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Seattle Noncommutative Alg Day
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On quasitriangular comodule algebras

joint w/ Monique Müller
2508.19845

<u>Q1</u>: What is a $\begin{cases}\text{categorical} \\ \text{Hopf}\end{cases}$ braided structure?

<u>Q2</u>: ... And when are two the same?

<u>A1</u> (rough) $\boxed{\text{Categorical Braided structures}}$
= a categorical structure whose objects represent a braid group.

Braid groups

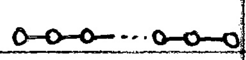
$A_n, BC_n, D_n, E_6, E_7, E_8, F_4, G_2, H_3, H_4, I_2(m)$

Ref: Kassel–Turaev (2008)

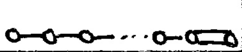
<u>Defn</u> Given a Coxeter matrix $(a_{ij}) \in \mathrm{Mat}_n(\mathbb{N})$ of type X_n, the <u>braid group of type X_n</u> is the group

$$\boxed{Br_n^X} = \langle \theta_1, \dots, \theta_n \mid \underbrace{\theta_i \theta_j \theta_i \cdots}_{a_{ij}\text{ factors}} = \underbrace{\theta_j \theta_i \theta_j \cdots}_{a_{ij}\text{ factors}} \rangle$$

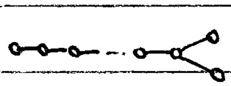
There's a "Coxeter graph of type X_n" ...



Ex. $$\boxed{Br_n^A} = \left\langle \sigma_1, \dots, \sigma_n \;\middle|\; \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, \dots, n-1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i-j| \geq 2 \end{array} \right\rangle$$



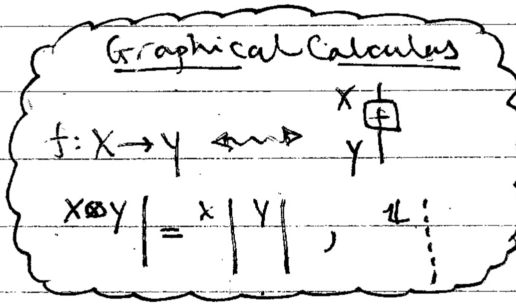
$$\boxed{Br_n^{BC}} = \left\langle \sigma_1, \dots, \sigma_{n-1}, t \;\middle|\; \begin{array}{l} \sigma_i\text{'s satisfy Type A relations} \\ \underset{i=1,\dots,n-2}{\sigma_i t = t \sigma_i}, \quad \sigma_{n-1} t \sigma_{n-1} t = t \sigma_{n-1} t \sigma_{n-1} \end{array} \right\rangle$$



$$\boxed{Br_n^{D}} = \left\langle \sigma_1, \dots, \sigma_{n-1}, t \;\middle|\; \begin{array}{l} \sigma_i\text{'s satisfy Type A relations,} \\ \underset{i=1,\dots,n-3,\,n-1}{\sigma_i t = t \sigma_i}, \quad \sigma_{n-2} t \sigma_{n-2} = t \sigma_{n-2} t \end{array} \right\rangle$$

A1 (Type A) $\boxed{\text{Prop: Braided monoidal categories are categorical braided structures of Type A.}}$

$(\mathcal{C}, \otimes, \mathbb{1}, c)$

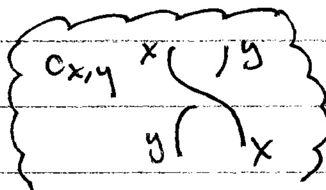


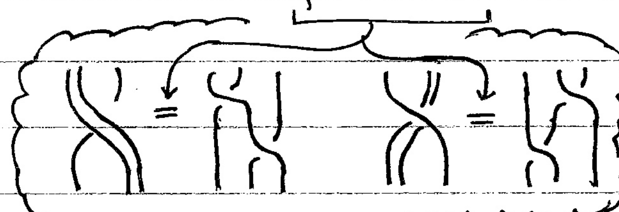
- $\mathcal{C}$ = category
- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ monoidal product (bifunctor)
- $\mathbb{1} \in \mathcal{C}$ disting. object

$\ni$ $(\mathcal{C}, \otimes, \mathbb{1})$ behaves like a monoid

(have associativity & unitality nat'l $\cong$...)

- $c = \{c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X\}_{X,Y \in \mathcal{C}}$ nat'l $\cong$ compatible with $\otimes$





$\underline{\underline{\text{Ex.}}}$ $(\mathrm{Vec}_{\Bbbk}, \otimes = \otimes_{\Bbbk}, \mathbb{1} = \Bbbk, \text{flip})$

$(\Bbbk G\text{-Mod}, \otimes_{\Bbbk}, \Bbbk, \text{flip})$ (group)

Ex. $R = 1 \otimes 1$ for $\Bbbk G$

$(H\text{-Mod}, \otimes_{\Bbbk}, \Bbbk, \text{via } R\text{-matrix})$ (bialgebra/Hopf alg)

$R = R^1 \otimes R^2 \in H \otimes H$ satisfying axioms

For $X, Y \in \mathcal{C}$:

$h \cdot (x \otimes y)$ $\qquad$ $h \cdot 1_{\Bbbk}$ $\qquad$ $c_{X,Y}(x \otimes y)$

$\Delta(h) = h_1 \otimes h_2 \to \quad \| \qquad\qquad \| \qquad\qquad \|$

$(h_1 \cdot x) \otimes (h_2 \cdot y)$ $\qquad$ $\varepsilon(h) 1_{\Bbbk}$ $\qquad$ $(R^2 \cdot y) \otimes (R^1 \cdot x)$

The tuple $(\underbrace{H, m, u, \Delta, \varepsilon}_{\text{bialgebra}}, R)$ is called a quasitriangular bialgebra

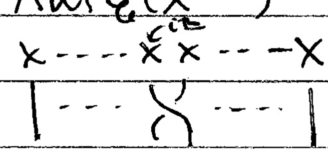
$\mathcal{C} \quad \otimes \quad \mathbb{1} \quad c$

(same for Hopf)

<u>pf of Prop:</u> For any object $X \in (\mathcal{C}, \otimes, \mathbb{1}, c)$,
$\exists$ group homomorphism:

$$\rho_n^X : Br_A^{n-1} \longrightarrow \mathrm{Aut}_{\mathcal{C}}(X^{\otimes n})$$

$$\sigma_i \longmapsto$$



[relations] $\longmapsto$ [braid axioms]

That is, $X^{\otimes n}$ yields a representation of Br_{n-1}^A.

A2 (Type A)

$\boxed{\text{Theorem [Shimizu] Can use } \rho_n^X \text{ to get a (braided) Morita invariant in the Hopf case.}}$

<u>Defn</u> Say f.d. (quasitriangular) Hopf algebras (H, R) & (H', R') <u>(braided) Morita equivalent</u> if $H\text{-Mod} \simeq H'\text{-Mod}$
(braided) categories.

<u>Result:</u> $H\text{-Mod} \simeq H'\text{-mod} \implies \rho_n^{H\mathrm{reg}} \simeq \rho_n^{H'\mathrm{reg}}$

braided categories — reps of Br_{n-1}^A

<u>Application:</u> If $\exists\, \theta \in Br_{n-1}^A \ni\ tr(\rho_n^{H\mathrm{reg}}(\theta)) \neq tr(\rho_n^{H'\mathrm{reg}}(\theta))$
then $H\text{-Mod} \not\simeq H'\text{-Mod}$.

<u>Another result</u> For f.d. Hopf algebras L, L' with quasitriangular Drinfeld doubles $D(L), D(L')$, get

$$L\text{-Mod} \simeq L'\text{-Mod} \implies \rho_n^{D(L)\mathrm{reg}} \simeq \rho_n^{D(L')\mathrm{reg}}.$$

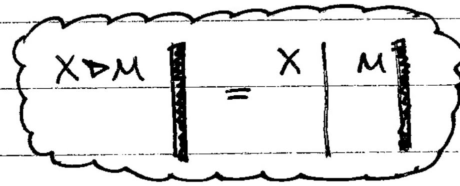
monoidal

A1 (Type BC)

Prop: $\boxed{\text{Braided module categories are categorical structures of Type B}}$

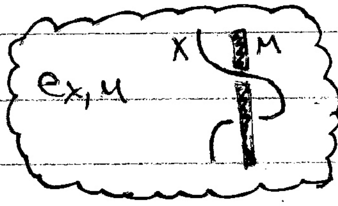
$(\mathcal{M}, \triangleright, e)$ over $(\mathcal{C}, \otimes, \mathbb{1}, c)$

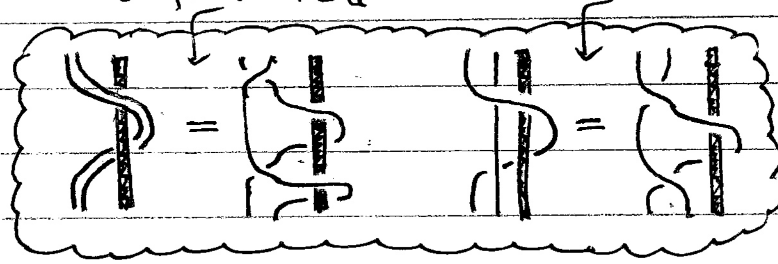
- $\mathcal{M}$ = category
- $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ left $\mathcal{C}$-action (bifunctor)



- $e = \{ e_{X,M} : X \triangleright M \xrightarrow{\sim} X \triangleright M \}_{X \in \mathcal{C}, M \in \mathcal{M}}$ nat'l $\cong$

compatible w/ $\otimes_{\mathcal{C}}$ & with $\triangleright$





Ex For G finite group, $R_u = \frac{1}{2}(1 \otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u)$ ← R-matrix on $\Bbbk G$ $\quad u \in Z(G)\ u^2 = 1$

Get $\mathcal{C} = (\Bbbk G\text{-Mod}, \otimes_{\Bbbk}, \Bbbk, c \leftrightarrow R_u)$ br. $\otimes$ cat.

For $L \leq G$ subgroup, get

$$\triangleright : \Bbbk G\text{-Mod} \times \Bbbk L\text{-Mod} \longrightarrow \Bbbk L\text{-Mod}$$

$$(X, \cdot), (M, *) \longmapsto \big(X \otimes M,\ \ell \cdot (x \otimes m) = (\ell \cdot x) \otimes (\ell * m)\big)$$

Also for $K = a \otimes 1 \in \Bbbk G \otimes \Bbbk L$, for $a \in C_G(L)$

$e_{X,M}(x \otimes m) = (a \cdot x) \otimes (1 * m)$ works.

In general, for $\mathcal{C} = (H, R)$-mod,

a left H-comodule algebra $(B, \delta : B \to H \otimes B)$

$\rightsquigarrow$ $\triangleright$ for B-mod (left $\mathcal{C}$-module category)

& "K-matrix" $K \in H \otimes B$ (satisfying axioms)

$\rightsquigarrow$ e for $(B\text{-mod}, \triangleright)$.

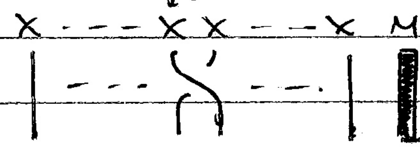
The tuple (B, δ, k) is called a quasitriangular (left H-) comodule algebra

$$\begin{matrix} (B, & \delta, & k) \\ \downarrow & \downarrow & \downarrow \\ \mathcal{M} & \triangleright & e \end{matrix}$$

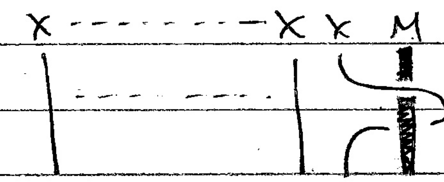
<u>Pf of Prop</u> For any objects $X \in (\mathcal{C}, \otimes, 1, c)$ & $M \in (\mathcal{M}, \triangleright, e)$ $\exists$ group homomorphism.

$$\rho_n^{X,M}: Br_n^{BC} \longrightarrow \mathrm{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$$

$\sigma_i \longmapsto$



$t \longmapsto$



That is, we get a rep'n of Br_n^{BC}.

A2 (type BC) **Theorem** [Müller-W] Can use $\rho_n^{X,M}$ to get a (braided) Morita invariant in the comodule algebra case.

<u>Def'n</u> Say f.d. (quasitriangular) left (H,R)-comodule algebras (B,k), (B',k') are <u>(braided) Morita equivalent</u> if

$$B\text{-Mod} \simeq B'\text{-Mod} \text{ as (braided) mod. categories}$$

<u>Result</u> $* \Rightarrow \rho_n^{H_{reg}, B_{reg}} \simeq \rho_n^{H_{reg}, B'_{reg}}$ as reps of type BC

* need B, B' "H-simple" & "augmented"

Ex. Coideal subalgebras satisfy this ✓

Similar { application w/ traces as type A
other result
⋮

replace Drinfeld double w/ "reflective algebra"

can'l quasitriang. comod. algebra attached to arbitrary comod. alg. (see Laugwitz-W-Yakimov)

$$\left[\begin{array}{c} A\text{-Mod} \overset{\otimes}{\simeq} A'\text{-Mod} \\ \Rightarrow \; RH(A)\text{-Mod} \overset{\text{br}\otimes}{\simeq} RH(A')\text{-Mod} \end{array}\right]$$

$\boxed{\text{Questions}}$

① What about other Coxeter types?

- A1, A2 in type D done by Müller-W. Here, Categorical braided structure ≡ "symmetric" mod. categ. ($e^2 = id$)

② Remove H-simplicity & augmentation assumptions on Type BC/D theorems

- They were needed to use key results of Andruskiewitsch-Mombelli on "equivariant bimodules" & Skryabin's freeness results on such structures

Why care?

[R-matrices ≡ solutions to QYBE]

&

[K-matrices ≡ solutions to quantum reflection equation]

③ Classify K-matrices for concrete examples of comodule algebras over quasitriangular Hopf algebras ... up to braided Morita equivalence

- This was done for coideal subalgebras of $\Bbbk G$ and of the Sweedler algebra by Müller-W

... more examples are needed!