MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #1

TOPICS:

I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

II. LOGISTICS OF COURSE & STUDENT INTROS

III. GROUPS, RINGS, VECTOR SPACES (§§ 1.1.1, 1.1.2, 1.1.3)

IV. STRUCTURE VS. PROPERTY (§ 1.1.1)

V. OPERATIONS ON VECTOR SPACES (§ 1.1.4)

I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

A SYMMETRY OF AN OBJECT X IS A (PROPERTY PRESERVING) TRANSFORMATION FROM X TO ITSELF

COLLECTION OF SYMMETRIES OF X, Sym(X), FORMS A MONOID UNDER COMPOSITION

SET EQUIPPED w/ OPERATION
& NEUTRAL/IDENTITY ELEMENT

w/ NEUTRAL ELEMENT = "DO NOTHING TO X"

I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

ALTERNATIVE DEFN

A **SYMMETRY** OF AN OBJECT X IS A (PROPERTY PRESERVING) REVERSIBLE TRANSFORMATION FROM X TO ITSELF

COLLECTION OF SYMMETRIES OF X, Sym(X), FORMS A ~~MONOID~~ GROUP UNDER COMPOSITION

SET EQUIPPED W/ OPERATION & NEUTRAL/IDENTITY ELEMENT

W/ NEUTRAL ELEMENT = "DO NOTHING TO X"

W/ INVERSE ELEMENT = THE REVERSE SYMMETRY

I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

A **SYMMETRY** OF AN OBJECT X
IS A (PROPERTY PRESERVING) REVERSIBLE TRANSFORMATION
FROM X TO ITSELF

COLLECTION OF SYMMETRIES OF X, $\mathrm{Sym}(X)$,
FORMS A <u>GROUP</u> UNDER COMPOSITION

EXAMPLE:

$$X = \mathbb{C}^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; x, y \in \mathbb{C}^2 \right\}$$

COMPLEX 2-SPACE

TAKE SYMMETRIES THAT
- SEND LINES TO LINES (LINEAR)
- PRESERVE ORIGIN

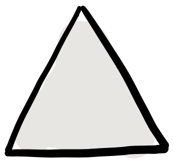
$$\mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}^2) \cong GL_2(\mathbb{C})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C}) \rightsquigarrow \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

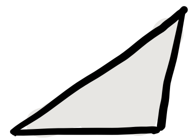
I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

SYMMETRIES UNDER DEFORMATION ARE NOT PRESERVED IN THIS FRAMEWORK ...

TOY EXAMPLE

$X =$ 

REGULAR TRIANGLE

$X_{def} =$ 

SKEW TRIANGLE

$$\mathrm{Sym}(X) \cong S_3 \ncong \mathrm{Sym}(X_{def})$$

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SYMMETRIES UNDER DEFORMATION ARE NOT PRESERVED IN THIS FRAMEWORK ...

OF USING GROUPS TO CAPTURE SYMMETRIES

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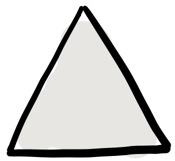
$X =$ 

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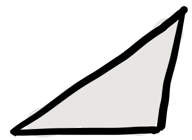
$X_{def} =$ 

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$$\mathrm{Sym}(X) \cong S_3 \not\cong \mathrm{Sym}(X_{def})$$

ALSO GET $\mathrm{Sym}^{lin}(\mathbb{C}^2) \cong GL_2(\mathbb{C}) \not\cong \mathrm{Sym}^{lin}(\mathbb{C}^2_{def})$

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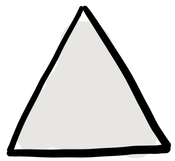
$X =$ 

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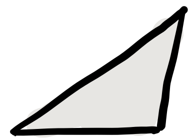
$X_{def} =$ 

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$$\mathrm{Sym}(X) \cong S_3 \not\cong \mathrm{Sym}(X_{def})$$

NEED TO USE ALGEBRAS TO DESCRIBE THIS

ALSO GET $\mathrm{Sym}^{lin}(\mathbb{C}^2) \cong GL_2(\mathbb{C}) \not\cong \mathrm{Sym}^{lin}(\mathbb{C}^2_{def})$

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ROUGHLY, AN ALGEBRA IS A COMBINATION OF A

UNITAL RING & VECTOR SPACE

HAVE: ADDITION & SUBTRACTION

ALSO HAVE: MULTIPLICATION SCALAR-MULTIPLICATION

EX. $\mathcal{O}(\mathbb{C}^2) := \mathbb{C}[x,y]$

COORDINATE ALGEBRA POLYNOMIAL ALGEBRA

NEED TO USE ALGEBRAS TO DESCRIBE THIS

ALSO GET $\mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}^2) \cong GL_2(\mathbb{C}) \not\cong \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}^2_{\mathrm{def}})$

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>
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> HAVE: ADDITION & SUBTRACTION
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> ALSO HAVE: MULTIPLICATION   SCALAR-MULTIPLICATION

EX. $\mathcal{O}(\mathbb{C}^2) := \mathbb{C}[x,y]$ $yx = xy$ $\mathcal{O}(\mathbb{C}_q^2) := \mathbb{C}_q[x,y]$ $yx = qxy$

COORDINATE ALGEBRA   POLYNOMIAL ALGEBRA   q-POLYNOMIAL ALGEBRA

QUANTUM 2-SPACE

ALSO GET $\mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}^2) \cong GL_2(\mathbb{C}) \not\cong \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}_q^2)$ $q \in \mathbb{C}^\times$

I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

SYMMETRIES UNDER DEFORMATION ARE NOT PRESERVED IN THIS FRAMEWORK ...

OF USING GROUPS TO CAPTURE SYMMETRIES

GROUP SYMMETRY OF $\mathbb{C}_q^2$ MUST PRESERVE $\mathbb{C}(yx - qxy)$ RELATION SPACE

USING $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

$$yx - qxy \longmapsto (1-q)acx^2 + (bc-qad)xy + (ad-qbc)yx + (1-q)bdy^2$$

EX. $\mathcal{O}(\mathbb{C}^2) := \mathbb{C}[x,y]$ $\underline{yx=xy}$ $\qquad$ $\mathcal{O}(\mathbb{C}_q^2) := \mathbb{C}_q[x,y]$ $\underline{yx=qxy}$

COORDINATE ALGEBRA $\qquad$ POLYNOMIAL ALGEBRA $\qquad$ q-POLYNOMIAL ALGEBRA

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ALSO GET $\mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}^2) \cong GL_2(\mathbb{C}) \not\cong \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}_q^2)$ $\quad q \in \mathbb{C}^\times$

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REQUIRING $\in \mathbb{C}(yx-qxy)$

YIELDS CONDS ON a, b, c, d

GET $\mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}_q^2) \cong \begin{cases} GL_2(\mathbb{C}) & q=1 \\ \text{ANTI/DIAG MATRICES} & q=-1 \\ \text{DIAG. MATS.} & q \neq \pm 1 \end{cases}$

$\mathcal{O}(\mathbb{C}_q^2) := \mathbb{C}_q[x,y]$ $\quad yx = qxy$

q-POLYNOMIAL ALGEBRA

QUANTUM 2-SPACE

ALSO GET $\mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}^2) \cong GL_2(\mathbb{C}) \not\cong \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}_q^2)$ $\quad q \in \mathbb{C}^\times$

I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

SYMMETRIES UNDER DEFORMATION ARE NOT PRESERVED IN THIS FRAMEWORK ...

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GEOMETRY

ALGEBRA

$$\mathbb{C}^2 \longleftrightarrow \mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x,y] \xrightarrow{\text{GROUP SYMMETRY}} \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}^2) \cong \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}[x,y]) \cong GL_2(\mathbb{C})$$

I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

SYMMETRIES UNDER DEFORMATION ARE NOT PRESERVED IN $\underbrace{\text{THIS FRAMEWORK}}$ ...

OF USING GROUPS TO CAPTURE SYMMETRIES

GEOMETRY | ALGEBRA | GROUP SYMMETRY

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, leftrightarrow] & \mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x,y] \arrow[d, "q\text{-deform}"'] \arrow[r, "\text{GROUP SYMMETRY}"] & \mathrm{Sym}^{lin}(\mathbb{C}^2) \cong \mathrm{Sym}^{lin}(\mathbb{C}[x,y]) \cong GL_2(\mathbb{C}) \\ \mathbb{C}^2_q \arrow[r, leftrightarrow] & \mathcal{O}(\mathbb{C}^2_q) = \mathbb{C}_q[x,y] & \end{tikzcd}$$

CAN'T SEE THIS

SO WORK ALGEBRAICALLY

I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

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GEOMETRY $\qquad$ ALGEBRA

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, leftrightarrow] & \mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x,y] \arrow[r, "\text{GROUP SYMMETRY}"] \arrow[d, "q\text{-DEFORM}"'] & \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}^2) \cong \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}[x,y]) \cong GL_2(\mathbb{C}) \\
\mathbb{C}^2_q \arrow[r, leftrightarrow] & \mathcal{O}(\mathbb{C}^2_q) = \mathbb{C}_q[x,y] \arrow[r, "\text{GROUP SYMMETRY}"] & \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}_q[x,y]) \cong \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}^2_q)
\end{tikzcd}$$

CAN'T SEE THIS

SO WORK ALGEBRAICALLY

$$\begin{cases} GL_2(\mathbb{C}) & q = 1 \\ \text{ANTI/DIAG MATRICES} & q = -1 \\ \text{DIAG. MATS.} & q \neq \pm 1 \end{cases} \cong \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}_q[x,y]) \cong \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}^2_q)$$

I. Introduction to "Symmetries of Algebras"

SYMMETRIES UNDER DEFORMATION ARE NOT PRESERVED IN THIS FRAMEWORK ...

of using groups to capture symmetries

GEOMETRY

$\mathbb{C}^2 \longleftrightarrow$ ALGEBRA $\mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x,y] \xrightarrow{\text{GROUP SYMMETRY}} \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}[x,y])$

$\mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}^2) \cong \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}[x,y]) \cong GL_2(\mathbb{C})$

q-DEFORM $\big\downarrow$

$\mathbb{C}^2_q$ CAN'T SEE THIS $\longleftrightarrow \mathcal{O}(\mathbb{C}^2_q) = \mathbb{C}_q[x,y]$ SO WORK ALGEBRAICALLY $\xrightarrow{\text{GROUP SYMMETRY}} \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}_q[x,y])$

CAN'T q-DEFORM !

$$\begin{cases} GL_2(\mathbb{C}) & q=1 \\ \text{ANTI/DIAG MATRICES} & q=-1 \\ \text{DIAG. MATS.} & q \neq \pm 1 \end{cases}$$

$\cong \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}_q[x,y]) \cong \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}^2_q)$

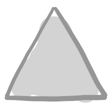
I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

SYMMETRIES UNDER DEFORMATION
ARE NOT PRESERVED IN THIS FRAMEWORK ...
OF USING GROUPS TO CAPTURE SYMMETRIES

LET'S EXPAND OUR NOTION OF SYMMETRY!

... BEYOND GROUPS TO GET

$$\begin{array}{ccc} X & \xrightarrow{\text{Sym.}} & \mathrm{Sym}(X) \\ \Big\downarrow{\scriptstyle \text{DEF}} & & \Big\downarrow{\scriptstyle \text{DEF}} \\ & & \mathrm{Sym}(X)_{\text{DEF}} \\ & & \wr\!\wr \; \checkmark \\ X_{\text{DEF}} & \xrightarrow{\text{Sym.}} & \mathrm{Sym}(X_{\text{DEF}}) \end{array}$$

$X =$ 

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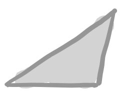
$X_{def} =$ 

SKEW TRIANGLE

$\mathrm{Sym}(X) \cong S_3 \ncong \mathrm{Sym}(X_{def})$

GEOMETRY

$\mathbb{C}^2 \longleftrightarrow \mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x,y] \xrightarrow{\text{GROUP SYMMETRY}} \mathrm{Sym}^{lin}(\mathbb{C}^2) \cong \mathrm{Sym}^{lin}(\mathbb{C}[x,y]) \cong GL_2(\mathbb{C})$

ALGEBRA

q-DEFORM

CAN'T q-DEFORM !

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CAN'T SEE THIS

SO WORK ALGEBRAICALLY

I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

SYMMETRIES UNDER DEFORMATION
ARE NOT PRESERVED IN THIS FRAMEWORK ...

OF USING GROUPS TO CAPTURE SYMMETRIES

LET'S EXPAND OUR NOTION OF SYMMETRY!

... BEYOND GROUPS TO GET

$$\begin{tikzcd}
X \arrow[r, "\text{Sym.}"] \arrow[dd, "\text{DEF}"'] & \mathrm{Sym}(X) \arrow[d, "\text{DEF}"] \\
 & \mathrm{Sym}(X)_{\text{DEF}} \arrow[d, phantom, "\cong" {rotate=90}] \\
X_{\text{DEF}} \arrow[r, "\text{Sym.}"] & \mathrm{Sym}(X_{\text{DEF}})
\end{tikzcd}$$

MAIN IDEA:

- IF X CAN BE REPLACED W/ AN ALGEBRA $A(X)$

  $\ni$. $\mathrm{Sym}(X) \cong \mathrm{Sym}(A(X))$

  THEN STUDY SYMMETRIES OF THE ALGEBRA $A(X)$ (& OF X)

  USING ALGEBRAS INSTEAD OF GROUPS.

I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

SYMMETRIES UNDER DEFORMATION ARE NOT PRESERVED IN $\underbrace{\text{THIS FRAMEWORK}}$ ...

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LET'S EXPAND OUR NOTION OF SYMMETRY!

USING RICHER STRUCTURES / TOOLS

HOPF ALGEBRAS, CATEGORY THEORY...

$$\begin{tikzcd}
\mathbb{C}[x,y] \arrow[r, "\text{Sym.}"] \arrow[dd, "q\text{-DEF}"'] & \mathrm{Sym}(\mathbb{C}[x,y]) \arrow[d, "q\text{-DEF}"] \\
 & \mathrm{Sym}(\mathbb{C}[x,y])_q \arrow[d, phantom, "\cong"] \\
\mathbb{C}_q[x,y] \arrow[r, "\text{Sym.}"] & \mathrm{Sym}(\mathbb{C}_q[x,y])
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MAIN IDEA:

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I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

JUST ONE WAY INTO
THE WORLD OF
"QUANTUM SYMMETRY"

LET'S EXPAND OUR
NOTION OF SYMMETRY!
USING RICHER STRUCTURES / TOOLS
HOPF ALGEBRAS, CATEGORY THEORY...

THESE ALGEBRAIC/
CATEGORICAL
GADGETS ARE
USEFUL IN THEIR
OWN RIGHT!

INTERSECTS MANY FIELDS OF
MATHEMATICS, PHYSICS, AND COMP SCI

ENJOY!

II. LOGISTICS OF COURSE & STUDENT INTROS

GRADE : POINTS BASED (SEE SYLLABUS)

HOMEWORK = SUBMIT ON GRADESCOPE

- CHAP 1 DUE FEB 1st @ 9 AM
- CHAP 2 DUE FEB 27th @ 9 AM
- CHAP 3 DUE MAR 26th @ 9 AM
- CHAP 4 DUE APR 11th @ 9 AM
- BEYOND CHAP 4 DUE APR 19th @ 9 AM

SOLUTIONS WORTH UP TO 3/4/5 POINTS (SEE TEACHING SCHEDULE)

EDITS = CHECK LOG, EMAIL CW

- Typo WORTH 1 POINT
- MATH ERROR WORTH 2 POINTS

UP TO 10 POINTS OF GRADE

A+	:	100+ PTS
A	:	60–99 PTS
A–	:	50–59 PTS
B+	:	40–49 PTS
B	:	30–39 PTS
B–	:	20–29 PTS
⋮		

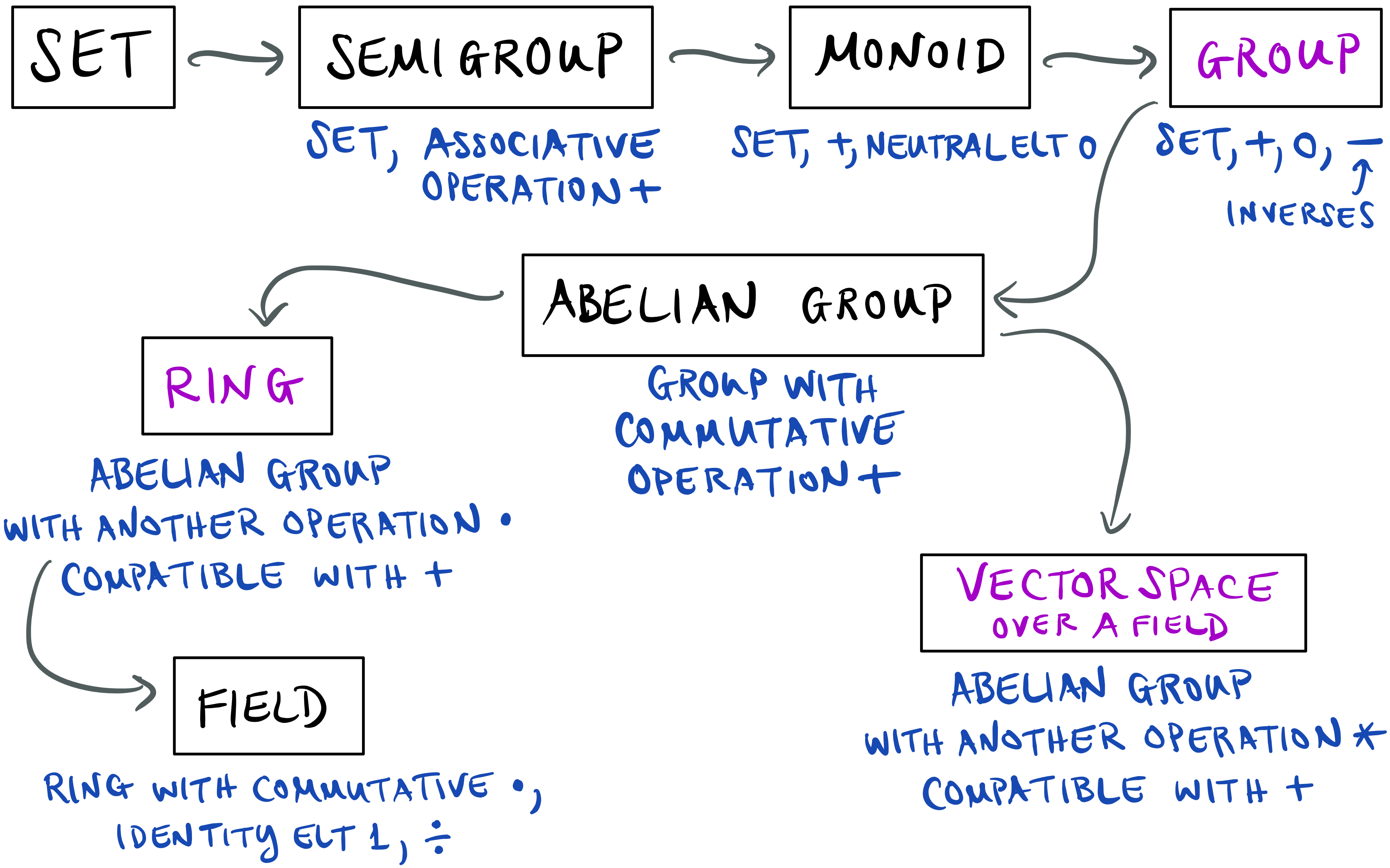
II. LOGISTICS OF COURSE & STUDENT INTROS

PLEASE INTRODUCE YOURSELF ☺

- NAME
- YEAR/ POSITION AT RICE
- WHY YOU'RE IN THE COURSE
- WHAT YOU DO FOR FUN

III. GROUPS, RINGS, VECTOR SPACES

ALGEBRAIC STRUCTURES —



III. GROUPS, RINGS, VECTOR SPACES

GROUP

SET, +, 0, $-$

INVERSES

> A **GROUP** IS A SET G EQUIPPED WITH
>
> AN ASSOCIATIVE OPERATION $\star : G \times G \longrightarrow G$
>
> $$(g, g') \longmapsto g \star g' =: gg'$$
>
> & AN IDENTITY ELEMENT e WITH RESPECT TO $\star$
>
> $$[ge = g = eg \quad \forall g \in G]$$
>
> & SUCH THAT $\forall g \in G \ \exists g^{-1} \in G$ WITH
>
> $$gg^{-1} = e = g^{-1}g.$$

EXAMPLES

$\mathrm{Sym}(X)$ (WHERE SYMMETRIES ARE REVERSIBLE)

$GL_n(\mathbb{C}) \cong \mathrm{Sym}^{\mathrm{lin}}(\mathbb{C}^n)$

$S_n \cong \mathrm{Sym}(n \text{ objects})$

$C_n \cong \mathrm{Sym}^{\mathrm{rot}}(n \text{ objects})$ JUST ROTATIONS

III. GROUPS, RINGS, VECTOR SPACES

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SET, +, 0, −
↑
INVERSES

A **GROUP** IS A SET G EQUIPPED WITH AN ASSOCIATIVE OPERATION

$$\star : G \times G \longrightarrow G \qquad (g, g') \longmapsto g \star g' =: g g'$$

& AN IDENTITY ELEMENT e WITH RESPECT TO $\star$

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SUBSTRUCTURES   **QUOTIENT STRUCTURES**   **MORPHISMS**

EXAMPLES

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SUBSTRUCTURES

QUOTIENT STRUCTURES

MORPHISMS

SUBGROUP $\equiv$

SUBSET $H \subseteq G$ THAT IS A GROUP UNDER $\star$

$h \star h' \in H \quad \forall h, h' \in H$

EXAMPLES

$\mathrm{Sym}(X)$

(WHERE SYMMETRIES ARE REVERSIBLE)

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QUOTIENT STRUCTURES

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SUBSET $H \subseteq G$ THAT IS A GROUP UNDER $\star$

$h \star h' \in H \quad \forall h, h' \in H$

$H \leq G$

NORMAL SUBGROUP $\equiv$

SUBGROUP N OF G s.t.

$gng^{-1} \in N$

$\forall g \in G, n \in N$

$N \trianglelefteq G$

III. GROUPS, RINGS, VECTOR SPACES

GROUP

SET, $+$, 0, $-$
INVERSES

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QUOTIENT STRUCTURES

QUOTIENT GROUP $\equiv$

FOR $N \trianglelefteq G$, AN INDUCED GROUP STRUCTURE ON

$$G/N := \{ gN \mid g \in G \}$$

$$gN \star g'N := gg'N$$

$$e_{G/N} := eN$$

$$(gN)^{-1} := g^{-1}N$$

MORPHISMS

III. GROUPS, RINGS, VECTOR SPACES

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INVERSES

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QUOTIENT STRUCTURES

QUOTIENT GROUP $\equiv$

FOR $N \trianglelefteq G$, AN INDUCED GROUP STRUCTURE ON

$$G/N := \{gN \mid g \in G\}$$

$$gN \star g'N := gg'N$$

$$e_{G/N} := eN$$

$$(gN)^{-1} := g^{-1}N$$

MORPHISMS

GROUP HOMOM.

$\phi : G \longrightarrow G'$

FUNCTION (SET MORPHISM)

SUCH THAT

$$\phi(g \star h) = \phi(g) \star' \phi(h)$$

$\forall g, h \in G$

III. GROUPS, RINGS, VECTOR SPACES

GROUP

SET, +, 0, −

↑ INVERSES

A **GROUP** is a set G equipped with an ASSOCIATIVE OPERATION

$$\star : G \times G \longrightarrow G \qquad (g, g') \longmapsto g \star g' =: gg'$$

& an identity element e with respect to $\star$

& such that $\forall g \in G \ \exists g^{-1} \in G$ with $gg^{-1} = e = g^{-1}g$.

COMMUTATIVE DIAGRAMS

ENCODING COMPOSITION OF FUNCTIONS

START

$$\begin{tikzcd}
G \times G \arrow[rrr, "\star"] \arrow[ddd, "\phi \times \phi"'] & & & G \arrow[ddd, "\phi"] \\
& (g,h) \arrow[r, mapsto] \arrow[d, mapsto] & g \star h \arrow[d, mapsto] & \\
& (\phi(g), \phi(h)) \arrow[r, mapsto] & \phi(g \star h) = \phi(g) \star' \phi(h) & \\
G' \times G' \arrow[rrr, "\star'"] & & & G'
\end{tikzcd}$$

FINISH

MORPHISMS

GROUP HOMOM.

$$\phi : G \longrightarrow G'$$

FUNCTION (SET MORPHISM)

SUCH THAT

$$\phi(g \star h) = \phi(g) \star' \phi(h) \quad \forall g, h \in G$$

III. GROUPS, RINGS, VECTOR SPACES

GROUP

SET, $+$, 0, $-$

↑ INVERSES

A **GROUP** IS A SET G EQUIPPED WITH AN ASSOCIATIVE OPERATION

$$\star : G \times G \longrightarrow G \qquad (g, g') \longmapsto g \star g' =: gg'$$

& AN IDENTITY ELEMENT e WITH RESPECT TO $\star$

& SUCH THAT $\forall g \in G \;\; \exists g^{-1} \in G$ WITH $gg^{-1} = e = g^{-1}g$.

COMMUTATIVE DIAGRAMS

ENCODING COMPOSITION OF FUNCTIONS

START

$$\begin{tikzcd}
G \times G \arrow[rr, "\star"] \arrow[dd, "\phi \times \phi"'] & & G \arrow[dd, "\phi"] \\
(g,h) \arrow[r, mapsto] \arrow[d, mapsto] & g \star h \arrow[d, mapsto] & \\
(\phi(g), \phi(h)) \arrow[r, mapsto] & \phi(g) \star' \phi(h) = \phi(g \star h) & \\
G' \times G' \arrow[rr, "\star'"] & & G'
\end{tikzcd}$$

FINISH

PATHS FROM START TO FINISH YIELD SAME RESULT

MORPHISMS

GROUP HOMOM.

$\phi : G \longrightarrow G'$

FUNCTION (SET MORPHISM)

SUCH THAT

$\phi(g \star h) = \phi(g) \star' \phi(h) \quad \forall g, h \in G$

III. GROUPS, RINGS, VECTOR SPACES

ALGEBRAIC STRUCTURES —

SET → SEMIGROUP → MONOID → GROUP

SEMIGROUP: SET, ASSOCIATIVE OPERATION +

MONOID: SET, +, NEUTRAL ELT 0

GROUP: SET, +, 0, −
INVERSES

GROUP → ABELIAN GROUP

ABELIAN GROUP: GROUP WITH COMMUTATIVE OPERATION +

ABELIAN GROUP → RING

RING: ABELIAN GROUP WITH ANOTHER OPERATION • COMPATIBLE WITH +

RING → FIELD

FIELD: RING WITH COMMUTATIVE •, IDENTITY ELT 1, ÷

ABELIAN GROUP → VECTOR SPACE OVER A FIELD

VECTOR SPACE OVER A FIELD: ABELIAN GROUP WITH ANOTHER OPERATION ∗ COMPATIBLE WITH +

III. GROUPS, RINGS, VECTOR SPACES

$\boxed{\text{RING}}$

ABELIAN GROUP WITH ANOTHER OPERATION $\bullet$ COMPATIBLE WITH $+$

A RING IS AN ABELIAN GROUP $(R,+,0)$ WITH AN ASSOCIATIVE OPERATION

$$\bullet : R \times R \longrightarrow R \qquad (r,r') \longmapsto r \bullet r' =: rr'$$

SUCH THAT $\forall r,s,t \in R$:

$$(r+s)\cdot t = (r\cdot s) + (r\cdot t), \qquad r\cdot(s+t) = (r\cdot s) + (r\cdot t)$$

SUBSTRUCTURES

QUOTIENT STRUCTURES

MORPHISMS

III. GROUPS, RINGS, VECTOR SPACES

RING

ABELIAN GROUP WITH ANOTHER OPERATION $\cdot$ COMPATIBLE WITH $+$

A RING IS AN ABELIAN GROUP $(R,+,0)$ WITH AN ASSOCIATIVE OPERATION

$$\bullet : R \times R \longrightarrow R \qquad (r,r') \longmapsto r \cdot r' =: rr'$$

SUCH THAT $\forall r,s,t \in R$:

$$(r+s)\cdot t = (r\cdot s) + (r \cdot t), \qquad r\cdot(s+t) = (r\cdot s) + (r\cdot t)$$

SUBSTRUCTURES

QUOTIENT STRUCTURES

MORPHISMS

SUBRING $\equiv$
SUBSET $S \subseteq R$ THAT IS A RING UNDER $+$ AND $\bullet$

$S \leq R$

IDEAL $\equiv$
SUBRING I OF R
SUCH $r\ell, \ell r \in I$
$\forall r \in R, \ell \in I$

$I \trianglelefteq R$

III. GROUPS, RINGS, VECTOR SPACES

$\boxed{\text{RING}}$

ABELIAN GROUP WITH ANOTHER OPERATION $\cdot$ COMPATIBLE WITH $+$

A RING IS AN ABELIAN GROUP $(R,+,0)$ WITH AN ASSOCIATIVE OPERATION

$$\cdot : R \times R \longrightarrow R \qquad (r,r') \mapsto r \cdot r' =: rr'$$

SUCH THAT $\forall r,s,t \in R$:

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SUBSTRUCTURES

SUBRING $\equiv$ SUBSET $S \subseteq R$ THAT IS A RING UNDER $+$ AND $\cdot$

$S \leq R$

IDEAL $\equiv$ SUBRING I OF R SUCH $rl, lr \in I$ $\forall r \in R, l \in I$

$I \trianglelefteq R$

QUOTIENT STRUCTURES

QUOTIENT RING $\equiv$ FOR $I \trianglelefteq R$, AN INDUCED RING STRUCTURE ON

$$R/I := \{r + I \mid r \in R\}$$

USING ADDITIVE COSETS:

$I \leq R$ SUBGROUP IS AUTOMATICALLY NORMAL (R ABELIAN GROUP)

MORPHISMS

III. GROUPS, RINGS, VECTOR SPACES

RING

ABELIAN GROUP WITH ANOTHER OPERATION $\cdot$ COMPATIBLE WITH $+$

A **RING** IS AN ABELIAN GROUP $(R,+,0)$ WITH AN ASSOCIATIVE OPERATION

$$\cdot : R \times R \longrightarrow R \qquad (r,r') \mapsto r\cdot r' =: rr'$$

SUCH THAT $\forall r,s,t \in R$:

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SUBSTRUCTURES

SUBRING $\equiv$

SUBSET $S \subseteq R$ THAT IS A RING UNDER $+$ AND $\cdot$

$S \leq R$

IDEAL $\equiv$

SUBRING I OF R SUCH $rl, lr \in I$ $\forall r \in R, l \in I$

$I \trianglelefteq R$

QUOTIENT STRUCTURES

QUOTIENT RING $\equiv$

FOR $I \trianglelefteq R$, AN INDUCED RING STRUCTURE ON

$$R/I := \{r + I \mid r \in R\}$$

USING ADDITIVE COSETS:

$I \leq R$ SUBGROUP IS AUTOMATICALLY NORMAL (R ABELIAN GROUP)

MORPHISMS

RING HOMOM.

$\phi : R \longrightarrow R'$ FUNCTION SUCH THAT

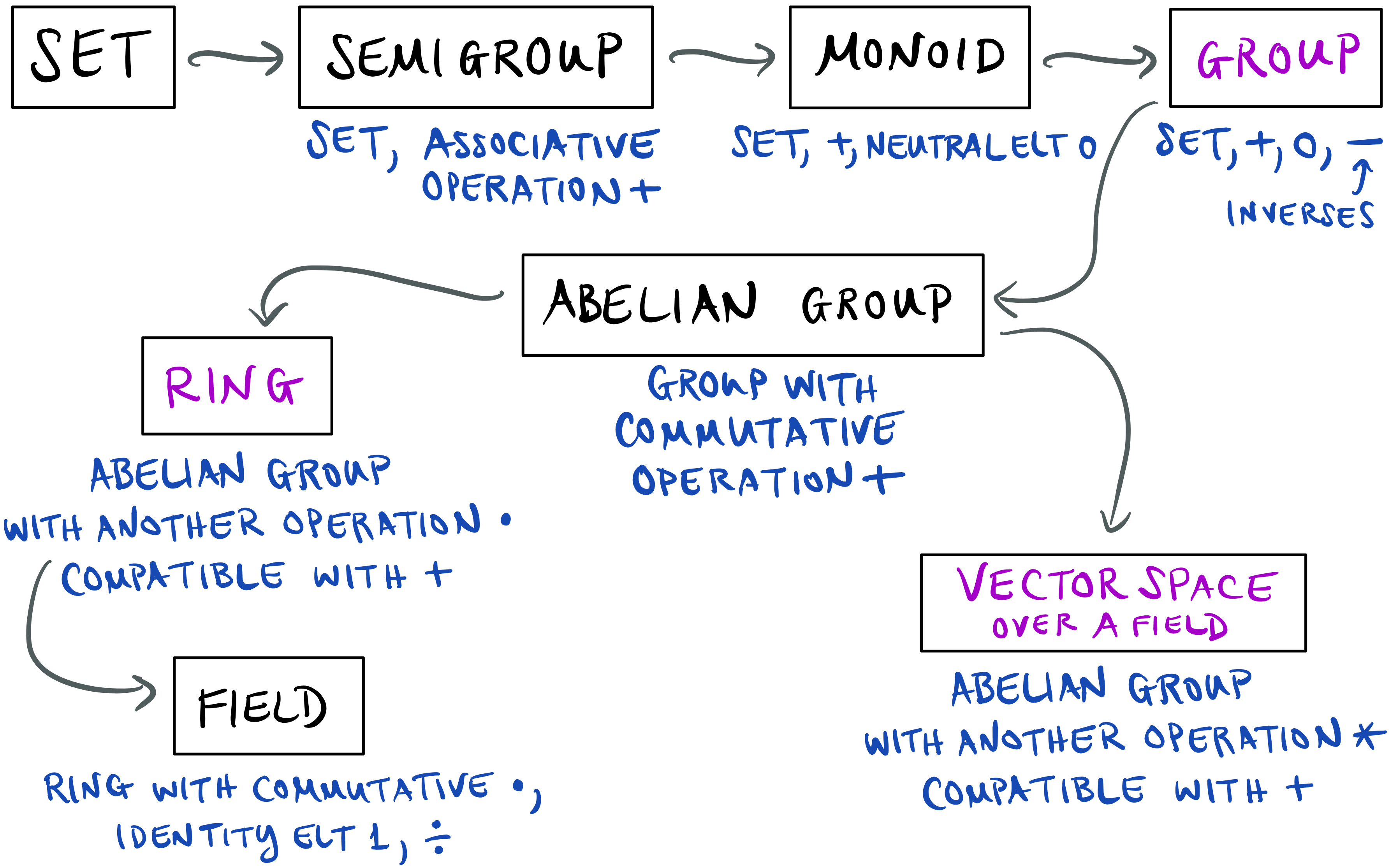
$$\phi(r+s) = \phi(r) + \phi(s)$$

$$\phi(rs) = \phi(r)\phi(s)$$

$\forall r,s \in R$

III. GROUPS, RINGS, VECTOR SPACES

ALGEBRAIC STRUCTURES —



III. GROUPS, RINGS, VECTOR SPACES

VECTOR SPACE OVER A FIELD $\Bbbk$

ABELIAN GROUP WITH ANOTHER OPERATION $*$ COMPATIBLE WITH $+$

A $\Bbbk$-VS IS AN ABELIAN GROUP $(V,+,0)$ WITH AN OPERATION

$$* : \Bbbk \times V \longrightarrow V \qquad (\lambda, v) \longmapsto \lambda * v =: \lambda v$$

SUCH THAT $\forall \lambda, \lambda' \in \Bbbk$ AND $v, v' \in V$:

$$\lambda(v+v') = \lambda v + \lambda v', \quad (\lambda+\lambda')v = \lambda v + \lambda' v, \quad (\lambda\lambda')v = \lambda(\lambda' v), \quad 1_{\Bbbk} v = v$$

SUBSTRUCTURES

QUOTIENT STRUCTURES

MORPHISMS

III. GROUPS, RINGS, VECTOR SPACES

VECTOR SPACE OVER A FIELD $\Bbbk$

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SUBSTRUCTURES

SUBSPACE $\equiv$

SUBSET $W \subseteq V$ THAT IS A $\Bbbk$-VS UNDER $+$ AND $*$

$W \leq V$

QUOTIENT STRUCTURES

QUOTIENT SPACE $\equiv$

FOR $W \leq V$, AN INDUCED $\Bbbk$-VS STRUCTURE ON

$$V/W := \{v + W \mid v \in V\}$$

USING ADDITIVE COSETS:
$W \leq V$ SUBGROUP
IS AUTOMATICALLY NORMAL
(V ABELIAN GROUP)

MORPHISMS

III. GROUPS, RINGS, VECTOR SPACES

VECTOR SPACE OVER A FIELD $\Bbbk$

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SUBSTRUCTURES

SUBSPACE $\equiv$

SUBSET $W \subseteq V$ THAT IS A $\Bbbk$-VS UNDER $+$ AND $*$

$W \leq V$

QUOTIENT STRUCTURES

QUOTIENT SPACE $\equiv$

FOR $W \leq V$, AN INDUCED $\Bbbk$-VS STRUCTURE ON

$$V/W := \{v + W \mid v \in V\}$$

USING ADDITIVE COSETS:

$W \leq V$ SUBGROUP IS AUTOMATICALLY NORMAL

(V ABELIAN GROUP)

MORPHISMS

$\Bbbk$-LINEAR MAP

$\phi: V \longrightarrow V'$

FUNCTION SUCH THAT

$$\phi(v+w) = \phi(v) + \phi(w)$$

$$\phi(\lambda v) = \lambda \phi(v)$$

$\forall \lambda \in \Bbbk, \forall v, w \in V$

IV. STRUCTURE VS. PROPERTY

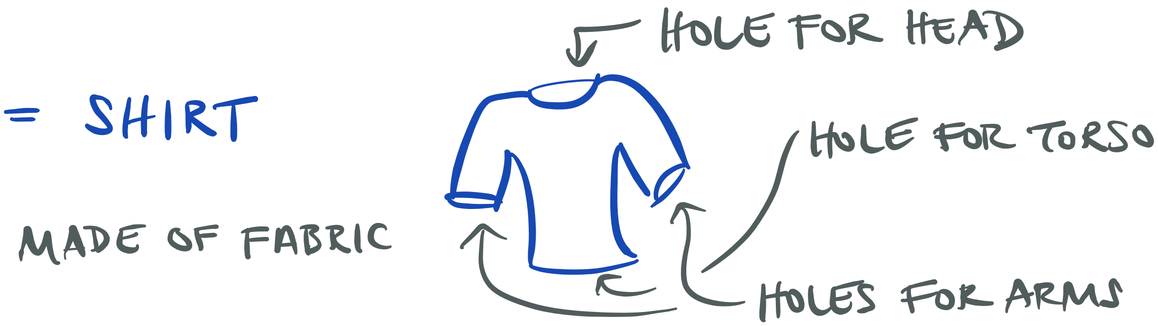
THE STRUCTURE OF A GADGET X ARE FEATURES THAT DEFINE X

Ex. X = SHIRT

IV. STRUCTURE VS. PROPERTY

THE STRUCTURE OF A GADGET X ARE FEATURES THAT DEFINE X

EX. X = SHIRT



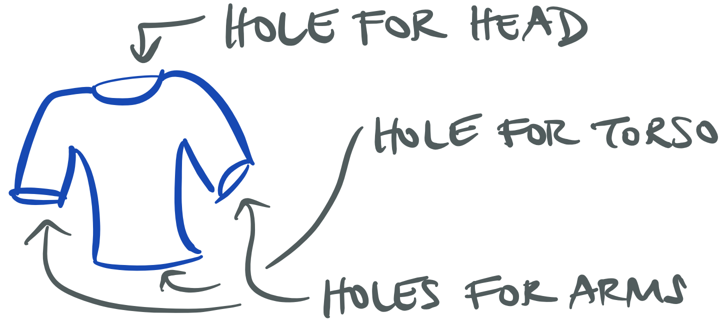
A PROPERTY OF X IS A CONDITION THAT COULD HOLD OR NOT

EX. CTNED SHIRT IS RED, IS SIZE 10, MADE OF LINEN

IV. STRUCTURE VS. PROPERTY

THE **STRUCTURE** OF A GADGET X ARE FEATURES THAT DEFINE X

EX. X = SHIRT



A **PROPERTY** OF X IS A CONDITION THAT COULD HOLD OR NOT

EX. CTNED SHIRT IS RED, IS SIZE 10, MADE OF LINEN

STRUCTURE IS TO NOUN AS
PROPERTY IS TO ADJECTIVE

IV. STRUCTURE VS. PROPERTY

THE STRUCTURE OF A GADGET X ARE FEATURES THAT DEFINE X

A PROPERTY OF X IS A CONDITION THAT COULD HOLD OR NOT

STRUCTURE IS TO NOUN
AS
PROPERTY IS TO ADJECTIVE

YOU DO! EXERCISE 1.1

DISCUSS THE "STRUCTURE" & SOME "PROPERTIES" OF

- GROUPS
- RINGS
- $\Bbbk$-V.SPACES

IV. STRUCTURE VS. PROPERTY

THE STRUCTURE OF A GADGET X ARE FEATURES THAT DEFINE X

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STRUCTURE IS TO NOUN
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YOU DO! EXERCISE 1.1

DISCUSS THE "STRUCTURE" & SOME "PROPERTIES" OF

- GROUPS
- RINGS
- $\mathbb{k}$-V.SPACES

THIS ENDS OUR CHAT ON §§1.1.1 – 1.1.3. ≡ PLEASE READ THE REST ≡

V. OPERATIONS ON VECTOR SPACES

VECTOR SPACE OVER A FIELD $\Bbbk$

ABELIAN GROUP WITH ANOTHER OPERATION $*$ COMPATIBLE WITH $+$

A $\Bbbk$-VS IS AN ABELIAN GROUP $(V,+,0)$ WITH AN OPERATION

$$* : \Bbbk \times V \longrightarrow V \qquad (\lambda, v) \mapsto \lambda * v =: \lambda v$$

SUCH THAT $\forall \lambda, \lambda' \in \Bbbk$ AND $v, v' \in V$:

$$\lambda(v+v') = \lambda v + \lambda v', \quad (\lambda+\lambda')v = \lambda v + \lambda' v, \quad (\lambda\lambda')v = \lambda(\lambda' v), \quad 1_{\Bbbk} v = v$$

DIRECT PRODUCT

SUM

DIRECT SUM

V. OPERATIONS ON VECTOR SPACES

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DIRECT PRODUCT

$$V_1 \times \cdots \times V_r$$
$$\|$$
$$\{(v_1, \ldots, v_r) \mid v_i \in V_i \ \forall i\}$$

A $\Bbbk$-VECTOR SPACE WITH COMPONENT-WISE $+$ & COMPONENT-WISE $*$

SUM

DIRECT SUM

V. OPERATIONS ON VECTOR SPACES

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DIRECT PRODUCT

$$V_1 \times \cdots \times V_r$$
$$\|$$
$$\{(v_1, \ldots, v_r) \mid v_i \in V_i \ \forall i\}$$

A $\Bbbk$-VECTOR SPACE WITH COMPONENT-WISE $+$ & COMPONENT-WISE $*$

SUM

$$V_1 + \cdots + V_r$$
$$\| \quad V_i \subseteq V \text{ SUBSPACES}$$
$$\{v_1 + \cdots + v_r \mid v_i \in V_i \ \forall i\}$$

A $\Bbbk$-VECTOR SPACE WITH SUMMAND-WISE $+$ & SUMMAND-WISE $*$

DIRECT SUM

V. OPERATIONS ON VECTOR SPACES

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$$V_1 \times \cdots \times V_r$$
$$\|$$
$$\{(v_1, \dots, v_r) \mid v_i \in V_i \ \forall i\}$$

A $\Bbbk$-VECTOR SPACE WITH COMPONENT-WISE $+$ & COMPONENT-WISE $*$

SUM

$$V_1 + \cdots + V_r$$
$$\| \quad V_i \subseteq V \text{ SUBSPACES}$$
$$\{v_1 + \cdots + v_r \mid v_i \in V_i \ \forall i\}$$

A $\Bbbk$-VECTOR SPACE WITH SUMMAND-WISE $+$ & SUMMAND-WISE $*$

DIRECT SUM

$$V_1 \oplus \cdots \oplus V_r$$
$$\|$$
$$V_1 + \cdots + V_r$$

SUCH THAT THE ONLY WAY TO GET $0_{V_1 \oplus \cdots \oplus V_r}$ IS VIA $0_{V_1} + \cdots + 0_{V_r}$

V. OPERATIONS ON VECTOR SPACES

VECTOR SPACE OVER A FIELD $\Bbbk$

ABELIAN GROUP WITH ANOTHER OPERATION $*$ COMPATIBLE WITH $+$

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EX. $V = \mathbb{R}^2_{(x,y)}$ $\binom{x}{0}$ x

$V_1 = \mathbb{R}^2_{(x,y)}$ $\quad V_2 = \mathbb{R}_{(x)}$ x-AXIS

GET $V_1 + V_2 \cong \mathbb{R}^2_{(x,y)}$

HERE $\binom{1}{0} + \binom{-1}{0} = \binom{0}{0}$

SUM IS NOT DIRECT

TO FORM DIRECT SUM

$V_2 \rightsquigarrow V_2' = \mathbb{R}^2_{(x')}$ A DISJOINT COPY

GET $V_1 \oplus V_2' \cong \mathbb{R}^3_{(x,y,x')}$

SUM VS. DIRECT SUM

SUM

$V_1 + \cdots + V_r$ $\quad V_i \subseteq V$ SUBSPACES

$$\| $$

$$\{ v_1 + \cdots + v_r \mid v_i \in V_i \ \forall i \}$$

DIRECT SUM

$V_1 \oplus \cdots \oplus V_r$

$$\|$$

A $\Bbbk$-VECTOR SPACE WITH SUMMAND-WISE $+$ & SUMMAND-WISE $*$

$V_1 + \cdots + V_r$ SUCH THAT THE ONLY WAY TO GET $0_{V_1 \oplus \cdots \oplus V_r}$ IS VIA $0_{V_1} + \cdots + 0_{V_r}$

V. OPERATIONS ON VECTOR SPACES

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HOM

DUAL

V. OPERATIONS ON VECTOR SPACES

VECTOR SPACE OVER A FIELD $\Bbbk$

ABELIAN GROUP WITH ANOTHER OPERATION $*$ COMPATIBLE WITH $+$

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HOM

TAKE $\Bbbk$-VSPACES V, W

$\mathrm{Hom}_{\Bbbk}(V,W)$
$$\parallel$$
$\{ \phi : V \to W \text{ LINEAR MAP} \}$

IS A $\Bbbk$-VECTOR SPACE WITH

$$(\phi + \phi')(v) := \phi(v) + \phi'(v)$$
$$(\lambda\phi)(v) := \lambda\,\phi(v) = \phi(\lambda v)$$

$\forall \phi, \phi' \in \mathrm{Hom}_{\Bbbk}(V,W), \quad \lambda \in \Bbbk$

DUAL

V. OPERATIONS ON VECTOR SPACES

VECTOR SPACE OVER A FIELD $\Bbbk$

ABELIAN GROUP WITH ANOTHER OPERATION $*$ COMPATIBLE WITH $+$

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HOM

TAKE $\Bbbk$-VSPACES V, W

$$\mathrm{Hom}_{\Bbbk}(V,W) = \{\phi : V \to W \text{ LINEAR MAP}\}$$

IS A $\Bbbk$-VECTOR SPACE WITH

HOW WE DEFINE $+$ OF $\mathrm{Hom}_{\Bbbk}(V,W)$

$$(\phi + \phi')(v) := \phi(v) + \phi'(v)$$

$$(\lambda\phi)(v) := \lambda\phi(v) = \phi(\lambda v)$$

HOW WE DEFINE $*$ OF $\mathrm{Hom}_{\Bbbk}(V,W)$

$\forall \phi, \phi' \in \mathrm{Hom}_{\Bbbk}(V,W), \quad \lambda \in \Bbbk$

DUAL

V. OPERATIONS ON VECTOR SPACES

VECTOR SPACE OVER A FIELD $\Bbbk$

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HOM

TAKE $\Bbbk$-VSPACES V, W

$$\mathrm{Hom}_{\Bbbk}(V,W) = \{\phi : V \to W \text{ LINEAR MAP}\}$$

IS A $\Bbbk$-VECTOR SPACE WITH

$$(\phi + \phi')(v) := \phi(v) + \phi'(v)$$

$$(\lambda\phi)(v) := \lambda\phi(v) = \phi(\lambda v)$$

$\forall \phi, \phi' \in \mathrm{Hom}_{\Bbbk}(V,W)$, $\lambda \in \Bbbk$

DUAL

$$V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$$

ELEMENTS OF V^* ARE CALLED

LINEAR FUNCTIONALS

OR

LINEAR FORMS

V. OPERATIONS ON VECTOR SPACES

VECTOR SPACE OVER A FIELD $\Bbbk$

ABELIAN GROUP WITH ANOTHER OPERATION $*$ COMPATIBLE WITH $+$

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HOM

TAKE $\Bbbk$-VSPACES V, W

$$\mathrm{Hom}_{\Bbbk}(V,W) \;=\; \{\phi: V\to W \text{ LINEAR MAP}\}$$

IS A $\Bbbk$-VECTOR SPACE WITH

$$(\phi+\phi')(v):=\phi(v)+\phi'(v)$$
$$(\lambda\phi)(v):=\lambda\phi(v)=\phi(\lambda v)$$

$\forall \phi,\phi'\in \mathrm{Hom}_{\Bbbk}(V,W),\ \lambda\in\Bbbk$

DUAL

$$V^* := \mathrm{Hom}_{\Bbbk}(V,\Bbbk)$$

GET $V^{**}\cong V$ WHEN V FINITE DIM

$\supseteq$ IN GENERAL

ELEMENTS OF V^* ARE CALLED

LINEAR FUNCTIONALS

OR

LINEAR FORMS

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #1

TOPICS:

✓ I. INTRODUCTION TO "SYMMETRIES OF ALGEBRAS"

✓ II. LOGISTICS OF COURSE & STUDENT INTROS

✓ III. GROUPS, RINGS, VECTOR SPACES (§§ 1.1.1, 1.1.2, 1.1.3)

✓ IV. STRUCTURE VS. PROPERTY (§1.1.1)

V. OPERATIONS ON VECTOR SPACES (§1.1.4)
- DIRECT PRODUCT
- SUM
- DIRECT SUM
- HOM & DUAL

↪ NEXT TIME: TENSOR PRODUCT ⊗

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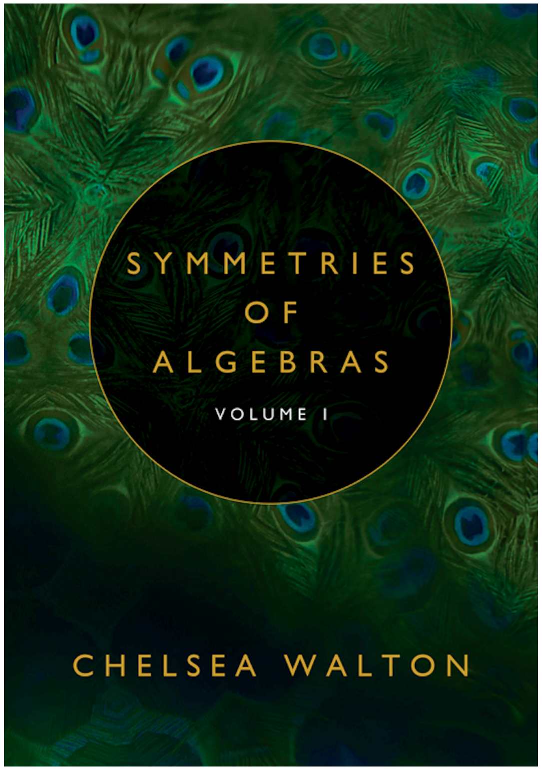
C. Walton's "Symmetries of Algebras, Volume 1" (2024)



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Lecture #1 keywords: group, ring, structure versus property, symmetry, vector space