MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #11

LAST TIME

- ADJUNCTION
- UNIVERSALITY REVISITED
- YONEDA'S LEMMA

TOPICS:

I. BUILDING BLOCK OBJECTS (§2.7)

II. EXACTNESS (§§2.8.1–2.8.2)

III. PROJECTIVITY & INJECTIVITY (§2.8.3)

IV. FINITENESS FOR LINEAR CATEGORIES (§2.9)

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- ADJUNCTION
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TOPICS: WRAPPING UP CATEGORY THEORY

INC. SNIPPET OF HOMOLOGICAL ALGEBRA

I. BUILDING BLOCK OBJECTS (§2.7)

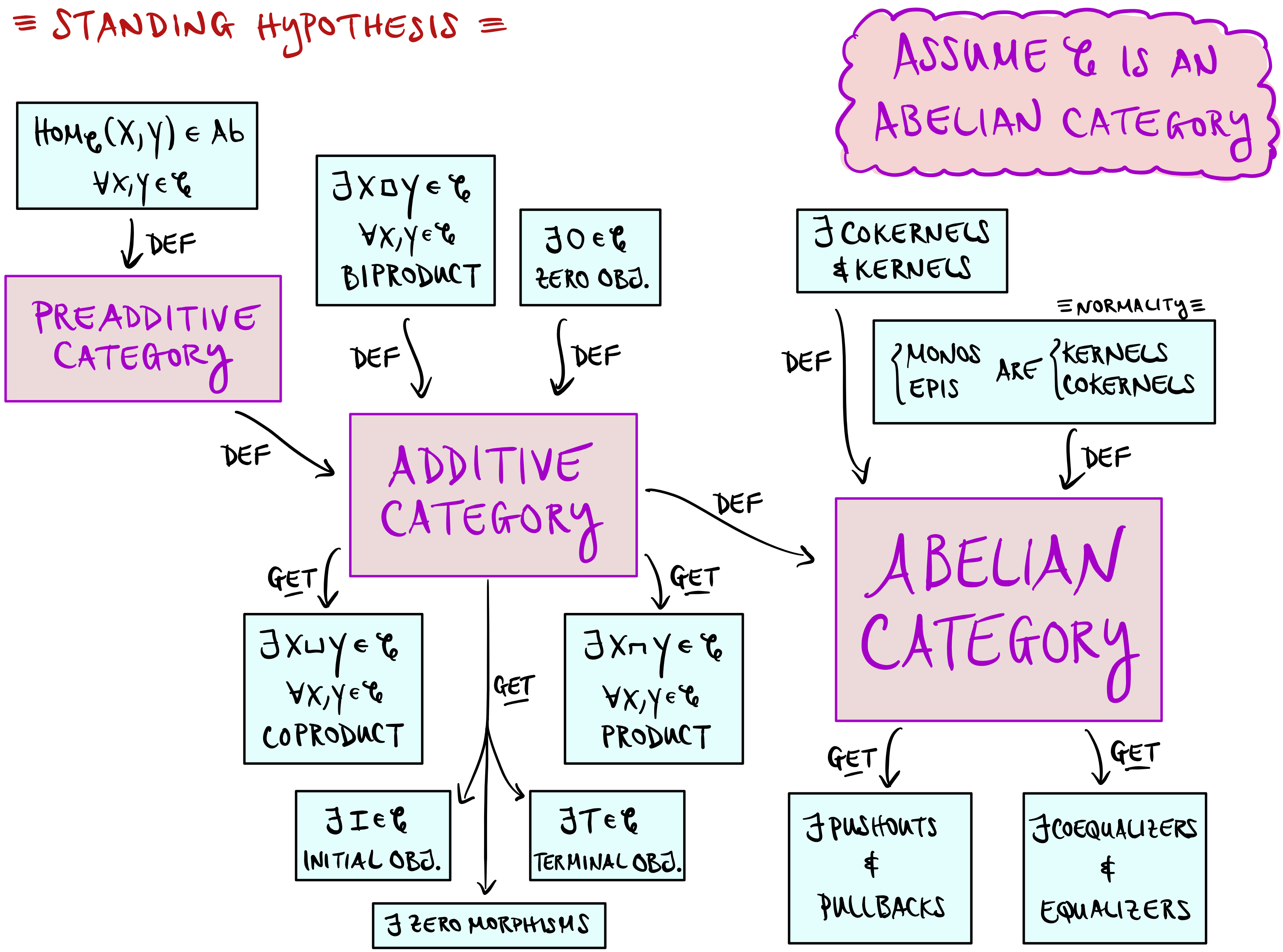
II. EXACTNESS (§§2.8.1–2.8.2)

III. PROJECTIVITY & INJECTIVITY (§2.8.3)

IV. FINITENESS FOR LINEAR CATEGORIES (§2.9)

≡ STANDING HYPOTHESIS ≡

ASSUME $\mathscr{C}$ IS AN ABELIAN CATEGORY


≡ STANDING HYPOTHESIS ≡
ASSUME $\mathscr{C}$ IS AN ABELIAN CATEGORY
$\mathrm{Hom}_{\mathscr{C}}(X,Y) \in \mathrm{Ab}$ $\forall X,Y \in \mathscr{C}$
DEF
PREADDITIVE CATEGORY
$\exists X \oplus Y \in \mathscr{C}$ $\forall X,Y \in \mathscr{C}$ BIPRODUCT
$\exists 0 \in \mathscr{C}$ ZERO OBJ.
DEF
DEF
DEF
ADDITIVE CATEGORY
$\exists$ COKERNELS & KERNELS
≡ NORMALITY ≡
MONOS, EPIS ARE KERNELS, COKERNELS
DEF
DEF
DEF
ABELIAN CATEGORY
GET
GET
GET
$\exists X \sqcup Y \in \mathscr{C}$ $\forall X,Y \in \mathscr{C}$ COPRODUCT
$\exists X \sqcap Y \in \mathscr{C}$ $\forall X,Y \in \mathscr{C}$ PRODUCT
$\exists I \in \mathscr{C}$ INITIAL OBJ.
$\exists T \in \mathscr{C}$ TERMINAL OBJ.
$\exists$ ZERO MORPHISMS
GET
GET
$\exists$ PUSHOUTS & PULLBACKS
$\exists$ COEQUALIZERS & EQUALIZERS

I. BUILDING BLOCK OBJECTS

$\mathcal{C} \equiv$ ABELIAN CATEG.

SIMPLE OBJECTS

FINITE LENGTH OBJECTS

INDECOMPOSABLE OBJECTS

SEMISIMPLE OBJECTS

SEMISIMPLE CATEGORIES

$\mathcal{C}$ ≡ ABELIAN CATEG.

INDECOMPOSABLE OBJECTS

I. BUILDING BLOCK OBJECTS

$\mathcal{C} \equiv$ ABELIAN CATEG.

INDECOMPOSABLE OBJECTS

$X^{\neq 0} \in \mathcal{C}$ IS INDECOMPOSABLE

IF $X \not\cong X_1 \sqcup X_2$

$\forall$ NONZERO SUBOBJ. X_1, X_2 OF X

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EXERCISE 2.49

$X^{\neq 0} \in \mathcal{C}$ IS INDECOMPOSABLE

$\Updownarrow$

THE ONLY IDEMPOTENT MORPHISMS IN $\mathrm{Hom}_{\mathcal{C}}(X,X)$ ARE $\vec{0}_{X,X}$ & id_X

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IN GENERAL $\not\Uparrow$ $\Downarrow$

INDECOMPOSABLE OBJECTS

$X \overset{\neq 0}{\in} \mathcal{C}$ IS INDECOMPOSABLE IF $X \not\cong X_1 \sqcup X_2$ $\forall$ NONZERO SUBOBJ. X_1, X_2 OF X

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IN GENERAL $\not\Leftarrow$

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$X^{\neq 0} \in \mathcal{C}$ IS INDECOMPOSABLE IF $X \not\cong X_1 \sqcup X_2$ $\forall$ NONZERO SUBOBJ. X_1, X_2 OF X

SCHUR'S LEMMA

IF $X, Y \in \mathcal{C}$ ARE SIMPLE, THEN $f : X \to Y \in \mathcal{C}$ IS AN ISO OR $\vec{0}$.

EXERCISE 2.49

$X^{\neq 0} \in \mathcal{C}$ IS INDECOMPOSABLE

$\Updownarrow$

THE ONLY IDEMPOTENT MORPHISMS IN $\mathrm{Hom}_{\mathcal{C}}(X,X)$ ARE $\vec{0}_{X,X}$ & id_X

I. BUILDING BLOCK OBJECTS

$\mathcal{C} \equiv$ ABELIAN CATEG.

SIMPLE OBJECTS	OBJECTS THAT ARE MEASURABLY CLOSE TO BEING SIMPLE $\longrightarrow$	FINITE LENGTH OBJECTS

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SIMPLE OBJECTS

OBJECTS THAT ARE MEASURABLY CLOSE TO BEING SIMPLE $\longrightarrow$

FINITE LENGTH OBJECTS

$X^{\neq 0} \in \mathcal{C}$ IS SIMPLE

IF THE ONLY SUBOBJS OF X

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A COMPOSITION SERIES FOR $X \in \mathcal{C}$

IS A SEQUENCE OF MONOS

$$0 = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \cdots \longrightarrow X$$

$\ni$. $\operatorname{coker}(f_i) = X_{i+1}/X_i$ IS SIMPLE $\forall i$

I. BUILDING BLOCK OBJECTS

$\mathcal{C} \equiv$ ABELIAN CATEG.

SIMPLE OBJECTS $\longrightarrow$ OBJECTS THAT ARE MEASURABLY CLOSE TO BEING SIMPLE $\longrightarrow$ **FINITE LENGTH OBJECTS**

$X^{\neq 0} \in \mathcal{C}$ IS SIMPLE IF THE ONLY SUBOBJS OF X ARE X & 0

$X \in \mathcal{C}$ HAS LENGTH n IF IT ADMITS A COMP. SERIES WITH $X = X_n$, BUT NOT WITH $X = X_d$ FOR $d < n$

A COMPOSITION SERIES FOR $X \in \mathcal{C}$ IS A SEQUENCE OF MONOS

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$X^{\neq 0} \in \mathcal{C}$ IS **SIMPLE** IF THE ONLY SUBOBJS OF X ARE X & 0

$X \in \mathcal{C}$ HAS **LENGTH n** IF IT ADMITS A COMP. SERIES WITH $X = X_n$, BUT NOT WITH $X = X_d$ FOR $d < n$

JORDAN–HÖLDER THEOREM

ANY TWO COMP. SERIES OF A FINITE LENGTH OBJ. HAVE THE SAME # OF COMPONENTS & SETS OF COKERNELS, UP TO PERMUTATION.

A **COMPOSITION SERIES** FOR $X \in \mathcal{C}$ IS A SEQUENCE OF MONOS

$$0 = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \cdots \longrightarrow X$$

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SIMPLE OBJECTS

OBJECTS THAT ARE MEASURABLY CLOSE TO BEING SIMPLE

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LENGTH IS WELL-DEFINED

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$\mathcal{C}$ ≡ ABELIAN CATEG.

SIMPLE OBJECTS —OBJECTS THAT ARE MEASURABLY CLOSE TO BEING SIMPLE⟶ **FINITE LENGTH OBJECTS**

$X \overset{\neq 0}{\in} \mathcal{C}$ IS SIMPLE IF THE ONLY SUBOBJS OF X ARE X & 0

OBJECTS OF LENGTH 1

$X \in \mathcal{C}$ HAS LENGTH n IF IT ADMITS A COMP. SERIES WITH $X = X_n$, BUT NOT WITH $X = X_d$ FOR $d < n$

LENGTH IS WELL-DEFINED

JORDAN–HÖLDER THEOREM

ANY TWO COMP. SERIES OF A FINITE LENGTH OBJ. HAVE THE SAME # OF COMPONENTS & SETS OF COKERNELS, UP TO PERMUTATION.

A COMPOSITION SERIES FOR $X \in \mathcal{C}$ IS A SEQUENCE OF MONOS

$$0 = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \cdots \longrightarrow X$$

s.t. $\operatorname{coker}(f_i) = X_{i+1}/X_i$ IS SIMPLE $\forall i$

I. BUILDING BLOCK OBJECTS

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SIMPLE OBJECTS

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FINITE LENGTH OBJECTS

$X \in \mathcal{C}$ ADMITS A COMP. SERIES
OF MINIMUM FINITE LENGTH

INDECOMPOSABLE OBJECTS

$X^{\neq 0} \in \mathcal{C}$ IS INDECOMPOSABLE
IF $X \not\cong X_1 \sqcup X_2$
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I. BUILDING BLOCK OBJECTS

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SIMPLE OBJECTS

IN GENERAL

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I. BUILDING BLOCK OBJECTS

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SIMPLE OBJECTS

$X^{\neq 0} \in \mathcal{C}$ IS SIMPLE IF THE ONLY SUBOBJS OF X ARE X & 0

IN GENERAL $\not\Rightarrow$

INDECOMPOSABLE OBJECTS

$X^{\neq 0} \in \mathcal{C}$ IS INDECOMPOSABLE IF $X \not\cong X_1 \sqcup X_2$ $\forall$ NONZERO SUBOBJ. X_1, X_2 OF X

CAN GET WHEN INVOLVING

FINITE LENGTH OBJECTS

$X \in \mathcal{C}$ ADMITS A COMP. SERIES OF MINIMUM FINITE LENGTH

SEMISIMPLE OBJECTS

SEMISIMPLE CATEGORIES

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$X^{\neq 0} \in \mathcal{C}$ IS SEMISIMPLE
IF $X \cong \coprod_{i \in I} X_i$
FOR SIMPLE OBJECTS X_i.

SEMISIMPLE OBJECTS

SEMISIMPLE CATEGORIES

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$\mathcal{C}$ IS SEMISIMPLE IF
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SEMISIMPLE OBJECTS

SEMISIMPLE CATEGORIES

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SIMPLE OBJECTS

$X^{\neq 0} \in \mathcal{C}$ IS SIMPLE IF THE ONLY SUBOBJS OF X ARE X & 0

EXAMPLES $A \equiv \Bbbk$-ALGEBRA ($\Bbbk = \overline{\Bbbk}$, CHAR 0)

A-Mod IS A SEMISIMPLE CATEGORY $\iff$ A IS A SEMISIMPLE ALGEBRA

$X^{\neq 0} \in \mathcal{C}$ IS SEMISIMPLE IF $X \cong \coprod_{i \in I} X_i$ FOR SIMPLE OBJECTS X_i.

$\mathcal{C}$ IS SEMISIMPLE IF ALL OBJECTS ARE SEMISIMPLE

SEMISIMPLE OBJECTS

SEMISIMPLE CATEGORIES

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EXAMPLES $A \equiv \Bbbk$-ALGEBRA ($\Bbbk = \bar{\Bbbk}$, CHAR 0)

A-Mod IS A SEMISIMPLE CATEGORY
$\iff$ A IS A SEMISIMPLE ALGEBRA

E.g. $\mathrm{Vec} \cong \Bbbk\text{-Mod}$

$A\text{-Bimod} \cong (A \otimes A^{op})\text{-Mod}$ FOR A SEPARABLE

$G\text{-Mod} \cong \Bbbk G\text{-Mod}$ WHEN $|G| < \infty$

$X^{\neq 0} \in \mathcal{C}$ IS SEMISIMPLE IF $X \cong \coprod_{i \in I} X_i$ FOR SIMPLE OBJECTS X_i.

$\mathcal{C}$ IS SEMISIMPLE IF ALL OBJECTS ARE SEMISIMPLE

SEMISIMPLE OBJECTS

SEMISIMPLE CATEGORIES

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SEMISIMPLE OBJECTS

SEMISIMPLE CATEGORIES

EXAMPLES $\quad A \equiv \Bbbk$-ALGEBRA $\quad (\Bbbk = \bar{\Bbbk}$, CHAR 0)

A-Mod IS A SEMISIMPLE CATEGORY
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E.g. $\quad$ Vec $\cong \Bbbk$-Mod
A-Bimod $\cong (A \otimes A^{op})$-Mod FOR A SEPARABLE
G-Mod $\cong \Bbbk G$-Mod WHEN $|G| < \infty$

Ab ABELIAN, NOT SS. WILL SEE LATER

I. BUILDING BLOCK OBJECTS

$\mathcal{C} \equiv$ ABELIAN CATEG.

NEED NOT EXIST FOR $|I| = \infty$

SIMPLE OBJECTS

$X \overset{\neq 0}{\in} \mathcal{C}$ is SIMPLE if the only subobjs of X are X & 0

$X \overset{\neq 0}{\in} \mathcal{C}$ is SEMISIMPLE if $X \cong \coprod_{i \in I} X_i$ for simple objects X_i.

$\mathcal{C}$ is SEMISIMPLE if all objects are semisimple

SEMISIMPLE OBJECTS

EXAMPLES $\quad A \equiv \Bbbk$-ALGEBRA $\quad (\Bbbk = \bar{\Bbbk}$, char $0)$

A-Mod is a semisimple category
$\iff$ A is a semisimple algebra

E.g. $\quad \mathrm{Vec} \cong \Bbbk\text{-Mod}$

$A\text{-Bimod} \cong (A \otimes A^{op})\text{-Mod}$ for A separable

$G\text{-Mod} \cong \Bbbk G\text{-Mod}$ when $|G| < \infty$

Ab abelian, NOT SS. Will see later

SEMISIMPLE CATEGORIES

I. BUILDING BLOCK OBJECTS

$\mathcal{C} \equiv$ ABELIAN CATEG.

HELPFUL TO IMPOSE FINITE LENGTH TO GET RESULTS...

SIMPLE OBJECTS

$X^{\neq 0} \in \mathcal{C}$ IS SIMPLE

IF THE ONLY SUBOBJS OF X ARE X & 0

$X^{\neq 0} \in \mathcal{C}$ IS SEMISIMPLE

IF $X \cong \coprod_{i \in I} X_i$

NEED NOT EXIST FOR $|I| = \infty$

FOR SIMPLE OBJECTS X_i.

$\mathcal{C}$ IS SEMISIMPLE IF ALL OBJECTS ARE SEMISIMPLE

SEMISIMPLE OBJECTS

EXAMPLES $\quad A \equiv \Bbbk$-ALGEBRA $\quad (\Bbbk = \overline{\Bbbk}$, CHAR 0)

A-Mod IS A SEMISIMPLE CATEGORY

$\iff$ A IS A SEMISIMPLE ALGEBRA

E.g. $\quad$ Vec $\cong \Bbbk$-Mod

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SEMISIMPLE CATEGORIES

Ab ABELIAN, NOT SS. WILL SEE LATER

I. BUILDING BLOCK OBJECTS

$\mathcal{C} \equiv$ ABELIAN CATEG.

SIMPLE OBJECTS

FINITE LENGTH OBJECTS

IN GENERAL

CAN GET WHEN INVOLVING

$X \overset{\neq 0}{\in} \mathcal{C}$ IS SIMPLE IF THE ONLY SUBOBJS OF X ARE X & 0

SEMISIMPLE OBJECTS

INDECOMPOSABLE OBJECTS

$X \overset{\neq 0}{\in} \mathcal{C}$ IS INDECOMPOSABLE IF $X \ncong X_1 \sqcup X_2$ $\forall$ NONZERO SUBOBJ. X_1, X_2 OF X

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IN GENERAL

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PROP: IN SEMISIMPLE CATEGORIES, INDECOMPOSABLE OBJECTS OF FINITE LENGTH ARE SIMPLE.

INDECOMPOSABLE OBJECTS

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SEMISIMPLE CATEGORIES

II. EXACTNESS

$\mathcal{C}$ ≡ ABELIAN CATEG.

THESE ARE
NICE CATEGORIES
IN WHICH A LOT OF
NICE RESULTS HOLD

⟶

SEMISIMPLE
CATEGORIES

NOW WE START TO WORK AWAY FROM THIS STRONG CONDITION

THESE ARE NICE CATEGORIES IN WHICH A LOT OF NICE RESULTS HOLD

SEMISIMPLE CATEGORIES

II. EXACTNESS

$\mathcal{C}$ ≡ ABELIAN CATEG.

ENTER THE WORLD OF
HOMOLOGICAL ALGEBRA

NOW WE START TO WORK AWAY
FROM THIS STRONG CONDITION

THESE ARE
NICE CATEGORIES
IN WHICH A LOT OF
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SEMISIMPLE
CATEGORIES

II. EXACTNESS

$\mathcal{C} \equiv$ ABELIAN CATEG.

ENTER THE WORLD OF
HOMOLOGICAL ALGEBRA

MAIN ENTITIES OF INTEREST :

EXACT SEQUENCES

II. EXACTNESS

$\mathcal{C} \equiv$ ABELIAN CATEG.

ENTER THE WORLD OF HOMOLOGICAL ALGEBRA

MAIN ENTITIES OF INTEREST :

EXACT SEQUENCES

A SEQUENCE OF MORPHISMS IN $\mathcal{C}$

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

IS EXACT AT X_i IF $\mathrm{Ker}(f_i) = \mathrm{im}(f_{i-1})$.

IT IS EXACT IF EXACT AT X_i $\forall i$

II. EXACTNESS

$\mathcal{C} \equiv$ ABELIAN CATEG.

A SEQUENCE OF MORPHISMS IN $\mathcal{C}$

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IT IS EXACT IF EXACT AT X_i $\forall i$

EXERCISE 2.51

- $0 \xrightarrow{\vec{0}_{X'}} X' \xrightarrow{f} X$ IS EXACT $\iff$
- $X \xrightarrow{g} X'' \xrightarrow{{}_{X''}\vec{0}} 0$ IS EXACT $\iff$

II. EXACTNESS

$\mathscr{C} \equiv$ ABELIAN CATEG.

A SEQUENCE OF MORPHISMS IN $\mathscr{C}$

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

IS EXACT AT X_i IF $\mathrm{Ker}(f_i) = \mathrm{im}(f_{i-1})$.

IT IS EXACT IF EXACT AT X_i $\forall i$

EXERCISE 2.51

- $0 \xrightarrow{\vec{0}_{X'}} X' \xrightarrow{f} X$ IS EXACT $\iff$ f IS MONIC.
- $X \xrightarrow{g} X'' \xrightarrow{{}_{X''}\vec{0}} 0$ IS EXACT $\iff$ g IS EPIC.

II. EXACTNESS

$\mathcal{C} \equiv$ ABELIAN CATEG.

A SEQUENCE OF MORPHISMS IN $\mathcal{C}$

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IT IS EXACT IF EXACT AT X_i $\forall i$

EXERCISE 2.51

- $0 \xrightarrow{\vec{0}_{X'}} X' \xrightarrow{f} X$ IS EXACT $\iff$ f IS MONIC.
- $X \xrightarrow{g} X'' \xrightarrow{{}_{X''}\vec{0}} 0$ IS EXACT $\iff$ g IS EPIC.
- $0 \xrightarrow{\vec{0}_X} X \xrightarrow{h} Y \xrightarrow{{}_Y\vec{0}} 0$ IS EXACT $\iff$ h IS AN ISO.

II. EXACTNESS

$\mathscr{C} \equiv$ ABELIAN CATEG.

A SEQUENCE OF MORPHISMS IN $\mathscr{C}$

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

IS EXACT AT X_i IF $\operatorname{Ker}(f_i) = \operatorname{im}(f_{i-1})$.

IT IS EXACT IF EXACT AT X_i $\forall i$

A SHORT EXACT SEQUENCE IN $\mathscr{C}$ IS AN EXACT SEQ. OF THE FORM

$$0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \longrightarrow 0.$$

EXERCISE 2.51

- $0 \xrightarrow{\vec{0}_{X'}} X' \xrightarrow{f} X$ IS EXACT $\iff$ f IS MONIC.
- $X \xrightarrow{g} X'' \xrightarrow{{}_{X''}\vec{0}} 0$ IS EXACT $\iff$ g IS EPIC.
- $0 \xrightarrow{\vec{0}_X} X \xrightarrow{h} Y \xrightarrow{{}_Y\vec{0}} 0$ IS EXACT $\iff$ h IS AN ISO.

II. EXACTNESS

$\mathcal{C} \equiv$ ABELIAN CATEG.

A SEQUENCE OF MORPHISMS IN $\mathcal{C}$

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

IS EXACT AT X_i IF $\ker(f_i) = \operatorname{im}(f_{i-1})$.

IT IS EXACT IF EXACT AT X_i $\forall i$

A SHORT EXACT SEQUENCE IN $\mathcal{C}$ IS AN EXACT SEQ. OF THE FORM

$$0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \longrightarrow 0.$$

$\therefore$ f MONIC, g EPIC, $\ker(g) = \operatorname{im}(f)$,

X' = SUBOBJ. OF X & $X'' \cong X/X'$

EXERCISE 2.51

- $0 \xrightarrow{\vec{0}_{X'}} X' \xrightarrow{f} X$ IS EXACT $\iff$ f IS MONIC.
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NICE SHORT EXACT SEQUENCES...

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$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "f"] & X \arrow[r, "g"] & X'' \arrow[r] \arrow[l, bend left, "s"] & 0. \end{tikzcd}$$

SECTION

ⓐ $\exists\, s : X'' \longrightarrow X$ s.t. $gs = \mathrm{id}_{X''}$.

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RETRACTION r

ⓐ $\exists\, s : X'' \longrightarrow X$ s.t. $gs = \mathrm{id}_{X''}$.

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CALL THE S.E.S. SPLIT.

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COOL FACT: ALL S.E.S. SPLIT IN SEMISIMPLE CATEGORIES.

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EXAMPLE Ab IS NOT SEMISIMPLE.

BY WAY OF CONTRADICTION, TAKE S.E.S.:

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

NICE SHORT EXACT SEQUENCES...

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IF $\exists r : \mathbb{Z} \to \mathbb{Z}$ s.t. $r(\cdot 2) = \mathrm{id}_{\mathbb{Z}}$,

THEN $r(\cdot 2)(1) = r(2) = 2n$

FOR SOME $n \in \mathbb{Z}$

NICE SHORT EXACT SEQUENCES...

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IF $\exists\, r : \mathbb{Z} \to \mathbb{Z}$ s.t. $r(\cdot 2) = \mathrm{id}_{\mathbb{Z}}$,

THEN $r(\cdot 2)(1) = r(2) = 2n \neq 1$.

FOR SOME $n \in \mathbb{Z}$ #

NICE SHORT EXACT SEQUENCES...

PROP TFAE FOR S.E.S.:

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "f"] & X \arrow[r, "g"] \arrow[l, bend left, "r"] & X'' \arrow[r] \arrow[l, bend left, "s"] & 0. \end{tikzcd}$$

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CALL THE S.E.S. SPLIT.

II. EXACTNESS

$\mathcal{C} \equiv$ ABELIAN CATEG.

LET'S STUDY HOW THESE ARE PRESERVED UNDER FUNCTORS...

A **SHORT EXACT SEQUENCE** IN $\mathcal{C}$ IS AN EXACT SEQ. OF THE FORM

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FUNCTOR $F: \mathcal{C} \longrightarrow \mathcal{D}$

- IS **LEFT EXACT** IF F SENDS $\star$ TO EX. SEQ:

$$0 \longrightarrow F(X') \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(X'')$$

- IS **RIGHT EXACT** IF F SENDS $\star$ TO EX. SEQ:

$$F(X') \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(X'') \longrightarrow 0$$

- IS **EXACT** IF LEFT & RIGHT EXACT.

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FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$ (RESP. CONTRAV'T)

- IS **LEFT EXACT** IF F SENDS $\star$ TO EX. SEQ:

$$0 \longrightarrow F(X') \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(X'')$$

(RESP. $0 \longrightarrow F(X'') \xrightarrow{F(g)} F(X) \xrightarrow{F(f)} F(X')$)

- IS **RIGHT EXACT** IF F SENDS $\star$ TO EX. SEQ:

$$F(X') \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(X'') \longrightarrow 0$$

(RESP. $F(X'') \xrightarrow{F(g)} F(X) \xrightarrow{F(f)} F(X') \longrightarrow 0$)

- IS **EXACT** IF LEFT & RIGHT EXACT.

II. EXACTNESS

$\mathcal{C} \equiv$ ABELIAN CATEG.

COOL FACTS: TAKE AN ADDITIVE FUNCTOR

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

$$f \longmapsto F(f)$$

IS A GROUP HOMOM. $\forall X, Y$

A SHORT EXACT SEQUENCE IN $\mathcal{C}$ IS AN EXACT SEQ. OF THE FORM

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COOL FACTS: TAKE AN ADDITIVE FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

- F IS LEFT EXACT $\Leftrightarrow$

  F PRESERVES KERNELS

  $F(\ker(f)) \cong \ker(F(f))$

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$\mathcal{C} \equiv$ ABELIAN CATEG.

COOL FACTS: TAKE AN ADDITIVE FUNCTOR
$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

- F IS LEFT EXACT $\Leftrightarrow$

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  $$F(\ker(f)) \cong \ker(F(f))$$

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- F PRESERVES SPLIT S.E.S.

A SHORT EXACT SEQUENCE IN $\mathscr{C}$ IS AN EXACT SEQ. OF THE FORM

$$\star \quad 0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0.$$

$\therefore$ f MONIC, g EPIC, $\ker(g) = \operatorname{im}(f)$

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$$F(\ker(f)) \cong \ker(F(f))$$

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- F PRESERVES SPLIT S.E.S.

- $\exists$ LEFT ADJOINT TO F $\Rightarrow$ F IS LEFT EXACT

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$\mathcal{C}$ ≡ ABELIAN CATEG.

A SHORT EXACT SEQUENCE IN $\mathcal{C}$ IS AN EXACT SEQ. OF THE FORM

$$\star \; 0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \longrightarrow 0.$$

∴ f MONIC, g EPIC, $\ker(g) = \operatorname{im}(f)$

FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$

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COOL FACTS: TAKE AN ADDITIVE FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$

- $\exists$ LEFT ADJOINT TO F $\Rightarrow$ F IS LEFT EXACT
- $\exists$ RIGHT ADJOINT TO F $\Rightarrow$ F IS RIGHT EXACT

II. EXACTNESS

$\mathcal{C}$ ≡ ABELIAN CATEG.

EXAMPLE TAKE $\mathbb{k}$-ALGS. A, B.
& $Q = {}_BQ_A$ BIMODULE.

GET ADDITIVE FUNCTORS:

$$Q \otimes_A - : A\text{-Mod} \longrightarrow B\text{-Mod}$$

$$\mathrm{Hom}_{B\text{-Mod}}(Q, -) : B\text{-Mod} \longrightarrow A\text{-Mod}$$

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COOL FACTS: TAKE AN ADDITIVE FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$

- $\exists$ LEFT ADJOINT TO $F \Longrightarrow$ F IS LEFT EXACT

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WITH $(Q \otimes_A -) \dashv (\mathrm{Hom}_{B\text{-Mod}}(Q, -))$

COOL FACTS: TAKE AN ADDITIVE FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$

- $\exists$ LEFT ADJOINT TO $F \Longrightarrow$ F IS LEFT EXACT
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A SHORT EXACT SEQUENCE IN $\mathcal{C}$ IS AN EXACT SEQ. OF THE FORM

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RIGHT EX. $\boxed{Q \otimes_A -}$: A-Mod $\longrightarrow$ B-Mod

$\boxed{\mathrm{Hom}_{B\text{-Mod}}(Q, -)}$: B-Mod $\longrightarrow$ A-Mod
LEFT EX.

WITH $\left(Q \otimes_A -\right) \dashv \left(\mathrm{Hom}_{B\text{-Mod}}(Q, -)\right)$

A SHORT EXACT SEQUENCE IN $\mathcal{C}$ IS AN EXACT SEQ. OF THE FORM

$$\star \quad 0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \longrightarrow 0.$$

$\therefore$ f MONIC, g EPIC, $\ker(g) = \mathrm{im}(f)$

COOL FACTS: TAKE AN ADDITIVE FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$

- $\exists$ LEFT ADJOINT TO $F \Rightarrow$ F IS LEFT EXACT
- $\exists$ RIGHT ADJOINT TO $F \Rightarrow$ F IS RIGHT EXACT

FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$

- IS LEFT EXACT IF F SENDS $\star$ TO EX. SEQ:

$$0 \longrightarrow F(X') \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(X'')$$

- IS RIGHT EXACT IF F SENDS $\star$ TO EX. SEQ:

$$F(X') \xrightarrow{F(f)} F(X) \xrightarrow{F(g)} F(X'') \longrightarrow 0$$

- IS EXACT IF LEFT & RIGHT EXACT.

II. EXACTNESS

$\mathcal{C} \equiv$ ABELIAN CATEG.

EXAMPLE TAKE $\Bbbk$-ALGS. A, B.
& $Q = {}_BQ_A$ BIMODULE.

GET ADDITIVE FUNCTORS:

RIGHT EX. $\boxed{Q \otimes_A -}$: $A\text{-Mod} \longrightarrow B\text{-Mod}$

$\boxed{\mathrm{Hom}_{B\text{-Mod}}(Q, -)}$: $B\text{-Mod} \longrightarrow A\text{-Mod}$

LEFT EX.

WITH $(Q \otimes_A -) \dashv (\mathrm{Hom}_{B\text{-Mod}}(Q, -))$

COOL FACTS: TAKE AN ADDITIVE FUNCTOR
$F : \mathcal{C} \longrightarrow \mathcal{D}$

- $\exists$ LEFT ADJOINT TO $F \implies$
  F IS LEFT EXACT

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EILENBERG–WATTS THEOREM

TAKE FINITE DIM'L $\Bbbk$-ALGEBRAS A, B.
FOR $\Bbbk$-LINEAR
$F : A\text{-FdMod} \longrightarrow B\text{-FdMod}$, GET:

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WITH $(Q \otimes_A -) \dashv (\mathrm{Hom}_{B\text{-Mod}}(Q, -))$

COOL FACTS: TAKE AN ADDITIVE FUNCTOR
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  F IS LEFT EXACT
- $\exists$ RIGHT ADJOINT TO $F \Longrightarrow$
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TAKE FINITE DIM'L $\Bbbk$-ALGEBRAS A, B.
FOR $\Bbbk$-LINEAR
$F : A\text{-FdMod} \longrightarrow B\text{-FdMod}$, GET:

F LEFT EXACT

$\Updownarrow$

F HAS A LEFT ADJOINT

$\Updownarrow$

$F \cong \mathrm{Hom}_{A\text{-FdMod}}(P, -)$
FOR SOME BIMOD. $P = {}_AP_B$.

II. EXACTNESS

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& $Q = {}_B Q_A$ BIMODULE.

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$F \cong \mathrm{Hom}_{A\text{-FdMod}}(P, -)$ FOR SOME BIMOD. $P = {}_A P_B$.	$F \cong Q \otimes_A -$ FOR SOME BIMOD. $Q = {}_B Q_A$.

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FOR $\Bbbk$-LINEAR
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STS

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PF/ F RIGHT EXACT $\Rightarrow$ $F \cong Q \otimes_A -$ FOR SOME ${}_B Q_A$:

EILENBERG–WATTS THEOREM

TAKE FINITE DIM'L $\Bbbk$-ALGEBRAS A, B.

FOR $\Bbbk$-LINEAR
$F: A\text{-FdMod} \longrightarrow B\text{-FdMod}$, GET:

F LEFT EXACT

$\Updownarrow$ ✓

F HAS A LEFT ADJOINT

$\Updownarrow$ ✓

$F \cong \mathrm{Hom}_{A\text{-FdMod}}(P, -)$

FOR SOME BIMOD. $P = {}_A P_B$.

STS

F RIGHT EXACT

$\Updownarrow$ ✓

F HAS A RIGHT ADJOINT

$\Updownarrow$ ✓

$F \cong Q \otimes_A -$

FOR SOME BIMOD. $Q = {}_B Q_A$.

II. EXACTNESS

$\mathcal{C} \equiv$ ABELIAN CATEG.

PF/ F RIGHT EXACT $\Rightarrow$ $F \cong Q \otimes_A -$ FOR SOME ${}_B Q_A$:

TAKE $Q := F({}_A A_{reg}) \in B\text{-FdMod}$.

GET $Q \in (B,A)\text{-FdBimod}$ (See pf of Morita's Thm)

EILENBERG-WATTS THEOREM

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PF/ F RIGHT EXACT $\Rightarrow$ $F \cong Q \otimes_A -$ FOR SOME ${}_B Q_A$:

TAKE $Q := F({}_A A_{reg}) \in B\text{-FdMod}$.

GET $Q \in (B, A)\text{-FdBimod}$ (SEE PF OF MORITA'S THM)

DEFINE $\phi_V : V \cong \mathrm{Hom}_{A\text{-FdMod}}(A_{reg}, V) \xrightarrow{F} \mathrm{Hom}_{B\text{-FdMod}}(Q, F(V))$.
FOR $V \in A\text{-FdMod}$

$$\therefore \phi_V \in \mathrm{Hom}_{A\text{-FdMod}}\left(V, \mathrm{Hom}_{B\text{-FdMod}}(Q, F(V))\right)$$

$$\cong$$

$$\mathrm{Hom}_{B\text{-FdMod}}\left(Q \otimes_A V,\ F(V)\right)$$

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TAKE FINITE DIM'L $\Bbbk$-ALGEBRAS A, B.
FOR $\Bbbk$-LINEAR
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F LEFT EXACT	F RIGHT EXACT
$\Updownarrow$ ✓	$\Updownarrow$ ✓
F HAS A LEFT ADJOINT	F HAS A RIGHT ADJOINT
$\Updownarrow$ ✓	$\Updownarrow$ ✓
$F \cong \mathrm{Hom}_{A\text{-FdMod}}(P, -)$	$F \cong Q \otimes_A -$
FOR SOME BIMOD. $P = {}_A P_B$.	FOR SOME BIMOD. $Q = {}_B Q_A$.

STS

II. EXACTNESS

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PF/ F RIGHT EXACT $\Rightarrow$ $F \cong Q \otimes_A -$ FOR SOME ${}_B Q_A$:

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FOR $V \in A\text{-FdMod}$

$\therefore \phi_V \in \mathrm{Hom}_{A\text{-FdMod}}(V, \mathrm{Hom}_{B\text{-FdMod}}(Q, F(V)))$

$\cong$ [BIMOD. $\otimes$-HOM ADJ'N]

$\mathrm{Hom}_{B\text{-FdMod}}(Q \otimes_A V, F(V))$

EILENBERG-WATTS THEOREM

TAKE FINITE DIM'L $\Bbbk$-ALGEBRAS A, B.

FOR $\Bbbk$-LINEAR

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YIELDS NAT'L TRANSF:

$$\begin{tikzcd} A\text{-FdMod} \arrow[r, bend left, "Q \otimes_A -"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow \phi"] & B\text{-FdMod} \end{tikzcd}$$

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STS IS AN ISO $\forall V$

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⇕	⇕
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⇕	⇕
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FOR $Q := F({}_A A_{reg})$

$\dim_{\Bbbk} V < \infty \Rightarrow \exists$ EPIMORPHISM $A^{\oplus n} \overset{g}{\twoheadrightarrow} V$ IN A-FdMod FOR SOME $n \in \mathbb{N}$.

$\therefore\ 0 \longrightarrow \ker(g) \longrightarrow A^{\oplus n} \xrightarrow{g} V \longrightarrow 0$ IS EXACT.

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APPLY $Q \otimes_A -$ & $F(-)$ TO YIELD:

RIGHT EXACT

$$\begin{tikzcd} Q\otimes_A \ker(g) \arrow[r] \arrow[d, "\phi_{\ker(g)}"'] \arrow[dr, phantom, "\circlearrowright"] & Q\otimes_A A^{\oplus n} \arrow[r] \arrow[d, "\phi_A^{\oplus n}"] \arrow[dr, phantom, "\circlearrowright"] & Q\otimes_A V \arrow[r] \arrow[d, "\phi_V"] \arrow[dr, phantom, "\circlearrowright"] & 0 \arrow[d, "\phi_0"] \\ F(\ker(g)) \arrow[r] & F(A^{\oplus n}) \arrow[r] & F(V) \arrow[r] & 0 \end{tikzcd}$$

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FOR $\Bbbk$-LINEAR $F: A\text{-FdMod} \to B\text{-FdMod}$, GET:

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$Q \otimes_A -$ COMMUTES WITH $\oplus$

EILENBERG-WATTS THEOREM

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FOR $\Bbbk$-LINEAR $F: A\text{-FdMod} \to B\text{-FdMod}$, GET:

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F RIGHT EX & PRESERVES COKERNELS

EILENBERG-WATTS THEOREM

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FOR $\Bbbk$-LINEAR

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$F \cong \mathrm{Hom}_{A\text{-FdMod}}(P, -)$	$F \cong Q \otimes_A -$
FOR SOME BIMOD.	FOR SOME BIMOD.
$P = {}_A P_B$.	$Q = {}_B Q_A$.

STS

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$\mathscr{C} \equiv$ ABELIAN CATEG.

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$$\begin{tikzcd} Q \otimes_A \ker(g) \arrow[r] \arrow[d, "\phi_{\ker(g)}"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \otimes_A A^{\oplus n} \arrow[r] \arrow[d, "\phi_{A^{\oplus n}}"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & Q \otimes_A V \arrow[r] \arrow[d, "\phi_V"'] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[d, "\phi_0"] \\ F(\ker(g)) \arrow[r] & F(A^{\oplus n}) \arrow[r, two heads, "F(g)"] & F(V) \arrow[r] & 0 \end{tikzcd}$$

$\therefore\ \phi_V$ IS EPIC $\quad \forall V$

EILENBERG-WATTS THEOREM

TAKE FINITE DIM'L $\Bbbk$-ALGEBRAS A, B.

FOR $\Bbbk$-LINEAR $F: A\text{-FdMod} \longrightarrow B\text{-FdMod}$, GET:

F LEFT EXACT	F RIGHT EXACT
$\Updownarrow$ ✓	$\Updownarrow$ ✓
F HAS A LEFT ADJOINT	F HAS A RIGHT ADJOINT
$\Updownarrow$ ✓	$\Updownarrow$ ✓
$F \cong \mathrm{Hom}_{A\text{-FdMod}}(P, -)$	$F \cong Q \otimes_A -$
FOR SOME BIMOD.	FOR SOME BIMOD.
$P = {}_A P_B$.	$Q = {}_B Q_A$.

STS

II. EXACTNESS

$\mathcal{C} \equiv$ ABELIAN CATEG.

PF/ F RIGHT EXACT $\Rightarrow$ $F \cong Q \otimes_A -$ FOR SOME ${}_BQ_A$:

HAVE

$$\begin{tikzcd} A\text{-FdMod} \arrow[rr, bend left, "Q\otimes_A -"] \arrow[rr, bend right, "F"'] \arrow[rr, phantom, "\Downarrow \phi"] & & B\text{-FdMod} \end{tikzcd}$$

FOR $Q := F({}_A A_{reg})$

$\dim_{\Bbbk} V < \infty \Rightarrow \exists$ EPIMORPHISM $A^{\oplus n} \overset{g}{\twoheadrightarrow} V$ IN A-FdMod FOR SOME $n \in \mathbb{N}$.

$\therefore\ 0 \to \ker(g) \to A^{\oplus n} \xrightarrow{g} V \to 0$ IS EXACT.

APPLY $\underbrace{Q \otimes_A -}$ & $\underbrace{F(-)}$ TO YIELD:

RIGHT EXACT

$$\begin{tikzcd} Q\otimes_A \ker(g) \arrow[r] \arrow[d, "\phi_{\ker(g)}"'] \arrow[dr, phantom, "\circlearrowright"] & Q\otimes_A A^{\oplus n} \arrow[r] \arrow[d, "\phi_{A^{\oplus n}}"', "\cong"] \arrow[dr, phantom, "\circlearrowright"] & Q\otimes_A V \arrow[r] \arrow[d, "\phi_V"'] \arrow[dr, phantom, "\circlearrowright"] & 0 \arrow[d, "\phi_0"] \\ F(\ker(g)) \arrow[r] & F(A^{\oplus n}) \arrow[r, two heads, "F(g)"] & F(V) \arrow[r] & 0 \end{tikzcd}$$

$\therefore\ \phi_{\ker(g)}$ IS EPIC

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HOMOLOGICAL "4-LEMMA" $\Rightarrow$ ϕ_V MONIC //

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INDEED, MONIC EPIS ARE ISOS IN AB. CATS. //

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F LEFT EXACT

$\Updownarrow$

F HAS A LEFT ADJOINT

$\Updownarrow$

$F \cong \mathrm{Hom}_{A\text{-FdMod}}(P,-)$

FOR SOME BIMOD. $P = {}_A P_B$.

F RIGHT EXACT

$\Updownarrow$

F HAS A RIGHT ADJOINT

$\Updownarrow$

$F \cong Q \otimes_A -$

FOR SOME BIMOD. $Q = {}_B Q_A$.

III. PROJECTIVITY & INJECTIVITY

$\mathcal{C} \equiv$ ABELIAN CATEG.

IN GENERAL:

$\mathrm{Hom}_{\mathcal{C}}(P,-) : \mathcal{C} \longrightarrow Ab$ (COVARIANT)

IS ALWAYS LEFT EXACT

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$\Updownarrow$

EVERY S.E.S. IN $\mathcal{C}$ OF THE FORM

$$0 \to X' \to X \to P \to 0$$

SPLITS

III. PROJECTIVITY & INJECTIVITY

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SPLITS

$$\Updownarrow$$

$$\begin{tikzcd}
 & P \arrow[dl, "\exists \tilde{f}"'] \arrow[d, "\forall f"] \arrow[dl, phantom, shift right=4, "\circlearrowleft"] & \\
Y \arrow[r, two heads, "\forall p"] & Z \arrow[r] & 0
\end{tikzcd}$$

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$$\Updownarrow$$

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 & P \arrow[dl, bend right, "\exists \tilde{f}"'] \arrow[d, "\forall f"] & \\
Y \arrow[r, two heads, "\forall p"] & Z \arrow[r] & 0
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IN THIS CASE,

P IS A PROJECTIVE OBJECT IN $\mathscr{C}$.

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<u>FACT</u>: $\mathrm{Hom}_{\mathcal{C}}(P,-)$ IS RIGHT EXACT

$$\Updownarrow$$

EVERY S.E.S. IN $\mathcal{C}$ OF THE FORM

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SPLITS

$$\Updownarrow$$

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists \tilde{f}"'] \arrow[d, "\forall f"] \arrow[dl, phantom, shift right=4, "2"] & \\ Y \arrow[r, two heads, "\forall p"] & Z \arrow[r] & 0 \end{tikzcd}$$

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$$\Updownarrow$$

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$$0 \to X' \to X \to P \to 0$$

SPLITS

$$\Updownarrow$$

$$\begin{tikzcd} & P \arrow[d, "\forall f"] \arrow[dl, dashed, bend right, "\exists \tilde{f}"'] \arrow[dl, phantom, "2", near end] \\ Y \arrow[r, two heads, "\forall p"] & Z \arrow[r] & 0 \end{tikzcd}$$

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FACT: $\mathrm{Hom}_{\mathscr{C}}(-,Q)$ IS RIGHT EXACT

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SPLITS

$\Updownarrow$

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists \tilde{f}"'] \arrow[d, "\forall f"] \arrow[dl, phantom, shift left=4, "\circlearrowleft"] & \\ Y \arrow[r, two heads, "\forall p"] & Z \arrow[r] & 0 \end{tikzcd}$$

IN THIS CASE,

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FACT: $\mathrm{Hom}_{\mathcal{C}}(-, Q)$ IS RIGHT EXACT

$\Updownarrow$

EVERY S.E.S. IN $\mathcal{C}$ OF THE FORM

$$0 \to Q \to X \to X'' \to 0$$

SPLITS

$\Updownarrow$

$$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r, hook, "\forall g"] \arrow[d, "\forall g"'] & Y \arrow[dl, dashed, "\exists \tilde{g}"] \\ & Q & \end{tikzcd}$$

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SPLITS

$\Updownarrow$

$$\begin{tikzcd}
 & & P \arrow[d, "\forall f"] \arrow[dll, dashed, bend right, "\exists \tilde{f}"'] & \\
Y \arrow[rr, two heads, "\forall p"] & & Z \arrow[r] & 0
\end{tikzcd}$$

IN THIS CASE,

P IS A PROJECTIVE OBJECT IN $\mathscr{C}$.

FACT: $\mathrm{Hom}_{\mathscr{C}}(-,Q)$ IS RIGHT EXACT

$\Updownarrow$

EVERY S.E.S. IN $\mathscr{C}$ OF THE FORM

$$0 \to Q \to X \to X'' \to 0$$

SPLITS

$\Updownarrow$

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r, hook, "\forall q"] \arrow[d, "\forall g"'] & Y \arrow[dl, dashed, bend left, "\exists \tilde{g}"] \\
 & Q &
\end{tikzcd}$$

IN THIS CASE,

Q IS AN INJECTIVE OBJECT IN $\mathscr{C}$.

III. PROJECTIVITY & INJECTIVITY

$\mathscr{C} \equiv$ ABELIAN CATEG.

TIEING BACK TO SEMISIMPLICITY...

FACT: $\mathrm{Hom}_{\mathscr{C}}(P, -)$ IS RIGHT EXACT

$$\Updownarrow$$

EVERY S.E.S. IN $\mathscr{C}$ OF THE FORM

$$0 \to X' \to X \to P \to 0$$

SPLITS

$$\Updownarrow$$

$$\begin{tikzcd}
 & & P \arrow[dll, dashed, "\exists \tilde{f}"'] \arrow[d, "\forall f"] \\
Y \arrow[rr, two heads, "\forall p"] & \arrow[u, phantom, "\circlearrowleft"] & Z \arrow[r] & 0
\end{tikzcd}$$

IN THIS CASE,

P IS A PROJECTIVE OBJECT IN $\mathscr{C}$.

FACT: $\mathrm{Hom}_{\mathscr{C}}(-, Q)$ IS RIGHT EXACT

$$\Updownarrow$$

EVERY S.E.S. IN $\mathscr{C}$ OF THE FORM

$$0 \to Q \to X \to X'' \to 0$$

SPLITS

$$\Updownarrow$$

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r, hook, "\forall q"] \arrow[d, "\forall g"'] & Y \arrow[dl, dashed, "\exists \tilde{g}"] \\
 & Q &
\end{tikzcd}$$

IN THIS CASE,

Q IS AN INJECTIVE OBJECT IN $\mathscr{C}$.

III. PROJECTIVITY & INJECTIVITY

$\mathscr{C} \equiv$ ABELIAN CATEG.

FACTS

IF $\mathscr{C}$ IS SEMISIMPLE, THEN EACH OBJECT IS PROJECTIVE & INJECTIVE.

FACT: $\mathrm{Hom}_{\mathscr{C}}(P,-)$ IS RIGHT EXACT

$$\Updownarrow$$

EVERY S.E.S. IN $\mathscr{C}$ OF THE FORM

$$0 \to X' \to X \to P \to 0$$

SPLITS

$$\Updownarrow$$

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\exists \tilde{f}"'] \arrow[d, "\forall f"] \arrow[dl, phantom, "2", shift right=3] & \\
Y \arrow[r, two heads, "\forall p"] & Z \arrow[r] & 0
\end{tikzcd}$$

IN THIS CASE,

P IS A PROJECTIVE OBJECT IN $\mathscr{C}$.

FACT: $\mathrm{Hom}_{\mathscr{C}}(-,Q)$ IS RIGHT EXACT

$$\Updownarrow$$

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FACTS

IF $\mathcal{C}$ IS SEMISIMPLE, THEN EACH OBJECT IS PROJECTIVE & INJECTIVE.

FOR AN $\mathbb{k}$-ALGEBRA A:
A SEMISIMPLE
$\iff$ EACH OBJ. OF A-Mod IS PROJ.
$\iff$ EACH OBJ. OF A-Mod IS INJ.

FACT: $\mathrm{Hom}_{\mathcal{C}}(P,-)$ IS RIGHT EXACT

$\Updownarrow$

EVERY S.E.S. IN $\mathcal{C}$ OF THE FORM

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SPLITS

$\Updownarrow$

$$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r, hook, "\forall q"] \arrow[d, "\forall g"'] & Y \arrow[dl, dashed, "\exists \tilde{g}"] \\ & Q & \end{tikzcd}$$

IN THIS CASE,
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IV. FINITENESS FOR LINEAR CATEGORIES

NICE CONDITIONS IMPOSED OFTEN IN LIEU OF SEMISIMPLICITY

$\mathcal{C}$ ≡ ABELIAN, $\Bbbk$-LINEAR CAT.

HOMS ARE ABELIAN GROUPS

... AND FURTHER ARE $\Bbbk$-VSPACES

IV. FINITENESS FOR LINEAR CATEGORIES

$\mathcal{C}$ = ABELIAN, $\Bbbk$-LINEAR CAT.

HOMS ARE ABELIAN GROUPS

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LOCALLY FINITE

FINITE

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$\boxed{\mathrm{Hom}_{\mathcal{C}}(X,Y) \text{ IS A FINITE DIM'L } \Bbbk\text{-VSPACE } \forall X,Y \in \mathcal{C}}$

$\Big\downarrow$ DEF

LOCALLY FINITE

FINITE

IV. FINITENESS FOR LINEAR CATEGORIES

$\mathcal{C}$ = ABELIAN, $\Bbbk$-LINEAR CAT.

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$\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS A FINITE DIM'L $\Bbbk$-VSPACE $\forall X,Y \in \mathcal{C}$

DEF

ALL OBJECTS IN $\mathcal{C}$ HAVE FINITE LENGTH

DEF

LOCALLY FINITE

FINITE

IV. FINITENESS FOR LINEAR CATEGORIES

$\mathcal{C}$ ≡ ABELIAN, $\Bbbk$-LINEAR CAT.

HOMS ARE ABELIAN GROUPS

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$\boxed{\text{Hom}_{\mathcal{C}}(X,Y) \text{ IS A FINITE DIM'L } \Bbbk\text{-VSPACE } \forall X,Y \in \mathcal{C}}$ $\xrightarrow{\text{DEF}}$ **LOCALLY FINITE**

$\boxed{\text{ALL OBJECTS IN } \mathcal{C} \text{ HAVE FINITE LENGTH}}$ $\xrightarrow{\text{DEF}}$ **LOCALLY FINITE**

LOCALLY FINITE $\xrightarrow{\text{DEF}}$ **FINITE**

IV. FINITENESS FOR LINEAR CATEGORIES

$\mathcal{C} \equiv$ ABELIAN, $\Bbbk$-LINEAR CAT.

HOMS ARE ABELIAN GROUPS

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$\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS A FINITE DIM'L $\Bbbk$-VSPACE $\forall X, Y \in \mathcal{C}$

DEF

ALL OBJECTS IN $\mathcal{C}$ HAVE FINITE LENGTH

DEF

LOCALLY FINITE

DEF

$\exists$ ONLY FINITELY MANY ISOCLASSES OF SIMPLE OBJECTS IN $\mathcal{C}$

DEF

FINITE

IV. FINITENESS FOR LINEAR CATEGORIES

$\mathcal{C} \equiv$ ABELIAN, $\Bbbk$-LINEAR CAT.

HOMS ARE ABELIAN GROUPS

... AND FURTHER ARE $\Bbbk$-VSPACES

$\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS A FINITE DIM'L $\Bbbk$-VSPACE $\forall X,Y \in \mathcal{C}$

DEF

ALL OBJECTS IN $\mathcal{C}$ HAVE FINITE LENGTH

DEF

LOCALLY FINITE

DEF

$\exists$ ENOUGH PROJECTIVES IN $\mathcal{C}$: $\forall Z \in \mathcal{C}$ $\exists$ PROJ. OBJ $P(Z) \in \mathcal{C}$ WITH EPI $P(Z) \twoheadrightarrow Z$ IN $\mathcal{C}$.

DEF

$\exists$ ONLY FINITELY MANY ISOCLASSES OF SIMPLE OBJECTS IN $\mathcal{C}$

DEF

FINITE

IV. Finiteness for linear categories

$\mathcal{C} \equiv$ abelian, $\Bbbk$-linear cat.

Homs are abelian groups ... and further are $\Bbbk$-vspaces

$\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a finite dim'l $\Bbbk$-vspace $\forall X,Y \in \mathcal{C}$

$\downarrow$ DEF

All objects in $\mathcal{C}$ have finite length

$\downarrow$ DEF

LOCALLY FINITE

$\downarrow$ DEF

$\exists$ enough projectives in $\mathcal{C}$: $\forall Z \in \mathcal{C}$ $\exists$ proj. obj $P(Z) \in \mathcal{C}$ with epi $P(Z) \twoheadrightarrow Z$ in $\mathcal{C}$.

$\downarrow$ DEF

$\exists$ only finitely many isoclasses of simple objects in $\mathcal{C}$

$\downarrow$ DEF

FINITE

<u>Prop</u>: $\mathcal{C}$ finite $\iff$ $\mathcal{C} \simeq A\text{-FdMod}$ for some f.d. $\Bbbk$-alg. A.

IV. FINITENESS FOR LINEAR CATEGORIES

$\mathcal{C} \equiv$ ABELIAN, $\Bbbk$-LINEAR CAT.

HOMS ARE ABELIAN GROUPS ... AND FURTHER ARE $\Bbbk$-VSPACES

$\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS A FINITE DIM'L $\Bbbk$-VSPACE $\forall X,Y \in \mathcal{C}$

ALL OBJECTS IN $\mathcal{C}$ HAVE FINITE LENGTH

DEF

DEF

LOCALLY FINITE

DEF

∃ ENOUGH PROJECTIVES IN $\mathcal{C}$: $\forall Z \in \mathcal{C}$ ∃ PROJ. OBJ $P(Z) \in \mathcal{C}$ WITH EPI $P(Z) \twoheadrightarrow Z$ IN $\mathcal{C}$.

DEF

∃ ONLY FINITELY MANY ISOCLASSES OF SIMPLE OBJECTS IN $\mathcal{C}$

DEF

FINITE

PROP: $\mathcal{C}$ FINITE $\iff$ $\mathcal{C} \simeq A$-FdMod FOR SOME F.D. $\Bbbk$-ALG. A.

PF SKETCH/ ($\Leftarrow$) ✓

($\Rightarrow$) TAKE ISOCLASS REPS $\{X_1, \dots, X_n\}$ OF SIMPLE OBJECTS OF $\mathcal{C}$.

THEN $A := \bigoplus_{i=1}^{n} \mathrm{End}_{\mathcal{C}}(P(X_i))$ WORKS...

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EILENBERG-WATTS THM

COR: $\mathcal{C}, \mathcal{D}$ FINITE. F LEFT (RIGHT) EXACT $\Longleftrightarrow$ F HAS A LEFT (RIGHT) ADJOINT.

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #11

THIS ENDS OUR INTRO TO CATEGORY THEORY

TOPICS:

I. BUILDING BLOCK OBJECTS (§2.7)

II. EXACTNESS (§§2.8.1–2.8.2)

III. PROJECTIVITY & INJECTIVITY (§2.8.3)

IV. FINITENESS FOR LINEAR CATEGORIES (§2.9)

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NEXT TIME: MONOIDAL CATEGORIES

TOPICS:

I. BUILDING BLOCK OBJECTS (§2.7)

II. EXACTNESS (§§2.8.1–2.8.2)

III. PROJECTIVITY & INJECTIVITY (§2.8.3)

IV. FINITENESS FOR LINEAR CATEGORIES (§2.9)

Lecture #11 keywords: Eilenberg-Watts Theorem, exact functor, finite category, indecomposable object, injective object, projective object, Schur's Lemma, semisimple category, short exact sequence, simple object