra:

A is an H-comodule via ρ, and m_A and u_A of A are H-morphisms.

We need boxed equations to hold below for any $a, b \in A$ and $h \in H$:



Classical examples of Hopf (co)actions on algebras

Action by cocommutative Hopf algebra on commutative algebra

Take group alg. $\Bbbk[SL_2]$ gen. by matrices $\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \in SL_2(\Bbbk)$ with $e_{11}e_{22} - e_{12}e_{21} = 1$

$\Bbbk[SL_2]$ acts on $\Bbbk[u, v]$ by $\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \cdot u = e_{11}u + e_{21}v$, $\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \cdot v = e_{12}u + e_{22}v$

Take universal enveloping algebra $U(\mathfrak{sl}_2)$, as an algebra:

$$(h, x, y \mid hx - xh = 2x,\ hy - yh = -2y,\ xy - yx = h)$$

$U(\mathfrak{sl}_2)$ acts on $\Bbbk[u, v, w]$ by

$$\begin{array}{lll} h \cdot u = -2u, & h \cdot v = 0, & h \cdot w = 2w \\ x \cdot u = v, & x \cdot v = 2w, & x \cdot w = 0 \\ y \cdot u = 0, & y \cdot v = 2u, & y \cdot w = v \end{array}$$

Coaction by commutative Hopf algebra on commutative algebra

Take coordinate alg. of algebraic group $\mathcal{O}(SL_2) = \Bbbk[e_{ij}]_{i,j=1}^{2}/(e_{11}e_{22} - e_{12}e_{21} = 1)$,
with $\Delta(e_{ij}) = \sum_{\ell=1}^{2} e_{i\ell} \otimes e_{\ell j}$, $\epsilon(e_{ij}) = \delta_{ij}$, $S(e_{ij}) = (-1)^{i-j} e_{i+1,j+1}$ (indices mod 2).

$\mathcal{O}(SL_2)$ coacts on $\Bbbk[u, v]$ by $u \mapsto u \otimes e_{11} + v \otimes e_{21}$, $v \mapsto u \otimes e_{12} + v \otimes e_{22}$

Hopf algebras

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H is cocommutative if $(H,\ \Delta,\ \varepsilon)$ is cocommutative: i.e., $\tau \circ \Delta = \Delta$.

Classical Examples:

- group algebra $\Bbbk G$: we have for $g \in G$
  m ✓ u ✓ $\Delta(g) = g \otimes g$ $\epsilon(g) = 1_{\Bbbk}$ $S(g) = g^{-1}$.
- universal enveloping algebra of a Lie algebra $U(\mathfrak{g})$: for $x \in \mathfrak{g}$
  m ✓ u ✓ $\Delta(x) = 1_H \otimes x + x \otimes 1_H$ $\epsilon(x) = 0_{\Bbbk}$ $S(x) = -x$.
- $\Bbbk G$ and $U(\mathfrak{g})$ are cocommutative.
- $\Bbbk G$ (resp., $U(\mathfrak{g})$) are commutative $\iff$ G (resp., $\mathfrak{g}$) is abelian
- $\mathcal{O}(G)$ is commutative.

Hopf actions on algebras

We say that a Hopf algebra $H = (H, m_H, u_H, \Delta, \epsilon, S)$ over $\mathbb{k}$ <u>acts</u> on an algebra $A = (A, m_A, u_A)$ over $\mathbb{k}$ if

A is an H-module algebra:

A is an H-module, and m_A and u_A of A are H-morphisms.

We need boxed equations to hold below for any $a, b \in A$ and $h \in H$ with $\Delta(h) = \sum h_1 \otimes h_2$ (Sweedler notation):

$$\begin{array}{ccc} H \otimes A \otimes A & \xrightarrow{\mathrm{id}_H \otimes m_A} & H \otimes A \\ {\scriptstyle h\text{-action}}\downarrow & & \downarrow{\scriptstyle h\text{-action}} \\ A \otimes A & \xrightarrow[m_A]{} & A \end{array} \qquad \begin{array}{ccc} h \otimes a \otimes b & \longmapsto & h \otimes ab \\ \downarrow & & \downarrow \\ \sum (h_1 \cdot a) \otimes (h_2 \cdot b) & \longmapsto & \boxed{h \cdot (ab) = \sum (h_1 \cdot a) \otimes (h_2 \cdot b)} \end{array}$$

$$\begin{array}{ccc} H \otimes \mathbb{k} & \xrightarrow{\mathrm{id}_H \otimes u_A} & H \otimes A \\ {\scriptstyle h\text{-action}}\downarrow & & \downarrow{\scriptstyle h\text{-action}} \\ \mathbb{k} & \xrightarrow[u_A]{} & A \end{array} \qquad \begin{array}{ccc} h \otimes 1_{\mathbb{k}} & \longmapsto & h \otimes 1_A \\ \downarrow & & \downarrow \\ \epsilon(h) 1_{\mathbb{k}} & \longmapsto & \boxed{h \cdot 1_A = \epsilon(h) 1_A} \end{array}$$

Hopf coactions on algebras

We say that a Hopf algebra $H = (H, m_H, u_H, \Delta, \epsilon, S)$ over $\Bbbk$ coacts on an algebra $A = (A, m_A, u_A)$ over $\Bbbk$ if

A is an H-comodule algebra:

A is an H-comodule via ρ, and m_A and u_A of A are H-morphisms.

We need boxed equations to hold below for any $a, b \in A$ and $h \in H$:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{m_A} & A \\ \downarrow {\scriptstyle h\text{-coaction}} & & \downarrow {\scriptstyle h\text{-coaction}} \\ A \otimes H \otimes A \otimes H & \xrightarrow[m_{A \otimes H}]{} & A \otimes H \end{array} \qquad \begin{array}{ccc} a \otimes b & \longmapsto & ab \\ \downarrow & & \downarrow \\ \rho(a) \otimes \rho(b) & \longmapsto & \boxed{\rho(ab) = \rho(a)\rho(b)} \end{array}$$

$$\begin{array}{ccc} \Bbbk & \xrightarrow{u_A} & A \\ \downarrow {\scriptstyle h\text{-coaction}} & & \downarrow {\scriptstyle h\text{-coaction}} \\ \Bbbk \otimes H & \xrightarrow[u_A \otimes \mathrm{id}_H]{} & A \otimes H \end{array} \qquad \begin{array}{ccc} 1_{\Bbbk} & \longmapsto & 1_A \\ \downarrow & & \downarrow \\ 1_{\Bbbk} \otimes 1_H & \longmapsto & \boxed{\rho(1_A) = 1_A \otimes 1_H} \end{array}$$

Classical examples of Hopf (co)actions on algebras

Action by cocommutative Hopf algebra on commutative algebra

Take group alg. $\Bbbk(SL_2)$ gen. by matrices $\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \in SL_2(\Bbbk)$ with $e_{11}e_{22} - e_{12}e_{21} = 1$

$\Bbbk(SL_2)$ acts on $\Bbbk[u,v]$ by $\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \cdot u = e_{11}u + e_{21}v$, $\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} \cdot v = e_{12}u + e_{22}v$

Take universal enveloping algebra $U(\mathfrak{sl}_2)$, as an algebra:

$$\langle h, x, y \mid hx - xh = 2x,\ hy - yh = -2y,\ xy - yx = h \rangle$$

$U(\mathfrak{sl}_2)$ acts on $\Bbbk[u,v,w]$ by
$$\begin{array}{lll} h \cdot u = -2u, & h \cdot v = 0, & h \cdot w = 2w \\ x \cdot u = v, & x \cdot v = 2w, & x \cdot w = 0 \\ y \cdot u = 0, & y \cdot v = 2u, & y \cdot w = v \end{array}$$

Coaction by commutative Hopf algebra on commutative algebra

Take coordinate alg. of algebraic group $\mathcal{O}(SL_2) = \Bbbk[e_{ij}]_{i,j=1}^2/(e_{11}e_{22} - e_{12}e_{21} = 1)$,
with $\Delta(e_{ij}) = \sum_{\ell=1}^2 e_{i\ell} \otimes e_{\ell j}$, $\epsilon(e_{ij}) = \delta_{ij}$, $S(e_{ij}) = (-1)^{j-i} e_{i+1,j+1}$ (indices mod 2).

$\mathcal{O}(SL_2)$ coacts on $\Bbbk[u,v]$ by $u \mapsto u \otimes e_{11} + v \otimes e_{21}$, $v \mapsto u \otimes e_{12} + v \otimes e_{22}$

PROTOTYPICAL EXAMPLES OF QUANTUM SYMMETRY: ACTIONS / COACTIONS ON THE QUANTUM PLANE

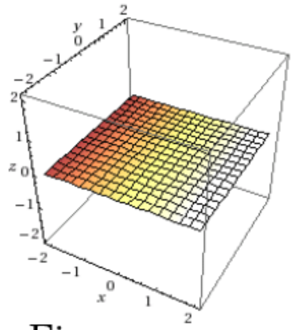

Fig:
Affine 2-space

replace plane with coordinate ring:

$$\Bbbk[x,y] = \frac{\Bbbk\langle x,y\rangle}{(xy-yx)}$$

polynomial algebra

$$\Bbbk_q[x,y] = \frac{\Bbbk\langle x,y\rangle}{(xy-qyx)}$$

$q \in \Bbbk^{\times}$

q-polynomial algebra

Fig:
Quantum 2-space

q-deform classical symmetries to get quantum symmetries....

Classical Symmetry:

Coaction by commutative Hopf algebra on commutative algebra

Take coordinate algebra of algebraic group

$$\mathcal{O}(SL_2) = \Bbbk[e_{ij}]_{i,j=1}^{2}/(e_{11}e_{22} - e_{12}e_{21} = 1),$$

with $\Delta(e_{ij}) = \sum_{k=1}^{2} e_{ik} \otimes e_{kj}$ $\quad \epsilon(e_{ij}) = \delta_{ij}$ $\quad S(e_{ij}) = (-1)^{i-j} e_{i+1,j+1}$ (indices mod 2).

$\mathcal{O}(SL_2)$ coacts on $\Bbbk[u,v]$ by $\quad u \mapsto u \otimes e_{11} + v \otimes e_{21}, \quad v \mapsto u \otimes e_{12} + v \otimes e_{22}$

Quantum Symmetry:

Coaction by noncom. Hopf algebra on noncommutative algebra

Take coordinate algebra of quantized algebraic group, for $q \in \Bbbk^{\times}$.

$$\mathcal{O}_q(SL_2) = \frac{\Bbbk\langle e_{11}, e_{12}, e_{21}, e_{22}\rangle}{\begin{pmatrix} e_{11}e_{12} = qe_{12}e_{11}, & e_{11}e_{21} = qe_{21}e_{11}, \\ e_{12}e_{22} = qe_{22}e_{12}, & e_{21}e_{22} = qe_{22}e_{21}, \\ e_{12}e_{21} = e_{21}e_{12}, & e_{11}e_{22} = e_{22}e_{11} + (q-q^{-1})e_{12}e_{21} \\ e_{11}e_{22} - qe_{12}e_{21} = 1 \end{pmatrix}},$$

with $\Delta(e_{ij}) = \sum_{k=1}^{2} e_{ik} \otimes e_{kj}$ $\quad \epsilon(e_{ij}) = \delta_{ij}$ $\quad S(e_{ij}) = (-q)^{i-j} e_{i+1,j+1}$ (indices mod 2).

$\mathcal{O}_q(SL_2)$ coacts on $\Bbbk_q[u,v]$ by $\quad u \mapsto u \otimes e_{11} + v \otimes e_{21}, \quad v \mapsto u \otimes e_{12} + v \otimes e_{22}$

Classical Symmetry:

Coaction by commutative Hopf algebra on commutative algebra

Take coordinate algebra of algebraic group

$$\mathcal{O}(SL_2) = \Bbbk[e_{ij}]_{i,j=1}^2/(e_{11}e_{22} - e_{12}e_{21} = 1),$$

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with $\Delta(e_{ij}) = \sum_{\ell=1}^2 e_{i\ell} \otimes e_{\ell j}$, $\epsilon(e_{ij}) = \delta_{ij}$, $S(e_{ij}) = (-q)^{j-i} e_{i+1,j+1}$ (indices mod 2).

$\mathcal{O}_q(SL_2)$ coacts on $\Bbbk_q[u, v]$ by $u \mapsto u \otimes e_{11} + v \otimes e_{21}$, $v \mapsto u \otimes e_{12} + v \otimes e_{22}$

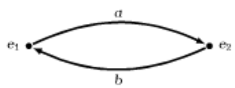
ANOTHER EXAMPLE OF QUANTUM SYMMETRY

A=path algebra of a quiver (directed graph)

$\Bbbk$-vector space basis of A = paths of quiver

Multiplication of A = concatenation of paths, 0 elsewhere

Example:



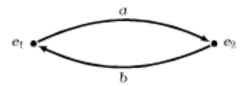
Eg. $e_1 a = a$ in A, $ab \in A$, $a^2 = 0$ in A

$\mathbb{Z}_2 = \langle g : g^2 = 1 \rangle$ acts on A:

$g \cdot e_1 = e_2, \quad g \cdot e_2 = e_1, \quad g \cdot a = b, \quad g \cdot b = a$

So A admits classical symmetry

Example continued: A also admits quantum symmetry



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$g \cdot e_1 = e_2, \quad g \cdot e_2 = e_1, \quad g \cdot a = b, \quad g \cdot b = a$

Extend to action of the Sweedler Hopf alg. (4-diml, noncom, noncocom)

$$H = \langle g, x \ : g^2 = 1, \ x^2 = 0, \ gx + xg = 0 \rangle$$
$$\text{with } \Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g$$
$$\Delta(x) = 1 \otimes x + x \otimes g, \quad \epsilon(x) = 0, \quad S(x) = -xg$$

$$x \cdot e_1 = -\gamma(e_1 + e_2), \qquad x \cdot e_2 = \gamma(e_1 + e_2)$$
$$x \cdot a = \gamma(a - b) + \lambda e_1, \qquad x \cdot b = \gamma(a - b) - \lambda e_2$$

for $\gamma, \lambda \in \Bbbk$

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$\Bbbk$-vector space basis of A = paths of quiver

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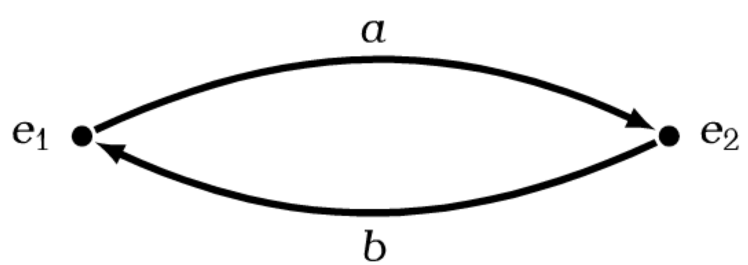
<u>Example:</u>



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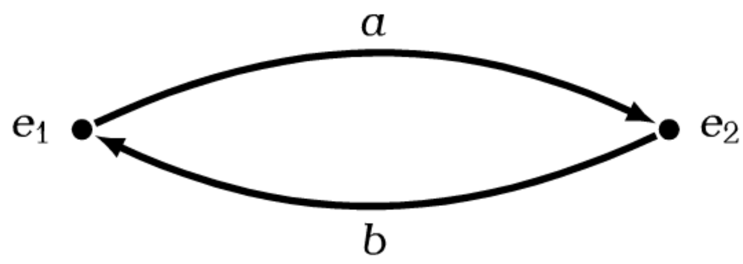

$\mathbb{Z}_2 = \langle g : g^2 = 1 \rangle$ acts on A:

$$g \cdot e_1 = e_2, \qquad g \cdot e_2 = e_1, \qquad g \cdot a = b, \qquad g \cdot b = a$$

Extend to action of the Sweedler Hopf alg. (4-diml, noncom, noncocom)

$$H = \langle g, x \;: g^2 = 1,\; x^2 = 0,\; gx + xg = 0 \rangle$$
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$$x \cdot a = \gamma(a - b) + \lambda e_1, \qquad x \cdot b = \gamma(a - b) - \lambda e_2$$

for $\gamma, \lambda \in \Bbbk$

MAIN QUESTION

When does there exist "genuine" quantum symmetry?

When are there actions (resp., coactions) of Hopf algebras that do not factor through actions (resp., coactions) of classical Hopf algebras?

Classical Hopf algebras = those that are com. or cocom.
e.g. group algebras, universal enveloping algebras, coordinate algebras of algebraic groups

USEFUL HYPOTHESES TO IMPOSE

TO ANSWER MAIN QUESTION

HYPOTHESES ON HOPF ALGEBRAS H

Could impose that H is:

- finite-dimensional as a vector space, or
- semisimple as an algebra (which implies finite-dimensionality), or
- cosemisimple as a coalgebra
  [each H-comodule = direct sum of simple H-subcomods], or
- involutory [the square of the antipode S of H is the identity].

If H is finite-dim'l and characteristic of ground field is 0, then
semisimple = cosemisimple = involutory.

pointed [every simple H-comodule is 1-dimensional]

There is a very active program to classify finite-dimensional Hopf algebras
in the semisimple (resp. pointed) settings.
Group theoretic (resp. Lie theoretic) techniques are employed.

HYPOTHESES ON ALGEBRAS A

Could impose that A is:

- homologically nice
  - finite global or injective dimension
  - a Koszulity condition (Koszul, N-Koszul, K2 condition)
  - a Calabi-Yau condition
- ring-theoretically nice
  - commutative
  - domain
  - Noetherian or coherent
  - if graded, polynomial growth of graded pieces (finite GK dim)
  - nice vector basis of monomials (PBW property)

There is a very active program to study
homological analogues of commutative polynomial rings:
"Artin-Schelter regular algebras", and more generally, "skew Calabi-Yau algebras".

HYPOTHESES ON ACTION OF HOPF ALGEBRA H ON ALGEBRA A

If A is graded (resp., filtered), then one could ask that the H-action on A preserves this grading (resp., filtration).

Could also use the homological (co)determinant of H-(co)action on A

Related to the quantum determinant in the literature

Eg., to get an analogue of a result involving group actions with $G<SL(V)$, impose trivial homological determinant

Avoiding technicalities here,

hdet(H,A) is an H-morphism from H to the ground field;
it is trivial if equal to counit map of H.
hcodet(H,A) arises as a "group-like element" in H;
it is trivial if equal to the unit element of H.

Main Results #1

No Quantum Symmetry

Recall:

Actions of groups G (or $\Bbbk G$) and Lie algebras $\mathfrak{g}$ (or $U(\mathfrak{g})$) are considered classical, and that $\Bbbk G$ and $U(\mathfrak{g})$ are cocommutative:
$\Delta = \tau \circ \Delta$, where $\tau(h \otimes \ell) = \ell \otimes h$, for $h, \ell \in H$.

Theorem (Cartier-Kostant-Milnor-Moore).
If H is a cocommutative Hopf algebra over an alg. closed field of characteristic 0, then $H \cong U(\mathfrak{g}) \# \Bbbk G$, for some $G \curvearrowright \mathfrak{g}$.
Further, if H is finite-dimensional, then $H \cong \Bbbk G$, for some group G.

Given an Hopf H-action on an algebra A, we say there is No Quantum Symmetry when this action must factor through the action of a cocommutative Hopf algebra.

No Quantum Symmetry results for
H-actions on commutative domains:

Below, Hopf actions must factor through the action of a cocom. Hopf algebra.

Conditions on k	on H	on A	on action	Reference
char 0 alg. closed	semisimple ($\Rightarrow$ fin-dim & coss)	commutative domain	(none)	[Etingof-W, 2014]
char > 0 alg. closed	semisimple & cosemisimple	commutative domain	(none)	[Etingof-W, 2014]
char > 0 alg. closed	finite-dim'l & cosemisimple	commutative domain	(none)	[Skryabin, 2016]

There are lots of No Quantum Symmetry results
in the analytic setting
(outside of the scope of this talk).

There, A is the function algebra of a geometric object (e.g. sphere, torus, certain manifolds).

So, A is a commutative domain.

Please see references.

No Quantum Symmetry results for
H-actions on quantizations of com. domains and other algebras:

Below, Hopf actions must factor through the action of a cocom. Hopf algebra.

Here, k is algebraically closed of characteristic 0.

Conditions on H	on A	on action	Reference
finite-dim'l	Weyl algebra $A_n(k)$	(none)	[Cuadra-Etingof-W, to appear]
semisimple	$U(\mathfrak{g})$ for $\mathfrak{g}$ fin.-dim'l, $\mathcal{D}(X)$ diff. op. on smooth aff var., generic Sklyanin algebras, twisted homog coord rings	(none)	[Etingof-W, submitted]
semisimple & cosemisimple	division algebra D	dim H & deg D are coprime	[Cuadra-Etingof-W, to appear]
finite-dim'l	$\frac{k\langle x_1,\dots,x_n\rangle}{(x_i x_j - q_{ij} x_j x_i)}$ $q_{ij} \in k^\times$ generic	pres. grading (none)	[Chan-W-Zhang] [Etingof-W, submitted]

Recall:

Actions of groups G (or $\Bbbk G$) and Lie algebras $\mathfrak{g}$ (or $U(\mathfrak{g})$) are considered <u>classical</u>, and that $\Bbbk G$ and $U(\mathfrak{g})$ are <u>cocommutative</u>:

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char > 0 alg. closed	semisimple & cosemisimple	commutative domain	(none)	[Etingof-W, 2014]
char > 0 alg. closed	finite-dim'l & cosemisimple	commutative domain	(none)	[Skryabin, 2016]

No Quantum Symmetry results for

H-actions on quantizations of com. domains and other algebras:

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Here, $\Bbbk$ is algebraically closed of characteristic 0.

Conditions on H	on A	on action	Reference
finite-dim'l	Weyl algebra $A_n(\Bbbk)$	(none)	[Cuadra-Etingof-W, to appear]
semisimple	$U(\mathfrak{g})$ for $\mathfrak{g}$ fin dim'l, $D(X)$ diff. op. on smooth aff var., generic Sklyanin algebras, twisted homog coord rings	(none)	[Etingof-W, submitted]
semisimple & cosemisimple	division algebra D	dim$H$ & (deg$D$)! are coprime	[Cuadra-Etingof-W, to appear]
finite-dim'l	$\frac{\Bbbk\langle x_1,\ldots,x_n\rangle}{(x_ix_j-q_{ij}x_jx_i)}$ $q_{ij}\in\Bbbk^\times$ generic	pres. grading (none)	[Chan-W-Zhang] [Etingof-W, submitted]

There are lots of No Quantum Symmetry results
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(outside of the scope of this talk).

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(e.g. sphere, torus, certain manifolds).

So, A is a commutative domain.

Please see references.

Main Results #2

Genuine Quantum Symmetry

Given an Hopf H-action on an algebra A, we say there is
Genuine Quantum Symmetry
when this action does "not" factor through
the action of a cocommutative Hopf algebra.

(Time permitting)
We discuss three occurrences of Genuine Quantum Symmetry...

Genuine Quantum Symmetry: on path algebras $\Bbbk Q$

Hopf actions below do not factor through actions of smaller Hopf alg. quotients

Conditions on k	on H	on Q	on action	Reference
contains a primitive n-th root of unity ζ for $n \geq 2$	pointed: $T_n(\zeta)$, Taft algebras [illegible], small quan. group $D(T_2(n))$, double of $T_2(n)$	finite loopless & no parallel arrows	preserves ascending path length filtration	[Kinser-W]

Example: We classify Sweedler Hopf $T(2)$-actions on the path algebra of Q to the right.

The action of $\mathbb{Z}_2$ is given by

Genuine Quantum Symmetry: on commutative domains (fields)

Take k an algebraically closed field of characteristic 0.

Let H be a Hopf algebra that acts on a field so that the action does "not" factor through a smaller Hopf algebra quotient; any such an H is Galois-theoretical.

Below are noncocom., noncom., finite-dim'l, non-ss, pointed Galois-th'l Hopf alg's.

H	"Cartan type"
Taft algebras $T_2(n)$	A_1
Nichols Hopf algebras $E(n)$	$A_1^{\times n}$
the book algebras $\mathbf{h}(\zeta, 1)$	$A_1 \times A_1$
the Hopf algebra H_{81} of dimension 81	A_2
[illegible]	$A_1 \times A_1$
[illegible]	$A_2 \times A_2$
Twists [illegible] for $n \geq 2$	$A_{n-1} \times A_{n-1}$
Twists [illegible] for $n \geq 3$	$A_{n-1} \times A_{n-1}$
Twists [illegible] of each J	same type as $\mathfrak{g}$

$\mathfrak{g}$ is a finite-dimensional simple Lie algebra

Reference: [Etingof-W2]

Genuine Quantum Symmetry: on noncommutative domains

The Hopf actions (that do not factor through smaller Hopf actions) in the setting below are classified:

k is an algebraically closed field of char. 0

H is a finite-dimensional Hopf algebra

A is an Artin-Schelter regular algebra of global dimension 2 (a homological analogue of $\Bbbk[u,v]$)

H-action preserves the grading of A, subject to trivial homol det.

...which is a generalization of the classical setting where $G \leq SL_2(\Bbbk)$ acts on $\Bbbk[u,v]$ linearly and faithfully

Reference: [Chan-Kirkman-W-Zhang]

Galois-theoretical property & Galois extensions

Take k an algebraically closed field of characteristic 0.
Say H is finite-dimensional, Galois-theoretical with H-module field L.

If, further, H is semisimple, then
$H \cong \Bbbk G$ and the extension $L^H = L^G \hookrightarrow L$ is Galois.

On the other hand, if, further, H is pointed, then
$L^H = L^{G(H)}$ and the extension $L^H \hookrightarrow L$ is Galois.

Here, $G(H)$ is the group of group-like elements of H:
$G(H) = \{h \in H \mid \Delta(h) = h \otimes h\}$

Reference: [Etingof-W2]

Genuine Quantum Symmetry: on commutative domains (fields)

The Galois-theoretical property is preserved under taking:

- Hopf subalgebra
- $\otimes$

... so this allows one to cook up more quantum symmetries

The Galois-theoretical property is "not" preserved under taking:

- Hopf dual
- 2-cocycle deformation (twisting the multiplication)
- dual 2-cocycle deformation (twisting the comultiplication)

Reference: [Etingof-W2]

Given an Hopf H-action on an algebra A, we say there is

Genuine Quantum Symmetry

when this action does *not* factor through

the action of a cocommutative Hopf algebra.

(Time permitting)

We discuss three occurrences of Genuine Quantum Symmetry...

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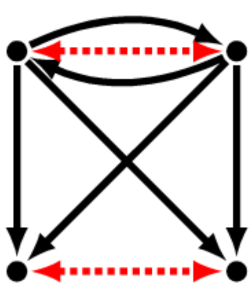
Hopf actions below do not factor through actions of smaller Hopf alg. quotients

Conditions on $\Bbbk$	on H	on Q	on action	Reference
contains a primitive n-th of root unity ζ for $n \geq 2$	pointed: $T_\zeta(n)$, Taft algebras $u_q(\mathfrak{sl}_2)$, small quan. group $D(T_\zeta(n))$, double of $T_\zeta(n)$	finite loopless & no parallel arrows	preserves ascending path length filtration	[Kinser-W]

Example: We classify Sweedler Hopf $T(2)$-actions on the path algebra of Q to the right.



The action of $\mathbb{Z}_2$ is given by 

Genuine Quantum Symmetry: on commutative domains (fields)

Take $\Bbbk$ an algebraically closed field of characteristic 0.

Let H be a Hopf algebra that acts on a field so that the action does *not* factor through a smaller Hopf algebra quotient; say such an H is Galois-theoretical.

Below are noncocom., noncom., finite-dim'l, non-ss, pointed Galois-th'l Hopf algs.

H	"Cartan type"
Taft algebras $T_\zeta(n)$	A_1
Nichols Hopf algebras $E(n)$	$A_1^{\times n}$
the book algebra $\mathbf{h}(\zeta, 1)$	$A_1 \times A_1$
the Hopf algebra H_{81} of dimension 81	A_2
$u_q(\mathfrak{sl}_2)$	$A_1 \times A_1$
$u_q(\mathfrak{gl}_2)$	$A_1 \times A_1$
Twists $u_q(\mathfrak{gl}_n)^{J^+}$, $u_q(\mathfrak{gl}_n)^{J^-}$ for $n \geq 2$	$A_{n-1} \times A_{n-1}$
Twists $u_q(\mathfrak{sl}_n)^{J^+}$, $u_q(\mathfrak{sl}_n)^{J^-}$ for $n \geq 2$	$A_{n-1} \times A_{n-1}$
Twists $u_q^{\geq 0}(\mathfrak{g})^J$ for $2^{\mathrm{rank}(\mathfrak{g})-1}$ of such J	same type as $\mathfrak{g}$

$\mathfrak{g}$ is a finite-dimensional simple Lie algebra

Reference: [Etingof-W(2)]

Genuine Quantum Symmetry: on commutative domains (fields)

The Galois-theoretical property is preserved under taking:

- Hopf subalgebra
- $\otimes$

... so this allows one to cook up more quantum symmetries

The Galois-theoretical property is *not* preserved under taking:

- Hopf dual
- 2-cocycle deformation (twisting the multiplication)
- dual 2-cocycle deformation (twisting the comultiplication)

Reference: [Etingof-W(2)]

Galois-theoretical property & Galois extensions

Take $\Bbbk$ an algebraically closed field of characteristic 0.
Say H is finite-dimensional, Galois-theoretical with H-module field L.

If, further, H is semisimple, then

$$H \cong \Bbbk G \text{ and the extension } L^H = L^G \hookrightarrow L \text{ is Galois.}$$

On the other hand, if, further, H is pointed, then

$$L^H = L^{G(H)} \text{ and the extension } L^H \hookrightarrow L \text{ is Galois.}$$

Here, $G(H)$ is the group of group-like elements of H.

$$G(H) = \{h \in H \mid \Delta(h) = h \otimes h\}$$

Reference: [Etingof-W(2)]

Genuine Quantum Symmetry: on noncommutative domains

The Hopf actions (that do not factor through smaller Hopf actions) in the setting below are classified:

$\Bbbk$ is an algebraically closed field of char. 0

H is a finite-dimensional Hopf algebra

A is an Artin-Schelter regular algebra of global dimension 2 (a homological analogue of $\Bbbk[u, v]$)

H-action preserves the grading of A, subject to trivial hom'l det.

....which is a generalization of the classical setting where $G \leq SL_2(\Bbbk)$ acts on $\Bbbk[u, v]$ linearly and faithfully

Reference: [Chan-Kirkman-W-Zhang]

More Results

Noncommutative Invariant Theory

given an H-action on A

study the invariant ring A^H ...

Deformation Theory

given an H-action on A

study the smash product algebra $A\#H$

and its deformations...

~Very happy to talk about these topics~

(If time does not permit, please email me!

I also have talk notes on my Research page.)

QUANTUM SYMMETRIES: GOTTA CATCH THEM ALL!

Universal Quantum Symmetry: set-up

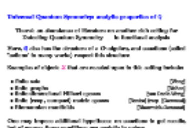
Given an algebra A, a universal quantum group $\mathcal{Q}(A)$ coacting on A is a Hopf algebra, so that for all Hopf coactions of H on A,

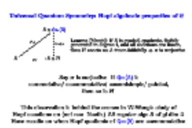
- we get a unique map $\pi : \mathcal{Q}(A) \to H$, with
- the following diagram commuting:

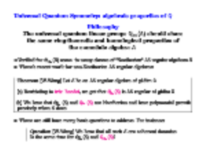
$$\begin{array}{ccc} & & A \otimes \mathcal{Q}(A) \\ & \overset{\rho_{\mathcal{Q}}}{\nearrow} & \Big\downarrow \mathrm{id}_A \otimes \pi \\ A & \xrightarrow{\rho_H} & A \otimes H \end{array}$$

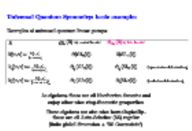
Similarly, could define the universal quantum linear group $\mathcal{Q}_{lin}(A)$ if

- A is graded and generated in degree 1, and
- we impose that all coactions on A preserve the grading of A.









Universal Quantum Symmetry: set-up

Given an algebra A, a universal quantum group $\mathbb{Q}(A)$ coacting on A is a Hopf algebra, so that for all Hopf coactions of H on A,

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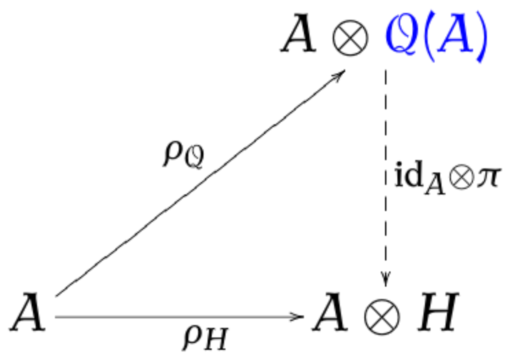


Similarly, could define the universal quantum linear group $\mathbb{Q}_{lin}(A)$ if

- A is graded and generated in degree 1, and
- we impose that all coactions on A preserve the grading of A.

Universal Quantum Symmetry: basic examples

Examples of universal quantum linear groups:

A	$\mathcal{Q}^c_{lin}(A)$ w/ central hcodet	$\mathcal{Q}_{lin}(A)$ w/ triv. hcodet	
$\Bbbk[u,v] = \frac{\Bbbk\langle u,v\rangle}{(uv-vu)}$	$\mathcal{O}(GL_2(\Bbbk))$	$\mathcal{O}(SL_2(\Bbbk))$	
$\Bbbk_q[u,v] := \frac{\Bbbk\langle u,v\rangle}{(uv-qvu)}$	$\mathcal{O}_q(GL_2(\Bbbk))$	$\mathcal{O}_q(SL_2(\Bbbk))$	(1-parameter deformation)
$\Bbbk_J[u,v] := \frac{\Bbbk\langle u,v\rangle}{(uv-vu-v^2)}$	$\mathcal{O}_J(GL_2(\Bbbk))$	$\mathcal{O}_J(SL_2(\Bbbk))$	(Jordanian deformation)

As algebras, these are all Noetherian domains and enjoy other **nice ring-theoretic properties**.

These algebras are also **nice homologically**– these are all Artin-Schelter (AS) regular (finite global dimension + "AS Gorenstein").

Universal Quantum Symmetry: algebraic properties of $\mathbb{Q}$

Philosophy

The universal quantum linear groups $\mathbb{Q}_{lin}(A)$ should share the same ring-theoretic and homological properties of the comodule algebra A.

∗ Verified for $\mathbb{Q}_{lin}(A)$ assoc. to many classes of *Noetherian* AS regular algebras A

∗ There's recent work for non-Noetherian AS regular algebras:

Theorem [W-Wang] Let S be an AS regular algebra of gl.dim 2.

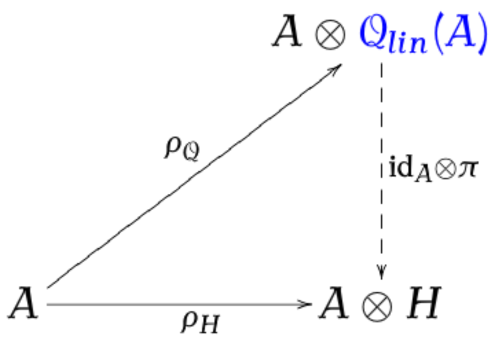
(a) Restricting to triv. hcodet, we get that $\mathbb{Q}_{lin}(S)$ is AS regular of gldim 3.

(b) We have that $\mathbb{Q}^c_{lin}(S)$ and $\mathbb{Q}_{lin}(S)$ are Noetherian and have polynomial growth precisely when S does.

∗ There are still have many basic questions to address. For instance:

Question [W-Wang] We have that all such S are coherent domains. Is the same true for $\mathbb{Q}^c_{lin}(S)$ and $\mathbb{Q}_{lin}(S)$?

Universal Quantum Symmetry: Hopf algebraic properties of $\mathbb{Q}$



Lemma [Manin]: If A is graded, quadratic, finitely generated in degree 1, and all coactions are linear, then H coacts on A inner-faithfully $\Leftrightarrow$ π is surjective.

Say π is surjective. If $\mathbb{Q}_{lin}(A)$ is commutative/ cocommutative/ cosemisimple/ pointed, then so is H.

This observation is behind the scenes in W-Wang's study of Hopf coactions on (not nec. Noeth.) AS regular algs S of gl.dim 2. Have results on when Hopf quotients of $\mathbb{Q}_{lin}(S)$ are cocommutative.

Universal Quantum Symmetry: analytic properties of $\mathcal{Q}$

There's an abundance of literature on another rich setting for Detecting Quantum Symmetry ... in functional analysis.

Here, $\mathcal{Q}$ also has the structure of a C^*-algebra, and coactions (called "actions" in many works) respect this structure.

Examples of objects X that are coacted upon in this setting include:

- finite sets [Wang]
- finite graphs [Bichon]
- finite-dimensional Hilbert spaces [van Daele-Wang]
- finite (resp., compact) metric spaces [Banica] (resp. [Goswami])
- Riemannian manifolds [Bhowmick-Goswami]

One may impose additional hypotheses on coactions to get results, but of course, these conditions are analytic in nature.

FURTHER QUESTIONS AND DIRECTIONS....

Computation.

Computations are a pain. Write a program to do this.

Classification results.

Pick a class of algebras. Pick a class of Hopf algebras. Perhaps impose some conditions on Hopf action. Is there quantum symmetry?

Fancy classification results.

Use the machinery of tensor categories/ fusion categories to understand Hopf actions.

Make connections to other fields.

This has been done in functional analysis & geometry. Topology?

Physical Applications.

This will be useful to physicists.
Investigate new applications.
Then tell me about this.

Thanks for listening!

References: Background on Hopf algebras and Hopf (co)actions

[Andruskiewitsch]
Nicolás Andruskiewitsch, On finite-dimensional Hopf algebras, Proc. ICM, 2014.

[Drinfeld]
Vladimir Drinfeld, Quantum Groups, Proc. ICM, 1986.

[Kassel]
Christian Kassel, Quantum groups. Springer Science Business Media, 2012.

[Majid]
Shahn Majid, What is a Quantum Group?, Notices AMS, 1995.

[Montgomery]
Susan Montgomery, Hopf algebras and their actions on rings, CBMS, 1993.

[Radford]
David Radford, Hopf algebras, World Scientific Publishing Co. Pte., 2012.

References: No Quantum Symmetry (incomplete list)

[Chan-W-Zhang]
Kenneth Chan, Chelsea Walton, and James Zhang, Hopf actions and Nakayama automorphisms, J. Alg., 2014

[Cuadra-Etingof-W]
Juan Cuadra, Pavel Etingof, and Chelsea Walton, Finite dimensional Hopf actions on Weyl algebras, to appear in Adv. Math.

[Cuadra-Etingof-W(2)]
Juan Cuadra, Pavel Etingof, and Chelsea Walton, Semisimple Hopf actions on Weyl algebras, Adv. Math, 2015.

[Etingof-W, 2014]
Pavel Etingof and Chelsea Walton, Semisimple Hopf actions on commutative domains, Adv. Math., 2014.

[Etingof-W, submitted]
Pavel Etingof and Chelsea Walton, Finite dimensional Hopf actions on algebraic quantizations, submitted.

[Skryabin]
Serge Skryabin, Finiteness of the number of coideal subalgebras, to appear in Proc AMS.

References: Genuine Quantum Symmetry (incomplete list)

[Chan-Kirkman-W-Zhang]
Kenneth Chan, Ellen Kirkman, Chelsea Walton, and James Zhang, Quantum binary polyhedral groups and their actions on quantum planes, to appear in J. Reine Angew. Math., arXiv:1303.7203.

[Etingof-W(2)]
Pavel Etingof and Chelsea Walton, Pointed Hopf actions on fields, I, Transform. Groups, 2015.

[Etingof-W(3)]
Pavel Etingof and Chelsea Walton, Pointed Hopf actions on fields, II, J. Algebra, 2016.

[Kinser-W]
Ryan Kinser and Chelsea Walton, Actions of some pointed Hopf algebras on path algebras of quivers, Algebra & Number Theory, 2016.

References: 'Universal' Results

[Brown-Goodearl]
Ken Brown and Ken Goodearl, Lectures on Algebraic Quantum Groups, Birkhäuser, 2002.

[Brown-Zhang]
Ken Brown and James Zhang, Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras, J. Alg., 1997.

[Manin]
Yuri Manin, Quantum groups and non-commutative geometry, Les. Publ. CRM, Univ. de Montréal, 1988.

[W-Wang] Chelsea Walton and Xingting Wang, On quantum groups associated to non-Noetherian regular algebras of dimension 2, arXiv:1503.09185.

References: Quantum Symmetry - analytic (incomplete list)

[Banica]
Teodor Banica, Quantum automorphism groups of small metric spaces, Pacific J. Math., 2005.

[Bichon]
Julien Bichon, Quantum automorphism groups of finite graphs, Proc. AMS, 2003.

[Bhowmick]
Jyotishman Bhowmick, Quantum Isometry Group of the n-tori, Proc. AMS, 2009

[Bhowmick-Goswami]
Jyotishman Bhowmick and Debashish Goswami, Quantum group of orientation-preserving Riemannian isometries. J. Funct. Anal., 2009.

[Chirvasitu]
Alexandru Chirvasitu, Quantum rigidity of negatively curved manifolds, arXiv:1503.07984.

[Goswami]
Debashish Goswami, Existence and examples of quantum isometry group for a class of compact metric spaces, to appear in Adv. Math.

[Wang]
Shuzhou Wang, Quantum symmetry groups of finite spaces, Comm. Math. Phys., 1998.

[Woronowicz]
Stanislaw Woronowicz, Compact matrix pseudogroups, Comm. Math. Phys., 1987.

References: Noncommutative Invariant Theory and Deformation Theory arising from Hopf actions

see also references within

[Kirkman]
Ellen Kirkman, Invariant theory of Artin-Schelter regular algebras: a survey, arXiv:1506.06121.

[Kirkman-Kuzmanovich-Zhang]
Ellen Kirkman, James Kuzmanovich, and James Zhang, Shephard-Todd-Chevalley theorem for skew polynomial rings, Alg Repr theory, 2010.

[Kirkman-Kuzmanovich-Zhang(2)]
Ellen Kirkman, James Kuzmanovich, and James Zhang, Gorenstein subrings of invariants under Hopf algebra actions, J. Algebra, 2009.

[Chan-Kirkman-W-Zhang(2, 3)]
Kenneth Chan, Ellen Kirkman, Chelsea Walton, and James Zhang, McKay Correspondence for semisimple Hopf actions on regular graded algebras, I and II, arXiv:1607.06977 and arXiv:1610.01220.

[W-Witherspoon]
Chelsea Walton and Sarah Witherspoon, Poincaré-Birkhoff-Witt deformations of smash product algebras from Hopf actions on Koszul algebras, Algebra & Number Theory, 2014.

[W-Witherspoon(2)]
Chelsea Walton and Sarah Witherspoon, PBW deformations of braided products, arXiv:1601.02274.

Thanks for listening!