MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #12

TOPICS:

I. MONOIDAL CATEGORIES (§§ 3.1.1, 3.1.2)

II. ADDITIVE MONOIDAL CATEGORIES (§ 3.1.3)

III. MONOIDAL FUNCTORS (§§ 3.2.1, 3.2.3)

I. MONOIDAL CATEGORIES

A CATEGORY $\mathcal{C}$ CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X, Y \in \mathcal{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY
$(hg)f = h(gf)$

UNITALITY
$id_X f = f$, $g\, id_X = g$

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THE STRUCTURE OF A GADGET X ARE FEATURES THAT DEFINE X

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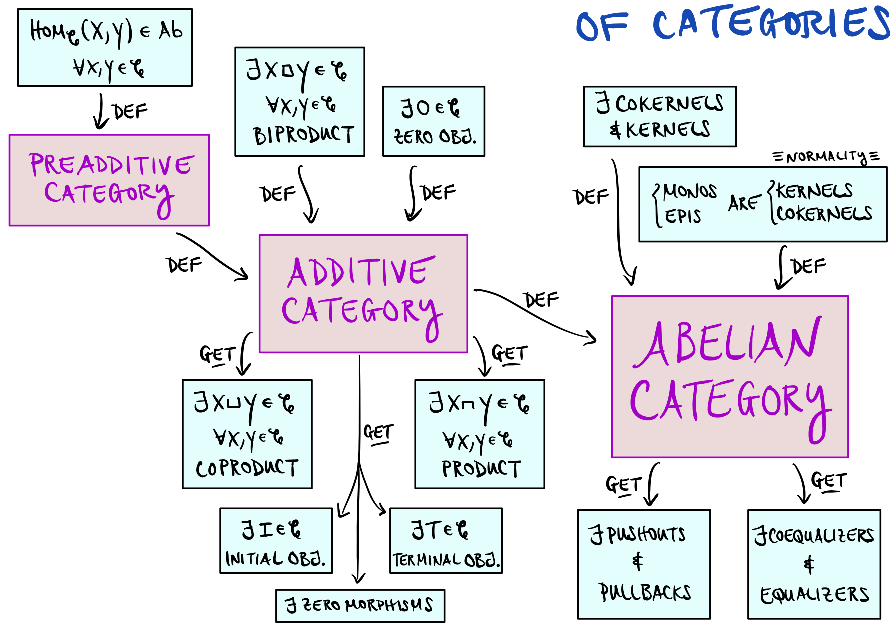

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NOW WE INTRODUCE A STRUCTURE ON A CATEGORY...

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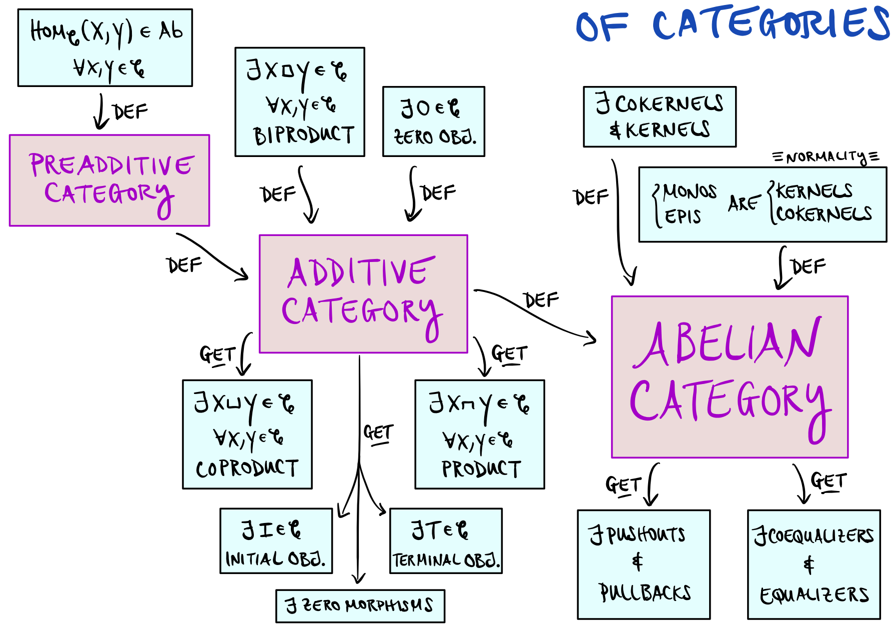

I. MONOIDAL CATEGORIES

A MONOIDAL CATEGORY IS LOOSELY

(a) A CATEGORY $\mathcal{C}$

(b) AN OPERATION $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

(c) AN OBJECT $\mathbb{1} \in \mathcal{C}$

SUCH THAT

$(\mathcal{C}, \otimes, \mathbb{1})$ MIMICS THE STRUCTURE OF A MONOID

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A NICE WAY OF COMBINING OBJECTS AND MORPHISMS IN $\mathcal{C}$

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(d, e, f) NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathscr{C} \times \mathscr{C} \times \mathscr{C} \arrow[r, bend left, "\otimes \circ (\otimes \times \mathrm{Id}_{\mathscr{C}})"] \arrow[r, bend right, "\otimes \circ (\mathrm{Id}_{\mathscr{C}} \times \otimes)"'] \arrow[r, phantom, "\sim\Downarrow a"] & \mathscr{C} \end{tikzcd}$$

$$a = \left\{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \right\}_{X,Y,Z \in \mathscr{C}}$$

NATURAL IN X, Y, Z

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$$\begin{tikzcd}
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(TRIANGLE AXIOM)

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WILL USE $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ MONOIDAL CATEG.
$\mathcal{A}$ FOR ORDINARY CATEGORY

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$$r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}} \text{ NATURAL IN } X$$

(RIGHT UNITALITY CONSTRAINT)

SATISFYING THE COMPATIBILITY CONDITIONS BELOW :

$\forall W, X, Y, Z \in \mathcal{C}$:

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes \mathrm{id}_Z"'] \arrow[dr, "a_{W \otimes X, Y, Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"'] \arrow[rr, phantom, "\circlearrowright"] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_W \otimes a_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

(PENTAGON AXIOM)

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & \arrow[d, phantom, "\circlearrowright"] & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y"] \\
 & X \otimes Y &
\end{tikzcd}$$

(TRIANGLE AXIOM)

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

(d, e, f) NATURAL ISOMS:

$$a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$$

$$l = \{ l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

$$r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

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$$r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

$(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ IS **STRICT** IF $\{ a_{X,Y,Z}, \ell_X, r_X \}_{X,Y,Z \in \mathcal{C}}$ ARE ALL IDENTITY MAPS

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

SUBSTRUCTURE —

A **MONOIDAL SUBCATEGORY** OF $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ CONSISTS OF A SUBCATEGORY $\mathcal{D}$ OF $\mathcal{C}$ $\ni$.

$(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ IS **STRICT** IF

$$\{ a_{X,Y,Z}, \; l_X, \; r_X \}_{X,Y,Z \in \mathcal{C}}$$

ARE ALL IDENTITY MAPS

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

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$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

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SATISFYING THE
PENTAGON AXIOM
& TRIANGLE AXIOM

SUBSTRUCTURE —

A **MONOIDAL SUBCATEGORY** OF $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ CONSISTS OF A SUBCATEGORY $\mathcal{D}$ OF $\mathcal{C}$ s.t.

- CLOSURE UNDER $\otimes$: $X \otimes Y \in \mathcal{D} \quad \forall X, Y \in \mathcal{D}$

$(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ IS **STRICT** IF
$\{ a_{X,Y,Z},\ l_X,\ r_X \}_{X,Y,Z \in \mathcal{C}}$
ARE ALL IDENTITY MAPS

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR

$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

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SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

SUBSTRUCTURE —

A MONOIDAL SUBCATEGORY OF $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ CONSISTS OF A SUBCATEGORY $\mathcal{D}$ OF $\mathcal{C}$ s.t.

- CLOSURE UNDER $\otimes$: $X \otimes Y \in \mathcal{D} \quad \forall X, Y \in \mathcal{D}$
- CLOSURE UNDER $\mathbb{1}$: $\mathbb{1} \in \mathcal{D}$

$(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ IS STRICT IF

$$\{ a_{X,Y,Z}, \ell_X, r_X \}_{X,Y,Z \in \mathcal{C}}$$

ARE ALL IDENTITY MAPS

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

SUBSTRUCTURE —

A **MONOIDAL SUBCATEGORY** OF $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ CONSISTS OF A SUBCATEGORY $\mathcal{D}$ OF $\mathcal{C}$ s.t.

- CLOSURE UNDER $\otimes$: $X \otimes Y \in \mathcal{D} \quad \forall X, Y \in \mathcal{D}$
- CLOSURE UNDER $\mathbb{1}$: $\mathbb{1} \in \mathcal{D}$
- a, ℓ, r RESTRICT TO $\mathcal{D}$ MAKING $(\mathcal{D}, \otimes, \mathbb{1}, a, \ell, r)$ MONOIDAL

$(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ IS **STRICT** IF $\{ a_{X,Y,Z}, \ell_X, r_X \}_{X,Y,Z \in \mathcal{C}}$ ARE ALL IDENTITY MAPS

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

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SATISFYING THE
PENTAGON AXIOM
& TRIANGLE AXIOM

SUBSTRUCTURE —

A **MONOIDAL SUBCATEGORY** OF $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ CONSISTS OF A SUBCATEGORY $\mathcal{D}$ OF $\mathcal{C}$ s.t.

- CLOSURE UNDER $\otimes$: $X \otimes Y \in \mathcal{D}$ $\forall X, Y \in \mathcal{D}$
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- a, ℓ, r RESTRICT TO $\mathcal{D}$ MAKING $(\mathcal{D}, \otimes, \mathbb{1}, a, \ell, r)$ MONOIDAL

IT IS **FULL** IF THE UNDERLYING CAT. $\mathcal{D}$ IS FULL.

$(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ IS **STRICT** IF
$\{ a_{X,Y,Z}, \ell_X, r_X \}_{X,Y,Z \in \mathcal{C}}$
ARE ALL IDENTITY MAPS

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

SUBSTRUCTURE —

A **MONOIDAL SUBCATEGORY** OF $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ CONSISTS OF A SUBCATEGORY $\mathcal{D}$ OF $\mathcal{C}$ s.t.

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- CLOSURE UNDER $\mathbb{1}$: $\mathbb{1} \in \mathcal{D}$
- a, ℓ, r MAKE $(\mathcal{D}, \otimes, \mathbb{1}, a, \ell, r)$ MONOIDAL

OPPOSITE STRUCTURES —

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

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SATISFYING THE **PENTAGON AXIOM** & **TRIANGLE AXIOM**

SUBSTRUCTURE —

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OPPOSITE STRUCTURES —

HAVE $\mathcal{A}^{op}$: $\mathrm{Ob}(\mathcal{A}^{op}) = \mathrm{Ob}(\mathcal{A})$, $\mathrm{Hom}_{\mathcal{A}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{A}}(Y,X)$

CAN TAKE $\otimes^{op}$: $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, $(X,Y) \longmapsto Y \otimes X$.

I. MONOIDAL CATEGORIES

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

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GET $\mathcal{C}^{op} := (\mathcal{C}^{op}, \otimes, \mathbb{1}, \{a^{-1}_{X,Y,Z}\}, \{l^{-1}_X\}, \{r^{-1}_X\})$

I. MONOIDAL CATEGORIES

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

SUBSTRUCTURE —

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OPPOSITE STRUCTURES —

HAVE $\mathcal{A}^{op}$: $Ob(\mathcal{A}^{op}) = Ob(\mathcal{A})$, $Hom_{\mathcal{A}^{op}}(X,Y) = Hom_{\mathcal{A}}(Y,X)$

CAN TAKE $\otimes^{op} : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, $(X,Y) \longmapsto Y \otimes X$.

GET $\mathcal{C}^{op} := (\mathcal{C}^{op}, \otimes, \mathbb{1}, \{a^{-1}_{X,Y,Z}\}, \{l^{-1}_X\}, \{r^{-1}_X\})$

$\mathcal{C}^{\otimes op} := (\mathcal{C}, \otimes^{op}, \mathbb{1}, \{a^{-1}_{Z,Y,X}\}, \{r_X\}, \{l_X\})$

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

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SATISFYING THE **PENTAGON AXIOM** & **TRIANGLE AXIOM**

SUBSTRUCTURE —

A **MONOIDAL SUBCATEGORY** OF $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ CONSISTS OF A SUBCATEGORY $\mathcal{D}$ OF $\mathcal{C}$ s.t.

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- CLOSURE UNDER $\mathbb{1}$: $\mathbb{1} \in \mathcal{D}$
- a, ℓ, r MAKE $(\mathcal{D}, \otimes, \mathbb{1}, a, \ell, r)$ MONOIDAL

OPPOSITE STRUCTURES —

HAVE $\mathcal{A}^{op}$: $Ob(\mathcal{A}^{op}) = Ob(\mathcal{A})$, $Hom_{\mathcal{A}^{op}}(X,Y) = Hom_{\mathcal{A}}(Y,X)$

CAN TAKE $\otimes^{op} : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, $(X,Y) \longmapsto Y \otimes X$.

GET $\mathcal{C}^{op} := (\mathcal{C}^{op}, \otimes, \mathbb{1}, \{a^{-1}_{X,Y,Z}\}, \{\ell^{-1}_X\}, \{r^{-1}_X\})$

$\mathcal{C}^{\otimes op} := (\mathcal{C}, \otimes^{op}, \mathbb{1}, \{a^{-1}_{Z,Y,X}\}, \{r_X\}, \{\ell_X\})$

$\mathcal{C}^{rev} := (\mathcal{C}^{op}, \otimes^{op}, \mathbb{1}, \{a_{Z,Y,X}\}, \{r^{-1}_X\}, \{\ell^{-1}_X\})$

ARE VERSIONS OF **OPPOSITE MONOIDAL CATEGS.**

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

SUBSTRUCTURE —

A MONOIDAL SUBCATEGORY OF $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ CONSISTS OF A SUBCATEGORY $\mathcal{D}$ OF $\mathcal{C}$ s.t.

- CLOSURE UNDER $\otimes$: $X \otimes Y \in \mathcal{D}$ $\forall X, Y \in \mathcal{D}$
- CLOSURE UNDER $\mathbb{1}$: $\mathbb{1} \in \mathcal{D}$
- a, ℓ, r MAKE $(\mathcal{D}, \otimes, \mathbb{1}, a, \ell, r)$ MONOIDAL

OPPOSITE STRUCTURES —

EXERCISE 3.2

HAVE $\mathcal{A}^{op}$: $Ob(\mathcal{A}^{op}) = Ob(\mathcal{A})$, $Hom_{\mathcal{A}^{op}}(X,Y) = Hom_{\mathcal{A}}(Y,X)$

CAN TAKE $\otimes^{op} : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, $(X,Y) \longmapsto Y \otimes X$.

GET $\mathcal{C}^{op} := (\mathcal{C}^{op}, \otimes, \mathbb{1}, \{a^{-1}_{X,Y,Z}\}, \{\ell^{-1}_X\}, \{r^{-1}_X\})$

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$\mathcal{C}^{rev} := (\mathcal{C}^{op}, \otimes^{op}, \mathbb{1}, \{a_{Z,Y,X}\}, \{r^{-1}_X\}, \{\ell^{-1}_X\})$

ARE VERSIONS OF OPPOSITE MONOIDAL CATEGS.

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF:

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SATISFYING THE
PENTAGON AXIOM
& TRIANGLE AXIOM

ALGEBRAIC EXAMPLES —

Vec CATEGORY OF $\Bbbk$-VECTOR SPACES

$\otimes :=$?? $\mathbb{1} :=$??

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

ALGEBRAIC EXAMPLES —

Vec CATEGORY OF $\Bbbk$-VECTOR SPACES

$\otimes := \otimes_{\Bbbk}$ $\qquad \mathbb{1} := \Bbbk$

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF:

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SATISFYING THE
PENTAGON AXIOM
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ALGEBRAIC EXAMPLES —

Vec CATEGORY OF $\Bbbk$-VECTOR SPACES

$\otimes := \otimes_{\Bbbk}$ $\quad$ $\mathbb{1} := \Bbbk$

?? STRICT ??

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF:

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

ALGEBRAIC EXAMPLES —

Vec CATEGORY OF $\Bbbk$-VECTOR SPACES

$\otimes := \otimes_{\Bbbk}$ $\mathbb{1} := \Bbbk$

≡ NOT STRICT ≡ E.G. $\Bbbk \otimes_{\Bbbk} V \cong V$ (DON'T HAVE $=$)

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF:

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$\mathrm{Vec}_{\oplus}$ CATEGORY OF $\Bbbk$-VECTOR SPACES

$\otimes := \oplus$ $\qquad \mathbb{1} := 0_{\mathrm{Vec}}$

FdVec

CATEGORY OF FINITE-DIM'L $\Bbbk$-VSPACES

$\otimes := \otimes_{\Bbbk}$ $\qquad \mathbb{1} := \Bbbk$

$\mathrm{FdVec}_{\oplus}$

CATEGORY OF FINITE-DIM'L $\Bbbk$-VSPACES

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

ALGEBRAIC EXAMPLES

- Vec
- $\mathrm{Vec}_{\oplus}$
- FdVec, $\mathrm{FdVec}_{\oplus}$

G-Mod CATEGORY OF LEFT MODULES $(V, \triangleright : G \times V \to V)$ OVER A GROUP G

$V \in \mathrm{Vec}$

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A-Bimod CATEGORY OF BIMODULES OVER A $\Bbbk$-ALG A

$$\otimes := ?? \qquad \mathbb{1} := ??$$

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A-Bimod CATEGORY OF BIMODULES OVER A $\Bbbk$-ALG A

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$\underline{G}$ FOR A GROUP G

$\text{Ob}(\underline{G})$ = ELEMENTS OF G

$\text{Hom}_{\underline{G}}(g,h) = \begin{cases} \text{id}_g & g = h \\ \emptyset & g \neq h \end{cases}$

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Vec_N DEFINED LIKEWISE FOR AN ADDITIVE MONOID N

$\underline{G}$ FOR A GROUP G

FOR AN ADDITIVE MONOID N DEFINED LIKEWISE $\underline{N}$

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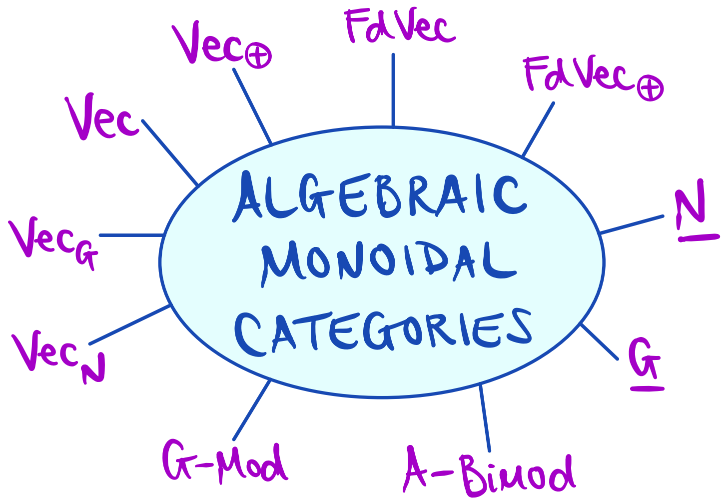

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

LOGICAL/CATEGORICAL MONOIDAL CATEGORIES

ALGEBRAIC MONOIDAL CATEGORIES

Vec, $\mathsf{Vec}_\oplus$, FdVec, $\mathsf{FdVec}_\oplus$, $\underline{\mathbb{N}}$, $\underline{G}$, A-Bimod, G-Mod, $\mathsf{Vec}_{\mathbb{N}}$, Vec_G

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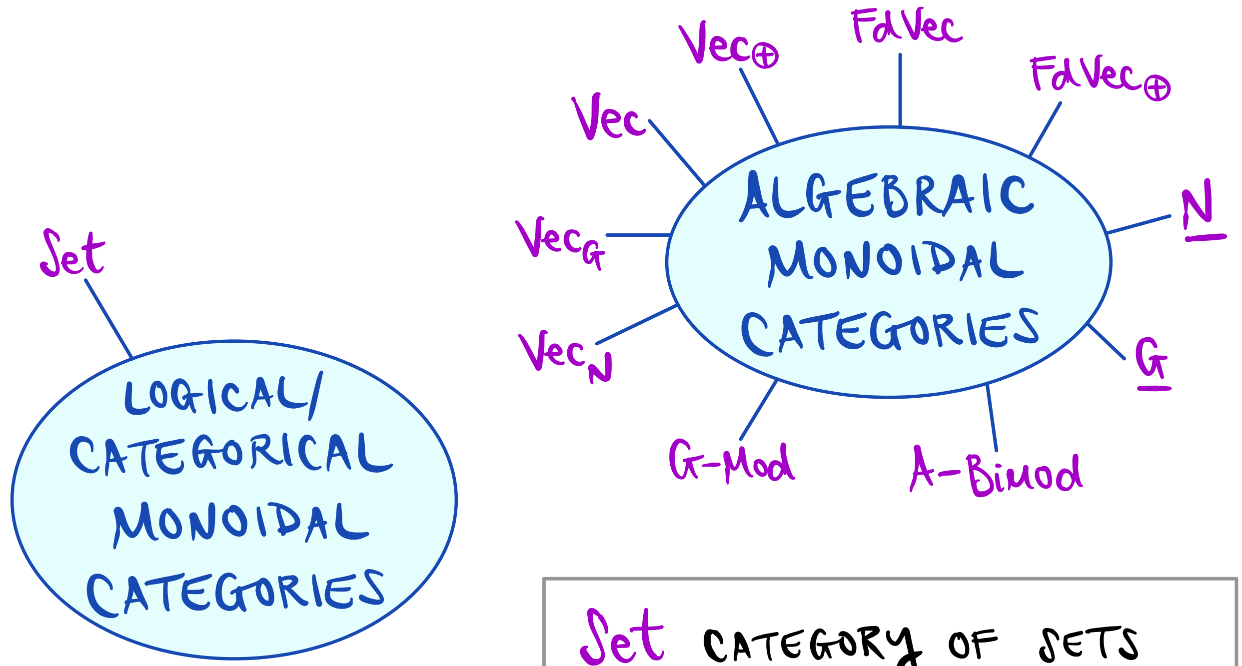

Set CATEGORY OF SETS

$\otimes := \times$ CARTESIAN PRODUCT

$\mathbb{1} := \{\bullet\}$ SINGLETON SET

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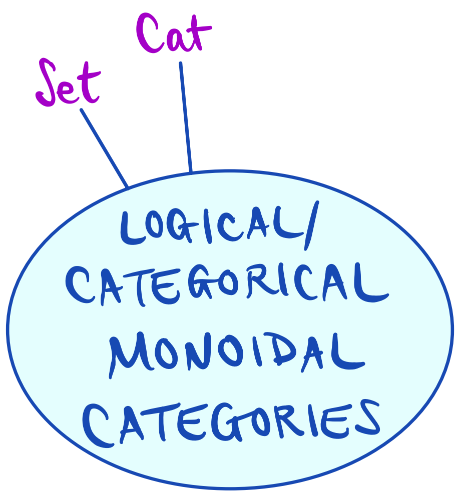
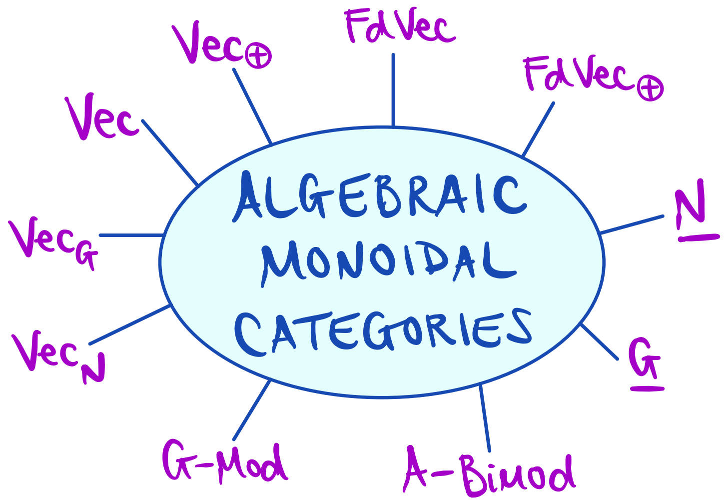
SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM





Set CATEGORY OF SETS

$\otimes := \times$ CARTESIAN PRODUCT

$\mathbb{1} := \{\bullet\}$ SINGLETON SET

Cat CATEG. OF SMALL CATEGORIES

$\otimes := \times$ PRODUCT OF CATEGORIES

$\mathbb{1} := \mathbf{1}$ $\quad Ob(\mathbf{1}) = \{*\}$

$Hom_{\mathbf{1}}(*,*) = \{id_*\}$

I. MONOIDAL CATEGORIES

MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

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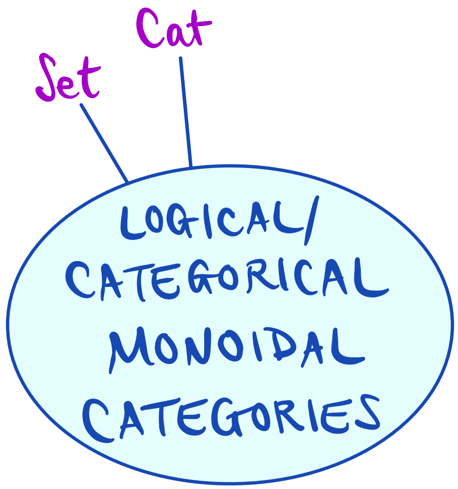
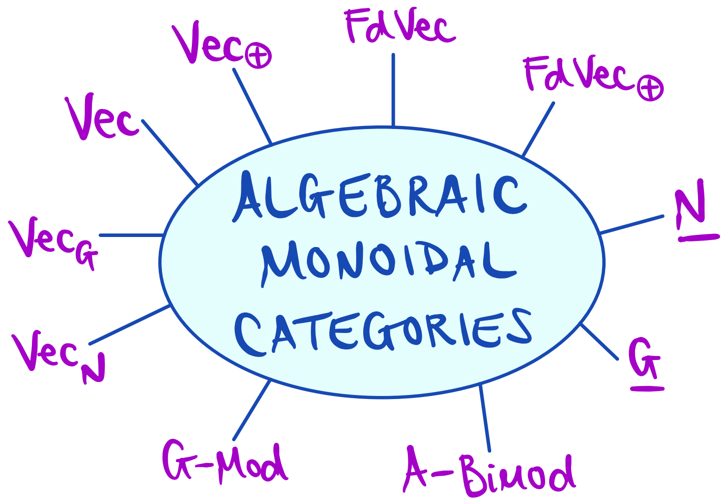
PENTAGON AXIOM
& TRIANGLE AXIOM





$\mathcal{A}$ ≡ CATEGORY

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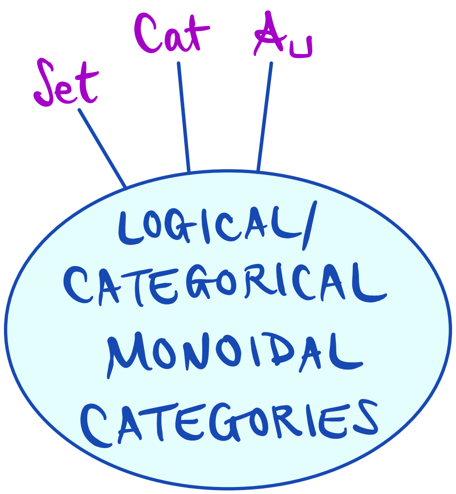

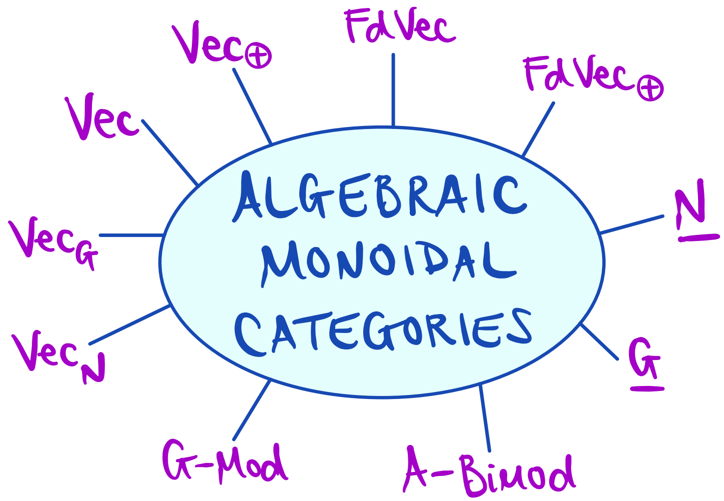

$\mathcal{A}$ = CATEGORY

$\mathcal{A}_\sqcup$ $\quad \otimes := \sqcup$ (COPRODUCT) $\quad \mathbb{1} := I$ (INITIAL OBJ.)

IF THEY EXIST

COCARTESIAN MONOIDAL CATEG.

I. MONOIDAL CATEGORIES

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SATISFYING THE
PENTAGON AXIOM
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LOGICAL/CATEGORICAL MONOIDAL CATEGORIES

Set

Cat

$\mathcal{A}_\sqcup$

ALGEBRAIC MONOIDAL CATEGORIES

Vec

$\mathrm{Vec}_\oplus$

FdVec

$\mathrm{FdVec}_\oplus$

$\underline{\mathbb{N}}$

$\underline{G}$

A-Bimod

G-Mod

$\mathrm{Vec}_{\mathbb{N}}$

Vec_G

E.G. $\mathrm{Vec}_\oplus$

$\otimes = \oplus$

$\mathbb{1} = 0_{vs}$

$\mathcal{A}$ = CATEGORY

$\mathcal{A}_\sqcup$	COPRODUCT $\otimes := \sqcup$	INITIAL OBJ. $\mathbb{1} := I$

IF THEY EXIST

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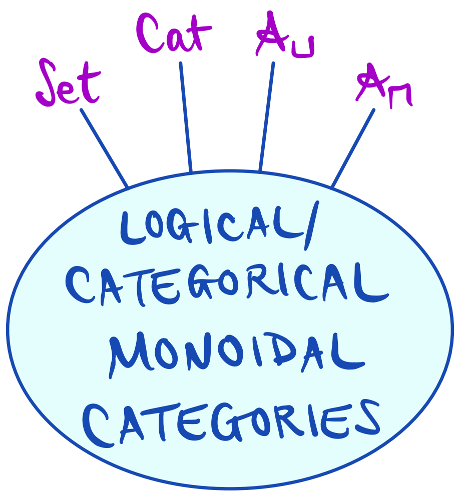

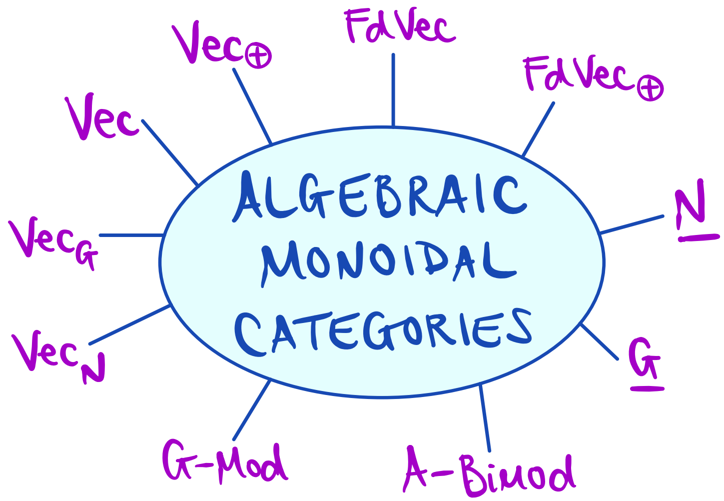

E.G. $\mathrm{Vec}_{\oplus}$ ⇝

$\otimes = \oplus$

$\mathbb{1} = 0_{vs}$

E.G. Set ⇝

$\otimes = \times$

$\mathbb{1} = \{\bullet\}$

$\mathcal{A}$ = CATEGORY		
$\mathcal{A}_{\sqcup}$	COPRODUCT $\otimes := \sqcup$	INITIAL OBJ. $\mathbb{1} := I$
	IF THEY EXIST	
COCARTESIAN MONOIDAL CATEG.		
$\mathcal{A}_{\sqcap}$	PRODUCT $\otimes := \sqcap$	TERMINAL OBJ. $\mathbb{1} := T$
	IF THEY EXIST	
CARTESIAN MONOIDAL CATEG.		

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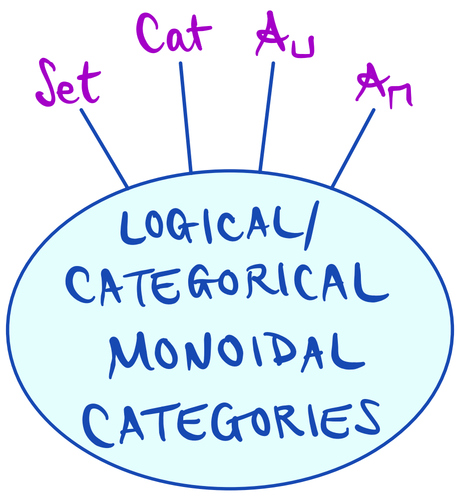

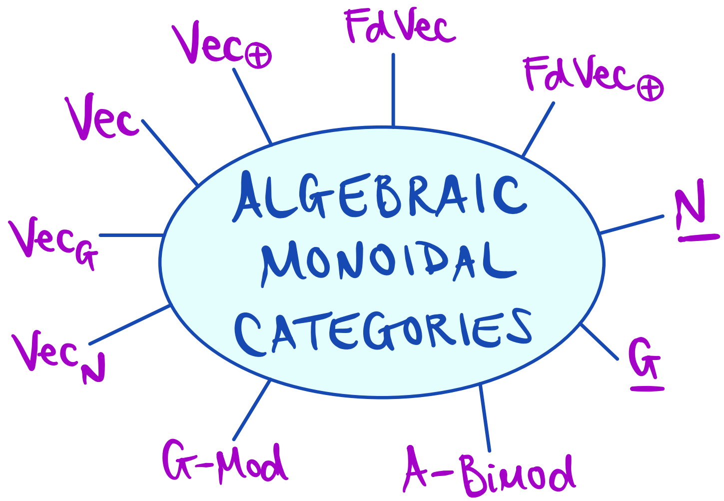

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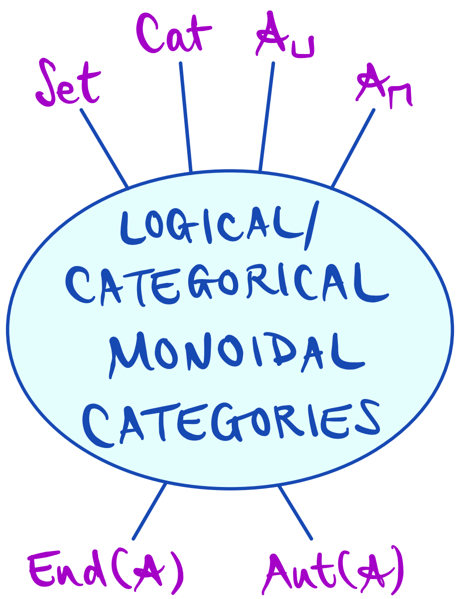

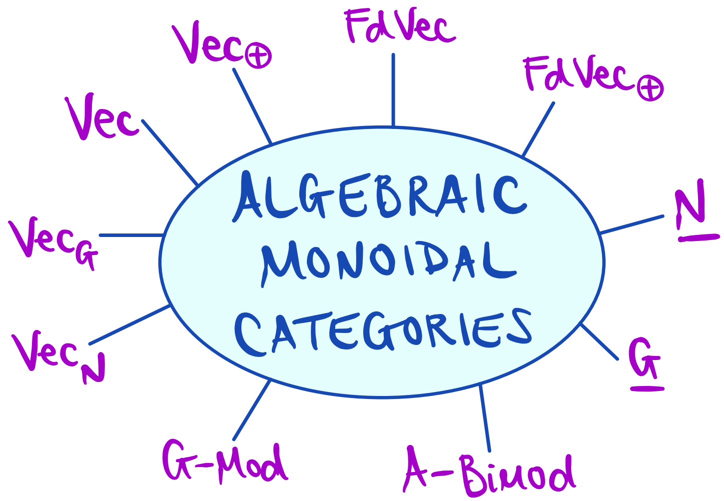

$\mathcal{A}$ = CATEGORY

End($\mathcal{A}$) ENDOFUNCTORS OF $\mathcal{A}$

$\otimes :=$ COMPOSITION $\quad \mathbb{1} := \mathrm{Id}_{\mathcal{A}}$

Aut($\mathcal{A}$) AUTOEQUIVALENCES OF $\mathcal{A}$

$\otimes :=$ COMPOSITION $\quad \mathbb{1} := \mathrm{Id}_{\mathcal{A}}$

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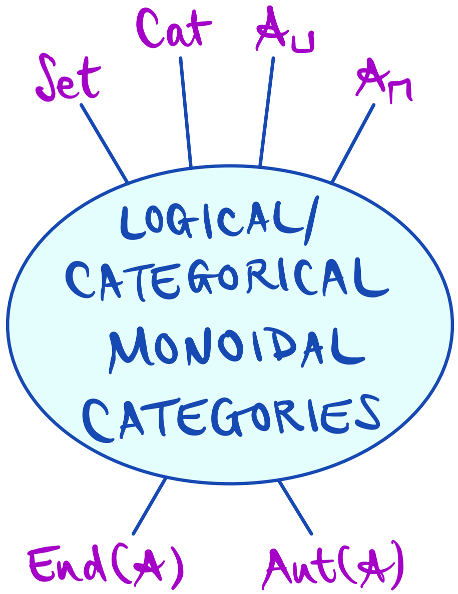

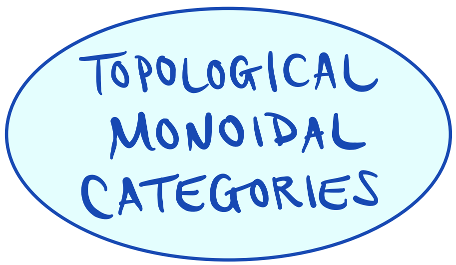

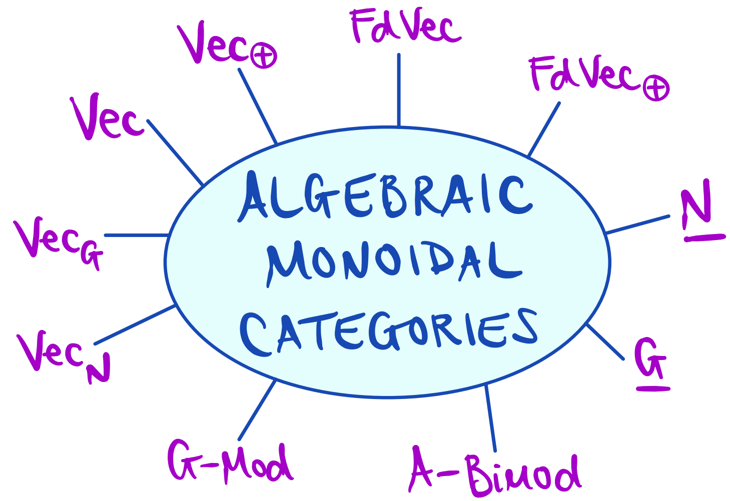

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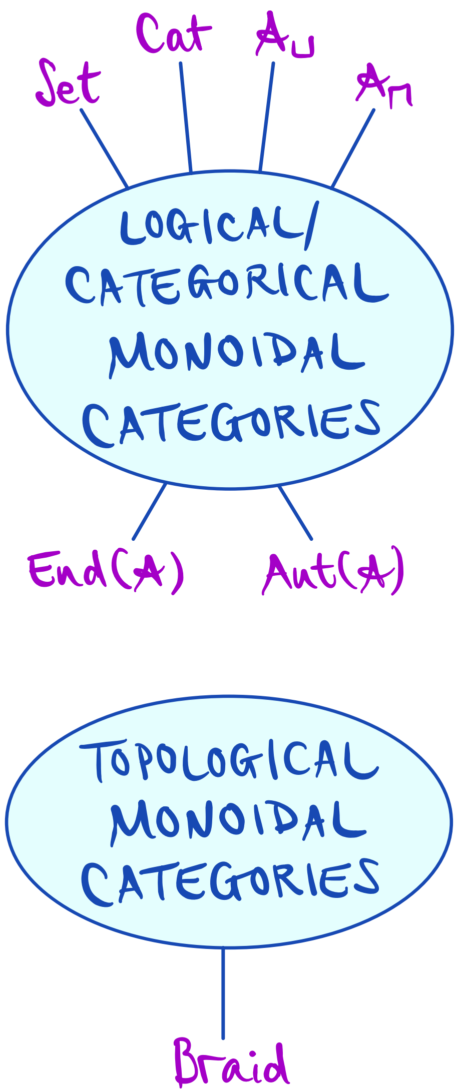

Braid

BRAID GROUP

$$B_n := \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ \forall i \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j|>1 \end{array} \right\rangle$$

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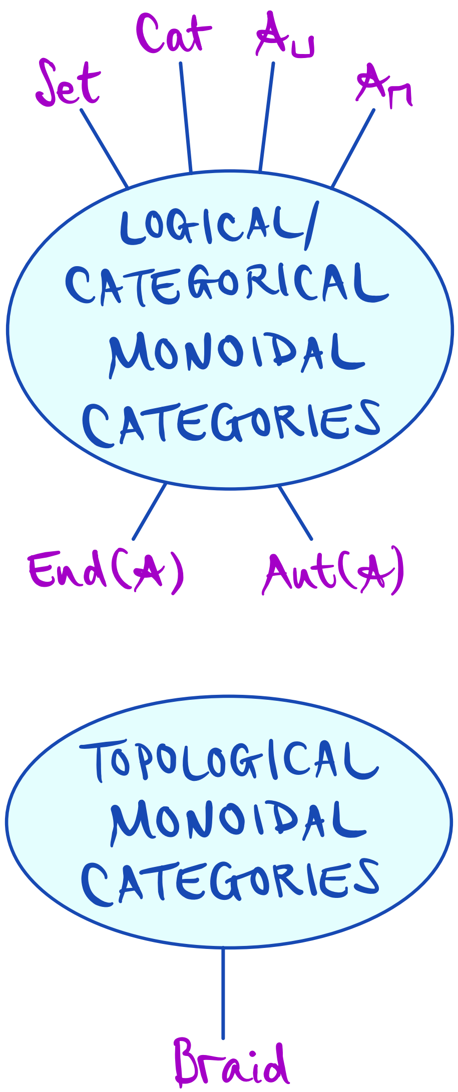

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OBJECTS $n \in \mathbb{N}$

$$\mathrm{Hom}_{\mathrm{Braid}}(n, m) = \begin{cases} B_n & n = m \\ \emptyset & n \neq m \end{cases}$$

VISUALIZED AS BRAIDS

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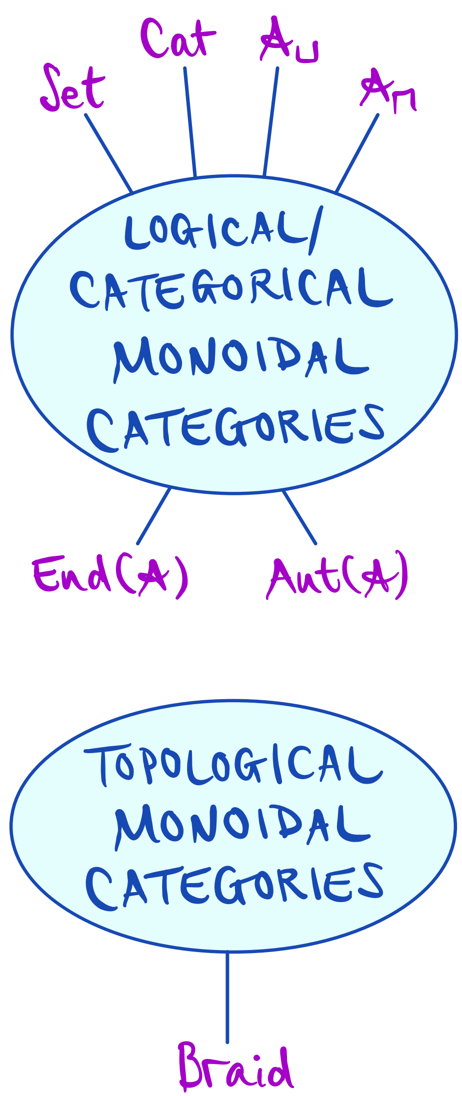

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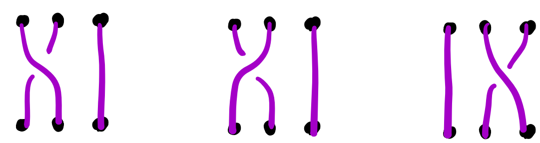
VISUALIZED AS BRAIDS (n=3)



σ_1 $\quad\sigma_1^{-1}$ $\quad\sigma_2$

AS MORPHISMS $3 \to 3$

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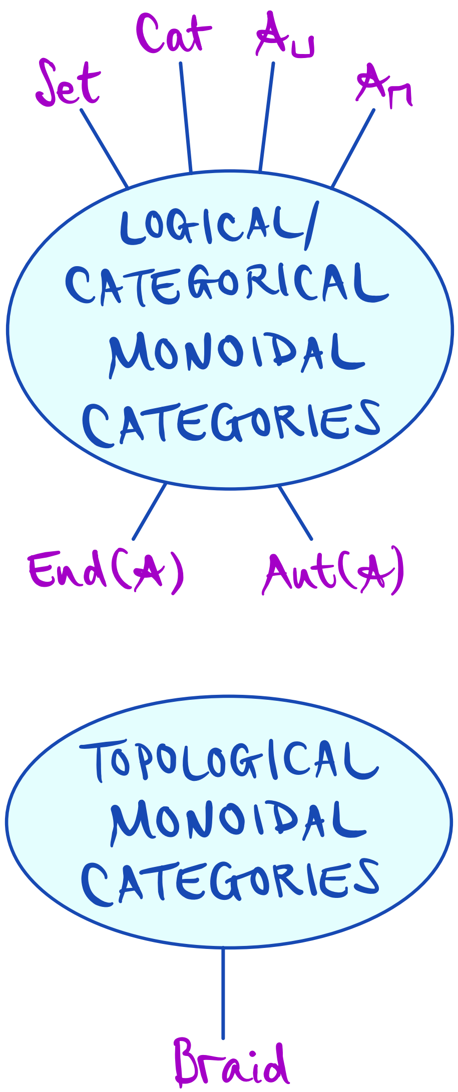

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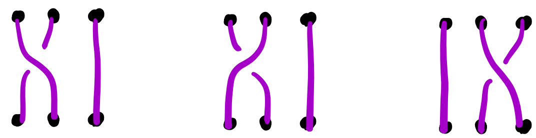

σ_1 σ_1^{-1} σ_2

AS MORPHISMS $3 \to 3$

$n \otimes m := n + m$ $\mathbb{1} := 0 \in \mathbb{N}$

$\otimes$ OF MORPHISMS := PUTTING BRAIDS SIDE-BY-SIDE

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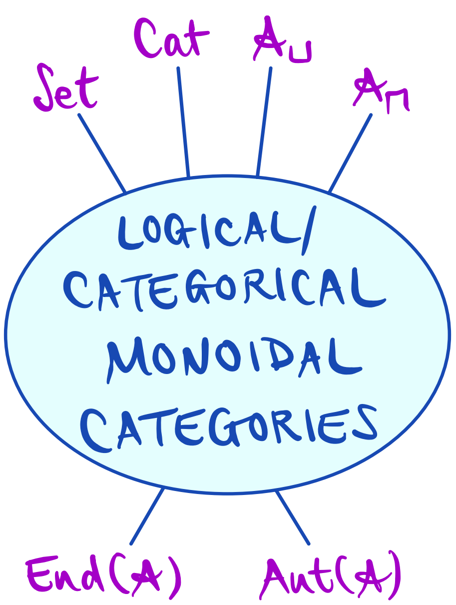

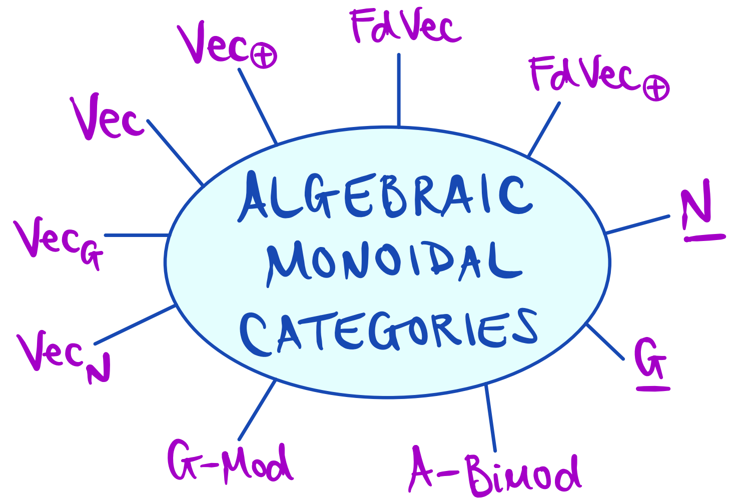

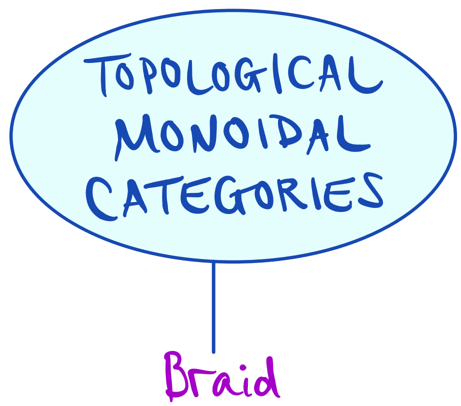

AND MANY MORE!

II. ADDITIVE MONOIDAL CATEGORIES

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

HOW DO THESE MESH?

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ab}$ $\forall X, Y \in \mathcal{C}$ $\xrightarrow{\text{DEF}}$ PREADDITIVE CATEGORY

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Vec}$ $\forall X, Y \in \mathcal{C}$ $\xrightarrow{\text{DEF}}$ LINEAR CATEGORY

PREADDITIVE CATEGORY + $\exists X \oplus Y \in \mathcal{C}$ $\forall X, Y \in \mathcal{C}$ BIPRODUCT + $\exists 0 \in \mathcal{C}$ ZERO OBJ. $\xrightarrow{\text{DEF}}$ ADDITIVE CATEGORY

ADDITIVE CATEGORY + $\exists$ COKERNELS & KERNELS + $\{$MONOS, EPIS$\}$ ARE $\{$KERNELS, COKERNELS$\}$ ($\equiv$ NORMALITY $\equiv$) $\xrightarrow{\text{DEF}}$ ABELIAN CATEGORY

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

HOW DO THESE MESH?

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ab}$ $\forall X,Y \in \mathcal{C}$ $\xrightarrow{\text{DEF}}$ PREADDITIVE CATEGORY

PREADDITIVE CATEGORY, $\exists X \oplus Y \in \mathcal{C}$ $\forall X,Y \in \mathcal{C}$ BIPRODUCT, $\exists 0 \in \mathcal{C}$ ZERO OBJ. $\xrightarrow{\text{DEF}}$ ADDITIVE CATEGORY

ADDITIVE CATEGORY, $\exists$ COKERNELS & KERNELS, $\left\{\begin{matrix}\text{MONOS}\\ \text{EPIS}\end{matrix}\right.$ ARE $\left\{\begin{matrix}\text{KERNELS}\\ \text{COKERNELS}\end{matrix}\right.$ (= NORMALITY =) $\xrightarrow{\text{DEF}}$ ABELIAN CATEGORY

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Vec}$ $\forall X,Y \in \mathcal{C}$ $\xrightarrow{\text{DEF}}$ LINEAR CATEGORY

A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ IS **ADDITIVE** IF $\mathcal{C}$ IS ADDITIVE & $(X \otimes -), (- \otimes X) : \mathcal{C} \to \mathcal{C}$ ARE ADDITIVE $\forall X \in \mathcal{C}$

II. ADDITIVE MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

(d, e, f) NATURAL ISOMS:

$$a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$$

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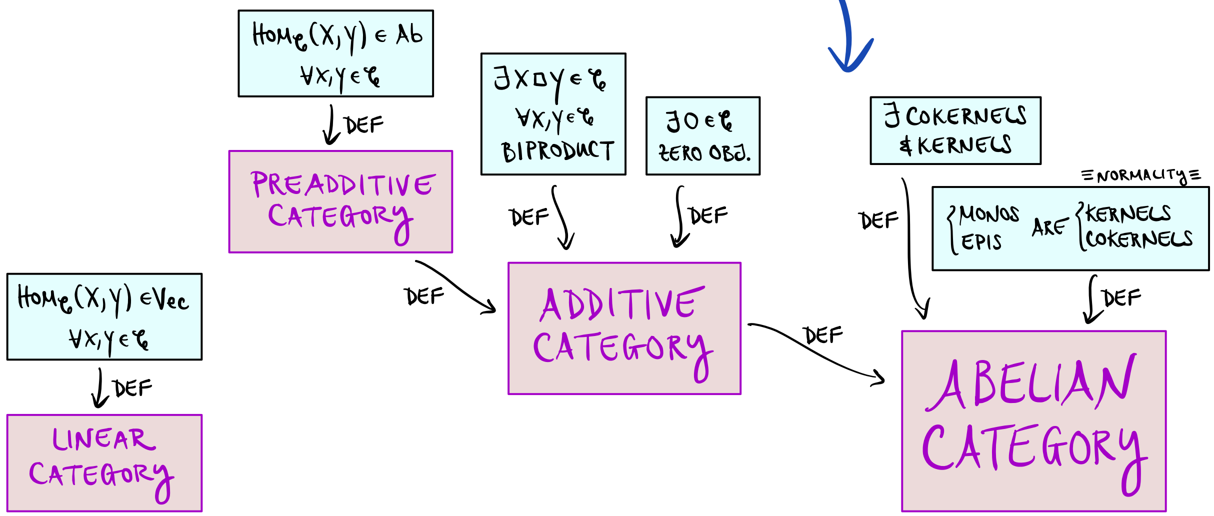
SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

HOW DO THESE MESH?



A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ IS **ADDITIVE** IF $\mathcal{C}$ IS ADDITIVE & $(X \otimes -), (- \otimes X) : \mathcal{C} \to \mathcal{C}$ ARE ADDITIVE $\forall X \in \mathcal{C}$

↳ $\forall$ MORPHISMS $f, f' \in \mathcal{C}$ GET

$$id_X \otimes (f + f') = (id_X \otimes f) + (id_X \otimes f')$$

$$(f + f') \otimes id_X = (f \otimes id_X) + (f' \otimes id_X)$$

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SATISFYING THE

PENTAGON AXIOM

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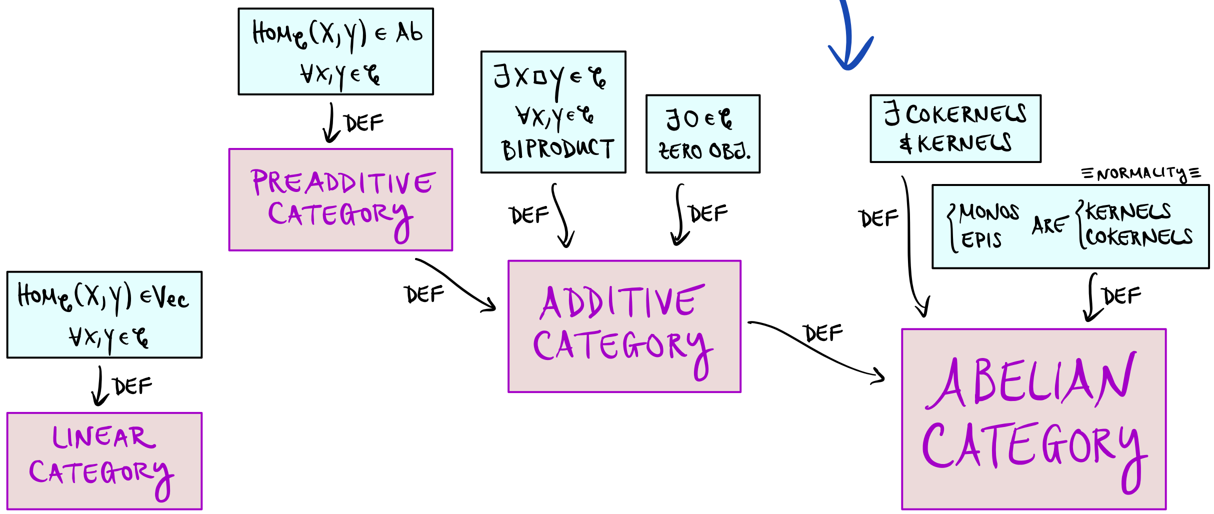
HOW DO THESE MESH?



A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ IS **ADDITIVE** IF $\mathcal{C}$ IS ADDITIVE & $(X \otimes -), (- \otimes X) : \mathcal{C} \to \mathcal{C}$ ARE ADDITIVE $\forall X \in \mathcal{C}$

<u>EXERCISE 2.19</u> $\Rightarrow$ WHEN $\mathcal{C}$ ADDITIVE MONOIDAL, GET:

$$X \otimes (Y \oplus Z) \cong (X \otimes Y) \oplus (X \otimes Z)$$
$$(Y \oplus Z) \otimes X \cong (Y \otimes X) \oplus (Z \otimes X) \qquad \forall X,Y,Z \in \mathcal{C}$$

II. ADDITIVE MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

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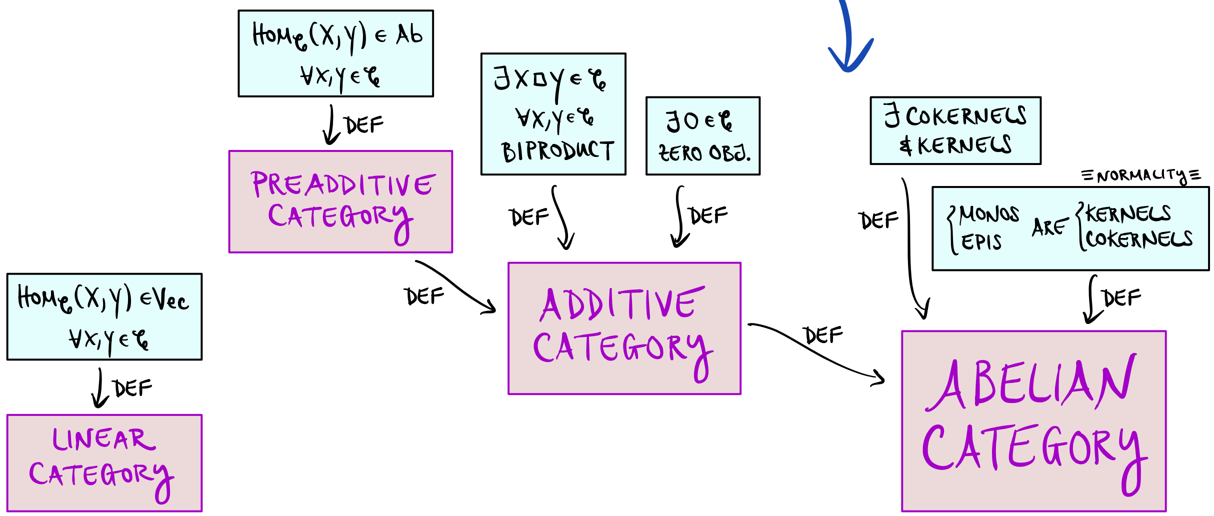

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

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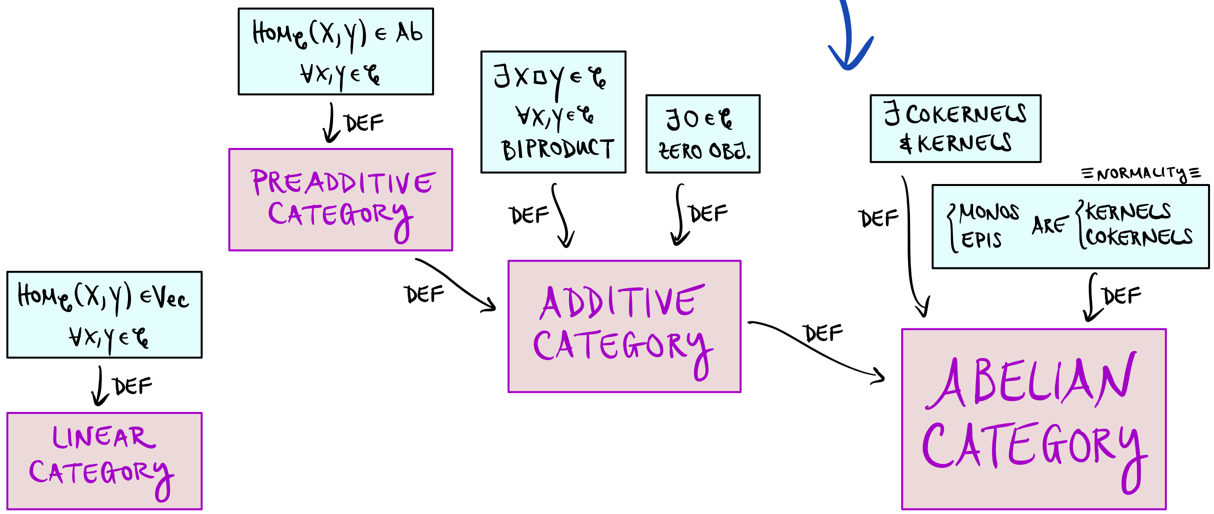

A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ IS **ABELIAN** IF $\mathcal{C}$ IS ABELIAN & $(X \otimes -), (- \otimes X) : \mathcal{C} \to \mathcal{C}$ ARE ADDITIVE $\forall X \in \mathcal{C}$

↳ $\forall$ MORPHISMS $f, f' \in \mathcal{C}$ GET

$$id_X \otimes (f + f') = (id_X \otimes f) + (id_X \otimes f')$$

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HOW DO THESE MESH?

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$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Vec}$ $\forall X,Y \in \mathcal{C}$ $\xrightarrow{\text{DEF}}$ LINEAR CATEGORY

A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ IS **LINEAR** IF $\mathcal{C}$ IS LINEAR & $(X \otimes -), (- \otimes X) : \mathcal{C} \to \mathcal{C}$ ARE LINEAR $\forall X \in \mathcal{C}$

↳ $\forall$ MORPHISMS $f, f' \in \mathcal{C}$ GET

$$\mathrm{id}_X \otimes (f + f') = (\mathrm{id}_X \otimes f) + (\mathrm{id}_X \otimes f'),$$

$$(f + f') \otimes \mathrm{id}_X = (f \otimes \mathrm{id}_X) + (f' \otimes \mathrm{id}_X),$$

$$\mathrm{id}_X \otimes \lambda f = \lambda(\mathrm{id}_X \otimes f), \quad \lambda f \otimes \mathrm{id}_X = \lambda(f \otimes \mathrm{id}_X)$$

III. MONOIDAL FUNCTORS

HOW TO MOVE FROM ONE MONOIDAL CATEGORY TO ANOTHER...

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

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SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

III. MONOIDAL FUNCTORS

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
>
> CONSISTS OF:
>
> (a) CATEGORY $\mathcal{C}$
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> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
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> SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

TAKE MONOIDAL CATEGORIES

$\mathcal{C} := (\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, a^{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}})$ & $\mathcal{D} := (\mathcal{D}, \otimes^{\mathcal{D}}, \mathbb{1}^{\mathcal{D}}, a^{\mathcal{D}}, \ell^{\mathcal{D}}, r^{\mathcal{D}})$.

A **MONOIDAL FUNCTOR** FROM $\mathcal{C}$ TO $\mathcal{D}$ CONSISTS OF:

III. MONOIDAL FUNCTORS

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
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A **MONOIDAL FUNCTOR** FROM $\mathcal{C}$ TO $\mathcal{D}$ CONSISTS OF:

(a) A FUNCTOR BTW UNDERLYING CATEGORIES

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

III. MONOIDAL FUNCTORS

> MONOIDAL CATEGORY
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>
> SATISFYING THE
> PENTAGON AXIOM
> & TRIANGLE AXIOM

TAKE MONOIDAL CATEGORIES

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A MONOIDAL FUNCTOR FROM $\mathcal{C}$ TO $\mathcal{D}$ CONSISTS OF:

(a) A FUNCTOR BTW UNDERLYING CATEGORIES

$$F:\mathcal{C}\longrightarrow\mathcal{D}$$

(b) A NATURAL TRANSFORMATION
(MONOIDAL PRODUCT CONSTRAINT)

$$\begin{tikzcd}
\mathcal{C}\times\mathcal{C} \arrow[r, bend left, "\otimes^{\mathcal{D}}\circ(F\times F)"] \arrow[r, bend right, "F\circ\otimes^{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow F^{(2)}"] & \mathcal{D}
\end{tikzcd}$$

$$F^{(2)}=\{F^{(2)}_{X,Y}:F(X)\otimes^{\mathcal{D}}F(Y)\longrightarrow F(X\otimes^{\mathcal{C}}Y)\}_{X,Y\in\mathcal{C}}$$

III. MONOIDAL FUNCTORS

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

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SATISFYING THE
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TAKE MONOIDAL CATEGORIES

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(MONOIDAL UNIT CONSTRAINT)

III. MONOIDAL FUNCTORS

MONOIDAL CATEGORY
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SATISFYING THE
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(MONOIDAL UNIT CONSTRAINT)

PRESERVING THE STRUCTURE OF THE MONOIDAL CATEGS.

III. MONOIDAL FUNCTORS

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PRESERVING THE STRUCTURE OF THE MONOIDAL CATEGS.

SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS ...

III. MONOIDAL FUNCTORS

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$$F^{(2)} = \{ F^{(2)}_{X,Y} : F(X) \otimes^{\mathcal{D}} F(Y) \longrightarrow F(X \otimes^{\mathcal{C}} Y) \}_{X,Y \in \mathcal{C}}.$$

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS ...

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>
> $a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$
>
> $l = \{ l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
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>
> SATISFYING THE **PENTAGON AXIOM** & **TRIANGLE AXIOM**

TAKE MONOIDAL CATEGORIES

$$\mathcal{C} := (\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, a^{\mathcal{C}}, l^{\mathcal{C}}, r^{\mathcal{C}}) \quad \& \quad \mathcal{D} := (\mathcal{D}, \otimes^{\mathcal{D}}, \mathbb{1}^{\mathcal{D}}, a^{\mathcal{D}}, l^{\mathcal{D}}, r^{\mathcal{D}}).$$

A **MONOIDAL FUNCTOR** FROM $\mathcal{C}$ TO $\mathcal{D}$ CONSISTS OF:

(a) A FUNCTOR BTW CATEGORIES $F : \mathcal{C} \longrightarrow \mathcal{D}$.

(b) A NATURAL TRANSFORMATION

$$F^{(2)} = \{ F^{(2)}_{X,Y} : F(X) \otimes^{\mathcal{D}} F(Y) \longrightarrow F(X \otimes^{\mathcal{C}} Y) \}_{X,Y \in \mathcal{C}}.$$

(c) A MORPHISM $F^{(0)} : \mathbb{1}^{\mathcal{D}} \longrightarrow F(\mathbb{1}^{\mathcal{C}})$ IN $\mathcal{D}$.

SATISFYING:

$$\begin{tikzcd}
(F(X) \otimes^{\mathcal{D}} F(Y)) \otimes^{\mathcal{D}} F(Z) \arrow[r, "F^{(2)}_{X,Y} \otimes \mathrm{id}"] \arrow[d, "a^{\mathcal{D}}_{F(X),F(Y),F(Z)}"'] & F(X \otimes^{\mathcal{C}} Y) \otimes^{\mathcal{D}} F(Z) \arrow[r, "F^{(2)}_{X \otimes Y, Z}"] \arrow[d, phantom, "\circlearrowright"] & F((X \otimes^{\mathcal{C}} Y) \otimes^{\mathcal{C}} Z) \arrow[d, "F(a^{\mathcal{C}}_{X,Y,Z})"] \\
F(X) \otimes^{\mathcal{D}} (F(Y) \otimes^{\mathcal{D}} F(Z)) \arrow[r, "\mathrm{id} \otimes F^{(2)}_{Y,Z}"] & F(X) \otimes^{\mathcal{D}} F(Y \otimes^{\mathcal{C}} Z) \arrow[r, "F^{(2)}_{X, Y \otimes Z}"] & F(X \otimes^{\mathcal{C}} (Y \otimes^{\mathcal{C}} Z))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1}^{\mathcal{D}} \otimes^{\mathcal{D}} F(X) \arrow[r, "l^{\mathcal{D}}_{F(X)}"] \arrow[d, "F^{(0)} \otimes \mathrm{id}"'] \arrow[dr, phantom, "\circlearrowright"] & F(X) \\
F(\mathbb{1}^{\mathcal{C}}) \otimes^{\mathcal{D}} F(X) \arrow[r, "F^{(2)}_{\mathbb{1},X}"] & F(\mathbb{1} \otimes^{\mathcal{C}} X) \arrow[u, "F(l^{\mathcal{C}}_X)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \otimes^{\mathcal{D}} \mathbb{1}^{\mathcal{D}} \arrow[r, "r^{\mathcal{D}}_{F(X)}"] \arrow[d, "\mathrm{id} \otimes F^{(0)}"'] \arrow[dr, phantom, "\circlearrowright"] & F(X) \\
F(X) \otimes^{\mathcal{D}} F(\mathbb{1}^{\mathcal{C}}) \arrow[r, "F^{(2)}_{X,\mathbb{1}}"] & F(X \otimes^{\mathcal{C}} \mathbb{1}) \arrow[u, "F(r^{\mathcal{C}}_X)"']
\end{tikzcd}$$

III. MONOIDAL FUNCTORS

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$
>
> CONSISTS OF:
>
> (a) CATEGORY $\mathcal{C}$
>
> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS ...

III. MONOIDAL FUNCTORS

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS ...

STRICT IF $\{F^{(2)}_{X,Y}\}$ & $F^{(0)}$ ARE ALL IDENTITY MORPHISMS IN $\mathcal{D}$

III. MONOIDAL FUNCTORS

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS ...

STRICT IF $\{F^{(2)}_{X,Y}\}$ & $F^{(0)}$ ARE ALL IDENTITY MORPHISMS IN $\mathcal{D}$

> <u>EX</u> $\mathrm{Id}_{(\mathcal{C},\otimes)} := (\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}^{(2)}, \mathrm{Id}^{(0)})$ WITH
>
> $\mathrm{Id}^{(2)}_{X,Y} := \mathrm{id}_{X \otimes^{\mathcal{C}} Y} \qquad \mathrm{Id}^{(0)} := \mathrm{id}_{\mathbb{1}^{\mathcal{C}}}$

III. MONOIDAL FUNCTORS

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS ...

STRICT IF $\{F^{(2)}_{X,Y}\}$ & $F^{(0)}$ ARE ALL IDENTITY MORPHISMS IN $\mathcal{D}$

STRONG IF $\{F^{(2)}_{X,Y}\}$ & $F^{(0)}$ ARE ALL ISOS IN $\mathcal{D}$

<u>EX</u> $Id_{(\mathcal{C},\otimes)} := (Id_{\mathcal{C}}, Id^{(2)}, Id^{(0)})$ WITH

$$Id^{(2)}_{X,Y} := id_{X \otimes^{\mathcal{C}} Y} \qquad Id^{(0)} := id_{\mathbb{1}^{\mathcal{C}}}$$

III. MONOIDAL FUNCTORS

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS ...

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$\Rightarrow$

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III. MONOIDAL FUNCTORS

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SATISFYING THE **PENTAGON AXIOM** & **TRIANGLE AXIOM**

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

EXAMPLE G = GROUP.

$$F := \mathrm{Forg} : G\text{-Mod} \longrightarrow \mathrm{Vec} \qquad (V, \triangleright) \longmapsto V$$

III. MONOIDAL FUNCTORS

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III. MONOIDAL FUNCTORS

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III. MONOIDAL FUNCTORS

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III. MONOIDAL FUNCTORS

MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$
CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR
$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

(d, e, f) NATURAL ISOMS:

$$a = \left\{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \right\}_{X,Y,Z \in \mathcal{C}}$$

$$l = \{ l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

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SATISFYING THE
PENTAGON AXIOM
& TRIANGLE AXIOM

A MONOIDAL FUNCTOR FROM $\mathcal{C}$ TO $\mathcal{D}$ CONSISTS OF:

(a) A FUNCTOR BTW CATEGORIES $F : \mathcal{C} \to \mathcal{D}$.

(b) A NATURAL TRANSF'N $F^{(2)} = \{ F^{(2)}_{X,Y} : F(X) \otimes^{\mathcal{D}} F(Y) \to F(X \otimes^{\mathcal{C}} Y) \}_{X,Y \in \mathcal{C}}$.

(c) A MORPHISM $F^{(0)} : \mathbb{1}^{\mathcal{D}} \to F(\mathbb{1}^{\mathcal{C}})$ IN $\mathcal{D}$.

SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

EXAMPLE G = GROUP.

$$F := \mathrm{Forg} : G\text{-Mod} \longrightarrow \mathrm{Vec} \qquad (V, \triangleright) \mapsto V$$

$$\begin{array}{ccccc} F^{(2)}_{(V,\triangleright),(V',\triangleright')} & : & F(V,\triangleright) \otimes^{\mathrm{Vec}} F(V',\triangleright') & \longrightarrow & F((V,\triangleright) \otimes^{G\text{-Mod}} (V',\triangleright')) \\ \| & & \| & & \| \\ & & F(V,\triangleright) \otimes_{\Bbbk} F(V',\triangleright') & & F(V \otimes_{\Bbbk} V', \blacktriangleright) \\ & & \| & & \| \\ \mathrm{id}_{V \otimes_{\Bbbk} V'} & & V \otimes_{\Bbbk} V' & & V \otimes_{\Bbbk} V' \end{array}$$

$$\begin{array}{ccccc} F^{(0)} & : & \mathbb{1}^{\mathrm{Vec}} & \longrightarrow & F(\mathbb{1}^{G\text{-Mod}}) \\ \| & & \| & & \| \\ \mathrm{id}_{\Bbbk} & & \Bbbk & & \Bbbk \end{array}$$

III. MONOIDAL FUNCTORS

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

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$$F^{(2)}_{(V,\triangleright),(V',\triangleright')} = \mathrm{id}_{V \otimes_{\Bbbk} V'}, \quad F(V,\triangleright) \otimes^{\mathrm{Vec}} F(V',\triangleright') = F(V,\triangleright) \otimes_{\Bbbk} F(V',\triangleright') = V \otimes_{\Bbbk} V', \quad F((V,\triangleright) \otimes^{G\text{-Mod}} (V',\triangleright')) = F(V \otimes_{\Bbbk} V', \blacktriangleright) = V \otimes_{\Bbbk} V'$$

$$F^{(0)} : \mathbb{1}^{\mathrm{Vec}} \longrightarrow F(\mathbb{1}^{G\text{-Mod}})$$

$$F^{(0)} = \mathrm{id}_{\Bbbk}, \quad \mathbb{1}^{\mathrm{Vec}} = \Bbbk, \quad F(\mathbb{1}^{G\text{-Mod}}) = \Bbbk$$

STRICT & STRONG MONOIDAL

III. MONOIDAL FUNCTORS

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

EXAMPLE

$$F := \mathrm{Forg} : \mathrm{FdVec} \longrightarrow \mathrm{Set}$$

$$(V_{set}, +, 0, *) \mapsto V_{set}$$

III. MONOIDAL FUNCTORS

MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF:

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

EXAMPLE $F := \mathrm{Forg} : \mathrm{FdVec} \longrightarrow \mathrm{Set}$

$$(V_{\mathrm{set}}, +, 0, *) \mapsto V_{\mathrm{set}}$$

$$F^{(2)}_{V,V'} : \underset{\substack{\| \\ V_{\mathrm{set}} \times V'_{\mathrm{set}}}}{F(V) \otimes^{\mathrm{Set}} F(V')} \longrightarrow \underset{\substack{\| \\ (V \otimes_{\mathbb{R}} V')_{\mathrm{set}}}}{F(V \otimes^{\mathrm{FdVec}} V')}$$

$$(v, v') \longmapsto v \otimes_{\mathbb{R}} v'$$

III. MONOIDAL FUNCTORS

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

EXAMPLE

$$F := \text{Forg} : \text{FdVec} \longrightarrow \text{Set}$$

$$(V_{\text{set}}, +, 0, *) \mapsto V_{\text{set}}$$

$$F^{(2)}_{V,V'} : \underbrace{F(V) \otimes^{\text{Set}} F(V')}_{= V_{\text{set}} \times V'_{\text{set}}} \longrightarrow \underbrace{F(V \otimes^{\text{FdVec}} V')}_{= (V \otimes_{\mathbb{k}} V')_{\text{set}}}$$

$$(v, v') \longmapsto v \otimes_{\mathbb{k}} v'$$

$$F^{(0)} : \underbrace{\mathbb{1}^{\text{Set}}}_{= \{\bullet\}} \longrightarrow \underbrace{F(\mathbb{1}^{\text{FdVec}})}_{= \mathbb{k}_{\text{set}}}$$

III. MONOIDAL FUNCTORS

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

EXAMPLE

$$F := \mathrm{Forg} : \mathrm{FdVec} \longrightarrow \mathrm{Set}$$

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$$(v, v') \longmapsto v \otimes_{\mathbb{k}} v'$$

$$F^{(0)} : \underset{\substack{\parallel \\ \{\bullet\}}}{\mathbb{1}^{Set}} \longrightarrow \underset{\substack{\parallel \\ \mathbb{k}_{set}}}{F(\mathbb{1}^{FdVec})}$$

MONOIDAL NOT STRONG

III. MONOIDAL FUNCTORS

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

EXAMPLE REGULAR LEFT ACTION OF $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ ON ITSELF IS BY DEF'N THE STRONG MONOIDAL FUNCTOR:

$$\rho : (\mathcal{C}, \otimes) \longrightarrow \mathrm{End}(\mathcal{C})$$

UNDERLYING CATEG.

$$X \longmapsto \left[\begin{array}{l} (X \otimes -) : \mathcal{C} \to \mathcal{C} \\ \qquad W \mapsto X \otimes W \end{array} \right]$$

III. MONOIDAL FUNCTORS

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

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III. MONOIDAL FUNCTORS

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

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$$\rho^{(2)}_{X,Y} : \underset{(X\otimes -) \circ (Y \otimes -)}{\underbrace{\rho(X) \otimes^{\mathrm{End}(\mathcal{C})} \rho(Y)}} \to \underset{[(X \otimes Y) \otimes -]}{\underbrace{\rho(X \otimes Y)}}$$

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III. MONOIDAL FUNCTORS

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$$\rightsquigarrow \boxed{\rho^{(2)}_{X,Y}(Z) = a^{-1}_{X,Y,Z}} : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z$$

III. MONOIDAL FUNCTORS

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

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$$X \longmapsto \left[(X \otimes -) : \mathcal{C} \to \mathcal{C}, \quad W \mapsto X \otimes W \right]$$

$$\rho^{(2)}_{X,Y} : \underbrace{\rho(X) \otimes^{\mathrm{End}(\mathcal{C})} \rho(Y)}_{= (X \otimes -) \circ (Y \otimes -)} \to \underbrace{\rho(X \otimes Y)}_{= [(X \otimes Y) \otimes -]}$$

$$\rho^{(0)} : \underbrace{\mathbb{1}^{\mathrm{End}(\mathcal{C})}}_{= \mathrm{Id}_{\mathcal{C}}} \to \underbrace{\rho(\mathbb{1}^{\mathcal{C}})}_{= \mathbb{1}^{\mathcal{C}} \otimes -}$$

$$\rightsquigarrow \boxed{\rho^{(2)}_{X,Y}(Z) = a^{-1}_{X,Y,Z}} : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z$$

$$\rightsquigarrow \rho^{(0)}(Z) : Z \longrightarrow \mathbb{1}^{\mathcal{C}} \otimes Z$$

III. MONOIDAL FUNCTORS

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

(d, e, f) NATURAL ISOMS:

$$a = \{a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)\}_{X,Y,Z \in \mathcal{C}}$$

$$l = \{l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X\}_{X \in \mathcal{C}}$$

$$r = \{r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X\}_{X \in \mathcal{C}}$$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A **MONOIDAL FUNCTOR** FROM $\mathcal{C}$ TO $\mathcal{D}$ CONSISTS OF:

(a) A FUNCTOR BTW CATEGORIES $F : \mathcal{C} \to \mathcal{D}$.

(b) A NATURAL TRANSF'N $F^{(2)} = \{F^{(2)}_{X,Y} : F(X) \otimes^{\mathcal{D}} F(Y) \to F(X \otimes^{\mathcal{C}} Y)\}_{X,Y \in \mathcal{C}}$.

(c) A MORPHISM $F^{(0)} : \mathbb{1}^{\mathcal{D}} \to F(\mathbb{1}^{\mathcal{C}})$ IN $\mathcal{D}$.

SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

EXAMPLE: REGULAR LEFT ACTION OF $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ ON ITSELF IS BY DEF'N THE STRONG MONOIDAL FUNCTOR:

$$\rho : (\mathcal{C}, \otimes) \longrightarrow \mathrm{End}(\mathcal{C})$$

UNDERLYING CATEG.

$$X \longmapsto \left[\begin{array}{l} (X \otimes -) : \mathcal{C} \to \mathcal{C} \\ \quad W \mapsto X \otimes W \end{array} \right]$$

$$\rho^{(2)}_{X,Y} : \underbrace{\rho(X) \otimes^{\mathrm{End}(\mathcal{C})} \rho(Y)}_{= (X \otimes -) \circ (Y \otimes -)} \to \underbrace{\rho(X \otimes Y)}_{= [(X \otimes Y) \otimes -]}$$

$$\rightsquigarrow \boxed{\rho^{(2)}_{X,Y}(Z) = a^{-1}_{X,Y,Z}} : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z$$

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$$\rightsquigarrow \boxed{\rho^{(0)}(Z) = l_Z^{-1}} : Z \longrightarrow \mathbb{1}^{\mathcal{C}} \otimes Z$$

III. MONOIDAL FUNCTORS

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF:

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

A LEFT ACTION OF A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1})$ ON A CATEG. $\mathcal{A}$ IS BY DEF'N A STRONG MON. FUNCTOR

$$(\rho, \rho^{(2)}, \rho^{(0)}) : (\mathcal{C}, \otimes, \mathbb{1}) \longrightarrow (\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$$

III. MONOIDAL FUNCTORS

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

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EXER 3.7 THESE ARE LIKE REP'NS OF $(\mathcal{C}, \otimes, \mathbb{1})$

III. MONOIDAL FUNCTORS

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EXER 3.7 THESE ARE LIKE REP'NS OF $(\mathcal{C}, \otimes, \mathbb{1})$

SHOW THAT THIS YIELDS A PAIR: $(\mathcal{A}, \underbrace{* : \mathcal{C} \times \mathcal{A} \to \mathcal{A}}_{\text{BIFUNCTOR}})$

WITH NATURAL ISOMORPHISMS CORRESP. TO $\rho^{(2)}$, $\rho^{(0)}$

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SHOW THAT THIS YIELDS A PAIR: $(\mathcal{A}, * : \mathcal{C} \times \mathcal{A} \to \mathcal{A})$ BIFUNCTOR

WITH NATURAL ISOMORPHISMS CORRESP. TO $\rho^{(2)}, \rho^{(0)}$

MIMICKING REP'NS ⇝ MODULES

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #12

TOPICS:

~~I~~. MONOIDAL CATEGORIES (§§ 3.1.1, 3.1.2)

~~II~~. ADDITIVE MONOIDAL CATEGORIES (§3.1.3)

~~III~~. MONOIDAL FUNCTORS (§§3.2.1, 3.2.3)

NEXT TIME: "MODULE CATEGORIES" OVER MONOIDAL CATEGORIES

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ALSO NEXT TIME:

WILL ALSO DISCUSS WHEN TWO MONOIDAL CATEGORIES ARE "THE SAME"

TOPICS:

I. MONOIDAL CATEGORIES (§§ 3.1.1, 3.1.2)

II. ADDITIVE MONOIDAL CATEGORIES (§3.1.3)

III. MONOIDAL FUNCTORS (§§3.2.1, 3.2.3)

NEXT TIME: "MODULE CATEGORIES" OVER MONOIDAL CATEGORIES

Lecture #12 keywords: monoidal category, monoidal functor, pentagon axiom, strict monoidal category, triangle axiom