HOPF ALGEBRA ACTIONS
ON
NONCOMMUTATIVE ALGEBRAS

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Outline

- Noncommutative Algebras
- Hopf Algebras
- Hopf Actions on Algebras
- Three types of results
- Examples of the three types of results
- References

The following document may be helpful:
`http://math.mit.edu/~notlaw/Examples_Hopf_ASregular.pdf`

Noncommutative Algebras

Noncommutative algebras are ubiquitous!

Examples include:

- skew polynomial rings $\Bbbk\langle v_1, v_2, \ldots, v_n\rangle/(v_i v_j = q_{ij} v_j v_i),\ q_{ij} \in \Bbbk^\times$
- matrix algebras $M_n(\Bbbk)$
- path algebras of a quiver $\Bbbk Q$
- Weyl algebras $A_n(\Bbbk)$ (the algebra of quantum mechanics)
- algebras of differential operators $D(X)$
- quaternions $\mathbb{H}$
- universal enveloping algebras of Lie algebras $U(\mathfrak{g})$
- free algebras $\Bbbk\langle v_1, v_2, \ldots, v_n\rangle$
- Clifford algebras $Cl(V, Q)$
- twisted homogeneous coordinate rings $B(X, \mathcal{L}, \sigma)$
- endomorphism algebras $\mathrm{End}(M)$
- division algebras D

Hopf Algebras

Groups (in geometry) arose historically as symmetries of various types of objects. These were Lie groups, studied by F. Klein, S. Lie, and others in the late 19th century.

Hopf algebras (or quantum groups) arise in several contexts. They are realized as:

- the algebraic structure of symmetries, where transformations are not necessarily invertible;
- deformations of Lie groups/ Lie algebras;
- etc.

Definition A Hopf algebra

$$H = (H, \mu, u, \Delta, \varepsilon, S)$$

is simultaneously an algebra (H, μ, u), a coalgebra (H, Δ, ε), with antipode S (playing the role of the inverse), satisfying several compatibility conditions.

Hopf Algebras

Examples include

$\Bbbk G$, group algebras of finite groups

algebra structure ✓, $\quad \Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1} \qquad \forall g \in G$

$U(\mathfrak{g})$, universal enveloping algebras

algebra structure ✓, $\quad \Delta(x) = 1 \otimes x + x \otimes 1, \quad \varepsilon(x) = 0, \quad S(x) = -x \qquad \forall x \in \mathfrak{g}$

$T(n)$, Taft algebras algebras generated by g, x with:

$g^n = 1, \quad x^n = 0, \quad gx = \zeta xg$ for ζ a primitive n-th root of unity

$$\Delta(g) = g \otimes g, \qquad \Delta(x) = 1 \otimes x + x \otimes g$$
$$\varepsilon(g) = 1, \qquad \varepsilon(x) = 0 \qquad S(g) = g^{-1}, \qquad S(x) = -xg^{-1}$$

Properties

- As with groups, there is an adjoint action of H on itself via $h \cdot \ell = \sum h_1 \ell S(h_2)$, where $\Delta(h) = \sum h_1 \otimes h_2$ (Sweedler's notation)
- There is a dual Hopf algebra structure for H. In the case where H is finite-dimensional (as a $\Bbbk$-vector space), one can take H^*, the $\Bbbk$-linear dual.
- The category of H-modules forms a $\otimes$ category:

$$\text{If } M, N \in H\text{-mod, then } M \otimes N \in H\text{-mod.}$$

Hopf Actions on Algebras

We say that a Hopf algebra H acts on an algebra A if

A is an H-*module algebra*:

A is an H-module, and the multiplication and unit maps of A are H-morphisms.

We also need a notion of faithfulness:

H acts on A *inner faithfully*

if there is not an induced action of H/I on A for any nonzero Hopf ideal I of H.
In other words, the Hopf action does not factor through a smaller Hopf quotient.

We need a notion of H-action having 'determinant 1'
– for analogues of results involving group actions with $G < SL(V)$

H-action on A has *trivial homological determinant*

Here, $\mathrm{hdet}_H A$: $H \to \Bbbk$ is a H-morphism; it is *trivial* if equal to counit map ε of H.

Two types of results

Fix a field $\Bbbk$.
Let $\mathcal{H}$ be a class of Hopf algebras over $\Bbbk$.
Let $\mathcal{A}$ be a class of (noncommutative) algebras over $\Bbbk$.

[No Quantum Symmetry] (Hopf action factors through a cocom. Hopf algebra)

If $H \in \mathcal{H}$ acts inner faithfully on any $A \in \mathcal{A}$,
then H must be cocommutative.

(e.g. H must be a group algebra in the finite dimensional, char. 0 setting)

[Honest Quantum Symmetry]

Classify all pairs (H, A) so that $H \in \mathcal{H}$ acts inner faithfully on $A \in \mathcal{A}$.

(possibly subject to conditions on the action, e.g. trivial hom. determinant)

The problem is more tractable when either:
the size of the class of Hopf algebras $\mathcal{H}$ is limited, or
the size of the class of algebras $\mathcal{A}$ is limited.

Examples of the two types of results

[No Quantum Symmetry] This occurs for:

$\Bbbk$ = algebraically closed of characteristic 0

$\mathcal{H}$ = the class of semisimple Hopf algebras (*)

$\mathcal{A}$ = the class of commutative domains [EW],

the class of Weyl algebras [CEW1], or

the class of other quantizations of commutative domains [CEW2].

(*) so, finite dimensional and isomorphic to a finite product of matrix algebras

Examples of the two types of results

[Honest Quantum Symmetry] (with $\mathcal{H}$ vast, $\mathcal{A}$ limited)
We achieved this for:

$\Bbbk$ = algebraically closed of characteristic 0

$\mathcal{H}$ = the class of finite dimensional Hopf algebras

$\mathcal{A}$ = the class of "Artin-Schelter regular algebras of dim. 2" [CKWZ]

($A \in \mathcal{A}$ is a graded homological analogue of $\Bbbk[x, y]$: gl.dim 2, poly'l growth, AS-Gorenstein)

Here, we assume that:

- the H-action on A preserves the grading of A, and
- the H-action on A has trivial homological determinant.

Examples of the two types of results

[Honest Quantum Symmetry] (with $\mathcal{H}$ limited, $\mathcal{A}$ vast)
This has been completed for:

$\Bbbk$ = containing a primitive n-th root of unity

H = Taft algebras $T(n)$ (some n^2-dim'l, pointed(**) Hopf algebras)

A = path algebra $\Bbbk Q$ of a finite, loopless, Schurian quiver [KW].

(**) so, every simple H-comodule is 1-dimensional

References

[CEW1] Juan Cuadra, Pavel Etingof, Chelsea Walton. Semisimple Hopf actions on Weyl algebras (preprint). arXiv:1409.1644.

[CEW2] Juan Cuadra, Pavel Etingof, Chelsea Walton. Semisimple Hopf actions on quantizations. In preparation.

[CKWZ] Kenneth Chan, Ellen Kirkman, Chelsea Walton, James Zhang. Quantum binary polyhedral groups and their actions on quantum planes. To appear in *in J. Reine Angew. Math.*

[EW] Pavel Etingof and Chelsea Walton. Semisimple Hopf actions on commutative domains. *Adv. Math.*, 251: 47–61, 2014.

[KW] Ryan Kinser and Chelsea Walton. Taft algebra actions on path algebras of quivers. In preparation.