MATH 466/566
SPRING 2024

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LECTURE #10

LAST TIME

- CATEGORY ISOMORPHISM
- CATEGORY EQUIVALENCE
- MORITA EQUIVALENCE

TOPICS:

I. ADJUNCTION (§2.5)

II. UNIVERSALITY REVISITED (§2.6.1)

III. YONEDA'S LEMMA (§2.6.2)

I. ADJUNCTION

NOTION OF SAMENESS FOR CATEGORIES

$$\begin{tikzcd}
\begin{array}{c} \mathcal{C} = \mathcal{D} \\ \text{EQUALITY} \\ \text{OF CATEGORIES} \end{array} \arrow[d, bend right, "\text{WEAKEN}"'] & \\
\begin{array}{c} \mathcal{C} \cong \mathcal{D} \\ \text{ISOMORPHISM} \\ \text{OF CATEGORIES} \end{array} \arrow[r, bend left, "\text{WEAKEN}"] & \begin{array}{c} \mathcal{C} \simeq \mathcal{D} \\ \text{EQUIVALENCE} \\ \text{OF CATEGORIES} \end{array}
\end{tikzcd}$$

I. ADJUNCTION

NOTION OF SAMENESS FOR CATEGORIES

$\mathcal{C} = \mathcal{D}$

EQUALITY OF CATEGORIES

WEAKEN

$\mathcal{C} \cong \mathcal{D}$

ISOMORPHISM OF CATEGORIES

WEAKEN

$\mathcal{C} \simeq \mathcal{D}$

EQUIVALENCE OF CATEGORIES

$$\begin{tikzcd}
\exists \text{ FUNCTORS } \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}$$

s.t.

$$\mathrm{Id}_{\mathcal{C}} = GF \ \& \ FG = \mathrm{Id}_{\mathcal{D}}$$

INVOLVING EQUALITIES OF FUNCTORS

I. ADJUNCTION

NOTION OF SAMENESS FOR CATEGORIES

$\mathcal{C} = \mathcal{D}$

EQUALITY OF CATEGORIES

WEAKEN

$\mathcal{C} \cong \mathcal{D}$

ISOMORPHISM OF CATEGORIES

WEAKEN

$\mathcal{C} \simeq \mathcal{D}$

EQUIVALENCE OF CATEGORIES

∃ FUNCTORS $$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$ s.t.

$\mathrm{Id}_{\mathcal{C}} = GF$ & $FG = \mathrm{Id}_{\mathcal{D}}$

INVOLVING EQUALITIES OF FUNCTORS

∃ FUNCTORS $$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$ s.t.

$\mathrm{Id}_{\mathcal{C}} \cong GF$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

INVOLVING ISOMORPHISMS OF FUNCTORS

I. ADJUNCTION

NOTION OF SAMENESS FOR CATEGORIES → RICH THEORY OF FUNCTORS

$\mathscr{C} = \mathscr{D}$

EQUALITY
OF CATEGORIES

WEAKEN

$\mathscr{C} \cong \mathscr{D}$

ISOMORPHISM
OF CATEGORIES

$$\begin{tikzcd}\mathscr{C} \arrow[r, bend left, "F"] & \mathscr{D} \arrow[l, bend left, "G"]\end{tikzcd}$$

∃ FUNCTORS $\mathscr{C} \rightleftarrows \mathscr{D}$ ∋. $\mathrm{Id}_{\mathscr{C}} = GF$ & $FG = \mathrm{Id}_{\mathscr{D}}$

INVOLVING
EQUALITIES
OF FUNCTORS

WEAKEN

$\mathscr{C} \simeq \mathscr{D}$

EQUIVALENCE
OF CATEGORIES

$$\begin{tikzcd}\mathscr{C} \arrow[r, bend left, "F"] & \mathscr{D} \arrow[l, bend left, "G"]\end{tikzcd}$$

∃ FUNCTORS $\mathscr{C} \rightleftarrows \mathscr{D}$ ∋. $\mathrm{Id}_{\mathscr{C}} \cong GF$ & $FG \cong \mathrm{Id}_{\mathscr{D}}$

INVOLVING
ISOMORPHISMS
OF FUNCTORS

WEAKEN

NOW:
ADJUNCTION

I. ADJUNCTION

$\mathcal{C} = \mathcal{D}$

EQUALITY OF CATEGORIES

NOTION OF SAMENESS FOR CATEGORIES → RICH THEORY OF FUNCTORS

WEAKEN

$\mathcal{C} \cong \mathcal{D}$

ISOMORPHISM OF CATEGORIES

WEAKEN

$\mathcal{C} \simeq \mathcal{D}$

EQUIVALENCE OF CATEGORIES

WEAKEN

NOW: ADJUNCTION

$\exists$ FUNCTORS
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$
$\ni$. $\mathrm{Id}_{\mathcal{C}} = GF$ & $FG = \mathrm{Id}_{\mathcal{D}}$

INVOLVING EQUALITIES OF FUNCTORS

$\exists$ FUNCTORS
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$
$\ni$. $\mathrm{Id}_{\mathcal{C}} \cong GF$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

INVOLVING ISOMORPHISMS OF FUNCTORS

$\exists$ FUNCTORS
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$
$\ni$. $\mathrm{Id}_{\mathcal{C}} \overset{\eta}{\Rightarrow} GF$ & $FG \overset{\varepsilon}{\Rightarrow} \mathrm{Id}_{\mathcal{D}}$

HAVE A NICE COMPATIBILITY CONDITION

INVOLVES CERTAIN TRANSF'NS OF FUNCTORS

I. ADJUNCTION

FUNCTORS $F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

IF ∃ NAT'L TRANSFORMATIONS

$$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF \quad \& \quad \varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$$

∋.

RICH THEORY OF FUNCTORS

NOW:
ADJUNCTION

∃ FUNCTORS
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

∋.

$$\mathrm{Id}_{\mathcal{C}} \overset{\eta}{\Rightarrow} GF \quad \& \quad FG \overset{\varepsilon}{\Rightarrow} \mathrm{Id}_{\mathcal{D}}$$

HAVE A NICE COMPATIBILITY CONDITION

INVOLVES CERTAIN TRANSF'NS OF FUNCTORS

I. ADJUNCTION

FUNCTORS $F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

IF ∃ NAT'L TRANSFORMATIONS

$$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF \quad \& \quad \varepsilon: FG\Rightarrow \mathrm{Id}_{\mathcal{D}}$$

∋.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=70, "F"{name=U}] \arrow[r, "FGF"{name=M}] \arrow[r, bend right=70, "F"'{name=L}] & \mathcal{D} & {=} & \mathcal{C} \arrow[r, bend left=70, "F"{name=U2}] \arrow[r, bend right=70, "F"'{name=L2}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=M, "F*\eta"]
\arrow[Rightarrow, from=M, to=L, "\varepsilon*F"]
\arrow[Rightarrow, from=U2, to=L2, "\mathrm{ID}_F"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend left=70, "G"{name=U}] \arrow[r, "GFG"{name=M}] \arrow[r, bend right=70, "G"'{name=L}] & \mathcal{C} & {=} & \mathcal{D} \arrow[r, bend left=70, "G"{name=U2}] \arrow[r, bend right=70, "G"'{name=L2}] & \mathcal{C}
\arrow[Rightarrow, from=U, to=M, "\eta*G"]
\arrow[Rightarrow, from=M, to=L, "G*\varepsilon"]
\arrow[Rightarrow, from=U2, to=L2, "\mathrm{ID}_G"]
\end{tikzcd}$$

RICH THEORY OF FUNCTORS

NOW:

ADJUNCTION

∃ FUNCTORS $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$

∋.

$\mathrm{Id}_{\mathcal{C}} \overset{\eta}{\Rightarrow} GF$ & $FG \overset{\varepsilon}{\Rightarrow} \mathrm{Id}_{\mathcal{D}}$

HAVE A NICE COMPATIBILITY CONDITION

INVOLVES CERTAIN TRANSF'NS OF FUNCTORS

I. ADJUNCTION

FUNCTORS $F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

IF ∃ NAT'L TRANSFORMATIONS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ & $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

$F * \eta :$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "\mathrm{Id}_{\mathcal{C}}"] \arrow[r, bend right=50, "GF"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd}$$

$\varepsilon * F :$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow \varepsilon"] & \mathcal{D} \end{tikzcd}$$

∋.

$$\begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=70, "F"] \arrow[r, "FGF" description] \arrow[r, bend right=70, "F"'] \arrow[r, phantom, shift left=4ex, "\Downarrow F*\eta"] \arrow[r, phantom, shift right=4ex, "\Downarrow \varepsilon*F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=70, "F"] \arrow[r, bend right=70, "F"'] \arrow[r, phantom, "\Downarrow \mathrm{ID}_F"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge] \mathcal{D} \arrow[r, bend left=70, "G"] \arrow[r, "GFG" description] \arrow[r, bend right=70, "G"'] \arrow[r, phantom, shift left=4ex, "\Downarrow \eta*G"] \arrow[r, phantom, shift right=4ex, "\Downarrow G*\varepsilon"] & \mathcal{C} \end{tikzcd} = \begin{tikzcd}[column sep=huge] \mathcal{D} \arrow[r, bend left=70, "G"] \arrow[r, bend right=70, "G"'] \arrow[r, phantom, "\Downarrow \mathrm{ID}_G"] & \mathcal{C} \end{tikzcd}$$

I. ADJUNCTION

FUNCTORS $F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

IF ∃ NAT'L TRANSFORMATIONS

$\eta : Id_{\mathcal{C}} \Rightarrow GF$ & $\varepsilon : FG \Rightarrow Id_{\mathcal{D}}$

ADJUNCTION UNIT

ADJUNCTION COUNIT

∋.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=70, "F"] \arrow[r, "FGF" description] \arrow[r, bend right=70, "F"'] \arrow[r, phantom, shift left=4ex, "\Downarrow F*\eta"] \arrow[r, phantom, shift right=4ex, "\Downarrow \varepsilon*F"] & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=70, "F"] \arrow[r, bend right=70, "F"'] \arrow[r, phantom, "\Downarrow ID_F"] & \mathcal{D}
\end{tikzcd}$$

$$\begin{tikzcd}
F*\eta : \mathcal{C} \arrow[r, bend left=60, "Id_{\mathcal{C}}"] \arrow[r, bend right=60, "GF"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}
\end{tikzcd}$$

$$\begin{tikzcd}
\varepsilon*F : \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left=60, "FG"] \arrow[r, bend right=60, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow \varepsilon"] & \mathcal{D}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend left=70, "G"] \arrow[r, "GFG" description] \arrow[r, bend right=70, "G"'] \arrow[r, phantom, shift left=4ex, "\Downarrow \eta*G"] \arrow[r, phantom, shift right=4ex, "\Downarrow G*\varepsilon"] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{D} \arrow[r, bend left=70, "G"] \arrow[r, bend right=70, "G"'] \arrow[r, phantom, "\Downarrow ID_G"] & \mathcal{C}
\end{tikzcd}$$

I. ADJUNCTION

FUNCTORS $F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN **ADJUNCTION**

IF ∃ NAT'L TRANSFORMATIONS

$$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF \quad \& \quad \varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$$

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$$\begin{tikzcd}
F * \eta : \mathcal{C} \arrow[r, bend left=60, "\mathrm{Id}_{\mathcal{C}}", ""{name=U, below}] \arrow[r, bend right=60, "GF"', ""{name=D, above}] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}
\arrow[Rightarrow, from=U, to=D, "\eta"]
\end{tikzcd}$$

$$\begin{tikzcd}
\varepsilon * F : \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left=60, "FG", ""{name=U, below}] \arrow[r, bend right=60, "\mathrm{Id}_{\mathcal{D}}"', ""{name=D, above}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=D, "\varepsilon"]
\end{tikzcd}$$

∋.

$$\begin{tikzcd}[column sep=huge]
\mathcal{C} \arrow[r, bend left=70, "F", ""{name=A, below}] \arrow[r, "FGF" description, ""{name=B}] \arrow[r, bend right=70, "F"', ""{name=C, above}] & \mathcal{D}
\arrow[Rightarrow, from=A, to=B, "F * \eta"]
\arrow[Rightarrow, from=B, to=C, "\varepsilon * F"]
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
\mathcal{C} \arrow[r, bend left=70, "F", ""{name=A, below}] \arrow[r, bend right=70, "F"', ""{name=C, above}] & \mathcal{D}
\arrow[Rightarrow, from=A, to=C, "\mathrm{ID}_F"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
\mathcal{D} \arrow[r, bend left=70, "G", ""{name=A, below}] \arrow[r, "GFG" description, ""{name=B}] \arrow[r, bend right=70, "G"', ""{name=C, above}] & \mathcal{C}
\arrow[Rightarrow, from=A, to=B, "\eta * G"]
\arrow[Rightarrow, from=B, to=C, "G * \varepsilon"]
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
\mathcal{D} \arrow[r, bend left=70, "G", ""{name=A, below}] \arrow[r, bend right=70, "G"', ""{name=C, above}] & \mathcal{C}
\arrow[Rightarrow, from=A, to=C, "\mathrm{ID}_G"]
\end{tikzcd}$$

ON COMPONENTS

$\forall x \in \mathcal{C}$:

$$\begin{tikzcd}
F(x) \arrow[r, "F(\eta_x)"] \arrow[dr, "\mathrm{id}_{F(x)}"'] & FGF(x) \arrow[d, "\varepsilon_{F(x)}"] \arrow[dl, phantom, "\circlearrowright"] \\
& F(x)
\end{tikzcd}$$

I. ADJUNCTION

FUNCTORS $F:\mathscr{C}\longrightarrow\mathscr{D}$ & $G:\mathscr{D}\longrightarrow\mathscr{C}$

FORM AN ADJUNCTION

IF $\exists$ NAT'L TRANSFORMATIONS

$\eta: Id_{\mathscr{C}} \Rightarrow GF$ & $\varepsilon: FG \Rightarrow Id_{\mathscr{D}}$

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$$F*\eta: \begin{tikzcd} \mathscr{C} \arrow[r, bend left=50, "Id_{\mathscr{C}}"{name=U}] \arrow[r, bend right=50, "GF"'{name=D}] & \mathscr{C} \arrow[r, "F"] & \mathscr{D} \arrow[from=U, to=D, Rightarrow, "\eta"] \end{tikzcd}$$

$$\varepsilon*F: \begin{tikzcd} \mathscr{C} \arrow[r, "F"] & \mathscr{D} \arrow[r, bend left=50, "FG"{name=U}] \arrow[r, bend right=50, "Id_{\mathscr{D}}"'{name=D}] & \mathscr{D} \arrow[from=U, to=D, Rightarrow, "\varepsilon"] \end{tikzcd}$$

$\ni$.

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left=70, "F"{name=U}] \arrow[r, "FGF"{name=M, description}] \arrow[r, bend right=70, "F"'{name=D}] & \mathscr{D} \arrow[from=U, to=M, Rightarrow, "F*\eta"] \arrow[from=M, to=D, Rightarrow, "\varepsilon*F"] \end{tikzcd} = \begin{tikzcd} \mathscr{C} \arrow[r, bend left=70, "F"{name=U}] \arrow[r, bend right=70, "F"'{name=D}] & \mathscr{D} \arrow[from=U, to=D, Rightarrow, "ID_F"] \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{D} \arrow[r, bend left=70, "G"{name=U}] \arrow[r, "GFG"{name=M, description}] \arrow[r, bend right=70, "G"'{name=D}] & \mathscr{C} \arrow[from=U, to=M, Rightarrow, "\eta*G"] \arrow[from=M, to=D, Rightarrow, "G*\varepsilon"] \end{tikzcd} = \begin{tikzcd} \mathscr{D} \arrow[r, bend left=70, "G"{name=U}] \arrow[r, bend right=70, "G"'{name=D}] & \mathscr{C} \arrow[from=U, to=D, Rightarrow, "ID_G"] \end{tikzcd}$$

ON COMPONENTS

$\forall x \in \mathscr{C}$:

$$\begin{tikzcd} F(x) \arrow[r, "F(\eta_x)"] \arrow[dr, "id_{F(x)}"'] \arrow[dr, phantom, "\circlearrowright"{above right}] & FGF(x) \arrow[d, "\varepsilon_{F(x)}"] \\ & F(x) \end{tikzcd}$$

$\forall y \in \mathscr{D}$:

$$\begin{tikzcd} G(y) \arrow[r, "\eta_{G(y)}"] \arrow[dr, "id_{G(y)}"'] \arrow[dr, phantom, "\circlearrowright"{above right}] & GFG(y) \arrow[d, "G(\varepsilon_y)"] \\ & G(y) \end{tikzcd}$$

I. ADJUNCTION

FUNCTORS $F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

IF ∃ NAT'L TRANSFORMATIONS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ & $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJUNCTION UNIT

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$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=70, "F"{name=U}] \arrow[r, "FGF"{name=M, description}] \arrow[r, bend right=70, "F"'{name=B}] & \mathcal{D} & = & \mathcal{C} \arrow[r, bend left=70, "F"{name=U2}] \arrow[r, bend right=70, "F"'{name=B2}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=M, "F * \eta"]
\arrow[Rightarrow, from=M, to=B, "\varepsilon * F"]
\arrow[Rightarrow, from=U2, to=B2, "\mathrm{ID}_F"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend left=70, "G"{name=U}] \arrow[r, "GFG"{name=M, description}] \arrow[r, bend right=70, "G"'{name=B}] & \mathcal{C} & = & \mathcal{D} \arrow[r, bend left=70, "G"{name=U2}] \arrow[r, bend right=70, "G"'{name=B2}] & \mathcal{C}
\arrow[Rightarrow, from=U, to=M, "\eta * G"]
\arrow[Rightarrow, from=M, to=B, "G * \varepsilon"]
\arrow[Rightarrow, from=U2, to=B2, "\mathrm{ID}_G"]
\end{tikzcd}$$

TRIANGLE IDENTITIES

ON COMPONENTS

$\forall x \in \mathcal{C}$:

$$\begin{tikzcd}
F(x) \arrow[r, "F(\eta_x)"] \arrow[dr, "\mathrm{id}_{F(x)}"'] \arrow[r, phantom, shift right=6, "\circlearrowright"] & FGF(x) \arrow[d, "\varepsilon_{F(x)}"] \\
 & F(x)
\end{tikzcd}$$

$\forall y \in \mathcal{D}$:

$$\begin{tikzcd}
G(y) \arrow[r, "\eta_{G(y)}"] \arrow[dr, "\mathrm{id}_{G(y)}"'] \arrow[r, phantom, shift right=6, "\circlearrowright"] & GFG(y) \arrow[d, "G(\varepsilon_y)"] \\
 & G(y)
\end{tikzcd}$$

I. ADJUNCTION

FUNCTORS $F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

IF $\exists$ NAT'L TRANSFORMATIONS

$$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF \quad \& \quad \varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$$

ADJUNCTION UNIT — ADJUNCTION COUNIT

$\ni$.

$$\begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=70, "F", ""{name=U, below}] \arrow[r, "FGF", ""{name=M, below}] \arrow[r, bend right=70, "F"', ""{name=D, above}] & \mathcal{D} \arrow[Rightarrow, from=U, to=M, "F*\eta"] \arrow[Rightarrow, from=M, to=D, "\varepsilon*F"] \end{tikzcd} = \begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=70, "F", ""{name=U2, below}] \arrow[r, bend right=70, "F"', ""{name=D2, above}] & \mathcal{D} \arrow[Rightarrow, from=U2, to=D2, "\mathrm{ID}_F"] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge] \mathcal{D} \arrow[r, bend left=70, "G", ""{name=U, below}] \arrow[r, "GFG", ""{name=M, below}] \arrow[r, bend right=70, "G"', ""{name=D, above}] & \mathcal{C} \arrow[Rightarrow, from=U, to=M, "\eta*G"] \arrow[Rightarrow, from=M, to=D, "G*\varepsilon"] \end{tikzcd} = \begin{tikzcd}[column sep=huge] \mathcal{D} \arrow[r, bend left=70, "G", ""{name=U2, below}] \arrow[r, bend right=70, "G"', ""{name=D2, above}] & \mathcal{C} \arrow[Rightarrow, from=U2, to=D2, "\mathrm{ID}_G"] \end{tikzcd}$$

WRITE $F \dashv G$

OR

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

TRIANGLE IDENTITIES

ON COMPONENTS

$\forall x \in \mathcal{C}$:

$$\begin{tikzcd} F(x) \arrow[r, "F(\eta_x)"] \arrow[dr, "id_{F(x)}"'] \arrow[dr, phantom, "\circlearrowleft" near end, shift left=3ex] & FGF(x) \arrow[d, "\varepsilon_{F(x)}"] \\ & F(x) \end{tikzcd}$$

$\forall y \in \mathcal{D}$:

$$\begin{tikzcd} G(y) \arrow[r, "\eta_{G(y)}"] \arrow[dr, "id_{G(y)}"'] \arrow[dr, phantom, "\circlearrowleft" near end, shift left=3ex] & GFG(y) \arrow[d, "G(\varepsilon_y)"] \\ & G(y) \end{tikzcd}$$

I. ADJUNCTION $\Leftarrow$ EQUIVALENCE (LOOSELY)

FUNCTORS $F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

IF ∃ NAT'L TRANSFORMATIONS

$$\eta : Id_{\mathcal{C}} \Rightarrow GF \quad \& \quad \varepsilon : FG \Rightarrow Id_{\mathcal{D}}$$

ADJUNCTION UNIT

ADJUNCTION COUNIT

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$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=70, "F", ""{name=U, below}] \arrow[r, "FGF", ""{name=M, below}] \arrow[r, bend right=70, "F"', ""{name=L, above}] & \mathcal{D} \arrow[r, phantom, "="] & \mathcal{C} \arrow[r, bend left=70, "F", ""{name=U2, below}] \arrow[r, bend right=70, "F"', ""{name=L2, above}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=M, "F*\eta"]
\arrow[Rightarrow, from=M, to=L, "\varepsilon*F"]
\arrow[Rightarrow, from=U2, to=L2, "ID_F"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend left=70, "G", ""{name=U, below}] \arrow[r, "GFG", ""{name=M, below}] \arrow[r, bend right=70, "G"', ""{name=L, above}] & \mathcal{C} \arrow[r, phantom, "="] & \mathcal{D} \arrow[r, bend left=70, "G", ""{name=U2, below}] \arrow[r, bend right=70, "G"', ""{name=L2, above}] & \mathcal{C}
\arrow[Rightarrow, from=U, to=M, "\eta*G"]
\arrow[Rightarrow, from=M, to=L, "G*\varepsilon"]
\arrow[Rightarrow, from=U2, to=L2, "ID_G"]
\end{tikzcd}$$

WRITE $F \dashv G$

OR

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}$$

TRIANGLE IDENTITIES

ON COMPONENTS

$\forall x \in \mathcal{C}$:

$$\begin{tikzcd}
F(x) \arrow[r, "F(\eta_x)"] \arrow[dr, "\circlearrowright", "id_{F(x)}"'] & FGF(x) \arrow[d, "\varepsilon_{F(x)}"] \\
 & F(x)
\end{tikzcd}$$

$\forall y \in \mathcal{D}$:

$$\begin{tikzcd}
G(y) \arrow[r, "\eta_{G(y)}"] \arrow[dr, "\circlearrowright", "id_{G(y)}"'] & GFG(y) \arrow[d, "G(\varepsilon_y)"] \\
 & G(y)
\end{tikzcd}$$

I. ADJUNCTION $\Leftarrow$ EQUIVALENCE (LOOSELY)

FUNCTORS $F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

IF $\exists$ NAT'L TRANSFORMATIONS

$\eta : Id_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF$ & $\varepsilon : FG \overset{\sim}{\Rightarrow} Id_{\mathcal{D}}$

ADJUNCTION UNIT

ADJUNCTION COUNIT

WRITE $F \dashv G$

OR

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

$\ni$.

$$\begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=70, "F", ""{name=U, below}] \arrow[r, "FGF" description, ""{name=M}] \arrow[r, bend right=70, "F"', ""{name=L, above}] & \mathcal{D} \arrow[Rightarrow, from=U, to=M, "F*\eta"] \arrow[Rightarrow, from=M, to=L, "\varepsilon*F"] \end{tikzcd} = \begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=70, "F", ""{name=U2, below}] \arrow[r, bend right=70, "F"', ""{name=L2, above}] & \mathcal{D} \arrow[Rightarrow, from=U2, to=L2, "ID_F"] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge] \mathcal{D} \arrow[r, bend left=70, "G", ""{name=U, below}] \arrow[r, "GFG" description, ""{name=M}] \arrow[r, bend right=70, "G"', ""{name=L, above}] & \mathcal{C} \arrow[Rightarrow, from=U, to=M, "\eta*G"] \arrow[Rightarrow, from=M, to=L, "G*\varepsilon"] \end{tikzcd} = \begin{tikzcd}[column sep=huge] \mathcal{D} \arrow[r, bend left=70, "G", ""{name=U2, below}] \arrow[r, bend right=70, "G"', ""{name=L2, above}] & \mathcal{C} \arrow[Rightarrow, from=U2, to=L2, "ID_G"] \end{tikzcd}$$

TRIANGLE IDENTITIES

ON COMPONENTS

$\forall x \in \mathcal{C}$:

$$\begin{tikzcd} F(x) \arrow[r, "F(\eta_x)", "\sim"'] \arrow[dr, "id_{F(x)}"', "\circlearrowleft"] & FGF(x) \arrow[d, "\varepsilon_{F(x)}", "\sim"'] \\ & F(x) \end{tikzcd}$$

$\forall y \in \mathcal{D}$:

$$\begin{tikzcd} G(y) \arrow[r, "\eta_{G(y)}", "\sim"'] \arrow[dr, "id_{G(y)}"', "\circlearrowleft"] & GFG(y) \arrow[d, "G(\varepsilon_y)", "\sim"'] \\ & G(y) \end{tikzcd}$$

I. ADJUNCTION ⟸ EQUIVALENCE (LOOSELY)

FUNCTORS $F : \mathscr{C} \longrightarrow \mathscr{D}$ & $G : \mathscr{D} \longrightarrow \mathscr{C}$

FORM AN ADJUNCTION

IF ∃ NAT'L TRANSFORMATIONS

$$\eta : \mathrm{Id}_{\mathscr{C}} \overset{\sim}{\Rightarrow} GF \quad \& \quad \varepsilon : FG \overset{\sim}{\Rightarrow} \mathrm{Id}_{\mathscr{D}}$$

ADJUNCTION UNIT — ADJUNCTION COUNIT

WRITE $F \dashv G$

OR

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "G"]
\end{tikzcd}$$

∋.

$$\begin{tikzcd}[column sep=huge]
\mathscr{C} \arrow[r, bend left=70, "F"{name=U}] \arrow[r, "FGF"{name=M}] \arrow[r, bend right=70, "F"'{name=D}] & \mathscr{D}
\arrow[Rightarrow, from=U, to=M, "F*\eta"] \arrow[Rightarrow, from=M, to=D, "\varepsilon*F"]
\end{tikzcd}
\overset{\checkmark}{=}
\begin{tikzcd}[column sep=huge]
\mathscr{C} \arrow[r, bend left=70, "F"{name=U}] \arrow[r, bend right=70, "F"'{name=D}] & \mathscr{D}
\arrow[Rightarrow, from=U, to=D, "\mathrm{ID}_F"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
\mathscr{D} \arrow[r, bend left=70, "G"{name=U}] \arrow[r, "GFG"{name=M}] \arrow[r, bend right=70, "G"'{name=D}] & \mathscr{C}
\arrow[Rightarrow, from=U, to=M, "\eta*G"] \arrow[Rightarrow, from=M, to=D, "G*\varepsilon"]
\end{tikzcd}
\overset{\checkmark}{=}
\begin{tikzcd}[column sep=huge]
\mathscr{D} \arrow[r, bend left=70, "G"{name=U}] \arrow[r, bend right=70, "G"'{name=D}] & \mathscr{C}
\arrow[Rightarrow, from=U, to=D, "\mathrm{ID}_G"]
\end{tikzcd}$$

TRIANGLE IDENTITIES

ON COMPONENTS

$\forall x \in \mathscr{C}$:

$$\begin{tikzcd}
F(x) \arrow[r, "F(\eta_x)", "\sim"'] \arrow[dr, "\mathrm{id}_{F(x)}"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowleft\checkmark"] & FGF(x) \arrow[d, "\varepsilon_{F(x)}", "\sim"'] \\
& F(x)
\end{tikzcd}$$

$\forall y \in \mathscr{D}$:

$$\begin{tikzcd}
G(y) \arrow[r, "\eta_{G(y)}", "\sim"'] \arrow[dr, "\mathrm{id}_{G(y)}"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowleft\checkmark"] & GFG(y) \arrow[d, "G(\varepsilon_y)", "\sim"'] \\
& G(y)
\end{tikzcd}$$

I. ADJUNCTION $\Leftarrow$ EQUIVALENCE (WILL REVISIT...)

FUNCTORS $F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

IF $\exists$ NAT'L TRANSFORMATIONS

$$\eta : \mathrm{Id}_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF \quad \& \quad \varepsilon : FG \overset{\sim}{\Rightarrow} \mathrm{Id}_{\mathcal{D}}$$

ADJUNCTION UNIT

ADJUNCTION COUNIT

WRITE $F \dashv G$

OR

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

$\ni.$

$$\begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=70, "F"{name=U}] \arrow[r, "FGF"{name=M}] \arrow[r, bend right=70, "F"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=M, "F*\eta"] \arrow[Rightarrow, from=M, to=D, "\varepsilon*F"] \end{tikzcd} \overset{\checkmark}{=} \begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=70, "F"{name=U}] \arrow[r, bend right=70, "F"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\mathrm{ID}_F"] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge] \mathcal{D} \arrow[r, bend left=70, "G"{name=U}] \arrow[r, "GFG"{name=M}] \arrow[r, bend right=70, "G"'{name=D}] & \mathcal{C} \arrow[Rightarrow, from=U, to=M, "\eta*G"] \arrow[Rightarrow, from=M, to=D, "G*\varepsilon"] \end{tikzcd} \overset{\checkmark}{=} \begin{tikzcd}[column sep=huge] \mathcal{D} \arrow[r, bend left=70, "G"{name=U}] \arrow[r, bend right=70, "G"'{name=D}] & \mathcal{C} \arrow[Rightarrow, from=U, to=D, "\mathrm{ID}_G"] \end{tikzcd}$$

TRIANGLE IDENTITIES

ON COMPONENTS

$\forall x \in \mathcal{C}$:

$$\begin{tikzcd} F(x) \arrow[r, "F(\eta_x)", "\sim"'] \arrow[dr, "\mathrm{id}_{F(x)}"'] & FGF(x) \arrow[d, "\varepsilon_{F(x)}", "\sim"'] \\ & F(x) \arrow[ul, phantom, "\circlearrowleft\checkmark" near start] \end{tikzcd}$$

$\forall y \in \mathcal{D}$:

$$\begin{tikzcd} G(y) \arrow[r, "\eta_{G(y)}", "\sim"'] \arrow[dr, "\mathrm{id}_{G(y)}"'] & GFG(y) \arrow[d, "G(\varepsilon_y)", "\sim"'] \\ & G(y) \arrow[ul, phantom, "\circlearrowleft\checkmark" near start] \end{tikzcd}$$

I. ADJUNCTION : CHARACTERIZATION

$F : \mathscr{C} \longrightarrow \mathscr{D}$ & $G : \mathscr{D} \longrightarrow \mathscr{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF $\exists$ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathscr{D}}$

ADJ'N COUNIT

∍. Δ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
 & F(X)
\end{tikzcd} \quad \forall X \in \mathscr{C}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\
 & G(Y)
\end{tikzcd} \quad \forall Y \in \mathscr{D}$$

I. ADJUNCTION : CHARACTERIZATION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

∍. △ IDENTITIES HOLD:

$$\begin{tikzcd} F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] \arrow[dr, phantom, shift left=4, "\circlearrowright"] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\ & F(X) \end{tikzcd} \quad \forall X \in \mathcal{C}$$

$$\begin{tikzcd} G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] \arrow[dr, phantom, shift left=4, "\circlearrowright"] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\ & G(Y) \end{tikzcd} \quad \forall Y \in \mathcal{D}$$

F ≡ LEFT ADJOINT OF G

G ≡ RIGHT ADJOINT OF F

I. ADJUNCTION : CHARACTERIZATION

$F: \mathscr{C} \longrightarrow \mathscr{D}$ & $G: \mathscr{D} \longrightarrow \mathscr{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta: Id_{\mathscr{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon: FG \Rightarrow Id_{\mathscr{D}}$

ADJ'N COUNIT

∋. Δ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "id_{F(X)}"'] \arrow[dr, phantom, "\circlearrowright", shift left=3] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] & \\
& F(X) & \forall X \in \mathscr{C}
\end{tikzcd}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "id_{G(Y)}"'] \arrow[dr, phantom, "\circlearrowright", shift left=3] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] & \\
& G(Y) & \forall Y \in \mathscr{D}
\end{tikzcd}$$

$F \equiv$ LEFT ADJOINT OF G

$G \equiv$ RIGHT ADJOINT OF F

NEED NOT EXIST,

UNIQUE UP TO NATURAL ≅ IF EXISTS

I. ADJUNCTION : CHARACTERIZATION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

∋. △ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] \arrow[dr, phantom, "\circlearrowright", shift left=4] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
 & F(X)
\end{tikzcd} \quad \forall X \in \mathcal{C}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] \arrow[dr, phantom, "\circlearrowright", shift left=4] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\
 & G(Y)
\end{tikzcd} \quad \forall Y \in \mathcal{D}$$

PROP:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd} \iff \forall X \in \mathcal{C} \text{ \& } Y \in \mathcal{D} : \exists \text{ BIJECTION}$$

$$\tau_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y

I. ADJUNCTION : CHARACTERIZATION

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

∋. △ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \arrow[dl, phantom, "\circlearrowright"] \\
 & F(X)
\end{tikzcd}$$

$\forall X \in \mathcal{C}$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \arrow[dl, phantom, "\circlearrowright"] \\
 & G(Y)
\end{tikzcd}$$

$\forall Y \in \mathcal{D}$

PROP:

$\forall X \in \mathcal{C}$ & $Y \in \mathcal{D}$: ∃ BIJECTION

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd} \iff \varphi_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y

$\forall f: X' \longrightarrow X''$ WITH Y FIXED GET:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(\bullet), Y) \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(F(f),Y)"] \arrow[d, "\varphi_{\bullet,Y}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_{\mathcal{D}}(F(\bullet), Y) \arrow[d, "\varphi_{\bullet,Y}"] \\
\mathrm{Hom}_{\mathcal{C}}(\bullet, G(Y)) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f, G(Y))"] & \mathrm{Hom}_{\mathcal{C}}(\bullet, G(Y))
\end{tikzcd}$$

I. ADJUNCTION : CHARACTERIZATION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF $\exists$ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

s.t. Δ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\circlearrowright", "\mathrm{id}_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
 & F(X)
\end{tikzcd} \quad \forall X \in \mathcal{C}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\circlearrowright", "\mathrm{id}_{G(Y)}"'] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\
 & G(Y)
\end{tikzcd} \quad \forall Y \in \mathcal{D}$$

PROP: $\forall X \in \mathcal{C}$ & $Y \in \mathcal{D}$: $\exists$ BIJECTION

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd} \iff \varsigma_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y

$\forall f : X' \longrightarrow X''$ WITH Y FIXED GET:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X''),Y) \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(F(f){,}Y)"] \arrow[d, "\varsigma_{X'',Y}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_{\mathcal{D}}(F(X'),Y) \arrow[d, "\varsigma_{X',Y}"] \\
\mathrm{Hom}_{\mathcal{C}}(X'',G(Y)) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f{,}G(Y))"] & \mathrm{Hom}_{\mathcal{C}}(X',G(Y))
\end{tikzcd}$$

RECALL $\mathrm{Hom}_{\mathcal{C}}(-,Z)$ IS CONTRAVARIANT

I. ADJUNCTION : CHARACTERIZATION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF$

ADJ'N UNIT

& $\varepsilon: FG\Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

$\ni.$ Δ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & F(X)
\end{tikzcd}$$

$\forall X\in\mathcal{C}$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & G(Y)
\end{tikzcd}$$

$\forall Y\in\mathcal{D}$

PROP:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}$$

$\Leftrightarrow$ $\forall X\in\mathcal{C}$ & $Y\in\mathcal{D}$: ∃ BIJECTION

$\varphi_{X,Y}:\mathrm{Hom}_{\mathcal{D}}(F(X),Y)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{C}}(X,G(Y))$

NATURAL IN X & Y

$\forall f: X'\longrightarrow X''$ WITH Y FIXED GET:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X''),Y) \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(F(f){,}Y)"] \arrow[d, "\varphi_{X'',Y}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\mathcal{D}}(F(X'),Y) \arrow[d, "\varphi_{X',Y}"] \\
\mathrm{Hom}_{\mathcal{C}}(X'',G(Y)) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f{,}G(Y))"] & \mathrm{Hom}_{\mathcal{C}}(X',G(Y))
\end{tikzcd}$$

RECALL $\mathrm{Hom}_{\mathcal{C}}(-,Z)$ IS CONTRAVARIANT

I. ADJUNCTION : CHARACTERIZATION

$F: \mathscr{C} \longrightarrow \mathscr{D}$ & $G: \mathscr{D} \longrightarrow \mathscr{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathscr{D}}$

ADJ'N COUNIT

∋. △ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] \arrow[dr, phantom, bend left=10, "\circlearrowright"] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
 & F(X)
\end{tikzcd}$$

$\forall X \in \mathscr{C}$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] \arrow[dr, phantom, bend left=10, "\circlearrowright"] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\
 & G(Y)
\end{tikzcd}$$

$\forall Y \in \mathscr{D}$

PROP: $\forall X \in \mathscr{C}$ & $Y \in \mathscr{D}$: ∃ BIJECTION

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "G"]
\end{tikzcd} \iff \varsigma_{X,Y} : \mathrm{Hom}_{\mathscr{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(X,G(Y))$$

NATURAL IN X & Y

$\forall f : X' \longrightarrow X''$ WITH Y FIXED GET:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(F(X''),Y) \arrow[r, "\mathrm{Hom}_{\mathscr{D}}(F(f),Y)"] \arrow[d, "\varsigma_{X'',Y}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_{\mathscr{D}}(F(X'),Y) \arrow[d, "\varsigma_{X',Y}"] \\
\mathrm{Hom}_{\mathscr{C}}(X'',G(Y)) \arrow[r, "\mathrm{Hom}_{\mathscr{C}}(f,G(Y))"] & \mathrm{Hom}_{\mathscr{C}}(X',G(Y))
\end{tikzcd}$$

RECALL $\mathrm{Hom}_{\mathscr{C}}(Z,-)$ IS COVARIANT

$\forall g : Y' \longrightarrow Y''$ WITH X FIXED GET:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(F(X),Y') \arrow[r, "\mathrm{Hom}_{\mathscr{D}}(F(X),g)"] \arrow[d, "\varsigma_{X,Y'}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_{\mathscr{D}}(F(X),Y'') \arrow[d, "\varsigma_{X,Y''}"] \\
\mathrm{Hom}_{\mathscr{C}}(X,G(Y')) \arrow[r, "\mathrm{Hom}_{\mathscr{C}}(X,G(g))"] & \mathrm{Hom}_{\mathscr{C}}(X,G(Y''))
\end{tikzcd}$$

I. ADJUNCTION : CHARACTERIZATION

$F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

∋. Δ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] \arrow[dr, phantom, "\circlearrowright"{above right, yshift=6pt}] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
 & F(X)
\end{tikzcd} \quad \forall X \in \mathcal{C}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] \arrow[dr, phantom, "\circlearrowright"{above right, yshift=6pt}] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\
 & G(Y)
\end{tikzcd} \quad \forall Y \in \mathcal{D}$$

PROP: ∀$X \in \mathcal{C}$ & $Y \in \mathcal{D}$: ∃ BIJECTION

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd} \iff \varphi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y

$\forall f : X' \longrightarrow X''$ WITH Y FIXED GET:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X''), Y) \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(F(f), Y)"] \arrow[d, "\varphi_{X'',Y}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_{\mathcal{D}}(F(X'), Y) \arrow[d, "\varphi_{X',Y}"] \\
\mathrm{Hom}_{\mathcal{C}}(X'', G(Y)) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f, G(Y))"] & \mathrm{Hom}_{\mathcal{C}}(X', G(Y))
\end{tikzcd}$$

$\forall g : Y' \longrightarrow Y''$ WITH X FIXED GET:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X), Y') \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(F(X), g)"] \arrow[d, "\varphi_{X,Y'}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_{\mathcal{D}}(F(X), Y'') \arrow[d, "\varphi_{X,Y''}"] \\
\mathrm{Hom}_{\mathcal{C}}(X, G(Y')) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(X, G(g))"] & \mathrm{Hom}_{\mathcal{C}}(X, G(Y''))
\end{tikzcd}$$

I. ADJUNCTION : CHARACTERIZATION

$F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF $\exists$ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

$\ni$. $\triangle$ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & F(X)
\end{tikzcd}$$

$\forall X \in \mathcal{C}$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & G(Y)
\end{tikzcd}$$

$\forall Y \in \mathcal{D}$

PROP:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}$$

$\Leftrightarrow$ $\forall X \in \mathcal{C}$ & $Y \in \mathcal{D}$: $\exists$ BIJECTION $\varphi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$

NATURAL IN X & Y

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathcal{D}}(F(-),-)"] \arrow[r, bend right, "\mathrm{Hom}_{\mathcal{C}}(-,G(-))"'] \arrow[r, phantom, "\sim \Downarrow \varphi"] & \mathrm{Set}
\end{tikzcd}$$

$\forall f : X' \longrightarrow X''$ WITH Y FIXED GET:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X''), Y) \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(F(f),Y)"] \arrow[d, "\varphi_{X'',Y}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\mathcal{D}}(F(X'), Y) \arrow[d, "\varphi_{X',Y}"] \\
\mathrm{Hom}_{\mathcal{C}}(X'', G(Y)) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f,G(Y))"] & \mathrm{Hom}_{\mathcal{C}}(X', G(Y))
\end{tikzcd}$$

$\forall g : Y' \longrightarrow Y''$ WITH X FIXED GET:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X), Y') \arrow[r, "\mathrm{Hom}_{\mathcal{D}}(F(X),g)"] \arrow[d, "\varphi_{X,Y'}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\mathcal{D}}(F(X), Y'') \arrow[d, "\varphi_{X,Y''}"] \\
\mathrm{Hom}_{\mathcal{C}}(X, G(Y')) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(X,G(g))"] & \mathrm{Hom}_{\mathcal{C}}(X, G(Y''))
\end{tikzcd}$$

I. ADJUNCTION : CHARACTERIZATION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN **ADJUNCTION**

$F \dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

∋. **△ IDENTITIES** HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowright"] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
& F(X) \quad \forall X \in \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowright"] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\
& G(Y) \quad \forall Y \in \mathcal{D}
\end{tikzcd}$$

PROP:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}
\iff \forall X \in \mathcal{C} \ \& \ Y \in \mathcal{D} : \exists \text{ BIJECTION } \varphi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{D} \arrow[r, bend left, "{\mathrm{Hom}_{\mathcal{D}}(F(-),-)}"] \arrow[r, bend right, "{\mathrm{Hom}_{\mathcal{C}}(-,G(-))}"'] \arrow[r, phantom, "\sim \Downarrow \varphi"] & \mathrm{Set}
\end{tikzcd}$$

PF/($\Longrightarrow$) GIVEN $(F \dashv G, \eta, \varepsilon)$,

DEFINE:

$$\mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\varphi_{X,Y}} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

I. ADJUNCTION : CHARACTERIZATION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$ FORM AN ADJUNCTION

$F \dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$
ADJ'N UNIT

& $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$
ADJ'N COUNIT

∋. △ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] \arrow[dr, phantom, "\circlearrowright", shift left=4] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
 & F(X)
\end{tikzcd}$$
$\forall X \in \mathcal{C}$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] \arrow[dr, phantom, "\circlearrowright", shift left=4] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\
 & G(Y)
\end{tikzcd}$$
$\forall Y \in \mathcal{D}$

PROP:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}$$

$\Leftrightarrow$ $\forall X \in \mathcal{C}$ & $Y \in \mathcal{D}$: ∃ BIJECTION $\varphi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$

NATURAL IN X & Y

$$\begin{tikzcd}
\mathcal{C}^{op}\times\mathcal{D} \arrow[r, bend left, "{\mathrm{Hom}_{\mathcal{D}}(F(-),-)}"] \arrow[r, bend right, "{\mathrm{Hom}_{\mathcal{C}}(-,G(-))}"'] \arrow[r, phantom, "\sim\Downarrow\varphi"] & \mathrm{Set}
\end{tikzcd}$$

PF/($\Longrightarrow$) GIVEN $(F \dashv G, \eta, \varepsilon)$,

DEFINE:

$$\mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\varphi_{X,Y}} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

$$[h: F(X) \to Y] \longmapsto$$

$$\longleftarrow\!\shortmid \left[X \xrightarrow{\ell} G(Y)\right]$$

I. ADJUNCTION : CHARACTERIZATION

$F:\mathscr{C}\longrightarrow\mathscr{D}$ & $G:\mathscr{D}\longrightarrow\mathscr{C}$

FORM AN ADJUNCTION

$F\dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta : Id_{\mathscr{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow Id_{\mathscr{D}}$

ADJ'N COUNIT

s.t. Δ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "id_{F(X)}"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] & \\
& F(X) & \forall X\in\mathscr{C}
\end{tikzcd}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "id_{G(Y)}"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] & \\
& G(Y) & \forall Y\in\mathscr{D}
\end{tikzcd}$$

PROP: $\forall X\in\mathscr{C}$ & $Y\in\mathscr{D}$: ∃ BIJECTION

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "G"]
\end{tikzcd} \iff \varphi_{X,Y} : Hom_{\mathscr{D}}(F(X),Y) \xrightarrow{\sim} Hom_{\mathscr{C}}(X,G(Y))$$

NATURAL IN X & Y

$$\begin{tikzcd}
\mathscr{C}^{op}\times\mathscr{D} \arrow[r, bend left, "{Hom_{\mathscr{D}}(F(-),-)}"] \arrow[r, bend right, "{Hom_{\mathscr{C}}(-,G(-))}"'] \arrow[r, phantom, "\sim\Downarrow\varphi"] & Set
\end{tikzcd}$$

PF/($\Rightarrow$) GIVEN $(F\dashv G, \eta, \varepsilon)$,

DEFINE:

$$Hom_{\mathscr{D}}(F(X),Y) \xrightarrow{\varphi_{X,Y}} Hom_{\mathscr{C}}(X,G(Y))$$

$$\left[h : F(X)\to Y\right] \longmapsto \left[X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(h)} G(Y)\right]$$

$$\left[F(X) \xrightarrow{F(\ell)} FG(Y) \xrightarrow{\varepsilon_Y} Y\right] \longmapsfrom \left[X \xrightarrow{\ell} G(Y)\right]$$

I. ADJUNCTION : CHARACTERIZATION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF $\exists$ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

$\ni$. $\triangle$ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] \arrow[dr, phantom, bend left, "\circlearrowright"] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
 & F(X) \quad \forall X \in \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] \arrow[dr, phantom, bend left, "\circlearrowright"] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\
 & G(Y) \quad \forall Y \in \mathcal{D}
\end{tikzcd}$$

PROP:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd} \iff \begin{array}{l} \forall X \in \mathcal{C} \ \& \ Y \in \mathcal{D} : \exists \text{ BIJECTION} \\ \varphi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y)) \end{array}$$

NATURAL IN X & Y

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathcal{D}}(F(-),-)"] \arrow[r, bend right, "\mathrm{Hom}_{\mathcal{C}}(-,G(-))"'] \arrow[r, phantom, "\sim\Downarrow\varphi"] & \mathbf{Set}
\end{tikzcd}$$

PF/$(\Longrightarrow)$ GIVEN $(F \dashv G, \eta, \varepsilon)$,

DEFINE:

$$\mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\varphi_{X,Y}} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

$$\left[h : F(X) \to Y\right] \longmapsto \left[X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(h)} G(Y)\right]$$

$$\left[F(X) \xrightarrow{F(\ell)} FG(Y) \xrightarrow{\varepsilon_Y} Y\right] \longleftarrow\!\shortmid \left[X \xrightarrow{\ell} G(Y)\right]$$

CHECK ASSIGNMENTS ARE MUTUALLY INVERSE & NATURAL IN X AND Y.

EXER. 2.40

I. ADJUNCTION : CHARACTERIZATION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

∋. △ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "id_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
\arrow[ur, phantom, "\circlearrowleft"] & F(X)
\end{tikzcd} \quad \forall X\in\mathcal{C}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "id_{G(Y)}"'] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\
\arrow[ur, phantom, "\circlearrowleft"] & G(Y)
\end{tikzcd} \quad \forall Y\in\mathcal{D}$$

<u>PROP</u>:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd} \iff \begin{array}{l} \forall X\in\mathcal{C} \ \& \ Y\in\mathcal{D} : \exists \text{ BIJECTION} \\ \mathcal{J}_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y)) \end{array}$$

NATURAL IN X & Y

$$\begin{tikzcd}
\mathcal{C}^{op}\times\mathcal{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathcal{D}}(F(-),-)"] \arrow[r, bend right, "\mathrm{Hom}_{\mathcal{C}}(-,G(-))"'] \arrow[r, phantom, "\sim\Downarrow\mathcal{J}"] & \mathrm{Set}
\end{tikzcd}$$

PF/(⟸) GIVEN BIJECTIONS $\mathcal{J}_{X,Y}$ ON RHS.

I. ADJUNCTION : CHARACTERIZATION

$F:\mathscr{C}\longrightarrow\mathscr{D}$ & $G:\mathscr{D}\longrightarrow\mathscr{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF $\exists$ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathscr{D}}$

ADJ'N COUNIT

s.t. Δ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & F(X)
\end{tikzcd}$$

$\forall X \in \mathscr{C}$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & G(Y)
\end{tikzcd}$$

$\forall Y \in \mathscr{D}$

PROP:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "G"]
\end{tikzcd}
\iff
\begin{array}{l}
\forall X \in \mathscr{C} \ \& \ Y \in \mathscr{D} : \exists \text{ BIJECTION} \\
\mathcal{J}_{X,Y} : \mathrm{Hom}_{\mathscr{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(X,G(Y))
\end{array}$$

NATURAL IN X & Y

$$\begin{tikzcd}
\mathscr{C}^{op} \times \mathscr{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathscr{D}}(F(-),-)"] \arrow[r, bend right, "\mathrm{Hom}_{\mathscr{C}}(-,G(-))"'] \arrow[r, phantom, "\sim \Downarrow \mathcal{J}"] & \mathrm{Set}
\end{tikzcd}$$

PF/($\Leftarrow$) GIVEN BIJECTIONS $\mathcal{J}_{X,Y}$ ON RHS.

DEFINE: $\forall X \in \mathscr{C}$ AND $Y \in \mathscr{D}$:

$$X \xrightarrow{\ \eta_X\ } GF(X) \quad \& \quad FG(Y) \xrightarrow{\ \varepsilon_Y\ } Y$$

I. ADJUNCTION : CHARACTERIZATION

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

∋. Δ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] & \\
& F(X) & \forall X \in \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] & \\
& G(Y) & \forall Y \in \mathcal{D}
\end{tikzcd}$$

PROP:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}
\iff \forall X \in \mathcal{C} \ \& \ Y \in \mathcal{D} : \exists \text{ BIJECTION } \Phi_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathcal{D}}(F(-),-)"] \arrow[r, bend right, "\mathrm{Hom}_{\mathcal{C}}(-,G(-))"'] \arrow[r, phantom, "\sim \Downarrow \Phi"] & \mathsf{Set}
\end{tikzcd}$$

PF/(⇐) GIVEN BIJECTIONS $\Phi_{X,Y}$ ON RHS.

DEFINE: $\forall X \in \mathcal{C}$ AND $Y \in \mathcal{D}$:

$$X \xrightarrow{\eta_X} G\underline{F(X)} \quad \& \quad FG(Y) \xrightarrow{\varepsilon_Y} Y$$

TAKE Y ABOVE
TO BE F(X) HERE

I. ADJUNCTION : CHARACTERIZATION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

∋. △ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \arrow[dl, phantom, "\circlearrowright"] \\
& F(X)
\end{tikzcd} \quad \forall X \in \mathcal{C}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \arrow[dl, phantom, "\circlearrowright"] \\
& G(Y)
\end{tikzcd} \quad \forall Y \in \mathcal{D}$$

PROP: $\forall X \in \mathcal{C}$ & $Y \in \mathcal{D}$: ∃ BIJECTION

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd} \iff \varphi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y

$$\begin{tikzcd}
\mathcal{C}^{op}\times\mathcal{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathcal{D}}(F(-){,}-)"] \arrow[r, bend right, "\mathrm{Hom}_{\mathcal{C}}(-{,}G(-))"'] \arrow[r, phantom, "\sim\Downarrow\varphi"] & \mathrm{Set}
\end{tikzcd}$$

PF/ ($\Leftarrow$) GIVEN BIJECTIONS $\varphi_{X,Y}$ ON RHS.

DEFINE: $\forall X \in \mathcal{C}$ AND $Y \in \mathcal{D}$:

$$X \xrightarrow[\substack{|||\\ \varphi_{X,F(X)}(\mathrm{id}_{F(X)})}]{\eta_X} G\underline{F(X)} \quad \& \quad FG(Y) \xrightarrow{\varepsilon_Y} Y$$

TAKE Y ABOVE TO BE F(X) HERE

I. ADJUNCTION : CHARACTERIZATION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

∋. △ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] \arrow[dr, phantom, "\circlearrowright", bend left=15] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
 & F(X) \quad \forall X\in\mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] \arrow[dr, phantom, "\circlearrowright", bend left=15] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\
 & G(Y) \quad \forall Y\in\mathcal{D}
\end{tikzcd}$$

PROP:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}
\iff
\forall X\in\mathcal{C} \ \& \ Y\in\mathcal{D} : \exists \text{ BIJECTION } \varsigma_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y

$$\begin{tikzcd}
\mathcal{C}^{op}\times\mathcal{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathcal{D}}(F(-),-)"] \arrow[r, bend right, "\mathrm{Hom}_{\mathcal{C}}(-,G(-))"'] \arrow[r, phantom, "\sim\Downarrow\varsigma"] & \mathrm{Set}
\end{tikzcd}$$

PF/ (⟸) GIVEN BIJECTIONS $\varsigma_{X,Y}$ ON RHS.

DEFINE: $\forall X\in\mathcal{C}$ AND $Y\in\mathcal{D}$:

$$X \xrightarrow[\equiv]{\eta_X} \underbrace{GF(X)}_{} \quad \& \quad FG(Y) \xrightarrow{\varepsilon_Y} Y$$

$$\varsigma_{X,F(X)}(\mathrm{id}_{F(X)})$$

TAKE Y ABOVE TO BE F(X) HERE

I. ADJUNCTION : CHARACTERIZATION

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN **ADJUNCTION**

$F \dashv G$

IF $\exists$ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

$\ni$. Δ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \arrow[dl, phantom, "\circlearrowright"] \\
& F(X)
\end{tikzcd} \quad \forall X \in \mathcal{C}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \arrow[dl, phantom, "\circlearrowright"] \\
& G(Y)
\end{tikzcd} \quad \forall Y \in \mathcal{D}$$

PROP:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd} \iff \forall X \in \mathcal{C} \text{ \& } Y \in \mathcal{D} : \exists \text{ BIJECTION } \varsigma_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathcal{D}}(F(-),-)"] \arrow[r, bend right, "\mathrm{Hom}_{\mathcal{C}}(-,G(-))"'] \arrow[r, phantom, "\sim \Downarrow \varsigma"] & \mathrm{Set}
\end{tikzcd}$$

PF/($\Leftarrow$) GIVEN BIJECTIONS $\varsigma_{X,Y}$ ON RHS.

DEFINE: $\forall X \in \mathcal{C}$ AND $Y \in \mathcal{D}$:

$$X \underset{\underset{\varsigma_{X,F(X)}(\mathrm{id}_{F(X)})}{|||}}{\xrightarrow{\eta_X}} \underbracket{GF(X)} \quad \& \quad \underbracket{FG(Y)} \xrightarrow{\varepsilon_Y} Y$$

TAKE Y ABOVE
TO BE F(X) HERE

TAKE X ABOVE
TO BE G(Y) HERE

I. ADJUNCTION : CHARACTERIZATION

$F:\mathscr{C}\longrightarrow\mathscr{D}$ & $G:\mathscr{D}\longrightarrow\mathscr{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathscr{D}}$

ADJ'N COUNIT

∋. △ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] \arrow[dr, phantom, "=", shift left=3ex] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
 & F(X)
\end{tikzcd} \quad \forall X \in \mathscr{C}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] \arrow[dr, phantom, "=", shift left=3ex] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\
 & G(Y)
\end{tikzcd} \quad \forall Y \in \mathscr{D}$$

PROP:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "G"]
\end{tikzcd}$$

$\Leftrightarrow$ $\forall X \in \mathscr{C}$ & $Y \in \mathscr{D}$: ∃ BIJECTION $\varsigma_{X,Y} : \mathrm{Hom}_{\mathscr{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(X,G(Y))$

NATURAL IN X & Y

$$\begin{tikzcd}
\mathscr{C}^{op} \times \mathscr{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathscr{D}}(F(-)\text{,}-)"] \arrow[r, bend right, "\mathrm{Hom}_{\mathscr{C}}(-\text{,}G(-))"'] \arrow[r, phantom, "\sim \Downarrow \varsigma"] & \mathrm{Set}
\end{tikzcd}$$

PF/ ($\Leftarrow$) GIVEN BIJECTIONS $\varsigma_{X,Y}$ ON RHS.

DEFINE: $\forall X \in \mathscr{C}$ AND $Y \in \mathscr{D}$:

$$X \xrightarrow{\eta_X} G\underline{F(X)} \qquad \& \qquad F\underline{G(Y)} \xrightarrow{\varepsilon_Y} Y$$

$$|||\qquad\qquad\qquad\qquad\qquad |||$$

$$\varsigma_{X,F(X)}(\mathrm{id}_{F(X)}) \qquad\qquad \varsigma^{-1}_{G(Y),Y}(\mathrm{id}_{G(Y)})$$

TAKE Y ABOVE TO BE F(X) HERE

TAKE X ABOVE TO BE G(Y) HERE

I. ADJUNCTION : CHARACTERIZATION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$

IF ∃ NAT'L TRANSF'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

ADJ'N UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

ADJ'N COUNIT

∋. △ IDENTITIES HOLD:

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] \arrow[dr, phantom, shift left=4ex, "2"] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
& F(X) \quad \forall X \in \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] \arrow[dr, phantom, shift left=4ex, "2"] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \\
& G(Y) \quad \forall Y \in \mathcal{D}
\end{tikzcd}$$

PROP:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}
\iff
\forall X \in \mathcal{C} \;\&\; Y \in \mathcal{D} : \exists \text{ BIJECTION } \varphi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y

$$\begin{tikzcd}
\mathcal{C}^{op} \times \mathcal{D} \arrow[r, bend left, "{\mathrm{Hom}_{\mathcal{D}}(F(-),-)}", ""{name=U, below}] \arrow[r, bend right, "{\mathrm{Hom}_{\mathcal{C}}(-,G(-))}"', ""{name=D, above}] & \mathrm{Set} \arrow[from=U, to=D, Rightarrow, "\varphi", "\sim"']
\end{tikzcd}$$

PF/ (⇐) GIVEN BIJECTIONS $\varphi_{X,Y}$ ON RHS.

DEFINE: $\forall X \in \mathcal{C}$ AND $Y \in \mathcal{D}$:

$$X \xrightarrow{\eta_X} G\underbrace{F(X)} \qquad \& \qquad \underbrace{FG(Y)} \xrightarrow{\varepsilon_Y} Y$$

$$\varphi_{X,F(X)}\left(\mathrm{id}_{F(X)}\right) \;|||\; \qquad\qquad \;|||\; \varphi^{-1}_{G(Y),Y}\left(\mathrm{id}_{G(Y)}\right)$$

TAKE Y ABOVE TO BE F(X) HERE

TAKE X ABOVE TO BE G(Y) HERE

CHECK THAT THESE FORM COMPONENTS OF THE DESIRED NAT'L TRANS'NS η & ε. //

EXER. 2.40

I. ADJUNCTION : CHARACTERIZATION

$F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$$\phi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

I. ADJUNCTION $\Leftarrow$ EQUIVALENCE

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF$ UNIT

& $\varepsilon: FG\Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)}\circ F(\eta_X)=id_{F(X)} \quad \forall X\in\mathcal{C}$$

$$G(\varepsilon_Y)\circ\eta_{G(Y)}=id_{G(Y)} \quad \forall Y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\tau_{X,Y}:\mathrm{Hom}_{\mathcal{D}}(F(X),Y)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X\in\mathcal{C}, Y\in\mathcal{D}$

PROP IF $F:\mathcal{C}\xrightarrow{\sim}\mathcal{D}$ IS AN EQUIV. OF CATEGS, THEN $\exists\, G:\mathcal{D}\longrightarrow\mathcal{C}$ WITH $F\dashv G$ & $G\dashv F$ WHERE UNIT & COUNIT ARE NAT'L ISOMS.

I. ADJUNCTION ⇐ EQUIVALENCE

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF$ UNIT

& $\varepsilon: FG\Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. Δ IDENTITIES HOLD:

$$\varepsilon_{F(x)}\circ F(\eta_x)=\mathrm{id}_{F(x)}\quad \forall x\in\mathcal{C}$$

$$G(\varepsilon_y)\circ\eta_{G(y)}=\mathrm{id}_{G(y)}\quad \forall y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\tau_{x,y}:\mathrm{Hom}_{\mathcal{D}}(F(x),y)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{C}}(x,G(y))$$

NATURAL IN X & Y $\forall x\in\mathcal{C}, y\in\mathcal{D}$

PROP IF $F:\mathcal{C}\xrightarrow{\sim}\mathcal{D}$ IS AN EQUIV. OF CATEGS, THEN $\exists\, G:\mathcal{D}\longrightarrow\mathcal{C}$ WITH $F\dashv G$ & $G\dashv F$ WHERE UNIT & COUNIT ARE NAT'L ISOMS.

PF/ $F:\mathcal{C}\xrightarrow{\sim}\mathcal{D}$ EQUIVALENCE

$\Rightarrow$ ∃ QUASI-INVERSE $G:\mathcal{D}\longrightarrow\mathcal{C}$.

THAT IS: ∃ NATURAL ISOMORPHISMS:

$$\phi:\mathrm{Id}_{\mathcal{C}}\xRightarrow{\sim} GF \quad \& \quad \psi: FG\xRightarrow{\sim}\mathrm{Id}_{\mathcal{D}}.$$

I. ADJUNCTION $\Leftarrow$ EQUIVALENCE

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN **ADJUNCTION**

$F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

$\ni$. Δ IDENTITIES HOLD:

$$\varepsilon_{F(x)} \circ F(\eta_x) = \mathrm{id}_{F(x)} \quad \forall x \in \mathcal{C}$$

$$G(\varepsilon_y) \circ \eta_{G(y)} = \mathrm{id}_{G(y)} \quad \forall y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$$\tau_{x,y}: \mathrm{Hom}_{\mathcal{D}}(F(x), y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(x, G(y))$$

NATURAL IN x & y $\forall x \in \mathcal{C}, y \in \mathcal{D}$

PROP IF $F: \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ IS AN EQUIV. OF CATEGS, THEN $\exists\, G: \mathcal{D} \longrightarrow \mathcal{C}$ WITH $F \dashv G$ & $G \dashv F$ WHERE UNIT & COUNIT ARE NAT'L ISOMS.

PF/ $F: \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ EQUIVALENCE

$\Rightarrow \exists$ QUASI-INVERSE $G: \mathcal{D} \longrightarrow \mathcal{C}$.

THAT IS: $\exists$ NATURAL ISOMORPHISMS:

$$\phi: \mathrm{Id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} GF \quad \& \quad \psi: FG \overset{\sim}{\Longrightarrow} \mathrm{Id}_{\mathcal{D}}.$$

NOW DEFINE:

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(GF(X), G(Y)) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

I. ADJUNCTION $\Longleftarrow$ EQUIVALENCE

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN **ADJUNCTION**

$F\dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF$ UNIT

& $\varepsilon: FG\Rightarrow\mathrm{Id}_{\mathcal{D}}$ COUNIT

$\ni$. Δ IDENTITIES HOLD:

$$\varepsilon_{F(x)}\circ F(\eta_x)=\mathrm{id}_{F(x)}\quad\forall x\in\mathcal{C}$$

$$G(\varepsilon_y)\circ\eta_{G(y)}=\mathrm{id}_{G(y)}\quad\forall y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$$\Im_{X,Y}:\mathrm{Hom}_{\mathcal{D}}(F(X),Y)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X\in\mathcal{C}, Y\in\mathcal{D}$

PROP IF $F:\mathcal{C}\xrightarrow{\sim}\mathcal{D}$ IS AN EQUIV. OF CATEGS, THEN $\exists\, G:\mathcal{D}\longrightarrow\mathcal{C}$ WITH $F\dashv G$ & $G\dashv F$ WHERE UNIT & COUNIT ARE NAT'L ISOMS.

PF/ $F:\mathcal{C}\xrightarrow{\sim}\mathcal{D}$ EQUIVALENCE

$\Rightarrow$ $\exists$ QUASI-INVERSE $G:\mathcal{D}\longrightarrow\mathcal{C}$.

THAT IS: $\exists$ NATURAL ISOMORPHISMS:

$$\phi:\mathrm{Id}_{\mathcal{C}}\xRightarrow{\sim}GF \quad \& \quad \psi: FG\xRightarrow{\sim}\mathrm{Id}_{\mathcal{D}}.$$

NOW DEFINE:

$$\mathrm{Hom}_{\mathcal{D}}(F(X),Y)\longrightarrow\mathrm{Hom}_{\mathcal{C}}(GF(X),G(Y))\longrightarrow\mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

$$[f:F(X)\to Y]\longmapsto[G(f):GF(X)\to G(Y)]$$

$$[h:GF(X)\to G(Y)]\longmapsto[X\xrightarrow{\phi_X}GF(X)\xrightarrow{h}G(Y)]$$

I. ADJUNCTION ⇐ EQUIVALENCE

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

$\ni$. $\triangle$ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = \mathrm{id}_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$$\varsigma_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

PROP IF $F: \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ IS AN EQUIV. OF CATEGS, THEN $\exists G: \mathcal{D} \longrightarrow \mathcal{C}$ WITH $F \dashv G$ & $G \dashv F$ WHERE UNIT & COUNIT ARE NAT'L ISOMS.

PF/ $F: \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ EQUIVALENCE

$\Rightarrow$ $\exists$ QUASI-INVERSE $G: \mathcal{D} \longrightarrow \mathcal{C}$.

THAT IS: $\exists$ NATURAL ISOMORPHISMS:

$$\phi: \mathrm{Id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} GF \quad \& \quad \psi: FG \overset{\sim}{\Longrightarrow} \mathrm{Id}_{\mathcal{D}}.$$

NOW DEFINE:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \arrow[r, "\cong\ G\ \text{FULLY FAITHFUL}"] \arrow[rr, bend left, "\varsigma_{X,Y}\ \text{AS COMPOSITION}"] & \mathrm{Hom}_{\mathcal{C}}(GF(X),G(Y)) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(X,G(Y)) \end{tikzcd}$$

$$[f: F(X) \to Y] \longmapsto [G(f): GF(X) \to G(Y)]$$

$$[h: GF(X) \to G(Y)] \longmapsto [X \xrightarrow[\cong]{\phi_X} GF(X) \xrightarrow{h} G(Y)]$$

I. ADJUNCTION $\Leftarrow$ EQUIVALENCE

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF$ UNIT

& $\varepsilon: FG\Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. $\triangle$ IDENTITIES HOLD:

$$\varepsilon_{F(x)}\circ F(\eta_x)=\mathrm{id}_{F(x)}\quad \forall x\in\mathcal{C}$$

$$G(\varepsilon_y)\circ\eta_{G(y)}=\mathrm{id}_{G(y)}\quad \forall y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\mathcal{S}_{x,y}:\mathrm{Hom}_{\mathcal{D}}(F(x),y)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{C}}(x,G(y))$$

NATURAL IN X & Y $\forall x\in\mathcal{C}, y\in\mathcal{D}$

PROP IF $F:\mathcal{C}\xrightarrow{\sim}\mathcal{D}$ IS AN EQUIV. OF CATEGS, THEN $\exists\, G:\mathcal{D}\longrightarrow\mathcal{C}$ WITH $F\dashv G$ & $G\dashv F$ WHERE UNIT & COUNIT ARE NAT'L ISOMS.

PF/ $F:\mathcal{C}\xrightarrow{\sim}\mathcal{D}$ EQUIVALENCE

$\Rightarrow$ ∃ QUASI-INVERSE $G:\mathcal{D}\to\mathcal{C}$.

THAT IS: ∃ NATURAL ISOMORPHISMS:

$$\phi:\mathrm{Id}_{\mathcal{C}}\overset{\sim}{\Longrightarrow}GF \quad \& \quad \psi: FG\overset{\sim}{\Longrightarrow}\mathrm{Id}_{\mathcal{D}}.$$

NOW DEFINE:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X),Y) \arrow[r, "\cong\ G\text{ FULLY FAITHFUL}"] \arrow[rr, bend left, "\mathcal{S}_{X,Y}\text{ AS COMPOSITION}"] & \mathrm{Hom}_{\mathcal{C}}(GF(X),G(Y)) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(X,G(Y))
\end{tikzcd}$$

$$[f:F(X)\to Y]\longmapsto[G(f):GF(X)\to G(Y)]$$

$$[h:GF(X)\to G(Y)]\longmapsto[X\xrightarrow[\cong]{\phi_X}GF(X)\xrightarrow{h}G(Y)]$$

DEFINE $\mathcal{S}_{X,Y}^{-1}$ USING ψ... //

EXER. 2.41

I. ADJUNCTION ⇐ EQUIVALENCE

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$ FORM AN **ADJUNCTION** $F \dashv G$ IF ∃ NAT. TRANS'NS

$\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. **Δ IDENTITIES** HOLD:

$$\varepsilon_{F(x)} \circ F(\eta_x) = \mathrm{id}_{F(x)} \quad \forall x \in \mathcal{C}$$

$$G(\varepsilon_y) \circ \eta_{G(y)} = \mathrm{id}_{G(y)} \quad \forall y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\xi_{x,y}: \mathrm{Hom}_{\mathcal{D}}(F(x),y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(x,G(y))$$

NATURAL IN x & y $\forall x \in \mathcal{C}, y \in \mathcal{D}$

PROP IF $F: \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ IS AN EQUIV. OF CATEGS, THEN $\exists\, G: \mathcal{D} \longrightarrow \mathcal{C}$ WITH $F \dashv G$ & $G \dashv F$ WHERE UNIT & COUNIT ARE NAT'L ISOMS.

PF/ $F: \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ EQUIVALENCE

$\Rightarrow$ ∃ QUASI-INVERSE $G: \mathcal{D} \longrightarrow \mathcal{C}$.

THAT IS: ∃ NATURAL ISOMORPHISMS:

$$\phi: \mathrm{Id}_{\mathcal{C}} \xRightarrow{\sim} GF \quad \& \quad \psi: FG \xRightarrow{\sim} \mathrm{Id}_{\mathcal{D}}.$$

⚠ NOT THE CO/UNIT OF $F \dashv G$

NOW DEFINE:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \arrow[r, "\cong\ G\text{ FULLY FAITHFUL}"] \arrow[rr, bend left, "\xi_{X,Y}\text{ AS COMPOSITION}"] & \mathrm{Hom}_{\mathcal{C}}(GF(X),G(Y)) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(X,G(Y)) \end{tikzcd}$$

$$[f: F(X) \to Y] \longmapsto [G(f): GF(X) \to G(Y)]$$

$$[h: GF(X) \to G(Y)] \longmapsto [X \xrightarrow[\cong]{\phi_X} GF(X) \xrightarrow{h} G(Y)]$$

DEFINE $\xi^{-1}_{X,Y}$ USING ψ... ///

EXER. 2.41

I. ADJUNCTION ⇐ EQUIVALENCE

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF$ UNIT

& $\varepsilon: FG\Rightarrow\mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. △ IDENTITIES HOLD:

$$\varepsilon_{F(x)}\circ F(\eta_x)=\mathrm{id}_{F(x)}\quad\forall x\in\mathcal{C}$$

$$G(\varepsilon_y)\circ\eta_{G(y)}=\mathrm{id}_{G(y)}\quad\forall y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

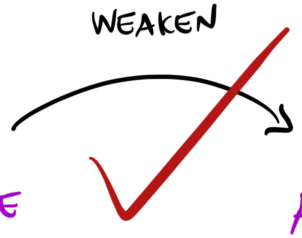
$$\tau_{x,y}:\mathrm{Hom}_{\mathcal{D}}(F(x),y)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{C}}(x,G(y))$$

NATURAL IN X & Y $\forall x\in\mathcal{C}, y\in\mathcal{D}$

PROP IF $F:\mathcal{C}\xrightarrow{\sim}\mathcal{D}$ IS AN EQUIV. OF CATEGS, THEN $\exists\, G:\mathcal{D}\longrightarrow\mathcal{C}$ WITH $F\dashv G$ & $G\dashv F$ WHERE UNIT & COUNIT ARE NAT'L ISOMS.

$\mathcal{C}\sim\mathcal{D}$

EQUIVALENCE OF CATEGORIES



NOW:

ADJUNCTION

∃ FUNCTORS
$$\begin{tikzcd}\mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"]\end{tikzcd}$$
∋.

$\mathrm{Id}_{\mathcal{C}}\cong GF$ & $FG\cong\mathrm{Id}_{\mathcal{D}}$

INVOLVING ISOMORPHISMS OF FUNCTORS

∃ FUNCTORS
$$\begin{tikzcd}\mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"]\end{tikzcd}$$
∋.

$\mathrm{Id}_{\mathcal{C}}\overset{\eta}{\Rightarrow}GF$ & $FG\overset{\varepsilon}{\Rightarrow}\mathrm{Id}_{\mathcal{D}}$

HAVE A NICE COMPATIBILITY CONDITION

INVOLVES CERTAIN TRANSF'NS OF FUNCTORS

I. ADJUNCTION : EXAMPLES

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

$\ni$. Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$$\tau_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

≡ TENSOR-HOM ADJUNCTION ≡ (LECT #2)

FOR $\Bbbk$-VECTOR SPACES U, V, W

I. ADJUNCTION : EXAMPLES

$F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

$\ni.$ Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = \mathrm{id}_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$$\tau_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

≡ TENSOR-HOM ADJUNCTION ≡ (LECT #2)

FOR $\mathbb{R}$-VECTOR SPACES U, V, W

$$\boxed{\mathrm{Hom}_{\mathbb{R}}(U \otimes_{\mathbb{R}} V, W) \cong \mathrm{Hom}_{\mathbb{R}}(U, \mathrm{Hom}_{\mathbb{R}}(V,W))}$$

$$f \longmapsto \left[\begin{array}{l} U \longrightarrow \mathrm{Hom}_{\mathbb{R}}(V,W) \\ u \longmapsto [v \longmapsto f(u \otimes v)] \end{array}\right]$$

$$\left[\begin{array}{l} U \otimes V \longrightarrow W \\ u \otimes v \longmapsto g(u)(v) \end{array}\right] \longleftarrow\!\shortmid\; g$$

I. ADJUNCTION : EXAMPLES

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF ∃ NAT. TRANS'NS

$\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. △ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = \mathrm{id}_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\tau_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

≡ TENSOR-HOM ADJUNCTION ≡

FOR $\mathbb{R}$-VECTOR SPACES U, V, W

(LECT #2)

$$\mathrm{Hom}_{\mathbb{R}}(U \otimes_{\mathbb{R}} V, W) \cong \mathrm{Hom}_{\mathbb{R}}(U, \mathrm{Hom}_{\mathbb{R}}(V, W))$$

$$f \longmapsto \left[\begin{array}{l} U \longrightarrow \mathrm{Hom}_{\mathbb{R}}(V,W) \\ u \longmapsto [v \longmapsto f(u \otimes v)] \end{array}\right]$$

$$\left[\begin{array}{l} U \otimes V \longrightarrow W \\ u \otimes v \longmapsto g(u)(v) \end{array}\right] \longleftarrow\!\!\mapstochar \; g$$

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "- \otimes_{\mathbb{R}} V"] \arrow[r, phantom, "\perp"] & \mathrm{Vec} \arrow[l, bend left, "\mathrm{Hom}_{\mathbb{R}}(V, -)"] \end{tikzcd}$$

I. ADJUNCTION : EXAMPLES

$F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF ∃ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. △ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = \mathrm{id}_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\Phi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

≡ TENSOR-HOM ADJUNCTION ≡ (LECT #2)

FOR $\mathbb{R}$-VECTOR SPACES U, V, W

$$\boxed{\mathrm{Hom}_{\mathbb{R}}(U \otimes_{\mathbb{R}} V, W) \cong \mathrm{Hom}_{\mathbb{R}}(U, \mathrm{Hom}_{\mathbb{R}}(V, W))}$$

$$f \longmapsto \left[\begin{array}{l} U \longrightarrow \mathrm{Hom}_{\mathbb{R}}(V,W) \\ u \longmapsto [v \longmapsto f(u \otimes v)] \end{array}\right]$$

$$\left[\begin{array}{l} U \otimes V \longrightarrow W \\ u \otimes v \longmapsto g(u)(v) \end{array}\right] \longleftarrow\!\shortmid \; g$$

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "- \otimes_{\mathbb{R}} V"] \arrow[r, phantom, "\perp"] & \mathrm{Vec} \arrow[l, bend left, "\mathrm{Hom}_{\mathbb{R}}(V, -)"] \end{tikzcd}$$

ALSO GET:

$$\boxed{\mathrm{Hom}_{\mathbb{R}}(U \otimes_{\mathbb{R}} V, W) \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathrm{Hom}_{\mathbb{R}}(U, W))}$$

I. ADJUNCTION : EXAMPLES

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. △ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\varphi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

≡ TENSOR-HOM ADJUNCTION ≡

(LECT #2)

FOR $\mathbb{R}$-VECTOR SPACES U,V,W

$$\boxed{\mathrm{Hom}_{\mathbb{R}}(U \otimes_{\mathbb{R}} V, W) \cong \mathrm{Hom}_{\mathbb{R}}(U, \mathrm{Hom}_{\mathbb{R}}(V,W))}$$

$$f \longmapsto \left[\begin{array}{l} U \longrightarrow \mathrm{Hom}_{\mathbb{R}}(V,W) \\ u \longmapsto [v \longmapsto f(u\otimes v)] \end{array}\right]$$

$$\left[\begin{array}{l} U\otimes V \longrightarrow W \\ u\otimes v \longmapsto g(u)(v) \end{array}\right] \longleftarrow\!\shortmid \; g$$

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "- \otimes_{\mathbb{R}} V"] \arrow[r, phantom, "\perp"] & \mathrm{Vec} \arrow[l, bend left, "\mathrm{Hom}_{\mathbb{R}}(V,-)"] \end{tikzcd}$$

ALSO GET:

$$\boxed{\mathrm{Hom}_{\mathbb{R}}(U \otimes_{\mathbb{R}} V, W) \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathrm{Hom}_{\mathbb{R}}(U,W))}$$

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "U \otimes_{\mathbb{R}} -"] \arrow[r, phantom, "\perp"] & \mathrm{Vec} \arrow[l, bend left, "\mathrm{Hom}_{\mathbb{R}}(U,-)"] \end{tikzcd}$$

I. ADJUNCTION : EXAMPLES

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF ∃ NAT. TRANS'NS

$\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\mathcal{J}_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

≡ BIMODULE TENSOR-HOM ADJ'N ≡ (LECT #4)

I. ADJUNCTION : EXAMPLES

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF$ UNIT

& $\varepsilon: FG\Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)}\circ F(\eta_X) = id_{F(X)} \quad \forall X\in\mathcal{C}$$

$$G(\varepsilon_Y)\circ\eta_{G(Y)} = id_{G(Y)} \quad \forall Y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\tau_{X,Y}:\mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X\in\mathcal{C}, Y\in\mathcal{D}$

≡ BIMODULE TENSOR-HOM ADJ'N ≡ (LECT #4)

$$\mathrm{Hom}_{B_1\text{-Mod}}(U\otimes_A V, W) \cong \mathrm{Hom}_{A\text{-Mod}}(V, \mathrm{Hom}_{B_1\text{-Mod}}(U,W))$$

$$U = {}_{B_1}U_A \qquad V = {}_AV \qquad W = {}_{B_1}W$$

I. ADJUNCTION : EXAMPLES

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF ∃ NAT. TRANS'NS

$\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. $\triangle$ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = \mathrm{id}_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\tau_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

≡ BIMODULE TENSOR-HOM ADJ'N ≡ (LECT #4)

$$\mathrm{Hom}_{B_1\text{-Mod}}(U \otimes_A V, W) \cong \mathrm{Hom}_{A\text{-Mod}}(V, \mathrm{Hom}_{B_1\text{-Mod}}(U, W))$$

$U = {}_{B_1}U_A \qquad V = {}_A V \qquad W = {}_{B_1}W$

$$F := U \otimes_A - \; : A\text{-Mod} \longrightarrow B_1\text{-Mod}$$

$$G := \mathrm{Hom}_{B_1\text{-Mod}}(U, -) : B_1\text{-Mod} \longrightarrow A\text{-Mod}$$

I. ADJUNCTION : EXAMPLES

$F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

$\ni.$ $\triangle$ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$$\vartheta_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

≡ BIMODULE TENSOR-HOM ADJ'N ≡ (LECT #4)

$$\mathrm{Hom}_{B_1\text{-Mod}}(U \otimes_A V, W) \cong \mathrm{Hom}_{A\text{-Mod}}(V, \mathrm{Hom}_{B_1\text{-Mod}}(U, W))$$

$U = {}_{B_1}U_A \qquad V = {}_AV \qquad W = {}_{B_1}W$

$$F := U \otimes_A - \; : A\text{-Mod} \longrightarrow B_1\text{-Mod}$$

$$G := \mathrm{Hom}_{B_1\text{-Mod}}(U, -) : B_1\text{-Mod} \longrightarrow A\text{-Mod}$$

$$\begin{tikzcd} A\text{-Mod} \arrow[r, bend left, "U \otimes_A -"] \arrow[r, phantom, "\perp"] & B_1\text{-Mod} \arrow[l, bend left, "\mathrm{Hom}_{B_1\text{-Mod}}(U,-)"] \end{tikzcd}$$

I. ADJUNCTION: EXAMPLES

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)}\circ F(\eta_X) = id_{F(X)} \quad \forall X\in\mathcal{C}$$

$$G(\varepsilon_Y)\circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\mathcal{J}_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X\in\mathcal{C}, Y\in\mathcal{D}$

≡ FREE-FORGET ADJUNCTION ≡

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "\text{Free}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\text{Forg}"]
\end{tikzcd}$$

I. ADJUNCTION : EXAMPLES

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF$ UNIT

& $\varepsilon: FG\Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. △ IDENTITIES HOLD:

$\varepsilon_{F(X)}\circ F(\eta_X) = \mathrm{id}_{F(X)}$ $\forall X\in\mathcal{C}$

$G(\varepsilon_Y)\circ\eta_{G(Y)} = \mathrm{id}_{G(Y)}$ $\forall Y\in\mathcal{D}$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$\tau_{X,Y}:\mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$

NATURAL IN X & Y $\forall X\in\mathcal{C}, Y\in\mathcal{D}$

≡ FREE - FORGET ADJUNCTION ≡

A FREE OBJECT ON $X\in\mathcal{C}$

IS AN OBJECT $\mathrm{Free}_X\in\mathcal{D}$

EQUIPPED WITH

A MONO $\alpha_X : X \longrightarrow \mathrm{Forg}(\mathrm{Free}_X)\in\mathcal{C}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\mathrm{Forg}"]
\end{tikzcd}$$

I. ADJUNCTION : EXAMPLES

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF$ UNIT

& $\varepsilon: FG\Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)}\circ F(\eta_X)=id_{F(X)} \quad \forall X\in\mathcal{C}$$

$$G(\varepsilon_Y)\circ\eta_{G(Y)}=id_{G(Y)} \quad \forall Y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\mathcal{T}_{X,Y}:\mathrm{Hom}_{\mathcal{D}}(F(X),Y)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X\in\mathcal{C}, Y\in\mathcal{D}$

≡ FREE-FORGET ADJUNCTION ≡

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\mathrm{Forg}"]
\end{tikzcd}$$

A FREE OBJECT ON $X\in\mathcal{C}$

IS AN OBJECT $\mathrm{Free}_X\in\mathcal{D}$

EQUIPPED WITH

A MONO $\alpha_X: X\longrightarrow\mathrm{Forg}(\mathrm{Free}_X)\in\mathcal{C}$

∋. $\forall\ \beta_X: X\longrightarrow\mathrm{Forg}(Z)\in\mathcal{C}$ WITH $Z\in\mathcal{D}$

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\,\beta_X"'] & \mathrm{Forg}(\mathrm{Free}_X) \\
 & \mathrm{Forg}(Z)
\end{tikzcd}$$

I. ADJUNCTION : EXAMPLES

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF$ UNIT

& $\varepsilon: FG\Rightarrow\mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. $\triangle$ IDENTITIES HOLD:

$$\varepsilon_{F(x)}\circ F(\eta_x)=\mathrm{id}_{F(x)}\quad\forall x\in\mathcal{C}$$

$$G(\varepsilon_y)\circ\eta_{G(y)}=\mathrm{id}_{G(y)}\quad\forall y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$$\tau_{x,y}:\mathrm{Hom}_{\mathcal{D}}(F(x),y)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{C}}(x,G(y))$$

NATURAL IN x & y $\forall x\in\mathcal{C}, y\in\mathcal{D}$

≡ FREE - FORGET ADJUNCTION ≡

$$\begin{tikzcd}\mathcal{C} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\mathrm{Forg}"]\end{tikzcd}$$

A FREE OBJECT ON $X\in\mathcal{C}$

IS AN OBJECT $\mathrm{Free}_X\in\mathcal{D}$

EQUIPPED WITH

A MONO $\alpha_X:X\longrightarrow\mathrm{Forg}(\mathrm{Free}_X)\in\mathcal{C}$

∋. $\forall\ \beta_X: X\longrightarrow\mathrm{Forg}(Z)\in\mathcal{C}$ WITH $Z\in\mathcal{D}$

$\exists!\ \gamma_X:\mathrm{Free}_X\longrightarrow Z\in\mathcal{D}$ WHERE

$$\begin{tikzcd}X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\ \beta_X"'] \arrow[dr, phantom, "\circlearrowright", shift left=3ex] & \mathrm{Forg}(\mathrm{Free}_X) \arrow[d, dashed, "\exists!\ \mathrm{Forg}(\gamma_X)"] & \sqsubset & \mathrm{Free}_X \arrow[d, dashed, "\exists!\ \gamma_X"] \\ & \mathrm{Forg}(Z) & \sqsubset & Z\end{tikzcd}$$

$\sqsubset$

$\equiv$

FORGET STRUCTURE

I. ADJUNCTION : EXAMPLES

$F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN **ADJUNCTION**

$F \dashv G$ IF ∃ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. △ IDENTITIES HOLD:

$$\varepsilon_{F(x)} \circ F(\eta_x) = \mathrm{id}_{F(x)} \quad \forall x \in \mathcal{C}$$

$$G(\varepsilon_y) \circ \eta_{G(y)} = \mathrm{id}_{G(y)} \quad \forall y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\varphi_{x,y} : \mathrm{Hom}_{\mathcal{D}}(F(x), y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(x, G(y))$$

NATURAL IN X & Y $\forall x \in \mathcal{C}, y \in \mathcal{D}$

≡ FREE - FORGET ADJUNCTION ≡

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\mathrm{Forg}"] \end{tikzcd}$$

A **FREE OBJECT ON** $X \in \mathcal{C}$

IS AN OBJECT $\mathrm{Free}_X \in \mathcal{D}$

EQUIPPED WITH

A MONO $\alpha_X : X \longrightarrow \mathrm{Forg}(\mathrm{Free}_X) \in \mathcal{C}$

∋. $\forall\, \beta_X : X \longrightarrow \mathrm{Forg}(Z) \in \mathcal{C}$ WITH $Z \in \mathcal{D}$

$\exists!\, \gamma_X : \mathrm{Free}_X \longrightarrow Z \in \mathcal{D}$ WHERE

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "Z", "\forall \beta_X"'] & \mathrm{Forg}(\mathrm{Free}_X) \arrow[d, dashed, "\exists!\, \mathrm{Forg}(\gamma_X)"] & \sqsubset & \mathrm{Free}_X \arrow[d, dashed, "\exists!\, \gamma_X"] \\ & \mathrm{Forg}(Z) & \sqsubset & Z \end{tikzcd}$$

$\sqsubset$
$\equiv$
FORGET STRUCTURE

$\mathrm{Free} : \mathcal{C} \longrightarrow \mathcal{D}$ IS A **FREE FUNCTOR**

$X \longmapsto \mathrm{Free}_X$ (IF IT INDEED IS A FUNCTOR)

I. ADJUNCTION : EXAMPLES

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃NAT. TRANS'NS

$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF$ UNIT

& $\varepsilon: FG\Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. △ IDENTITIES HOLD:

$$\varepsilon_{F(x)}\circ F(\eta_x)=\mathrm{id}_{F(x)}\quad \forall x\in\mathcal{C}$$

$$G(\varepsilon_y)\circ\eta_{G(y)}=\mathrm{id}_{G(y)}\quad \forall y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\xi_{x,y}:\mathrm{Hom}_{\mathcal{D}}(F(x),y)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{C}}(x,G(y))$$

NATURAL IN x & y $\forall x\in\mathcal{C}, y\in\mathcal{D}$

≡ FREE - FORGET ADJUNCTION ≡

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\mathrm{Forg}"]
\end{tikzcd}$$

A FREE OBJECT ON $X\in\mathcal{C}$

IS AN OBJECT $\mathrm{Free}_X\in\mathcal{D}$

EQUIPPED WITH

A MONO $\alpha_X:X\longrightarrow\mathrm{Forg}(\mathrm{Free}_X)\in\mathcal{C}$

∋. $\forall\ \beta_X:X\longrightarrow\mathrm{Forg}(z)\in\mathcal{C}$ WITH $z\in\mathcal{D}$

$\exists!\ \gamma_X:\mathrm{Free}_X\longrightarrow z\in\mathcal{D}$ WHERE

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\,\beta_X"'] \arrow[dr, phantom, "\circlearrowleft", shift left=4] & \mathrm{Forg}(\mathrm{Free}_X) \arrow[d, dashed, "\exists!\,\mathrm{Forg}(\gamma_X)"] & \sqsubset & \mathrm{Free}_X \arrow[d, dashed, "\exists!\,\gamma_X"] \\
 & \mathrm{Forg}(z) & \sqsubset & z
\end{tikzcd}$$

$\sqsubset$ ≡ FORGET STRUCTURE

$\mathrm{Free}:\mathcal{C}\longrightarrow\mathcal{D}$ IS A FREE FUNCTOR

$X\longmapsto\mathrm{Free}_X$ (IF IT INDEED IS A FUNCTOR)

GET: $\gamma_X\xleftrightarrow{1-1}\beta_X$

YIELDS: $\boxed{\mathrm{Hom}_{\mathcal{D}}(\mathrm{Free}(X),z)\cong\mathrm{Hom}_{\mathcal{C}}(X,\mathrm{Forg}(z))}$

I. ADJUNCTION : EXAMPLES

≡ FREE - FORGET ADJUNCTION ≡

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\mathrm{Forg}"] \end{tikzcd}$$

A FREE OBJECT ON $X \in \mathcal{C}$

IS AN OBJECT $\mathrm{Free}_X \in \mathcal{D}$

EQUIPPED WITH

A MONO $\alpha_X : X \longrightarrow \mathrm{Forg}(\mathrm{Free}_X) \in \mathcal{C}$

$\ni$ $\forall\ \beta_X : X \longrightarrow \mathrm{Forg}(Z) \in \mathcal{C}$ WITH $Z \in \mathcal{D}$

$\exists!\ \gamma_X : \mathrm{Free}_X \longrightarrow Z \in \mathcal{D}$ WHERE

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, shift left=4, "\circlearrowright"] & \mathrm{Forg}(\mathrm{Free}_X) \arrow[d, dashed, "\exists!\, \mathrm{Forg}(\gamma_X)"] & \sqsubset & \mathrm{Free}_X \arrow[d, dashed, "\exists!\, \gamma_X"] \\ & \mathrm{Forg}(Z) & \sqsubset & Z \end{tikzcd}$$

$\sqsubset$

$|||$

FORGET STRUCTURE

$\mathrm{Free} : \mathcal{C} \longrightarrow \mathcal{D}$ IS A FREE FUNCTOR

$X \longmapsto \mathrm{Free}_X$ (IF IT INDEED IS A FUNCTOR)

GET: $\gamma_X \overset{1-1}{\longleftrightarrow} \beta_X$

YIELDS: $\boxed{\mathrm{Hom}_{\mathcal{D}}(\mathrm{Free}(X), Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{Forg}(Z))}$

I. ADJUNCTION : EXAMPLES

$A = \Bbbk$-ALG.

$$\begin{tikzcd} \mathrm{Vec} & A\text{-}\mathrm{Mod} \arrow[l, bend left, "\text{FORGET } A\text{-ACTION}"] \end{tikzcd}$$

≡ FREE-FORGET ADJUNCTION ≡

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\mathrm{Forg}"] \end{tikzcd}$$

A FREE OBJECT ON $X \in \mathcal{C}$

IS AN OBJECT $\mathrm{Free}_X \in \mathcal{D}$

EQUIPPED WITH

A MONO $\alpha_X : X \longrightarrow \mathrm{Forg}(\mathrm{Free}_X) \in \mathcal{C}$

s.t. $\forall\ \beta_X : X \longrightarrow \mathrm{Forg}(Z) \in \mathcal{C}$ WITH $Z \in \mathcal{D}$

$\exists!\ \gamma_X : \mathrm{Free}_X \longrightarrow Z \in \mathcal{D}$ WHERE

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\ \beta_X"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowright"] & \mathrm{Forg}(\mathrm{Free}_X) \arrow[d, dashed, "\exists!\ \mathrm{Forg}(\gamma_X)"] & \sqsubset & \mathrm{Free}_X \arrow[d, dashed, "\exists!\ \gamma_X"] \\ & \mathrm{Forg}(Z) & \sqsubset & Z \end{tikzcd}$$

$\sqsubset$

$\equiv$

FORGET STRUCTURE

$\mathrm{Free} : \mathcal{C} \longrightarrow \mathcal{D}$ IS A FREE FUNCTOR

$X \longmapsto \mathrm{Free}_X$ (IF IT INDEED IS A FUNCTOR)

GET: $\gamma_X \xleftrightarrow{1\text{-}1} \beta_X$

YIELDS: $\boxed{\mathrm{Hom}_{\mathcal{D}}(\mathrm{Free}(X), Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{Forg}(Z))}$

I. ADJUNCTION : EXAMPLES

$A = \Bbbk\text{-ALG.}$

$$\begin{tikzcd}
\mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = A \otimes_{\Bbbk} -"] \arrow[r, phantom, "\perp"] & A\text{-}\mathrm{Mod} \arrow[l, bend left, "\text{FORGET } A\text{-ACTION}"]
\end{tikzcd}$$

≡ FREE-FORGET ADJUNCTION ≡

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\mathrm{Forg}"]
\end{tikzcd}$$

A FREE OBJECT ON $X \in \mathcal{C}$

IS AN OBJECT $\mathrm{Free}_X \in \mathcal{D}$

EQUIPPED WITH

A MONO $\alpha_X : X \longrightarrow \mathrm{Forg}(\mathrm{Free}_X) \in \mathcal{C}$

s.t. $\forall\ \beta_X : X \longrightarrow \mathrm{Forg}(Z) \in \mathcal{C}$ WITH $Z \in \mathcal{D}$

$\exists!\ \gamma_X : \mathrm{Free}_X \longrightarrow Z \in \mathcal{D}$ WHERE

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, "\circlearrowright", shift left=3ex] & \mathrm{Forg}(\mathrm{Free}_X) \arrow[d, dashed, "\exists!\ \mathrm{Forg}(\gamma_X)"] & \sqsubset & \mathrm{Free}_X \arrow[d, dashed, "\exists!\ \gamma_X"] \\
& \mathrm{Forg}(Z) & \sqsubset & Z
\end{tikzcd}$$

$\sqsubset$

$\equiv$

FORGET STRUCTURE

$\mathrm{Free} : \mathcal{C} \longrightarrow \mathcal{D}$ IS A FREE FUNCTOR

$X \longmapsto \mathrm{Free}_X$ (IF IT INDEED IS A FUNCTOR)

GET: $\gamma_X \xleftrightarrow{1-1} \beta_X$

YIELDS: $\boxed{\mathrm{Hom}_{\mathcal{D}}(\mathrm{Free}(X), Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{Forg}(Z))}$

I. ADJUNCTION : EXAMPLES

≡ FREE - FORGET ADJUNCTION ≡

$A = \Bbbk\text{-ALG.}$

$\mathrm{Free} = A \otimes_{\Bbbk} -$

$$\begin{tikzcd}
\mathrm{Vec} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & A\text{-Mod} \arrow[l, bend left, "\text{FORGET A-ACTION}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Vec} & \mathrm{Alg} \arrow[l, bend left, "\text{FORGET MULTIPLICATION}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\mathrm{Forg}"]
\end{tikzcd}$$

A FREE OBJECT ON $X \in \mathcal{C}$

IS AN OBJECT $\mathrm{Free}_X \in \mathcal{D}$

EQUIPPED WITH

A MONO $\alpha_X : X \longrightarrow \mathrm{Forg}(\mathrm{Free}_X) \in \mathcal{C}$

$\ni$ $\forall\ \beta_X : X \longrightarrow \mathrm{Forg}(Z) \in \mathcal{C}$ WITH $Z \in \mathcal{D}$

$\exists!\ \gamma_X : \mathrm{Free}_X \longrightarrow Z \in \mathcal{D}$ WHERE

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\ \beta_X"'] \arrow[dr, phantom, bend left=30, "\circlearrowright"] & \mathrm{Forg}(\mathrm{Free}_X) \arrow[d, dashed, "\exists!\ \mathrm{Forg}(\gamma_X)"] & \sqsubset & \mathrm{Free}_X \arrow[d, dashed, "\exists!\ \gamma_X"] \\
 & \mathrm{Forg}(Z) & \sqsubset & Z
\end{tikzcd}$$

$\sqsubset$
$\equiv$
FORGET STRUCTURE

$\mathrm{Free} : \mathcal{C} \longrightarrow \mathcal{D}$ IS A FREE FUNCTOR

$X \longmapsto \mathrm{Free}_X$ (IF IT INDEED IS A FUNCTOR)

GET: $\gamma_X \xleftrightarrow{1-1} \beta_X$

YIELDS: $\boxed{\mathrm{Hom}_{\mathcal{D}}(\mathrm{Free}(X), Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{Forg}(Z))}$

I. ADJUNCTION : EXAMPLES

$A = \Bbbk\text{-ALG.}$

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = A\otimes_{\Bbbk} -"] \arrow[r, phantom, "\perp"] & A\text{-Mod} \arrow[l, bend left, "\text{FORGET } A\text{-ACTION}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = T(-) \text{ TENSOR ALGEBRA}"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left, "\text{FORGET MULTIPLICATION}"] \end{tikzcd}$$

≡ FREE - FORGET ADJUNCTION ≡

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\mathrm{Forg}"] \end{tikzcd}$$

A FREE OBJECT ON $X \in \mathcal{C}$

IS AN OBJECT $\mathrm{Free}_X \in \mathcal{D}$

EQUIPPED WITH

A MONO $\alpha_X : X \longrightarrow \mathrm{Forg}(\mathrm{Free}_X) \in \mathcal{C}$

s.t. $\forall\ \beta_X : X \longrightarrow \mathrm{Forg}(z) \in \mathcal{C}$ WITH $z \in \mathcal{D}$

$\exists!\ \gamma_X : \mathrm{Free}_X \longrightarrow z \in \mathcal{D}$ WHERE

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\ \beta_X"'] \arrow[dr, phantom, shift left=4, "\circlearrowleft"] & \mathrm{Forg}(\mathrm{Free}_X) \arrow[d, dashed, "\exists!\ \mathrm{Forg}(\gamma_X)"] \arrow[r, phantom, "\sqsubset"] & \mathrm{Free}_X \arrow[d, dashed, "\exists!\ \gamma_X"] \\ & \mathrm{Forg}(z) \arrow[r, phantom, "\sqsubset"] & z \end{tikzcd}$$

$\sqsubset$ $\equiv$ FORGET STRUCTURE

$\mathrm{Free} : \mathcal{C} \longrightarrow \mathcal{D}$ IS A FREE FUNCTOR

$X \longmapsto \mathrm{Free}_X$ (IF IT INDEED IS A FUNCTOR)

GET: $\gamma_X \xleftrightarrow{1\text{-}1} \beta_X$

YIELDS: $\boxed{\mathrm{Hom}_{\mathcal{D}}(\mathrm{Free}(X), z) \cong \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{Forg}(z))}$

I. ADJUNCTION : EXAMPLES

$A = \Bbbk$-ALG.

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = A\otimes_{\Bbbk} -"] \arrow[r, phantom, "\perp"] & A\text{-}\mathrm{Mod} \arrow[l, bend left, "\text{FORGET A-ACTION}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = T(-)\ \text{TENSOR ALGEBRA}"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left, "\text{FORGET MULTIPLICATION}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Set} & \mathrm{Group} \arrow[l, bend left, "\text{FORGET OPERATION}"] \end{tikzcd}$$

≡ FREE-FORGET ADJUNCTION ≡

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\mathrm{Forg}"] \end{tikzcd}$$

A FREE OBJECT ON $X \in \mathcal{C}$

IS AN OBJECT $\mathrm{Free}_X \in \mathcal{D}$

EQUIPPED WITH

A MONO $\alpha_X : X \longrightarrow \mathrm{Forg}(\mathrm{Free}_X) \in \mathcal{C}$

$\ni$ $\forall\ \beta_X : X \longrightarrow \mathrm{Forg}(Z) \in \mathcal{C}$ WITH $Z \in \mathcal{D}$

$\exists!\ \gamma_X : \mathrm{Free}_X \longrightarrow Z \in \mathcal{D}$ WHERE

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\ \beta_X"'] \arrow[dr, phantom, "\circlearrowright", shift left=3ex] & \mathrm{Forg}(\mathrm{Free}_X) \arrow[d, dashed, "\exists!\ \mathrm{Forg}(\gamma_X)"] & \sqsubset & \mathrm{Free}_X \arrow[d, dashed, "\exists!\ \gamma_X"] \\ & \mathrm{Forg}(Z) & \sqsubset & Z \end{tikzcd}$$

$\sqsubset$

$\equiv$

FORGET STRUCTURE

$\mathrm{Free} : \mathcal{C} \longrightarrow \mathcal{D}$ IS A FREE FUNCTOR

$X \longmapsto \mathrm{Free}_X$ (IF IT INDEED IS A FUNCTOR)

GET: $\gamma_X \xleftrightarrow{1-1} \beta_X$

YIELDS: $\boxed{\mathrm{Hom}_{\mathcal{D}}(\mathrm{Free}(X), Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{Forg}(Z))}$

I. ADJUNCTION : EXAMPLES

≡ FREE-FORGET ADJUNCTION ≡

$A = \Bbbk$-ALG.

$\text{Free} = A \otimes_{\Bbbk} -$

$$\begin{tikzcd}
\text{Vec} \arrow[r, bend left, "\text{Free}"] \arrow[r, phantom, "\perp"] & A\text{-Mod} \arrow[l, bend left]
\end{tikzcd}$$

FORGET A-ACTION

$\text{Free} = T(-)$ TENSOR ALGEBRA

$$\begin{tikzcd}
\text{Vec} \arrow[r, bend left, "\text{Free}"] \arrow[r, phantom, "\perp"] & \text{Alg} \arrow[l, bend left]
\end{tikzcd}$$

FORGET MULTIPLICATION

$\text{Free} = F(-)$ FREE GROUP

$$\begin{tikzcd}
\text{Set} \arrow[r, bend left, "\text{Free}"] \arrow[r, phantom, "\perp"] & \text{Group} \arrow[l, bend left]
\end{tikzcd}$$

FORGET OPERATION

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "\text{Free}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\text{Forg}"]
\end{tikzcd}$$

A FREE OBJECT ON $X \in \mathcal{C}$

IS AN OBJECT $\text{Free}_X \in \mathcal{D}$

EQUIPPED WITH

A MONO $\alpha_X : X \longrightarrow \text{Forg}(\text{Free}_X) \in \mathcal{C}$

$\ni$: $\forall\ \beta_X : X \longrightarrow \text{Forg}(z) \in \mathcal{C}$ WITH $z \in \mathcal{D}$

$\exists!\ \gamma_X : \text{Free}_X \longrightarrow z \in \mathcal{D}$ WHERE

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \text{Forg}(\text{Free}_X) \arrow[d, dashed, "\exists!\, \text{Forg}(\gamma_X)"] \arrow[dl, phantom, "\circlearrowright"] & \sqsubset & \text{Free}_X \arrow[d, dashed, "\exists!\, \gamma_X"] \\
 & \text{Forg}(z) & \sqsubset & z
\end{tikzcd}$$

$\sqsubset$

$\equiv$

FORGET STRUCTURE

$\text{Free} : \mathcal{C} \longrightarrow \mathcal{D}$ IS A FREE FUNCTOR

$X \longmapsto \text{Free}_X$ (IF IT INDEED IS A FUNCTOR)

GET: $\gamma_X \overset{1\text{-}1}{\longleftrightarrow} \beta_X$

YIELDS: $\boxed{\text{Hom}_{\mathcal{D}}(\text{Free}(X), z) \cong \text{Hom}_{\mathcal{C}}(X, \text{Forg}(z))}$

I. ADJUNCTION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$ FORM AN **ADJUNCTION** $F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF$ UNIT

& $\varepsilon: FG\Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. △ IDENTITIES HOLD:

$$\varepsilon_{F(x)}\circ F(\eta_x)=id_{F(x)} \quad \forall x\in\mathcal{C}$$

$$G(\varepsilon_y)\circ\eta_{G(y)}=id_{G(y)} \quad \forall y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\xi_{x,y}:\mathrm{Hom}_{\mathcal{D}}(F(x),y)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{C}}(x,G(y))$$

NATURAL IN x & y $\forall x\in\mathcal{C}, y\in\mathcal{D}$

COOL FACT: FOR $F\dashv G$

I. ADJUNCTION

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF ∃ NAT. TRANS'NS

$\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = \mathrm{id}_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\tau_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

COOL FACT: FOR $F \dashv G$

LEFT ADJOINTS

F

PRESERVE

I. ADJUNCTION

$F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

$\ni.$ Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y \in \mathcal{D}$$

$\equiv$ OR EQUIVALENTLY $\equiv$

$\exists$ BIJECTIONS

$$\tau_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

COOL FACT: FOR $F \dashv G$

LEFT ADJOINTS

F

PRESERVE

- PUSHOUTS

PUSHOUT OF $f : Z \to X$ & $g : Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] \arrow[dr, phantom, "\circlearrowleft"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \gamma"'] & \\
& & C
\end{tikzcd}$$

I. ADJUNCTION

$F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN **ADJUNCTION**

$F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta : Id_{\mathcal{C}} \Rightarrow GF$ **UNIT**

& $\varepsilon : FG \Rightarrow Id_{\mathcal{D}}$ **COUNIT**

$\ni.$ **Δ IDENTITIES** HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$$\tau_{X,Y} : Hom_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} Hom_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

COOL FACT: FOR $F \dashv G$

LEFT ADJOINTS

F

PRESERVE

- PUSHOUTS
- COEQUALIZERS

PUSHOUT OF $f : Z \to X$ & $g : Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] \arrow[dr, phantom, "\circlearrowright"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] \arrow[dr, phantom, "\circlearrowright"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists! \gamma"'] & \\
& & C
\end{tikzcd}$$

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] \arrow[dr, phantom, shift left=3, "\circlearrowright"] & coeq(f,g) \arrow[d, dashed, "\exists! \gamma"] \\
& & C
\end{tikzcd}$$

I. ADJUNCTION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta: Id_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon: FG \Rightarrow Id_{\mathcal{D}}$ COUNIT

∋. △ IDENTITIES HOLD:

$$\varepsilon_{F(X)}\circ F(\eta_X) = id_{F(X)} \quad \forall X\in\mathcal{C}$$

$$G(\varepsilon_Y)\circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\tau_{X,Y}: Hom_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} Hom_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X\in\mathcal{C}, Y\in\mathcal{D}$

COOL FACT: FOR F⊣G

LEFT ADJOINTS
F
PRESERVE

- PUSHOUTS
- COEQUALIZERS
- COKERNELS

PUSHOUT OF $f: Z\to X$ & $g: Z\to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[rdd, bend left, "\forall\beta_X"] \arrow[rd, phantom, "\circlearrowleft"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[rrd, bend right, "\forall\beta_Y"'] & X\sqcup_Z Y \arrow[rd, dashed, "\exists!\gamma"] \arrow[d, phantom, "\circlearrowleft"] & \\
 & \, & C
\end{tikzcd}$$

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[rd, "\forall\beta"'] \arrow[rd, phantom, shift left=3, "\circlearrowleft"] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists!\gamma"] \\
 & & C
\end{tikzcd}$$

COKERNEL OF $f: X\to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[rrd, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[rd, "\forall\beta"'] \arrow[rd, phantom, shift left=3, "\circlearrowleft"] & \mathrm{coker}(f) \arrow[d, dashed, "\exists!\gamma"] \\
 & & C
\end{tikzcd}$$

I. ADJUNCTION

$F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$ FORM AN **ADJUNCTION** $F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

$\ni$. Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$$\phi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

COOL FACT: FOR $F \dashv G$

LEFT ADJOINTS F PRESERVE

- PUSHOUTS
- COEQUALIZERS
- COKERNELS

PUSHOUT OF $f : Z \to X$ & $g : Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] & X \sqcup_Z Y \arrow[dr, dashed, "\exists! \gamma"] \arrow[d, phantom, "\circlearrowleft"] & \\
 & {} & C
\end{tikzcd}$$

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] \arrow[r, phantom, shift right=6, "\circlearrowleft"] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists! \gamma"] \\
 & & C
\end{tikzcd}$$

COKERNEL OF $f : X \to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] \arrow[r, phantom, shift right=6, "\circlearrowleft"] & \mathrm{coker}(f) \arrow[d, dashed, "\exists! \gamma"] \\
 & & C
\end{tikzcd}$$

E.G. COKERNEL OF $F(f)$

$= F(\alpha : Y \to \mathrm{coker}(f))$

I. ADJUNCTION

$F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN **ADJUNCTION**

$F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

$\ni$. Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = \mathrm{id}_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$$\Phi_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

COOL FACT: FOR $F \dashv G$

LEFT ADJOINTS

F

PRESERVE

- PUSHOUTS
- COEQUALIZERS
- COKERNELS

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

PUSHOUT OF $f : Z \to X$ & $g : Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] \arrow[ddr, phantom, "\circlearrowright"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] \arrow[drr, phantom, "\circlearrowright"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \gamma"'] & \\
& & C
\end{tikzcd}$$

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] \arrow[dr, phantom, bend left, "\circlearrowright"] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists ! \gamma"] \\
& & C
\end{tikzcd}$$

COKERNEL OF $f : X \to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] \arrow[dr, phantom, bend left, "\circlearrowright"] & \mathrm{coker}(f) \arrow[d, dashed, "\exists ! \gamma"] \\
& & C
\end{tikzcd}$$

E.G. COKERNEL OF $F(f)$

$= F(\alpha : Y \longrightarrow \mathrm{coker}(f))$

I. ADJUNCTION

$F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. Δ IDENTITIES HOLD:

$\varepsilon_{F(X)} \circ F(\eta_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$

$G(\varepsilon_Y) \circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y \in \mathcal{D}$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$\tau_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$

$\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

COOL FACT: FOR $F \dashv G$

LEFT ADJOINTS

F

PRESERVE

- PUSHOUTS
- COEQUALIZERS
- COKERNELS

⋮

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, shift left=4, "2"] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

RIGHT ADJOINTS

G

PRESERVE

I. ADJUNCTION

$F : \mathcal{C} \longrightarrow \mathcal{D}$ & $G : \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN **ADJUNCTION**

$F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ **UNIT**

& $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ **COUNIT**

$\ni$. $\triangle$ **IDENTITIES** HOLD:

$$\varepsilon_{F(x)} \circ F(\eta_x) = id_{F(x)} \quad \forall x \in \mathcal{C}$$

$$G(\varepsilon_y) \circ \eta_{G(y)} = id_{G(y)} \quad \forall y \in \mathcal{D}$$

$\equiv$ OR EQUIVALENTLY $\equiv$

$\exists$ BIJECTIONS

$$\varsigma_{x,y} : \mathrm{Hom}_{\mathcal{D}}(F(x), y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(x, G(y))$$

NATURAL IN x & y $\forall x \in \mathcal{C}, y \in \mathcal{D}$

COOL FACT: FOR $F \dashv G$

LEFT ADJOINTS F PRESERVE

- PUSHOUTS
- COEQUALIZERS
- COKERNELS

$\vdots$

$$\begin{tikzcd}
X \arrow[r, "\alpha_x"] \arrow[rd, "\forall \beta_x"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_x"] \arrow[ld, phantom, "\circlearrowright"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

RIGHT ADJOINTS G PRESERVE

- PULLBACKS
- EQUALIZERS
- KERNELS

$\vdots$

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_x"'] \arrow[rd, "\forall \beta'_x"] \arrow[rd, phantom, "\circlearrowright"', near start] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_x"] & X
\end{tikzcd}$$

FORM II

I. ADJUNCTION

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)}\circ F(\eta_X) = id_{F(X)} \quad \forall X\in\mathcal{C}$$

$$G(\varepsilon_Y)\circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\tau_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X\in\mathcal{C}, Y\in\mathcal{D}$

COOL FACT: FOR $F\dashv G$

LEFT ADJOINTS F PRESERVE	RIGHT ADJOINTS G PRESERVE
• PUSHOUTS	• PULLBACKS
• COEQUALIZERS	• EQUALIZERS
• COKERNELS ⋮	• KERNELS ⋮
"COLIMITS"	"LIMITS"

≡ READ ABOUT THIS ≡

II. UNIVERSALITY REVISITED

$F: \mathscr{C} \longrightarrow \mathscr{D}$ & $G: \mathscr{D} \longrightarrow \mathscr{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta : Id_{\mathscr{C}} \Rightarrow GF$ UNIT

& $\varepsilon : FG \Rightarrow Id_{\mathscr{D}}$ COUNIT

$\ni.$ Δ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = id_{F(X)} \quad \forall X \in \mathscr{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y \in \mathscr{D}$$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$$\tau_{X,Y} : Hom_{\mathscr{D}}(F(X), Y) \xrightarrow{\sim} Hom_{\mathscr{C}}(X, G(Y))$$

NATURAL IN X & Y $\forall X \in \mathscr{C}, Y \in \mathscr{D}$

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"', "2"] & Univ(X) \arrow[d, dashed, "\exists ! \gamma_X"] \\
 & Arb(X)
\end{tikzcd}$$

FORM I

II. UNIVERSALITY REVISITED

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$
FORM AN ADJUNCTION
$F \dashv G$ IF ∃ NAT. TRANS'NS
$\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT
& $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT
∋. Δ IDENTITIES HOLD:

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$$G(\varepsilon_Y) \circ \eta_{G(Y)} = id_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\tau_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, "2" very near end, shift left=3ex] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

GET:

$$\gamma_X \xleftrightarrow{1-1} \beta_X$$

YIELDS:

$$\mathrm{Hom}_{\text{Structure}}(\mathrm{Univ}(X), \mathrm{Arb}(X))$$

$$\cong$$

$$\mathrm{Hom}_{\text{Gadget}}(X, \mathrm{Arb}(X)_{\text{Gadget}})$$

II. UNIVERSALITY REVISITED

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow GF$ UNIT

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$$G(\varepsilon_y)\circ\eta_{G(y)}=\mathrm{id}_{G(y)}\quad \forall y\in\mathcal{D}$$

≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\tau_{x,y}:\mathrm{Hom}_{\mathcal{D}}(F(x),y)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{C}}(x,G(y))$$

NATURAL IN x & y $\forall x\in\mathcal{C}, y\in\mathcal{D}$

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma_X"] \arrow[dl, phantom, "2"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Gadget} \arrow[r, bend left, "\mathrm{Univ}"] \arrow[r, phantom, "\perp"] & \mathrm{Structure} \arrow[l, bend left, "(-)_{\mathrm{Gadget}}"]
\end{tikzcd}$$

Underlying Gadget

GET:

$$\gamma_X \xleftrightarrow{1-1} \beta_X$$

YIELDS:

$$\mathrm{Hom}_{\mathrm{Structure}}(\mathrm{Univ}(X),\ \mathrm{Arb}(X))$$

$$\cong$$

$$\mathrm{Hom}_{\mathrm{Gadget}}(X,\ \mathrm{Arb}(X)_{\mathrm{Gadget}})$$

II. UNIVERSALITY REVISITED

(SIMILAR FOR FORM II...)

$F:\mathcal{C}\longrightarrow\mathcal{D}$ & $G:\mathcal{D}\longrightarrow\mathcal{C}$

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$F \dashv G$ IF ∃ NAT. TRANS'NS

$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

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∋. △ IDENTITIES HOLD:

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≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\tau_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Gadget} \arrow[r, bend left, "\mathrm{Univ}"] \arrow[r, phantom, "\perp"] & \mathrm{Structure} \arrow[l, bend left, "(-)_{\mathrm{Gadget}}"]
\end{tikzcd}$$

Underlying Gadget

GET:

$$\gamma_X \xleftrightarrow{1-1} \beta_X$$

YIELDS:

$$\mathrm{Hom}_{\mathrm{Structure}}(\mathrm{Univ}(X), \mathrm{Arb}(X))$$

$$\cong$$

$$\mathrm{Hom}_{\mathrm{Gadget}}(X, \mathrm{Arb}(X)_{\mathrm{Gadget}})$$

II. UNIVERSALITY REVISITED

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF ∃ NAT. TRANS'NS

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≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\tau_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

FORM I

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "2"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Gadget} \arrow[r, bend left, "\mathrm{Univ}"] \arrow[r, phantom, "\perp"] & \mathrm{Structure} \arrow[l, bend left, "(-)_{\mathrm{Gadget}}"]
\end{tikzcd}$$

Underlying Gadget

EXAMPLES —

II. UNIVERSALITY REVISITED

$F:\mathscr{C}\longrightarrow\mathscr{D}$ & $G:\mathscr{D}\longrightarrow\mathscr{C}$

FORM AN ADJUNCTION

$F\dashv G$ IF ∃ NAT. TRANS'NS

$\eta:\mathrm{Id}_{\mathscr{C}}\Rightarrow GF$ UNIT

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∋. $\triangle$ IDENTITIES HOLD:

$$\varepsilon_{F(x)}\circ F(\eta_x)=\mathrm{id}_{F(x)}\quad \forall x\in\mathscr{C}$$

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≡ OR EQUIVALENTLY ≡

∃ BIJECTIONS

$$\tau_{x,y}:\mathrm{Hom}_{\mathscr{D}}(F(x),y)\xrightarrow{\sim}\mathrm{Hom}_{\mathscr{C}}(x,G(y))$$

NATURAL IN x & y $\forall x\in\mathscr{C}, y\in\mathscr{D}$

FORM I

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\beta_X"'] \arrow[dr, phantom, "2", shift left=4ex] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma_X"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Gadget} \arrow[r, bend left, "\mathrm{Univ}"] \arrow[r, phantom, "\perp"] & \text{Structure} \arrow[l, bend left, "(-)_{\text{Gadget}}"]
\end{tikzcd}$$

Underlying Gadget

$A=\Bbbk$-ALG.

$$\begin{tikzcd}
\mathrm{Vec} \arrow[r, bend left, "\mathrm{Free}\,=\,A\otimes_{\Bbbk}-"] \arrow[r, phantom, "\perp"] & A\text{-}\mathrm{Mod} \arrow[l, bend left]
\end{tikzcd}$$

FORGET A-ACTION

$$\begin{tikzcd}
\mathrm{Vec} \arrow[r, bend left, "\mathrm{Free}\,=\,T(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left]
\end{tikzcd}$$

TENSOR ALGEBRA

FORGET MULTIPLICATION

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, bend left, "\mathrm{Free}\,=\,F(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Group} \arrow[l, bend left]
\end{tikzcd}$$

FREE GROUP

FORGET OPERATION

II. UNIVERSALITY REVISITED

$F: \mathcal{C} \longrightarrow \mathcal{D}$ & $G: \mathcal{D} \longrightarrow \mathcal{C}$

FORM AN ADJUNCTION

$F \dashv G$ IF $\exists$ NAT. TRANS'NS

$\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ UNIT

& $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ COUNIT

∋. △ IDENTITIES HOLD:

$$\varepsilon_{F(X)} \circ F(\eta_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$$

$$G(\varepsilon_Y) \circ \eta_{G(Y)} = \mathrm{id}_{G(Y)} \quad \forall Y \in \mathcal{D}$$

≡ OR EQUIVALENTLY ≡

$\exists$ BIJECTIONS

$$\tau_{X,Y}: \mathrm{Hom}_{\mathcal{D}}(F(X),Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X,G(Y))$$

NATURAL IN X & Y $\forall X \in \mathcal{C}, Y \in \mathcal{D}$

FORM I

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \\
\arrow[ur, phantom, "\circlearrowright"] & \mathrm{Arb}(X)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Gadget} \arrow[r, bend left, "\mathrm{Univ}"] \arrow[r, phantom, "\perp"] & \mathrm{Structure} \arrow[l, bend left, "(-)_{\mathrm{Gadget}}"]
\end{tikzcd}$$

Underlying Gadget

$A = \Bbbk$-ALG.

$$\begin{tikzcd}
\mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = A \otimes_{\Bbbk} -"] \arrow[r, phantom, "\perp"] & A\text{-}\mathrm{Mod} \arrow[l, bend left]
\end{tikzcd}$$

FORGET A-ACTION

$$\begin{tikzcd}
\mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = T(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left]
\end{tikzcd}$$

TENSOR ALGEBRA

FORGET MULTIPLICATION

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, bend left, "\mathrm{Free} = F(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Group} \arrow[l, bend left]
\end{tikzcd}$$

FREE GROUP

FORGET OPERATION

$$\begin{tikzcd}
\mathrm{Group} \arrow[r, bend left, "\Bbbk(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left, "(-)^{\times}"]
\end{tikzcd}$$

$\Bbbk(-)$ = GROUP ALG.

$(-)^{\times}$ = GROUP OF UNITS

II. UNIVERSALITY REVISITED

$H : \mathcal{D} \longrightarrow \mathsf{Set}$

COVARIANT

IS REPRESENTABLE IF

$$H(-) \cong \mathrm{Hom}_{\mathcal{D}}(U, -)$$

FOR SOME $U \in \mathcal{D}$

↑

UNIVERSAL OBJECT THAT REPRESENTS H

FORM I

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "2"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Gadget} \arrow[r, bend left, "\mathrm{Univ}"] \arrow[r, phantom, "\perp"] & \mathrm{Structure} \arrow[l, bend left, "(-)_{\mathrm{Gadget}}"] \end{tikzcd}$$

Underlying Gadget

$A = \Bbbk$-ALG.

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = A \otimes_{\Bbbk} -"] \arrow[r, phantom, "\perp"] & A\text{-}\mathrm{Mod} \arrow[l, bend left] \end{tikzcd}$$

FORGET A-ACTION

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = T(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left] \end{tikzcd}$$

TENSOR ALGEBRA

FORGET MULTIPLICATION

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "\mathrm{Free} = F(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Group} \arrow[l, bend left] \end{tikzcd}$$

FREE GROUP

FORGET OPERATION

$$\begin{tikzcd} \mathrm{Group} \arrow[r, bend left, "\Bbbk(-) = \text{GROUP ALG.}"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left, "(-)^{\times}"] \end{tikzcd}$$

$(-)^{\times}$ = GROUP OF UNITS

II. UNIVERSALITY REVISITED

$H: \mathcal{D} \longrightarrow \mathsf{Set}$

COVARIANT

(RESP., CONTRAVARIANT)

IS **REPRESENTABLE** IF

$$H(-) \cong \mathrm{Hom}_{\mathcal{D}}(u, -)$$

(RESP., $H(-) \cong \mathrm{Hom}_{\mathcal{D}}(-, u)$)

FOR SOME $u \in \mathcal{D}$

$\uparrow$

UNIVERSAL OBJECT THAT REPRESENTS H

FORM I

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "2"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Gadget} \arrow[r, bend left, "\mathrm{Univ}"] \arrow[r, phantom, "\perp"] & \mathrm{Structure} \arrow[l, bend left, "(-)_{\mathrm{Gadget}}"]
\end{tikzcd}$$

Underlying Gadget

$A = \Bbbk\text{-ALG.}$

$$\begin{tikzcd}
\mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = A \otimes_{\Bbbk} -"] \arrow[r, phantom, "\perp"] & A\text{-}\mathrm{Mod} \arrow[l, bend left]
\end{tikzcd}$$

FORGET A-ACTION

$$\begin{tikzcd}
\mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = T(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left]
\end{tikzcd}$$

TENSOR ALGEBRA

FORGET MULTIPLICATION

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, bend left, "\mathrm{Free} = F(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Group} \arrow[l, bend left]
\end{tikzcd}$$

FREE GROUP

FORGET OPERATION

$$\begin{tikzcd}
\mathrm{Group} \arrow[r, bend left, "\Bbbk(-) = \text{GROUP ALG.}"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left, "(-)^{\times}"]
\end{tikzcd}$$

$(-)^{\times}$ ≈ GROUP OF UNITS

II. UNIVERSALITY REVISITED

$H: \mathcal{D} \longrightarrow \mathrm{Set}$

COVARIANT

(RESP., CONTRAVARIANT)

IS REPRESENTABLE IF

$H(-) \cong \mathrm{Hom}_{\mathcal{D}}(u, -)$

(RESP.,

$H(-) \cong \mathrm{Hom}_{\mathcal{D}}(-, u)$)

FOR SOME $u \in \mathcal{D}$

$\uparrow$

UNIVERSAL OBJECT THAT REPRESENTS H

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

FORM I

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "2"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Gadget} \arrow[r, bend left, "\mathrm{Univ}"] \arrow[r, phantom, "\perp"] & \mathrm{Structure} \arrow[l, bend left, "(-)_{\mathrm{Gadget}}"] \end{tikzcd}$$

Underlying Gadget

$A = \Bbbk$-ALG.

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = A \otimes_{\Bbbk} -"] \arrow[r, phantom, "\perp"] & A\text{-}\mathrm{Mod} \arrow[l, bend left] \end{tikzcd}$$

FORGET A-ACTION

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = T(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left] \end{tikzcd}$$

TENSOR ALGEBRA

FORGET MULTIPLICATION

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "\mathrm{Free} = F(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Group} \arrow[l, bend left] \end{tikzcd}$$

FREE GROUP

FORGET OPERATION

$$\begin{tikzcd} \mathrm{Group} \arrow[r, bend left, "\Bbbk(-) = \text{GROUP ALG.}"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left, "(-)^{\times}"] \end{tikzcd}$$

$\cong$ GROUP OF UNITS

II. UNIVERSALITY REVISITED

$H: \mathcal{D} \longrightarrow \mathsf{Set}$

COVARIANT

(RESP., CONTRAVARIANT)

IS REPRESENTABLE IF

$H(-) \cong \mathrm{Hom}_{\mathcal{D}}(U, -)$

(RESP., $H(-) \cong \mathrm{Hom}_{\mathcal{D}}(-, U)$)

FOR SOME $U \in \mathcal{D}$

↑

UNIVERSAL OBJECT THAT REPRESENTS H

$$\mathrm{Hom}_{\mathcal{D}}(F(X), -) \;\cong\; \mathrm{Hom}_{\mathcal{C}}(X, G(-))$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "2"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Gadget} \arrow[r, bend left, "\text{Univ}"] \arrow[r, phantom, "\perp"] & \text{Structure} \arrow[l, bend left, "(-)_{\text{Gadget}}"]
\end{tikzcd}$$

Underlying Gadget

$A = \Bbbk$-ALG.

$$\begin{tikzcd}
\text{Vec} \arrow[r, bend left, "\text{Free} = A \otimes_{\Bbbk} -"] \arrow[r, phantom, "\perp"] & A\text{-Mod} \arrow[l, bend left]
\end{tikzcd}$$

FORGET A-ACTION

$$\begin{tikzcd}
\text{Vec} \arrow[r, bend left, "\text{Free} = T(-)"] \arrow[r, phantom, "\perp"] & \text{Alg} \arrow[l, bend left]
\end{tikzcd}$$

TENSOR ALGEBRA

FORGET MULTIPLICATION

$$\begin{tikzcd}
\text{Set} \arrow[r, bend left, "\text{Free} = F(-)"] \arrow[r, phantom, "\perp"] & \text{Group} \arrow[l, bend left]
\end{tikzcd}$$

FREE GROUP

FORGET OPERATION

$$\begin{tikzcd}
\text{Group} \arrow[r, bend left, "\Bbbk(-)"] \arrow[r, phantom, "\perp"] & \text{Alg} \arrow[l, bend left, "(-)^{\times}"]
\end{tikzcd}$$

$\Bbbk(-)$ = GROUP ALG.

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II. UNIVERSALITY REVISITED

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COVARIANT

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IS REPRESENTABLE IF

$$H(-) \cong \mathrm{Hom}_{\mathcal{D}}(U, -)$$

(RESP.,

$$H(-) \cong \mathrm{Hom}_{\mathcal{D}}(-, U))$$

FOR SOME $U \in \mathcal{D}$

↑

UNIVERSAL OBJECT THAT REPRESENTS H

$$\mathrm{Hom}_{\mathcal{D}}(F(X), -)$$

$$\wr\wr$$

$$\mathrm{Hom}_{\mathcal{C}}(X, G(-)) : \mathcal{D} \to \mathsf{Set}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

$=: H_X$ REPRESENTED BY $F(X) \in \mathcal{D}$

FORM I

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "2"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

$$\begin{tikzcd} \text{Gadget} \arrow[r, bend left, "\mathrm{Univ}"] \arrow[r, phantom, "\perp"] & \text{Structure} \arrow[l, bend left, "(-)_{\text{Gadget}}"] \end{tikzcd}$$

Underlying Gadget

$A = \Bbbk$-ALG.

$$\begin{tikzcd} \mathsf{Vec} \arrow[r, bend left, "\text{Free} = A \otimes_{\Bbbk} -"] \arrow[r, phantom, "\perp"] & A\text{-}\mathsf{Mod} \arrow[l, bend left] \end{tikzcd}$$

FORGET A-ACTION

$$\begin{tikzcd} \mathsf{Vec} \arrow[r, bend left, "\text{Free} = T(-)"] \arrow[r, phantom, "\perp"] & \mathsf{Alg} \arrow[l, bend left] \end{tikzcd}$$

TENSOR ALGEBRA

FORGET MULTIPLICATION

$$\begin{tikzcd} \mathsf{Set} \arrow[r, bend left, "\text{Free} = F(-)"] \arrow[r, phantom, "\perp"] & \mathsf{Group} \arrow[l, bend left] \end{tikzcd}$$

FREE GROUP

FORGET OPERATION

$$\begin{tikzcd} \mathsf{Group} \arrow[r, bend left, "\Bbbk(-)"] \arrow[r, phantom, "\perp"] & \mathsf{Alg} \arrow[l, bend left, "(-)^{\times}"] \end{tikzcd}$$

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COVARIANT

(RESP., CONTRAVARIANT)

IS **REPRESENTABLE** IF

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(RESP.,

$$H(-) \cong \mathrm{Hom}_{\mathcal{D}}(-, U)\text{)}$$

FOR SOME $U \in \mathcal{D}$

$\uparrow$

UNIVERSAL OBJECT THAT REPRESENTS H

FORM I

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

2

$$\begin{tikzcd}
\mathrm{Gadget} \arrow[r, bend left, "\mathrm{Univ}"] \arrow[r, phantom, "\perp"] & \mathrm{Structure} \arrow[l, bend left, "(-)_{\mathrm{Gadget}}"]
\end{tikzcd}$$

Underlying Gadget

$A = \Bbbk$-ALG.

$$\mathrm{Free} = A \otimes_{\Bbbk} -$$

$$\begin{tikzcd}
\mathrm{Vec} \arrow[r, bend left, "\mathrm{Free}"] \arrow[r, phantom, "\perp"] & A\text{-}\mathrm{Mod} \arrow[l, bend left, "\text{FORGET A-ACTION}"]
\end{tikzcd}$$

$$\mathrm{Hom}_{\mathcal{D}}(F(X), -)$$

$$\cong$$

$$\mathrm{Hom}_{\mathcal{C}}(X, G(-)) : \mathcal{D} \to \mathsf{Set}$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}$$

H_X REPRESENTED BY $F(X) \in \mathcal{D}$

$\Bbbk(-) =$ GROUP ALG.

$$\begin{tikzcd}
\mathrm{Group} \arrow[r, bend left, "\Bbbk(-)"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left, "(-)^{\times}"]
\end{tikzcd}$$

$(-)^{\times} =$ GROUP OF UNITS

II. UNIVERSALITY REVISITED

$H: \mathcal{D} \longrightarrow \mathrm{Set}$

COVARIANT

(RESP., CONTRAVARIANT)

IS REPRESENTABLE IF

$H(-) \cong \mathrm{Hom}_{\mathcal{D}}(U, -)$

(RESP., $H(-) \cong \mathrm{Hom}_{\mathcal{D}}(-, U)$)

FOR SOME $U \in \mathcal{D}$

↑

UNIVERSAL OBJECT THAT REPRESENTS H

$\mathrm{Hom}_{\mathcal{D}}(F(X), -)$

$\wr\wr$

$\mathrm{Hom}_{\mathcal{C}}(X, G(-)) : \mathcal{D} \to \mathrm{Set}$

$\therefore\!/\!/$

H_X REPRESENTED BY $F(X) \in \mathcal{D}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "2"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Gadget} \arrow[r, bend left, "\mathrm{Univ}"] \arrow[r, phantom, "\perp"] & \mathrm{Structure} \arrow[l, bend left, "(-)_{\mathrm{Gadget}}"]
\end{tikzcd}$$

Underlying Gadget

$A = \Bbbk$-ALG.

$$\begin{tikzcd}
\mathrm{Vec} \arrow[r, bend left, "\mathrm{Free} = A \otimes_{\Bbbk} -"] \arrow[r, phantom, "\perp"] & A\text{-Mod} \arrow[l, bend left, "\text{FORGET } A\text{-ACTION}"]
\end{tikzcd}$$

$\wr$

$\mathrm{Hom}_{A\text{-Mod}}(A \otimes_{\Bbbk} V, {}_A W)$

$\wr\wr$

$\mathrm{Hom}_{\mathrm{Vec}}(V, W)$

$$\begin{tikzcd}
\mathrm{Group} \arrow[r, bend left, "\Bbbk(-) = \text{GROUP ALG.}"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left, "(-)^{\times} = \text{GROUP OF UNITS}"]
\end{tikzcd}$$

II. UNIVERSALITY REVISITED

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(RESP., CONTRAVARIANT)

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FOR SOME $U \in \mathcal{D}$

↑

UNIVERSAL OBJECT THAT REPRESENTS H

$$\mathrm{Hom}_{\mathcal{D}}(F(X), -) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(-)) : \mathcal{D} \to \mathsf{Set}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

$=: H_X$ REPRESENTED BY $F(X) \in \mathcal{D}$

FORM I

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "2"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

$$\begin{tikzcd} \text{Gadget} \arrow[r, bend left, "\text{Univ}"] \arrow[r, phantom, "\perp"] & \text{Structure} \arrow[l, bend left, "(-)_{\text{Gadget}}"] \end{tikzcd}$$

Underlying Gadget

$A = \Bbbk\text{-ALG.}$

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "\text{Free} = A \otimes_{\Bbbk} -"] \arrow[r, phantom, "\perp"] & A\text{-}\mathrm{Mod} \arrow[l, bend left, "\text{FORGET A-ACTION}"] \end{tikzcd}$$

$$\Downarrow$$

$$\mathrm{Hom}_{A\text{-}\mathrm{Mod}}(A \otimes_{\Bbbk} V, {}_A W) \cong \mathrm{Hom}_{\mathrm{Vec}}(V, W)$$

$H_V := \mathrm{Hom}_{\mathrm{Vec}}(V, \mathrm{Forg}(-))$ IS REPRESENTED BY $A \otimes_{\Bbbk} V$

$$\begin{tikzcd} \mathrm{Group} \arrow[r, bend left, "\Bbbk(-) = \text{GROUP ALG.}"] \arrow[r, phantom, "\perp"] & \mathrm{Alg} \arrow[l, bend left, "(-)^{\times} = \text{GROUP OF UNITS}"] \end{tikzcd}$$

II. UNIVERSALITY REVISITED

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FOR SOME $U \in \mathcal{D}$

↑

UNIVERSAL OBJECT THAT REPRESENTS H

FORM I

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

$$\begin{tikzcd}
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Underlying Gadget

$A = \Bbbk\text{-ALG.}$

$$\begin{tikzcd}
\mathsf{Vec} \arrow[r, bend left, "\mathrm{Free} \,=\, A \otimes_{\Bbbk} -"] \arrow[r, phantom, "\perp"] & A\text{-}\mathsf{Mod} \arrow[l, bend left, "\text{FORGET } A\text{-ACTION}"]
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$$\mathrm{Hom}_{A\text{-Mod}}(A \otimes_{\Bbbk} V, {}_A W) \cong \mathrm{Hom}_{\mathsf{Vec}}(V, W)$$

$H_V := \mathrm{Hom}_{\mathsf{Vec}}(V, \mathrm{Forg}(-))$ IS REPRESENTED BY $A \otimes_{\Bbbk} V$

$$\mathrm{Hom}_{\mathsf{Alg}}(\Bbbk G, A) \cong \mathrm{Hom}_{\mathsf{Group}}(G, A^{\times})$$

↑

$\Bbbk(-) =$ GROUP ALG.

$$\begin{tikzcd}
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\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
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$=:$ H_X REPRESENTED BY $F(X) \in \mathcal{D}$

II. UNIVERSALITY REVISITED

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(RESP.,

$$H(-) \cong \mathrm{Hom}_{\mathcal{D}}(-, U)\text{)}$$

FOR SOME $U \in \mathcal{D}$

$\uparrow$

UNIVERSAL OBJECT THAT REPRESENTS H

FORM I

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "2"] \\
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Underlying Gadget

$A = \Bbbk\text{-ALG.}$

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\mathrm{Vec} \arrow[r, bend left, "\text{Free}"] \arrow[r, phantom, "\perp"] & A\text{-Mod} \arrow[l, bend left, "\text{FORGET A-ACTION}"]
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$\text{Free} = A \otimes_{\Bbbk} -$

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$$\mathrm{Hom}_{A\text{-Mod}}(A \otimes_{\Bbbk} V, {}_A W) \cong \mathrm{Hom}_{\mathrm{Vec}}(V, W)$$

$H_V := \mathrm{Hom}_{\mathrm{Vec}}(V, \mathrm{Forg}(-))$

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$$\mathrm{Hom}_{\mathrm{Alg}}(\Bbbk G, A) \cong \mathrm{Hom}_{\mathrm{Group}}(G, A^{\times})$$

$H_G := \mathrm{Hom}_{\mathrm{Group}}(G, (-)^{\times})$

IS REPRESENTED BY $\Bbbk G$

$\uparrow$

$\Bbbk(-) = \text{GROUP ALG.}$

$$\begin{tikzcd}
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$H_X =:$ REPRESENTED BY $F(X) \in \mathcal{D}$

II. UNIVERSALITY REVISITED

$H : \mathcal{D} \longrightarrow \mathsf{Set}$

COVARIANT

(RESP., CONTRAVARIANT)

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$\uparrow$

UNIVERSAL OBJECT THAT REPRESENTS H

FORM I

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
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$$\mathrm{Hom}_{A\text{-}\mathrm{Mod}}(A \otimes_{\Bbbk} V, {}_A W) \cong \mathrm{Hom}_{\mathrm{Vec}}(V, W)$$

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\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}$$

$H_X =:$ REPRESENTED BY $F(X) \in \mathcal{D}$

III. YONEDA'S LEMMA

SOME CALL THIS THE IMPORTANT RESULT
IN CATEGORY THEORY

$$H: \mathcal{D} \longrightarrow \mathsf{Set}$$

COVARIANT

(RESP., CONTRAVARIANT)

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$H(-) \cong \mathrm{Hom}_{\mathcal{D}}(-, U)$)

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UNIVERSAL OBJECT
THAT REPRESENTS H

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↑

UNIVERSAL OBJECT
THAT REPRESENTS H

SOME CALL THIS THE IMPORTANT RESULT
IN CATEGORY THEORY

YONEDA'S LEMMA:

TAKE A FUNCTOR $F: \mathcal{D} \longrightarrow \mathsf{Set}$

THEN $\forall U \in \mathcal{D}$, $\exists$ BIJECTION :

$$\Phi_{F,U} : \mathrm{Nat}_{\mathcal{D},\mathsf{Set}}(\mathrm{Hom}_{\mathcal{D}}(U,-), F) \xrightarrow{\sim} F(U).$$

III. YONEDA'S LEMMA

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$H: \mathcal{D} \longrightarrow \mathsf{Set}$

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UNIVERSAL OBJECT THAT REPRESENTS H

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PF/ DEFINE

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend left=50, "\mathrm{Hom}_{\mathcal{D}}(U,-)"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, "\Downarrow \phi"] & \mathsf{Set}
\end{tikzcd} \quad \xmapsto{\Phi_{F,U}} \qquad \in F(U).$$

III. YONEDA'S LEMMA

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UNIVERSAL OBJECT THAT REPRESENTS H

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PF/ DEFINE

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathcal{D}}(U,-)"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathrm{Set} \arrow[r, mapsto, "\Phi_{F,U}"] & \phi_U(\mathrm{id}_U) \in F(U). \end{tikzcd}$$

$$[\phi_U : \mathrm{Hom}_{\mathcal{D}}(U,U) \longrightarrow F(U)]$$

III. YONEDA'S LEMMA

SOME CALL THIS THE IMPORTANT RESULT IN CATEGORY THEORY

$H: \mathcal{D} \longrightarrow \mathsf{Set}$

COVARIANT

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$$\left(\text{RESP., } H(-) \cong \mathrm{Hom}_{\mathcal{D}}(-, U)\right)$$

FOR SOME $\underset{\substack{\uparrow \\ \text{UNIVERSAL OBJECT THAT REPRESENTS } H}}{U} \in \mathcal{D}$

YONEDA'S LEMMA:

TAKE A FUNCTOR $F: \mathcal{D} \longrightarrow \mathsf{Set}$

THEN $\forall U \in \mathcal{D}$, $\exists$ BIJECTION:

$$\Phi_{F,U} : \mathrm{Nat}_{\mathcal{D},\mathsf{Set}}\left(\mathrm{Hom}_{\mathcal{D}}(U,-), F\right) \xrightarrow{\sim} F(U).$$

PF/ DEFINE

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend left=50, "\mathrm{Hom}_{\mathcal{D}}(U,-)"{name=A}] \arrow[r, bend right=50, "F"'{name=B}] & \mathsf{Set} \arrow[r, mapsto, "\Phi_{F,U}"] & \phi_U(\mathrm{id}_U) \in F(U).
\arrow[from=A, to=B, Rightarrow, "\phi"]
\end{tikzcd}$$

$$\left[\phi_U : \mathrm{Hom}_{\mathcal{D}}(U,U) \longrightarrow F(U)\right]$$

DEFINE $\Psi_{F,U} : F(U) \longrightarrow \mathrm{Nat}_{\mathcal{D},\mathsf{Set}}\left(\mathrm{Hom}_{\mathcal{D}}(U,-), F\right)$

$$\begin{tikzcd}
\rotatebox{90}{$\in$} & & \\
x \arrow[r, mapsto] & \mathcal{D} \arrow[r, bend left=50, "\mathrm{Hom}_{\mathcal{D}}(U,-)"{name=C}] \arrow[r, bend right=50, "F"'{name=E}] & \mathsf{Set}
\arrow[from=C, to=E, Rightarrow, "\Psi_{F,U}(x)"]
\end{tikzcd}$$

III. YONEDA'S LEMMA

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↑

UNIVERSAL OBJECT THAT REPRESENTS H

YONEDA'S LEMMA: TAKE A FUNCTOR $F: \mathcal{D} \longrightarrow \mathsf{Set}$

THEN $\forall u \in \mathcal{D}$, $\exists$ BIJECTION:

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PF/ DEFINE

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathcal{D}}(u,-)"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow \phi"] & \mathsf{Set} \arrow[r, mapsto, "\Phi_{F,u}"] & \phi_u(\mathrm{id}_u) \in F(u). \end{tikzcd}$$

$[\phi_u: \mathrm{Hom}_{\mathcal{D}}(u,u) \longrightarrow F(u)]$

DEFINE $\Psi_{F,u}: F(u) \longrightarrow \mathrm{Nat}_{\mathcal{D},\mathsf{Set}}(\mathrm{Hom}_{\mathcal{D}}(u,-), F)$

$$\begin{tikzcd} \rotatebox{90}{$\in$} & & \\ x \arrow[r, mapsto] & \mathcal{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathcal{D}}(u,-)"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow \Psi_{F,u}(x)"] & \mathsf{Set} \end{tikzcd}$$

WHERE

$(\Psi_{F,u}(x))_z: \mathrm{Hom}_{\mathcal{D}}(u,z) \longrightarrow F(z)$

$f \longmapsto F(f)(x)$

III. YONEDA'S LEMMA

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UNIVERSAL OBJECT
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PF/ DEFINE

$$\begin{tikzcd}
\mathcal{D} \arrow[r, bend left=50, "\mathrm{Hom}_{\mathcal{D}}(u,-)", ""{name=U, below}] \arrow[r, bend right=50, "F"', ""{name=L, above}] & \mathsf{Set} \arrow[Rightarrow, from=U, to=L, "\phi"] \arrow[r, mapsto, "\Phi_{F,u}"] & \phi_u(\mathrm{id}_u) \in F(u).
\end{tikzcd}$$

$$\left[\phi_u : \mathrm{Hom}_{\mathcal{D}}(u,u) \longrightarrow F(u)\right]$$

CHECK THESE
ARE MUTUALLY
INVERSE

///

DEFINE $\Psi_{F,u} : F(u) \longrightarrow \mathrm{Nat}_{\mathcal{D},\mathsf{Set}}(\mathrm{Hom}_{\mathcal{D}}(u,-), F)$

$$\begin{tikzcd}
\rotatebox{90}{$\in$} & & \\
x \arrow[r, mapsto] & \mathcal{D} \arrow[r, bend left=50, "\mathrm{Hom}_{\mathcal{D}}(u,-)", ""{name=U, below}] \arrow[r, bend right=50, "F"', ""{name=L, above}] & \mathsf{Set} \arrow[Rightarrow, from=U, to=L, "\Psi_{F,u}(x)"]
\end{tikzcd}$$

WHERE

$$(\Psi_{F,u}(x))_z : \mathrm{Hom}_{\mathcal{D}}(u,z) \longrightarrow F(z)$$
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III. YONEDA'S LEMMA

SOME CALL THIS THE IMPORTANT RESULT IN CATEGORY THEORY

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FOR SOME $u \in \mathcal{D}$

↑

UNIVERSAL OBJECT THAT REPRESENTS H

YONEDA'S LEMMA:

TAKE A FUNCTOR $F: \mathcal{D} \longrightarrow \mathsf{Set}$

THEN $\forall u \in \mathcal{D}$, $\exists$ BIJECTION:

$$\Phi_{F,u} : \mathrm{Nat}_{\mathcal{D},\mathsf{Set}}(\mathrm{Hom}_{\mathcal{D}}(u,-), F) \xrightarrow{\sim} F(u).$$

MAIN CONSEQUENCE:

REPRESENTABLE FUNCTORS ARE DETERMINED BY THEIR UNIV. OBJECTS

III. YONEDA'S LEMMA

$H: \mathcal{D} \longrightarrow \mathsf{Set}$

COVARIANT

(RESP., CONTRAVARIANT)

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$$H(-) \cong \mathrm{Hom}_{\mathcal{D}}(u, -)$$

$$\left(\text{RESP.,}\ H(-) \cong \mathrm{Hom}_{\mathcal{D}}(-, u)\right)$$

FOR SOME $u \in \mathcal{D}$

↑

UNIVERSAL OBJECT THAT REPRESENTS H

SOME CALL THIS THE IMPORTANT RESULT IN CATEGORY THEORY

YONEDA'S LEMMA:

TAKE A FUNCTOR $F: \mathcal{D} \longrightarrow \mathsf{Set}$

THEN $\forall u \in \mathcal{D}$, ∃ BIJECTION:

$$\Phi_{F,u} : \mathrm{Nat}_{\mathcal{D},\mathsf{Set}}(\mathrm{Hom}_{\mathcal{D}}(u,-), F) \xrightarrow{\sim} F(u).$$

MAIN CONSEQUENCE:

REPRESENTABLE FUNCTORS ARE DETERMINED BY THEIR UNIV. OBJECTS

COROLLARY:

$$\begin{array}{ccc} \text{IN SET} & & \text{IN } \mathcal{D} \\ \mathrm{Hom}_{\mathcal{D}}(u,-) \cong \mathrm{Hom}_{\mathcal{D}}(u',-) & \Leftrightarrow & u \cong u'. \\ \mathrm{Hom}_{\mathcal{D}}(-,u) \cong \mathrm{Hom}_{\mathcal{D}}(-,u') & \Leftrightarrow & u \cong u'. \end{array}$$

III. YONEDA'S LEMMA

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FOR SOME $u \in \mathcal{D}$

↑

UNIVERSAL OBJECT THAT REPRESENTS H

PROOF ≡ EXERCISE 2.48

SOME CALL THIS THE IMPORTANT RESULT IN CATEGORY THEORY

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TAKE A FUNCTOR $F: \mathcal{D} \longrightarrow \mathsf{Set}$

THEN $\forall u \in \mathcal{D}$, $\exists$ BIJECTION :

$$\Phi_{F,u} : \mathrm{Nat}_{\mathcal{D},\mathsf{Set}}(\mathrm{Hom}_{\mathcal{D}}(u,-), F) \xrightarrow{\sim} F(u).$$

MAIN CONSEQUENCE :

REPRESENTABLE FUNCTORS ARE DETERMINED BY THEIR UNIV. OBJECTS

COROLLARY:

IN SET		IN $\mathcal{D}$
$\mathrm{Hom}_{\mathcal{D}}(u,-) \cong \mathrm{Hom}_{\mathcal{D}}(u',-)$	$\Leftrightarrow$	$u \cong u'$.
$\mathrm{Hom}_{\mathcal{D}}(-,u) \cong \mathrm{Hom}_{\mathcal{D}}(-,u')$	$\Leftrightarrow$	$u \cong u'$.

MATH 466/566
SPRING 2024

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RICE U.

LECTURE #10

TOPICS:

- ✓ I. ADJUNCTION (§2.5)
- ✓ II. UNIVERSALITY REVISITED (§2.6.1)
- ✓ III. YONEDA'S LEMMA (§2.6.2)

NEXT TIME: BITS OF HOMOLOGICAL ALGEBRA

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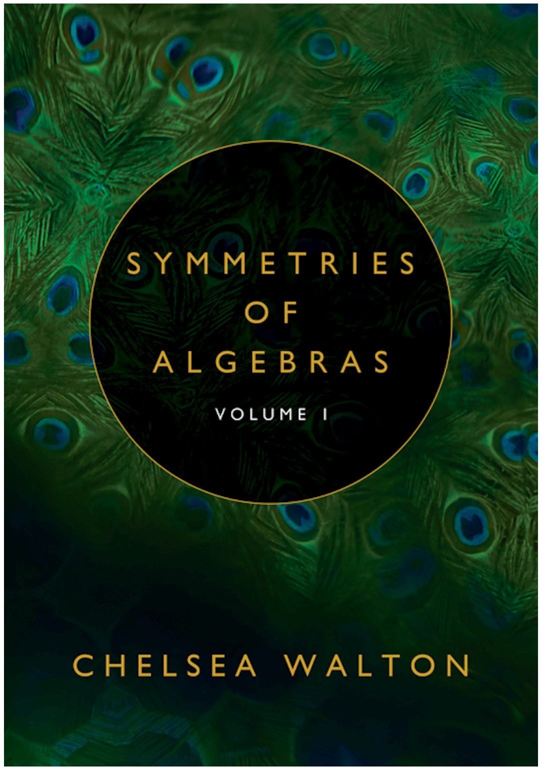
You’ll enjoy the textbook!

C. Walton’s “Symmetries of Algebras, Volume 1” (2024)



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Lecture #10 keywords: adjoint functors, adjunction, Free-Forgetful adjunction, representable functor, Tensor-Hom adjunction, universal object, Yoneda’s Lemma