QUANTUM SYMMETRY

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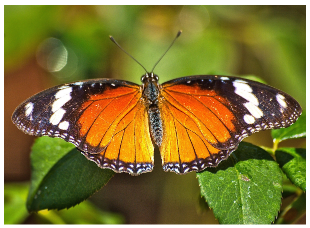
slides for Prezi presentation to be found here:
http://prezi.com/j34lw08mkenu/?utm_campaign=share&utm_medium=copy

Symmetry is a classic notion as it arises everywhere in nature

One of the historical origins of groups G is axiomatize the collection of symmetries of a given (geometric) object X



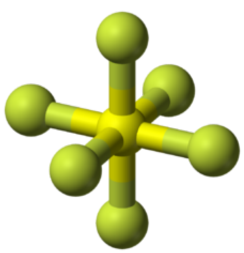
Hypolimnas misippus
(Butterfly)

admits 1 reflection symmetry

$G = \mathbb{Z}_2$

Cyclic group

Ophrys apifera
(Bee orchid)

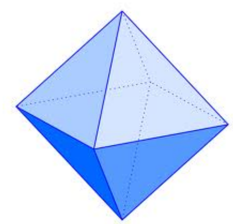


Sulfur hexafluoride molecule

admits 24 rotational symmetries

$G =: O_h \cong S_4$

Octahedral Group



Octahedron (Platonic solid)

This prompts the study of group actions of commutative algebras.

Example: X could be a space, variety, or a manifold.

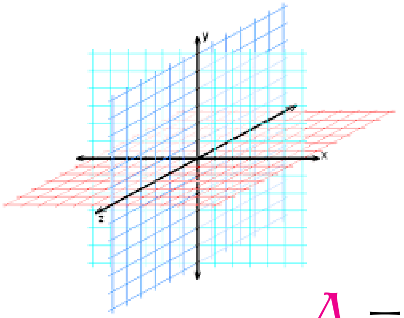
To study symmetries algebraically:

Replace X with the algebra A of functions on X

Example (continued): A could be, respectively,

- a polynomial ring $k[\underline{x}]$,
- coordinate ring $\mathcal{O}(X)$, or
- algebra of continuous functions $\mathcal{C}(X)$

X = affine 3-space over $\mathbb{R}$



$A = \mathbb{R}[x, y, z]$ polynomial ring

Symmetries of quantum objects?

We're motivated by study the symmetries of quantum objects X

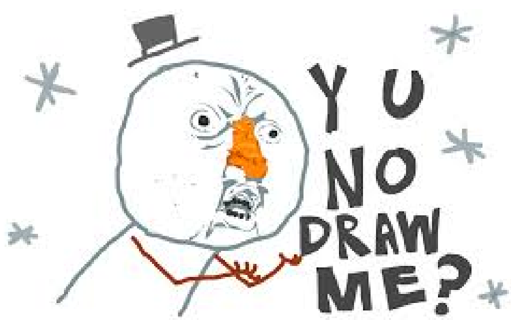
Done with the same approach:

Replace X with the algebra A of functions on X.

Here, A is a noncommutative algebra.

Toy Example:

X = quantum affine 2-space over a field $\boldsymbol{k}$



$A = \boldsymbol{k}\langle u, v\rangle/(uv - qvu)$ for $q \in \boldsymbol{k}^\times$

q-polynomial ring

Toy Example continued:

Take $A = k_q[u,v] = k\langle u,v\rangle/(uv - qvu)$ for $q \in k^\times$.

Consider symmetries of A = (invertible) linear automorphisms of A

$$u \mapsto au + cv, \qquad v \mapsto bu + dv$$

for $a,b,c,d \in k$ with $ad - bc \neq 0$.

	Group G of linear automorphisms of $k_q[u,v]$
$q = 1$	$G = GL_2(k)$
$q = -1$	$G = \langle \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \rangle \leq GL_2(k)$
$q \neq \pm 1$	$G = \langle \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \rangle \leq GL_2(k)$

As A gets more noncommutative, A has less classic (linear) symmetry

... need to investigate the quantum (linear) symmetries of A.

Symmetries of quantum objects?

There are at least two approaches to studying
quantum symmetries of noncommutative algebras.

I. Deform the Classic/Commutative Theory
to get a Quantum/Noncommutative Theory

II. Realize the Classic/Commutative Theory
as a special case
of the Quantum/Noncommutative Theory

Reference for I: [Drinfeld]

We'll use viewpoint II as noncommutative algebras are ubiquitous!

Examples include:

- skew polynomial rings $\Bbbk\langle v_1, v_2, \ldots, v_n\rangle/(v_i v_j = q_{ij} v_j v_i)$, $q_{ij} \in \Bbbk^\times$
- matrix algebras $M_n(\Bbbk)$
- path algebras of a quiver $\Bbbk Q$
- Weyl algebras $A_n(\Bbbk)$ (the algebra of quantum mechanics)
- algebras of differential operators $D(X)$
- quaternions $\mathbb{H}$
- universal enveloping algebras of Lie algebras $U(\mathfrak{g})$
- free algebras $\Bbbk\langle v_1, v_2, \ldots, v_n\rangle$
- Clifford algebras $Cl(V, q)$
- twisted homogeneous coordinate rings $B(X, \mathcal{L}, \sigma)$
- endomorphism algebras $\mathrm{End}(M)$
- division algebras D

... and not all are deformations/ quantizations/ alterations of commutative algebras

We will act on various classes of noncommutative algebras.

With what, you ask? A quantum group = Hopf algebra H.

A Hopf algebra $H = (H, \mu, u, \Delta, \varepsilon, S)$ is simultaneously

- an associative algebra (H, μ, u),
- a coassociative coalgebra (H, Δ, ε), with
- antipode S (playing the role of the inverse),

satisfying several compatibility conditions.

See handout for examples of Hopf algebras. Available at:
`http://math.mit.edu/~notlaw/Examples_Hopf_ASregular.pdf`

Reference: [Radford]

Hopf algebras (or quantum groups) arise in several contexts. They are realized as:

- the algebraic structure of symmetries, where transformations are not necessarily invertible;
- deformations of Lie groups/ Lie algebras;
- etc.

In particular, Hopf algebras arose via Viewpoint I [altering classic actions to get quantum actions]. Various alterations (*) of groups and Lie algebras, that (co)act naturally on commutative algebras, are examples of Hopf algebras.

(*) Such alterations include:

- cocycle and Drinfeld twists of group algebras $(\Bbbk G)_\sigma$, $(\Bbbk G)^J$
- quantizations of enveloping algebras of Lie algebras $U_q(\mathfrak{g})$
- quantizations of coordinate rings of algebraic groups $\mathcal{O}_q(G)$.

References: [Drinfeld] [Majid]

Hopf Algebras: Properties

- As with groups, there is an adjoint action of H on itself via $h \cdot \ell = \sum h_1 \ell S(h_2)$, where $\Delta(h) = \sum h_1 \otimes h_2$ (Sweedler's notation)

- There is a dual Hopf algebra structure for H. In the case where H is finite-dimensional (as a $\Bbbk$-vector space), one can take H^*, the $\Bbbk$-linear dual.

- The category of H-modules forms a $\otimes$ category:

$$\text{If } M, N \in H\text{-mod, then } M \otimes N \in H\text{-mod.}$$

Hopf Actions on Algebras

We say that a Hopf algebra H over $\mathbb{k}$ acts on an algebra $A = (A, \mu_A, u_A)$ over $\mathbb{k}$ if

A is an H-module algebra:

A is an H-module, and the multiplication map and unit map of A are H-morphisms.

Take $h \in H$ with $\Delta(h) = \sum h_1 \otimes h_2$, and $a, b \in A$.

$$\begin{tikzcd}
A \otimes A \arrow[r, "\mu_A"] \arrow[d, "h\text{-action}"'] & A \arrow[d, "h\text{-action}"] \\
A \otimes A \arrow[r, "\mu_A"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
a \otimes b \arrow[r, mapsto] \arrow[d, mapsto] & ab \arrow[d, mapsto] \\
\sum (h_1 \cdot a) \otimes (h_2 \cdot b) \arrow[r, mapsto] & \boxed{h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{k} \arrow[r, "u_A"] \arrow[d, "h\text{-action}"'] & A \arrow[d, "h\text{-action}"] \\
\mathbb{k} \arrow[r, "u_A"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{\mathbb{k}} \arrow[r, mapsto] \arrow[d, mapsto] & 1_A \arrow[d, mapsto] \\
\epsilon(h) 1_{\mathbb{k}} \arrow[r, mapsto] & \boxed{h \cdot 1_A = \epsilon(h) 1_A}
\end{tikzcd}$$

Reference: [Montgomery]

Hopf Actions on Algebras: Toy Example continued.

Take $A = k_i[u,v] = k\langle u,v\rangle/(uv - ivu)$ for $i = \sqrt{-1}$.

$$\mathrm{Aut}_{linear}(k_i[u,v]) = \left\langle \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\rangle \cong k^\times \times k^\times$$

Two examples of Hopf actions on $k_i[u,v]$ preserving the grading of $k_i[u,v]$.
See handout for presentations of the Hopf algebras H_8 and $T(4)$ (with $\zeta = i$)

H_8	$T(4)$
$x \cdot u = -u$ $x \cdot u = v$ $y \cdot u = u$ $y \cdot v = -v$ $z \cdot u = v$ $z \cdot v = u$	$g \cdot u = u$ $g \cdot v = iv$ $x \cdot u = v$ $x \cdot v = 0$

The algebra $k_i[u,v]$ admits more symmetries if we consider Hopf actions.
In other words, we have honest quantum symmetries on a quantum 2-space.

Hopf Actions on Algebras: Notion of faithfulness

H acts on A inner-faithfully

if there is not an induced action of H/I on A for any nonzero Hopf ideal I of H. In other words, the Hopf action does not factor through a smaller Hopf quotient.

H coacts on A inner-faithfully

if there is not an induced coaction of H' on A for any proper Hopf subalgebra H' of H, that is, the Hopf coaction does not factor through a smaller Hopf subalgebra.

Difficulties in studying Hopf actions on Algebras

0. Can depend on choice of base field k.

1. Calculations can be nasty.

2. Techniques depend on choice of class of module algebras.

3. Depends on classification of Hopf algebras
   (which will not be complete in the foreseeable future).

Can remedy some of these issues by imposing
additional hypotheses

Optional Hypotheses on Hopf algebras

Could impose that H is

finite-dimensional as a k-vector space, or

semisimple as an algebra (which implies finite-dimensionality), or

cosemisimple as a coalgebra [each H-comodule $= \bigoplus$ simple H-subcomodules], or

involutory [the square of the antipode of H is the identity].

(If H is finite-dim'l and char(k) =0, then semisimple $\equiv$ cosemisimple $\equiv$ involutory.)

pointed [every simple H-comodule is 1-dimensional]$\}$

There is a very active program to classify finite-dimensional Hopf algebras in the semisimple (resp. pointed) settings.
Group theoretic (resp. Lie theoretic) techniques are employed.

Reference: [Larson-Radford], [Andruskiewitsch]

Optional Hypotheses on Hopf actions/ coactions

If the module algebra A is graded, then we could assume that the H-(co)action on A preserves this grading (e.g. linear (co)actions).

Could use the homological (co)determinant of H-(co)action on A.

Often arises as the quantum determinant in the literature.

For instance, to get an analogue of a result involving group actions with $G < SL(V)$, impose trivial homological determinant.

Avoiding technicalities here, $\mathrm{hdet}_H A$: $H \to k$ is a H-morphism; it is *trivial* if equal to counit map ε of H. Moreover, $\mathrm{hcodet}_H A$ arises as a group-like element D in H

hdet and hcodet are also useful because there are various 'homological identities' relating these notions with properties of both H and A.

References: [Jørgensen-Zhang], [Kirkman-Kuzmanovich-Zhang]

No Quantum Symmetry

Actions of groups G (or kG) and Lie algebras $\mathfrak{g}$ (or $U(\mathfrak{g})$) are considered **classical**.

Note that kG and $U(\mathfrak{g})$ are cocommutative:
$\Delta = \tau \circ \Delta$, where $\tau(h \otimes \ell) = \ell \otimes h$, for $h, \ell \in H$.

Theorem (Cartier-Kostant-Milnor-Moore). If H is a cocommutative Hopf algebra over an algebraically closed field of characteristic 0, then $H \cong U(\mathfrak{g})\#kG$, for some group G acting on some Lie algebra $\mathfrak{g}$. If, further, H is finite-dimensional, then $H \cong kG$, for some group G.

Given an Hopf H-action on an algebra A, we say there is **No Quantum Symmetry** when this action must factor through the action of a cocommutative Hopf algebra.

No Quantum Symmetry: RESULTS

In the settings below, Hopf actions must factor through the action of a **cocommutative** Hopf algebra.

Conditions on k	on H	on A	on action	Reference
char 0 alg. closed	semisimple	commutative domain	(none)	[Etingof-W]
char > 0 alg. closed	semisimple & cosemisimple	commutative domain	(none)	[Etingof-W]
char 0 alg. closed	semisimple	Weyl algebra $A_n(k)$	(none)	[Cuadra-Etingof-W]
char ≥ 0 alg. closed	semisimple & cosemisimple	division algebra D	$\dim H$ & $(\deg D)!$ are coprime	[Cuadra-Etingof-W]
char 0 alg. closed	finite-dim'l	$\frac{k\langle x_1,\dots,x_n\rangle}{(x_ix_j - q_{ij}x_jx_i)}$ $q_{ij} \in k^\times$ generic	preserves grading	[Chan-W-Zhang]

No Quantum Symmetry: IN PROGRESS

In the settings below, we conjecture that Hopf actions must factor through the action of a **cocommutative** Hopf algebra.

(When char(k)=0, finite-dim'l, cocommutative Hopf algebras are group algebras.)

Conditions on k	on H	on A	on action	Collaborators
char 0 alg. closed	finite-dim'l	Weyl algebra $A_n(k)$	(none)	Cuadra-Etingof-W
char 0 alg. closed	semisimple	various quantizations of commutative domains, including $U(\mathfrak{g})$ for $\mathfrak{g}$ fin dim'l, $D(X)$ diff. op. on smooth aff var.	(none)	Etingof-W
char 0 alg. closed	semisimple	generic Sklyanin algebras & twisted hom. coord rings	(none)	Etingof-W
char 0 alg. closed	semisimple	$\dfrac{k\langle x_1,\ldots,x_n\rangle}{(x_ix_j - q_{ij}x_jx_i)}$ $q_{ij} \in k^\times$ generic	(none)	Etingof-W

Universal Quantum Symmetry: set-up

Given an algebra A, a universal quantum group $\mathbb{Q}(A)$ coacting on A is a Hopf algebra, so that for all Hopf coactions of H on A,

- we get a unique map $\pi : \mathbb{Q}(A) \to H$, with
- the following diagram commuting:

$$\begin{tikzcd} & A \otimes \mathbb{Q}(A) \arrow[d, dashed, "\mathrm{id}_A \otimes \pi"] \\ A \arrow[ur, "\rho_{\mathbb{Q}}"] \arrow[r, "\rho_H"'] & A \otimes H \end{tikzcd}$$

Similarly, could define the universal quantum linear group $\mathbb{Q}_{lin}(A)$ if

- A is graded and generated in degree 1, and
- we impose that all coactions on A preserve the grading of A.

Reference for $\mathbb{Q}_{lin}(A)$: [Manin]

Universal Quantum Symmetry: basic examples

Examples of universal quantum linear groups:

A	$\mathcal{Q}_{lin}(A)$	$\mathcal{Q}_{lin}(A)$ w/ triv. hcodet	
$\Bbbk[u,v]$	$\mathcal{O}(GL_2(\Bbbk))$	$\mathcal{O}(SL_2(\Bbbk))$	
$\Bbbk_q[u,v] := \frac{\Bbbk\langle u,v\rangle}{(uv-qvu)}$	$\mathcal{O}_q(GL_2(\Bbbk))$	$\mathcal{O}_q(SL_2(\Bbbk))$	(1-parameter deformation)
$\Bbbk_J[u,v] := \frac{\Bbbk\langle u,v\rangle}{(uv-vu-v^2)}$	$\mathcal{O}_J(GL_2(\Bbbk))$	$\mathcal{O}_J(SL_2(\Bbbk))$	(Jordanian deformation)

As algebras, these are all Noetherian domains and enjoy other **nice ring-theoretic properties**.

These algebras are also **nice homologically**– these are all Artin-Schelter (AS) regular (finite global dimension + "AS Gorenstein").

References: [Brown-Goodearl, Sec I.2 and Prop. II.9.12] and [Brown-Zhang, Lemma 6.1]

Universal Quantum Symmetry: algebraic properties of $\mathbb{Q}$

Philosophy

The universal quantum linear groups $\mathbb{Q}_{lin}(A)$ should share the same ring-theoretic and homological properties of the comodule algebra A.

$*$ Verified for $\mathbb{Q}_{lin}(A)$ assoc. to many classes of *Noetherian* AS regular algebras A
$*$ There's recent work for non-Noetherian AS regular algebras:

> **Theorem** [W-Wang] Let S be an AS regular algebra of gl.dim 2.
>
> (a) Restricting to triv. hcodet, we get that $\mathbb{Q}_{lin}(S)$ is AS regular of gldim 3.
>
> (b) We have that $\mathbb{Q}_{lin}(S)$ and $\mathbb{Q}_{lin}(S)$ are Noetherian and have polynomial growth precisely when S does.

$*$ There are still have many basic questions to address. For instance:

> **Question** [W-Wang] We have that all such S are coherent domains. Is the same true for $\mathbb{Q}_{lin}(S)$ and $\mathbb{Q}_{lin}(S)$?

Universal Quantum Symmetry: Hopf algebraic properties of $\mathbb{Q}$

$$\begin{tikzcd}
 & A \otimes \mathbb{Q}_{lin}(A) \arrow[d, dashed, "\mathrm{id}_A \otimes \pi"] \\
A \arrow[ur, "\rho_{\mathbb{Q}}"] \arrow[r, "\rho_H"'] & A \otimes H
\end{tikzcd}$$

Lemma [Manin]: If A is graded, quadratic, finitely generated in degree 1, and all coactions are linear, then H coacts on A inner-faithfully $\Leftrightarrow$ π is surjective.

Say π is surjective. If $\mathbb{Q}_{lin}(A)$ is commutative/ cocommutative/ cosemisimple/ pointed, then so is H.

This observation is behind the scenes in W-Wang's study of Hopf coactions on (not nec. Noeth.) AS regular algs S of gl.dim 2. Have results on when Hopf quotients of $\mathbb{Q}_{lin}(S)$ are cocommutative.

Universal Quantum Symmetry: analytic properties of $\mathbb{Q}$

There's an abundance of literature on another rich setting for Detecting Quantum Symmetry ... in functional analysis.

Here, $\mathbb{Q}$ also has the structure of a C^*-algebra, and coactions (called "actions" in many works) respect this structure.

Examples of objects X that are coacted upon in this setting include:

- finite sets [Wang]
- finite graphs [Bichon]
- finite-dimensional Hilbert spaces [van Daele-Wang]
- finite (resp., compact) metric spaces [Banica] (resp. [Goswami])
- Riemannian manifolds [Bhowmick-Goswami]

One may impose additional hypotheses on coactions to get results, but of course, these conditions are analytic in nature.

General References: [Connes] [Wang] [Woronowicz]

No Quantum Symmetry: some RESULTS in the analytic setting

In the following, C^*-Hopf coactions of a C^*-Hopf algebra H on a C^*-algebra A must factor through the coaction of a **commutative** C^*-Hopf algebra

(through the coaction of some classical $C(G)$)

- $\boldsymbol{k} = \mathbb{C}$,
- H is a compact quantum group, and
- $A = C(X)$ for some geometric object X
- all coactions are isometric (so that we're detecting quantum isometries)

Geometric object X	Reference
Sphere S^n	[Bhowmick-Goswami(2)]
Torus $\mathbb{T}^k$	[Bhowmick]
smooth, compact, connected Riemannian manifold	[Goswami-Joarder]
compact, connected Riemannian manifold with negative sectional curvature	[Chirvasitu]

Honest Quantum Symmetry: on path algebras kQ

There exist inner-faithful **noncocommutative** Hopf actions in the setting below.

Conditions on k	on H	on Q	on action	Reference
contains a primitive n-th of root unity for $n \geq 2$	pointed: $T(n)$, $u_q(\mathfrak{sl}_2)$, $D(T(n))$	finite loopless Schurian	preserves asc. path length filtration	[Kinser-W]

Let's give some details...

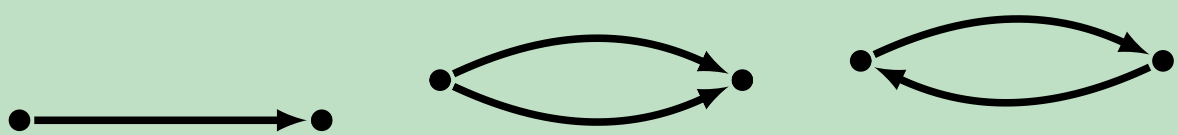
Set-up & Hypotheses

Quiver Q = a directed graph with vertex set Q_0, arrow set Q_1,
and start/ target maps $s, t : Q_1 \to Q_0$.

Path algebra kQ = k-vector space basis elts are paths in Q, where
multiplication is composition of paths if defined, 0 otherwise.

<u>We assume:</u>

- The quiver Q is finite $(|Q_0|, |Q_1| < \infty)$
- Q is loopless
- Q is Schurian $(\forall\ i, j \in Q_0, \exists$ at most one $a \in Q_1$ with $s(a) = i$ and $t(a) = j)$



Schurian   Not Schurian   Schurian

- The action of $T(n)$ preserves the ascending path length filtration on kQ
  (e.g. for $x \in T(n)$ and $a \in Q_1$, we allow $x \cdot a \in kQ_0$)

Strategy for getting $T(n)$-actions on kQ

Start of group action on Q.
Build Taft action on kQ from this.

There's a natural group associated to a Hopf algebra

$$G(H) = \{h \in H \mid \Delta(h) = h \otimes h\}$$

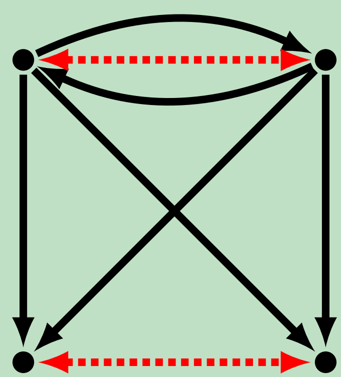
the group of group-like elements of H.

Example: $G(T(n))$ is a cyclic group $\mathbb{Z}_n = \langle g \rangle$.

Theorem ($T(n)$-actions on kQ)

> Given any quiver Q that admits a faithful action of $\mathbb{Z}_n$
> (by quiver automorphism),
>
> we have a classification of (e.g. precise formulae for)
> inner faithful actions of $T(n)$ on kQ
> that extend the given $\mathbb{Z}_n$-action on Q.

Example: We classify Sweedler ($T(2)$)-actions on the path algebra of Q below.



Here, the action of $\mathbb{Z}_2$ is given by $\bullet \longleftrightarrow \bullet$

Theorem (Extended actions of $u_q(\mathfrak{sl}_2)$, $D(T(n))$ on kQ)

Since $u_q(\mathfrak{sl}_2)$ and $D(T(n))$ are both generated by Hopf subalgebras that are isomorphic to Taft algebras,

we have the following result.

Fix an action of $\mathbb{Z}_n$ on a quiver Q.

Additional restraints on parameters are determined so that the Taft actions on kQ produced in Theorem above extend to an action of $u_q(\mathfrak{sl}_2)$ and to an action of $D(T(n))$.

Semisimple Hopf actions on kQ is a work in progress.

Honest Quantum Symmetry: on commutative domains (fields)

Take k an algebraically closed field of characteristic 0.

Let H be a Hopf algebra that acts on a field inner-faithfully;
we say that such an H is Galois-theoretical.

The following noncocommutative, noncommutative, finite-dimensional, pointed Galois-theoretical Hopf algebras.

H	"Cartan type"
Taft algebras $T(n)$	A_1
Nichols Hopf algebras $E(n)$	$A_1^{\times n}$
the book algebra $\mathbf{h}(\zeta, 1)$	$A_1 \times A_1$
the Hopf algebra H_{81} of dimension 81	A_2
$u_q(\mathfrak{sl}_2)$	$A_1 \times A_1$
$u_q(\mathfrak{gl}_2)$	$A_1 \times A_1$
Twists $u_q(\mathfrak{gl}_n)^{J^+}$, $u_q(\mathfrak{gl}_n)^{J^-}$ for $n \geq 2$	$A_{n-1} \times A_{n-1}$
Twists $u_q(\mathfrak{sl}_n)^{J^+}$, $u_q(\mathfrak{sl}_n)^{J^-}$ for $n \geq 2$	$A_{n-1} \times A_{n-1}$
Twists $u_q^{\geq 0}(\mathfrak{g})^J$ for $2^{\mathrm{rank}(\mathfrak{g})-1}$ of such J	same type as $\mathfrak{g}$

$\mathfrak{g}$ is a finite-dimensional simple Lie algebra

Reference: [Etingof-W(2)]

Honest Quantum Symmetry: on commutative domains (fields)

The Galois-theoretical property is preserved under taking:

- Hopf subalgebra
- $\otimes$

... so this allows one to cook up more quantum symmetries

The Galois-theoretical property is *not* preserved under taking:

- Hopf dual
- 2-cocycle deformation (twisting the multiplication)
- dual 2-cocycle deformation (twisting the comultiplication)

Reference: [Etingof-W(2)]

Galois-theoretical property & Galois extensions

Take k an algebraically closed field of characteristic 0.
Say H is finite-dimensional, Galois-theoretical with H-module field L.

If, further, H is semisimple, then

$$H \cong kG \text{ and the extension } L^H = L^G \hookrightarrow L \text{ is Galois.}$$

On the other hand, if, further, H is pointed, then

$$L^H = L^{G(H)} \text{ and the extension } L^H \hookrightarrow L \text{ is Galois.}$$

Here, $G(H)$ is the group of group-like elements of H.

$$G(H) = \{h \in H \mid \Delta(h) = h \otimes h\}$$

Reference: [Etingof-W(2)]

Honest Quantum Symmetry: on noncommutative domains

The inner-faithful Hopf actions in the setting below are classified

k is an algebraically closed field of char. 0

H is a finite-dimensional Hopf algebra

A is an Artin-Schelter regular algebra of gl.dim 2
(a homological analogue of $k[u, v]$)

action preserves the grading of A, subject to trivial hom. det.

which is a generalization of the setting where
$G \leq SL_2(k)$ acts on $k[u, v]$ linearly and faithfully

Reference: [Chan-Kirkman-W-Zhang]

Noncommutative Invariant Theory

The inner-faithful Hopf actions in the setting below are classified

k is an algebraically closed field of char. 0

H is a finite-dimensional semisimple Hopf algebra

A is an Artin-Schelter regular algebra of gl.dim 2
(a homological analogue of $k[u, v]$)

action preserves the grading of A, subject to trivial hom. det.

This is a generalization of the setting
of the classic McKay correspondence

Constructing a McKay correspondence in this setting is a WORK IN PROGRESS
(with Chan, Kirkman, Zhang)

If you would like to know more about
Noncommutative Invariant Theory
(via Hopf actions on Artin-Schelter regular algebras),

please see Ellen Kirkman's Auslander conference
lectures from 2013 and 2014.

... Or feel free to ask for more details after the talk.

Reference: [Kirkman:AuslanderTalks]

Directions for Research on Quantum Symmetry

Computation. Computations are a pain. Write a program to do this.

Classification results. Pick a class of algebras. Pick a class of Hopf algebras. Perhaps impose some conditions on Hopf action. Is there quantum symmetry?

Fancy classification results. Use the machinery of tensor categories/ fusion categories to understand Hopf actions.

Make connections to other fields. This has been done in functional analysis & geometry. Topology?

Physical Applications. There's a sneaking suspicion that this will be useful to physicists. Investigate new applications.