CMI Jan 2026 {Questions in red} C. Walton

Representations of braided ⊗ categories

based on ArXiv: 2307.14764, 2411.18453, 2508.19845
joint w/ R. Laugwitz & M. Yakimov, H. Yadav, M. Müller.

Why care about reps / modules?

Structure of interest $\mathcal{S}$ understood as symmetries of an object $\mathrm{Sym}(\mathcal{U})$

rep'n ≡ structure map $\mathcal{S} \to \mathrm{Sym}(\mathcal{U})$

$\mathcal{S}$-module ≡ $\mathcal{U}$ w/ action map $\triangleright : \mathcal{S} \times \mathcal{U} \to \mathcal{U}$

} repack

Here, $\boxed{\mathcal{S} = \text{braided monoidal category}}$

$= (\underbrace{\mathcal{C}, \otimes, \mathbb{1}}_{\text{monoidal category}}, c)$ ← braiding: nat'l ≅ $c = \{c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X\}_{X,Y \in \mathcal{C}}$

read ↓

such that

braid axioms.

- Introduced by Joyal & Street (1993), used to construct knot & link invariants

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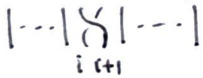
- Also used to represent Artin braid groups of Type A

$\rho_n^X : Br_{A_{n-1}} \longrightarrow \mathrm{Aut}_{\mathcal{C}}(X^{\otimes n})$ $(X \in \mathcal{C})$

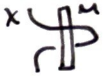
$\sigma_i \longmapsto$



Here, $Br_{A_{n-1}} = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i-j|>1) \end{array} \right\rangle$

$\mathcal{S}$-module = $\boxed{\text{braided module category over } \mathcal{C}}$

$= (\underbrace{\mathcal{M}, \triangleright}_{\text{left } \mathcal{C}\text{-module category}}, e)$ ← braiding: nat'l ≅ $e = \{e_{X,M} : X \triangleright M \xrightarrow{\sim} X \triangleright M\}_{X \in \mathcal{C},\, M \in \mathcal{M}}$



such that

braid axioms:

- Especially when $\mathcal{C}$ is ribbon, introduced by Brochier (2013) to study "B-tangles" (tangles around a pole) & related topological invariants.

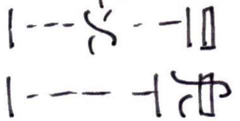
↳ Earlier versions introduced by Enriquez, Häring-Oldenburg

- Used to represent Artin braid groups of Type B

$\rho_n^{X,M} : Br_{B_n} \longrightarrow \mathrm{Aut}_{\mathcal{M}}(X^{\otimes n} \triangleright M)$ $(X \in \mathcal{C}, M \in \mathcal{M})$

$\sigma_i \longmapsto$

$t \longmapsto$



Here, $Br_{B_n} = \left\langle \sigma_1, \dots, \sigma_{n-1}, t \;\middle|\; \begin{array}{l} \text{Type A relations for } \sigma_i \\ t\sigma_i = \sigma_i t \quad i=1, \dots, n-2 \\ t\sigma_{n-1} t \sigma_{n-1} = \sigma_{n-1} t \sigma_{n-1} t \end{array} \right\rangle$

- Independent interest —

↳ Davydov-Nikshych (2021): ∃ 2-equiv. $\mathcal{Z}(\mathcal{C}\text{-Mod}) \simeq \mathcal{C}\text{-BrMod}$

↑ center of a ⊗ 2-categ ↑ 2-cat of br. mod cat

↳ Kolb (2020): used as a framework to examine quantum symmetric pairs in quantum group theory

Reps of braided ⊗ cats

Hopf examples

Take a bialgebra $(H, m, u, \Delta, \varepsilon)$
& equipped it with an **R-matrix**
$R = R^1 \otimes R^2 \in H \otimes H$ invertible
& satisfying axioms

$(H, R) \equiv$ "quasitriangular" bialgebra

- $\mathcal{C} = H\text{-Mod}$ is braided with c given by R.
- R-matrices are of independent interest
  $\equiv$ solutions to the quantum Yang-Baxter equation
  ↳ classification results for fixed H are of interest.
- Notion of sameness: braided Morita equivalence:
  $\equiv (H,R)\text{-Mod} \simeq (H',R')\text{-Mod}$ as braided ⊗ cats
- Convenient braided Morita invariants:

Shimizu (2010): Take H, H' f.d. quasiΔ Hopf algs.
If $H\text{-Mod} \simeq H'\text{-Mod}$ as braided ⊗ categories,
then $\rho_n^H \simeq \rho_n^{H'}$ as representations of $Br_{A_{n-1}}$. ///

Pf relied on Schauenburg's work on bi-Galois objects under Morita-Takeuchi equivalence.

Hopf examples

Take a quasiΔ Hopf alg H & a left H-comodule algebra A
→ $A\text{-Mod} \equiv$ left $(H\text{-mod})$-module category.

Now equip A with a **K-matrix**:
$K = K^1 \otimes K^2 \in H \otimes A$ invertible
& satisfying axioms

(A, K) = "quasitriangular H-comod. alg" due to Kolb (2020)

- $\mathcal{M} = A\text{-Mod}$ is braided with e given by K
- K-matrices are of independent interest
  $\equiv$ solutions to the quantum reflection equation

Classification results are of interest

Müller-W (2025) K-matrices are classified for coideal subalgebras of finite group algebras & the Sweedler Hopf alg. ///

(*) Just got ref report last night. Need to add a case. Thanks!

- Notion of sameness: braided Morita equivalence
  $\equiv (A,K)\text{-Mod} \simeq (A',K')\text{-Mod}$ as braided mod. cats / H-mod.
- Theorem (Müller-W) Take f.d. quasiΔ Hopf algebra H.
  Take quasiΔ H-comod. algs A, A' that are $\underbrace{H\text{-simple \& augmented}}_{(*)}$
  If $A\text{-Mod} \simeq A'\text{-Mod}$ as braided mod. cats
  then $\rho_n^{H,A} \simeq \rho_n^{H,A'}$ as representations of Br_{B_n}.

Pf/ relied on Andruskiewitsch-Mombelli's study of equivariant bimodules & Skryabin's freeness results for such bimodules ← (*) needed.

holds for coideal subalgs ⟺
H-simple ≡ ∄ nontrivial H-costable ideals
augmented ≡ ∃ algebra map $A \to \Bbbk$

(*) due to Nakamura, Shibata, Shimizu 2503.21265 Lemma 4.5.

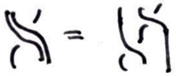
Reps of braided ⊗ cats

Categorical centers

Given a, not-nec. braided, ⊗ category $(\mathcal{C}, \otimes, \mathbb{1})$

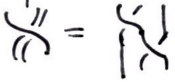
can construct its **Drinfeld center** $\mathcal{Z}(\mathcal{C})$

objects: $(V \in \mathcal{C},\ c^V := \{c^V_X : X \otimes V \xrightarrow{\sim} V \otimes X\}_{X \in \mathcal{C}})$

↑ "half-braiding" satisfying 

maps: $V \to W \in \mathcal{C}$ compatible with c^V & c^W

⊗: $(V, c^V) \otimes (W, c^W) := (V \otimes W,\ c^{V \otimes W})$

satisfying 

braiding: $(V, c^V) \otimes (W, c^W) \xrightarrow[c^W_V]{\sim} (W, c^W) \otimes (V, c^V)$.

Hopf case for a, not-nec quasi Δ, f.d. Hopf algebra H

$$\mathcal{Z}(H\text{-mod}) \underset{br\otimes}{\simeq} {}^H_H\mathcal{YD} \simeq \underbrace{D(H)}-\text{Mod}$$

${}^H_H\mathcal{YD}$: Yetter–Drinfeld modules

$(M, *, \delta)$

M ∈ H-mod, H-comod + compatibility

$D(H)$: **Drinfeld double** quasi Δ Hopf alg. $= H^* \otimes H$ as vspaces ///

- R-matrix for $D(H) = \sum_\alpha \varepsilon_H \otimes h_\alpha \otimes h_\alpha^* \otimes 1_H \in D(H)^{\otimes 2}$ for $\{h_\alpha, h_\alpha^*\}$ dual basis of H.

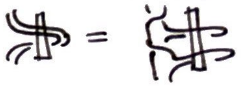
Shimizu (2010) If H-Mod $\simeq H'$-Mod as ⊗ cats, then $\rho_n^{D(H)} \simeq \rho_n^{D(H')}$ as reps of $Br_{A_{n-1}}$. ///

Categorical centers

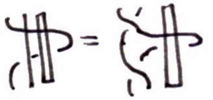
Given a braided ⊗ cat $\mathcal{C}$, & a not-nec-braided $\mathcal{M} \in \mathcal{C}$-mod,

can construct its **reflective center** $\mathcal{E}_{\mathcal{C}}(\mathcal{M})$

objects: $(M \in \mathcal{M},\ e^M = \{e^M_X : X \triangleright M \xrightarrow{\sim} X \triangleright M\}_{X \in \mathcal{C}})$

↑ "reflection" satisfying 

maps $M \to N \in \mathcal{M}$ compatible with e^M, e^N

▷: $X \triangleright (M, e^M) := (X \triangleright M,\ e^{X \triangleright M})$ satisfying 

braiding: $X \triangleright (M, e^M) \xrightarrow[e^M_X]{\sim} X \triangleright (M, e^M)$

Similar to "$\mathcal{Z}(\mathcal{M})$" appearing in previous works, but analogy with $\mathcal{Z}(\mathcal{C})$ is clearer packaged this way.

Hopf case Take a f.d. quasi Δ Hopf alg H & left H-comod alg A

Theorem (LWY) $\mathcal{E}_{H\text{-mod}}(A\text{-Mod}) \simeq \underbrace{{}^{\hat{H}}_{A}DH} \simeq \underbrace{R_H(A)}-\text{Mod}$ as braided mod. categs / H-mod

${}^{\hat{H}}_{A}DH$: a certain category of triples $(M, *, \delta)$, $(M, *)$ ∈ A-mod, ∈ $\hat{H}$-Mod, $\hat{H}$ "transmuted H"

$R_H(A)$: **reflective algebra** quasi Δ H-comod. alg $= H^* \otimes A$ as a vspace

- K-matrix for $R_H(A) = \sum_\alpha h_\alpha \otimes h_\alpha^* \otimes 1_A \in H \otimes R_H(A)$

Theorem (Müller–W) Take a f.d. quasi Δ Hopf H & left H-comod. algs A, A' that are H-simple & augmented & $R_H(A), R_H(A')$ H-simple. Then

? see prev. page (✱) (e.g. CSAs)

A-Mod $\simeq A'$-Mod as module cats / H-mod $\implies \rho_n^{H, R_H(A)} \simeq \rho_n^{H, R_H(A')}$ as reps of Br_{B_n} ///

Nondegeneracy used especially for topological invariants/ TQFTs.

Shimizu's Theorem (2019)

TFAE for a braided finite tensor category $(\mathcal{C}, \otimes, \mathbb{1}, c)$

$\mathcal{C}$ is nondegenerate:

$C_{\mathcal{C}} := \int^{X \in \mathcal{C}} X^* \otimes X$

$\in \mathrm{HopfAlg}(\mathcal{C})$

has a nondeg. Hopf pairing

$\omega_{\mathcal{C}} : C_{\mathcal{C}} \otimes C_{\mathcal{C}} \to \mathbb{1}$

$\Longrightarrow$

$\mathcal{C}$ is weakly factorizable:

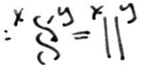
$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, C_{\mathcal{C}}) \to \mathrm{Hom}_{\mathcal{C}}(C_{\mathcal{C}}, \mathbb{1})$

$f \longmapsto [\omega_{\mathcal{C}} \circ (f \otimes \mathrm{id})]$

is bijective

$\Downarrow$

$\mathcal{C}$ has trivial Müger center:

$\{X \in \mathcal{C} :$  $\forall Y \in \mathcal{C}\}$

$:= \mathcal{Z}_2(\mathcal{C}) \simeq \mathrm{Vec}$

$\Longleftrightarrow$ ⊛

$\Updownarrow$

$\mathcal{C}$ is factorizable:

Embeddings

$\mathcal{C} := (\mathcal{C}, c) \to \mathcal{Z}(\mathcal{C})$

$\overline{\mathcal{C}} := (\mathcal{C}, c^{-1}) \to \mathcal{Z}(\mathcal{C})$

yield $\mathcal{C} \boxtimes \overline{\mathcal{C}} \simeq \mathcal{Z}(\mathcal{C})$

⊛ uses FP dimension argument

Nondegeneracy similar applications expected ⊛ (outside my realm of expertise)

Theorem (Yadav–W)

$\mathcal{C}$ braided finite ⊗ category

$\mathcal{M} \in \mathcal{C}\text{-BrMod}$ exact & indecomp.

$\mathcal{M}$ is nondegenerate: Ⓠ1

$E_{\mathcal{M}} := \int_{M \in \mathcal{M}} \underline{\mathrm{Hom}}(M, M) \in \mathrm{Alg}(\mathcal{C})$

has a nondegenerate pairing

$\omega_{\mathcal{M}} : \mathbb{1} \to C_{\mathcal{C}}^* \otimes E_{\mathcal{M}}$

$\Longrightarrow$

$\mathcal{M}$ is weakly factorizable:

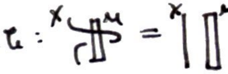
$\mathrm{Hom}_{\mathcal{C}}(C_{\mathcal{C}}^*, \mathbb{1}) \to \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, E_{\mathcal{M}})$

$f \longmapsto [(f \otimes \mathrm{id})\, \omega_{\mathcal{M}}]$

is bijective

$\Downarrow$

$\mathcal{M}$ has trivial symmetric center

Ⓠ2 $\{X \in \mathcal{C} :$  $\forall M \in \mathcal{M}\}$

$:= \mathcal{Z}_2(\mathcal{M}) \simeq \mathrm{Vec}$

$\Updownarrow$ ⊛

$\mathcal{M}$ is factorizable:

$\mathcal{M} \boxtimes \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) \to \mathcal{E}_{\mathcal{C}}(\mathcal{M})$

$M \boxtimes (F, s) \longmapsto (F(M), e^{F(M)})$ ...

is an equivalence

⊛ Pf involves

$$\begin{tikzcd}
\mathcal{M} \boxtimes \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) \arrow[r] \arrow[d, "\text{produced } \simeq"'] & \mathcal{E}_{\mathcal{C}}(\mathcal{M}) \\
E_{\mathcal{M}}\text{-Mod}(\mathcal{M}) \arrow[r] & E_{\mathcal{C}}^*\text{-Mod}(\mathcal{M}) \arrow[u, "\simeq \text{ produced}"']
\end{tikzcd}$$

$\mathcal{E}_{\mathcal{C}}(\mathcal{M})$ ← equiv ⇔ factoriz.

$E_{\mathcal{C}}^*\text{-Mod}(\mathcal{M})$ ← equiv ⇔ nondeg.

(Eilenberg–Moore cat.)

Ⓠ1 $E_{\mathcal{M}}$ should be a comodule algebra

Ⓠ2 Should get $\Longleftarrow$

Maybe different definition of symmetric is needed

Ex. In Müller–W:  is used.

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Nondegeneracy ctned

Main Example: $\mathcal{Z}(\mathcal{C})$ is nondegenerate

Hopf case H-Mod nondegenerate ⇔ H is "factorizable"

Ex. $\mathcal{Z}(\underset{\uparrow}{H}\text{-Mod}) \simeq \underbrace{D(H)}\text{-Mod}$

f.d. Hopf alg ↖ factorizable Hopf algebra

Ex. Quantum groups ...

Nondegeneracy ctned

(Q3) When is $\mathcal{E}_\mathcal{C}(\mathcal{M})$ nondegenerate? Expected often

Hopf case ∃ notion of "factorizable" comodule algebra

(Yadav-W)

Ex. H f.d. quasi△ Hopf algebra

→ Reflective algebra $R_H(A)$ is factorizable when A is $\underbrace{H\text{-simple}}$

(Q4) Can H-simplicity be removed?

(Q5) Quantum groups examples...

AN ANALOGY

Structure / monoid	module \ symmetries
(braided) monoidal category	(braided) module category
(quasitriangular) bi/ Hopf algebra	(quasitriangular) comodule algebra
R-matrices	K-matrices
Solutions to the quantum YBE	Solutions to the quantum reflection equation
Reps of braid groups of Type A	Reps of braid groups of Type B
Drinfeld centers $\mathcal{Z}(\mathcal{C})$	Reflective centers $\mathcal{E}_\mathcal{C}(\mathcal{M})$
Drinfeld doubles $D(H)$	Reflective algebras $R_H(A)$
Quantum groups $U_q(\mathfrak{g})$	Quantum symmetric pairs (CSAs) ← see intro to LWY for references. for comodule algebras?