LECTURE \#13

TopIcs:
I. ISOMORPHISMS AND Equivalence of monoidal categories
II. MODULE CATEGORIES ( $\{\$ 3.3 .1,3.3 .2,3.3 .4$ )
II. bimodule categories (\$3.3.3)
I.ISOM. AND Equiv. of monoidal categories

MONOIDAL CATEGORY

$$
(\varphi, \otimes, \Psi, a, l, r)
$$

CONSISTS OF:
(a) CATEGORY C
(b) BIFUNCTOR

$$
\begin{aligned}
& \otimes: \zeta \times \zeta \rightarrow \zeta \\
& \text { (c) OBJECT } \mathbb{1} \in \zeta \\
& \text { ( } d, e, f \text { ) NATURAL ( } \text { (oms: } \\
& \begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \simeq x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned} \\
& x, y, z \in \zeta \\
& l=\left\{l_{x}: 1 \otimes x \stackrel{\sim}{\longrightarrow} x\right\}_{x \in G} \\
& r=\left\{r_{x}: x \otimes \mathbb{L} \xrightarrow{\sim} x\right\}_{x \in e} \\
& \text { SATISFYING THE } \\
& \text { pentagon axiom } \\
& \text { \& Triangle axiom }
\end{aligned}
$$

I.ISOM. AND Equiv. of monoidal categories

MONOIDAL CATEGORY

$$
(\varphi, \otimes, \mathbb{1}, a, l, r)
$$

CONSISTS OF:
(a) CATEGORY C
(b) BIFUNCTOR
$\otimes: \zeta \times 6 \rightarrow \zeta$
(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) NATURAL ( (oms:

$$
\begin{aligned}
& a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\} \\
& x, y, z \in \zeta \\
& l=\left\{\ell_{x}: 1 \sim X \xrightarrow{\sim} X\right\}_{x \in G} \\
& r=\left\{r_{x}: x \otimes \mathbb{1} \xrightarrow{\sim} x\right\}_{x \in G^{\prime}}
\end{aligned}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom

I.ISOM. AND Equiv. of monoidal categories

MONOIDAL CATEGORY $(\varphi, \otimes, \Psi, a, l, r)$
CONSISTS OF:
(a) CATEGORY C
(b) BIFUNCTOR
$\otimes: \zeta \times \zeta \rightarrow \zeta$
(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) NATURAL (8 0MS:

$$
\begin{aligned}
& a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\} \\
& x, y, z \in \zeta \\
& l=\left\{l_{x}: 1 \otimes x \xrightarrow{\sim} X\right\}_{x \in G} \\
& r=\left\{r_{x}: x \otimes \mathscr{L} \xrightarrow{\sim} x\right\}_{x \in G}
\end{aligned}
$$

Satisfying the
PENTAGON AXIOM \& Triangle axiom

A MONOIDAL FUNCTOR FROM $\zeta$ To $\theta$ CONSISTS OF:
(a) A FUNCTOR BTW CATEGORIES $F: \zeta \rightarrow \theta$.
(b) A NATURAL TRANSF'N $F^{(2)}=\left\{F_{x, y}^{(2)}: F(x) \otimes^{\theta} F(y) \rightarrow F\left(x \otimes^{i} y\right)\right\}_{x, y \in \varphi .}$.
(c) A MORPHISM $F^{(0)}: \mathbb{L}^{\theta} \rightarrow F\left(\mathbb{L}^{\varphi}\right)$ in $\theta$.

SUBJECT TO ASSOCIATIVITY \& UNITALITY AXIOMS

I.ISOM. AND Equiv. of monoidal categories

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF:
(a) CATEGORY Ce
(b) BIFUNCTOR
$\otimes: 6 \times 6 \rightarrow$ C
(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) natural (sons:

$$
\begin{aligned}
& \begin{aligned}
& a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\} \\
& x, z, z=
\end{aligned} \\
& x, y, z \in \boldsymbol{\zeta} \\
& l=\left\{l_{x}: 1 \otimes x \leadsto x\right\}_{x \in G} \\
& r=\left\{r_{x}: x \otimes \mathbb{L} \xrightarrow{\sim} x\right\}_{x \in e}
\end{aligned}
$$

Satisfying The
pentagon axiom \& Triangle axiom

A MONOIDAL FUNCTOR FROM $\xi$ TO $\theta$ CONSISTS OF:
(a) A Functor bTw categories $F: \zeta \rightarrow \theta$.
(b) A NATURAL TRANSF'N $F^{(2)}=\left\{F_{x, y}^{(2)}: F(x) \otimes F(y) \rightarrow F\left(x \otimes^{8} y\right)\right\}_{x, y \in e}$.
(c) A MORPHISM $F^{(0)}: \mathbb{L}^{\theta} \rightarrow F\left(\mathbb{L}^{(\varphi)}\right)$ N $\theta$.

SUBJECT TO ASSOCIATIVITY \& UNITALITY AXIOMS

I. Isom. And equiv. of monoidal categories

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF:
(a) CATEGORY C
(b) BIFUNCTOR
$\otimes: \zeta \times \zeta \rightarrow \zeta$
(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) NATURAL (8 0MS:

$$
\begin{aligned}
& a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\} \\
& x, y, z \in \zeta \\
& l=\left\{l_{x}: 1 \otimes x \xrightarrow{\sim} X\right\}_{x \in G} \\
& r=\left\{r_{x}: x \otimes \mathscr{L} \xrightarrow{\sim} x\right\}_{x \in G}
\end{aligned}
$$

Satisfying the
pentagon axiom \& Triangle axiom

A STRONG MONOIDAL FUNCTOR FROM $\zeta_{e}$ To $\theta$ CONSISTS OF:
(a) A FUNCTOR BTW CATEGORIES $F: \zeta \rightarrow \theta$.
(b) A NATURAL $\stackrel{\text { (SOMORPHISM }}{\text { (GANOF }}$

SUBJECT TO ASSOCIATIVITY \& UNITALITY AXIOMS

I. Isom. And equiv. of monoidal categories

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF:
(a) CATEGORY Ce
(b) BIFUNCTOR

$$
\begin{aligned}
& \text { (c) OBJECT } \mathbb{L} \in \zeta \\
& \text { ( } d, e, f \text { ) natural (toms: } \\
& \begin{aligned}
& a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\} \\
& x, z,
\end{aligned} \\
& x, y, z \in \zeta \\
& l=\left\{l_{x}: 1 \otimes x \leadsto x\right\}_{x \in G} \\
& r=\left\{r_{x}: x \otimes \mathbb{\sim} \xrightarrow{\longrightarrow} x\right\}_{x \in \varepsilon} \\
& \text { Satisfying The } \\
& \text { pentagon axiom } \\
& \text { \& Triangle axiom }
\end{aligned}
$$

A STRONG MONOIDAL FUNCTOR FROM $\xi$ To $\theta$ CONSISTS OF:
(a) A Functor bTw categories $F: C \rightarrow \theta$.
(b) A NATURAL (SOMORPHISM $F^{(2)}=\left\{F_{x, y}^{(2)}: F(x) \otimes F(y) \rightarrow F\left(x \otimes^{8} y\right)\right\}_{x, y \in \varphi}$.
(c) AN 150

SUBJECT TO ASSOCIATIVITY \& UNITALITY AXIOMS

I. Isom. And equiv. of monoidal categories

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF:
(a) CATEGORY Ce
(b) BIFUNCTOR


A STRONG MONOIDAL FUNCTOR FROM $\xi$ To $\theta$ CONSISTS OF:
(a) A Functor bTW categories $F: \zeta \rightarrow \theta$.
(b) A NATURAL (SOMORPHISM $F^{(2)}=\left\{F_{x, y}^{(2)}: F(x) \otimes F(y) \rightarrow F\left(x \otimes^{8} y\right)\right\}_{x, y \in \varphi}$.
(c) AN 150 $F^{(0)}: \mathbb{L}^{\theta} \rightarrow F\left(\mathbb{L}^{\varphi}\right) \mathbb{N} \theta$.

SUBJECT TO ASSOCIATIVITY \& UNITALITY AXIOMS


Q: When are monoidal categs the same? Braid
$\equiv$ ACTIONS $\equiv$
$(\xi, \otimes, L)^{2} A$ BY DEF IS:
STRONG MONOIDAL FUNCTOR

$$
p:(\zeta, \otimes, L) \rightarrow\left(\operatorname{End}(A), 0, I d_{A}\right)
$$

I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES
$\equiv \operatorname{RECAL} \equiv$
Two categories $\zeta$ द and $\Delta$ are the same via

Q: When are monoidal categs the same?
I. Isom. And equiv. of monoidal categories
$\equiv$ RECAU $\equiv$
Two categories $\zeta$ and $\theta$ are the same via
CATEGORY 1 iSOMORPHISM $\zeta \cong \theta$ :
Ffunctors $F: \zeta \rightarrow \theta, G: \theta \rightarrow \varphi$.. $\quad$.

$$
G F=I d_{G} \neq F G=I d_{\theta}
$$

Q: When are monoidal categs the same?
I. ISOM. AND Equiv. of monoidal categories
$\equiv$ RECAU $\equiv$
Two categories $\zeta$ and $\theta$ are the same via
CATEGORY 1 iSOMORPHISM $\zeta \cong \theta$ :
Ffunctors $F: \zeta \rightarrow \theta, G: \theta \rightarrow \varphi$.. $\quad \boldsymbol{\varphi}$

$$
G F=I d_{G} \neq F G=I d_{\theta}
$$

Category equivalence $\zeta \simeq \theta$ :
Jfunctors $F: \zeta \rightarrow \theta, G: \theta \rightarrow \varphi$..

$$
G F \cong I d_{G} \notin F G \cong I d_{\theta}
$$

Q: When are monoidal categs the same?
I. Isom. And equiv. of monoidal categories

三RECAU $\equiv$
Two categories $\zeta$ and $\theta$ are the same via
CATEGORY 1 iSOMORPHISM $\zeta \cong \theta$ :
Ffunctors $F: \zeta \rightarrow \theta, G: \theta \rightarrow \varphi$.. $\quad \boldsymbol{\varphi}$

$$
G F=I d_{G} \neq F G=I d_{\theta}
$$

Category equivalence $\zeta \simeq \theta$ :
F Functors $F: \zeta \rightarrow \theta, G: \theta \rightarrow \varphi, \ldots$

$$
G F \cong I d_{G} \neq F G \cong I d_{\theta}
$$

Q: When are monoidal categs the same?

ESSENTIALLY SUR. FUNCTOR $F: C \rightarrow \theta$
I. Isom. And equiv. of monoidal categories

三RECAU $\equiv$
Two categories $\zeta$ and $\theta$ are the same via
CATEGORY 1 iSOMORPHISM $\zeta \cong \theta$ :
Ffunctors $F: \zeta \rightarrow \theta, G: \theta \rightarrow \varphi$.. $\quad \boldsymbol{\varphi}$

$$
G F=I d_{G} \neq F G=I d_{\theta}
$$

Category equivalence $\zeta \simeq \theta$ :
F Functors $F: \zeta \rightarrow \theta, G: \theta \rightarrow \varphi, \ldots$

$$
G F \cong I d_{G} \notin F G \cong I d_{\theta}
$$

FIRST TAKING FULLY FAITHFUL, $\xrightarrow{\text { THIS APPROACH }}$ ESSENTIALLy surd.
I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

TAKE MONOIDAL CATEGS.

$$
\zeta:=\left(\zeta, \otimes^{\varepsilon}, l^{\varepsilon}, a^{\xi}, l^{\xi}, r^{\varepsilon}\right)
$$

$\ddagger$

$$
\theta:=\left(A, \otimes^{8}, L^{A}, a^{8}, l^{8}, r^{8}\right)
$$

A STRONG MONOIDAL FUNCTOR FROM $\boldsymbol{b}_{6}$ to $\theta$ consISTS OF:
(a) A FUnctor bTW categories $F: C \rightarrow B$.
(b) A NATURAL (SOMORPHISM $F^{(2)}=\left\{F_{x, y}^{(2)} ; F(x) \otimes F(y) \xrightarrow{\sim} F\left(x \otimes^{8} y\right)\right)_{x, y \in e}$.
(c) $A N$ iso $F^{(0)}: \mathbb{1}^{\theta} \xrightarrow{\sim} F\left(\mathbb{L}^{(\varphi)}\right) \mathbb{N} \theta$.

SUBJECT TO ASSOCIATIVITY \& UNITALITY AXIOMS

$$
\zeta \simeq \theta
$$ categs the same? FUNCTOR $F: C \rightarrow \theta$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

$\zeta a n d$ a are
MONOIDALLY ISOMORPHIC

$$
C \stackrel{\otimes}{=} D
$$

IF $\exists$ STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow D
$$

$$
.
$$

F IS A CATEGORY ISOMORPHISM.
I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

$\zeta$ and $A$ are
MONOIDALLY ISOMORPHIC

$$
\zeta \stackrel{\otimes}{\cong} \theta
$$

IF J STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$$
.
$$

F IS A CATEGORY ISOMORPHISM.

EXERCISE 3.6 G GROUP. GET:

$$
G-M o d \stackrel{\otimes}{=} \mathbb{k} G-M o d
$$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

$\zeta$ and $D$ are
MONOIDALLY ISOMORPHIC

$$
\zeta \stackrel{\otimes}{=} \theta
$$

IF J STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$$
\Rightarrow
$$

F IS A CATEGORY ISOMORPHISM.

EXERCISE 3.6 G GROUP. GET:

$$
G-M o d \stackrel{\otimes}{=} \mathbb{R} G-M o d
$$

$\left(k_{G} G-\operatorname{Mod}, \otimes, \mathbb{L}\right)$ DEFINED By:

$$
\begin{aligned}
& \left(\Sigma_{g \in G} \lambda_{g} g\right)-\left(v \otimes_{k^{\prime}} V^{\prime}\right):= \\
& \sum_{g \in G} \lambda_{g}(g \nabla v) \otimes_{\|_{k}}\left(g \nabla v^{\prime}\right) \\
& \text { FOR }(V, D),\left(V^{\prime}, P^{\prime}\right) \in \mathbb{R} G-M O d
\end{aligned}
$$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

$\zeta$ and $\theta$ are
MONOIDALLY ISOMORPHIC

$$
\zeta \stackrel{\otimes}{=} \theta
$$

IF J STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$\rightarrow$
F IS A CATEGORY ISOMORPHISM.

EXERCISE 3.6 G GROUP. GET:

$$
G-M o d \stackrel{\otimes}{=} \mathbb{R} G-M o d
$$

$$
(\| k G-M o d, \otimes, \mathbb{L}) \text { DEFINED By: }
$$

$$
\left(\sum_{g \in G} \lambda_{g} g\right)-\left(v \otimes_{k} v^{\prime}\right):=
$$

$\sum_{g \in G} \lambda g(g \nabla v) \otimes_{\|_{k}}\left(g \nabla v^{\prime}\right)$
FOR $(V, D),\left(V^{\prime}, P^{\prime}\right) \in \mathbb{R} G-M 0 d$

$$
\begin{aligned}
\left(\sum_{g \in G} \lambda_{g} g\right)>L_{\mathbb{k}} & :=\sum_{g \in G} \lambda_{g}\left(g>1_{\mathbb{R}}\right) \\
\text { FOR } \mathbb{1}=\mathbb{k} \quad & =\sum_{g \in G} \lambda_{g}
\end{aligned}
$$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

$\zeta$ and $\theta$ are
monoidally equivalent

$$
\zeta \stackrel{\otimes}{\approx} \theta
$$

IF J STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$\rightarrow$
F is a category equivalence.

EXERCISE 3.6 G GROUP. GET:

$$
G-M o d \stackrel{\otimes}{=} \mathbb{R} G-M o d
$$

$$
(\mathbb{R G}-\operatorname{Mod}, \otimes, \mathbb{L}) \text { DEFINED By: }
$$

$$
\left(\sum_{g \in G} \lambda_{g} g\right)-\left(v \otimes_{k k} v^{\prime}\right):=
$$

$\sum_{g \in G} \lambda g(g \nabla v) \otimes_{\|_{k}}\left(g \nabla v^{\prime}\right)$
FOR $(V, D),\left(V^{\prime}, P^{\prime}\right) \in \mathbb{R} G-M 0 d$

$$
\begin{aligned}
\left(\sum_{g \in G} \lambda_{g} g\right)>L_{\mathbb{k}} & :=\sum_{g \in G} \lambda_{g}\left(g>1_{\mathbb{R}}\right) \\
\text { FOR } \mathbb{1}=\mathbb{k} \quad & =\sum_{g \in G} \lambda_{g}
\end{aligned}
$$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

$\zeta a n d$ a are
MONOIDALLY EQUIVALENT

$$
C \stackrel{\otimes}{=} D
$$

IF $\exists$ STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow D
$$

$$
.
$$

F IS A CATEGORY EqUiVALENCE.

EXAMPLE A $\mathbb{R}$-ALGEBRA Not necessarily $\}$ Monoidac $\}$-Mod
I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

$\zeta$ and a are
MONOIDALLY EQUIVALENT

$$
C \stackrel{\otimes}{=} D
$$

IF $\exists$ STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow D
$$

$$
.
$$

F is A CATEGORY EqUIVALENCE.

EXAMPLE A $\mathbb{R}$-ALGEBRA
Is MONOIDAL End (A-Mod)

$$
\begin{aligned}
\text { WITH } \otimes & =\text { COMPOSITION } \\
\mathbb{L} & =I d_{A-\mu \operatorname{Mod}}
\end{aligned}
$$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

$\zeta a n d$ a are
MONOIDALLY EQUIVALENT

$$
\zeta \stackrel{\otimes}{=} \Delta
$$

IF J STRONG- MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$$
.
$$

F IS A CATEGORY EqUiVALENCE.

EXAMPLE A $\mathbb{R}$-ALGEBRA

$$
A \text {-Bimod } \stackrel{\otimes}{=} \text { End (A-Mod) }
$$

WITH $\otimes=\otimes_{A}$
WITH $\otimes=$ Composition
$\mathbb{L}=\operatorname{A}_{A}\left(\right.$ Arg $_{A}$
I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

$C$ and $a$ are
MONOIDALLY EQUIVALENT

$$
C \stackrel{\otimes}{=} D
$$

IF F STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow D
$$

$$
.
$$

F is a category equivalence.

EXAMPLE A R-ALGEBRA

$$
A \text {-Bipod } \stackrel{\otimes}{=} \text { End (A-Mod) }
$$

WITH $\otimes=\otimes_{A}$
WITH $\otimes=$ Composition
$\mathcal{L}=A_{A}\left(\right.$ reg $_{A}$ $\mathcal{L}=I d_{A-\mu_{0 d}}$
VIA $\rho: A$-Bimod $\longrightarrow$ End (A-Mod)

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

$\zeta A N D$ a are
MONOIDALLY EQUIVALENT

$$
C \stackrel{\otimes}{=} D
$$

IF F STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow D
$$

$$
\rightarrow
$$

F is a category equivalence.

EXAMPLE A IR-ALGEBRA

$$
A \text {-Bimod } \stackrel{\otimes}{=} \text { End (A-Mod) }
$$

WITH $\otimes=\otimes_{A}$
WITH $\otimes=$ Con position

$$
\mathbb{L}=A_{A}(\text { reg })_{A}
$$

$$
\mathbb{L}=I_{A-\mu_{0 d}}
$$

$V I A \rho: A-$ Bimod $\longrightarrow E n d(A-M o d)$

$$
V \longmapsto V_{\otimes_{A}}-
$$

$$
\begin{gathered}
p_{V, w}^{(2)}: \rho(V) \rho p(W) \longrightarrow p\left(V \otimes_{A} W\right) \\
\left(a_{V, w, z}^{A}\right)^{-1} \stackrel{\text { Def }}{=} p_{V, W}^{(2)}(z): V \otimes_{A}\left(W \otimes_{A} z\right) \rightarrow\left(V \otimes_{A} W\right) \otimes_{A} z
\end{gathered}
$$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

$\zeta A N D$ a are
MONOIDALLY EQUIVALENT

$$
C \stackrel{\otimes}{=} D
$$

IF F STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow D
$$

$$
.
$$

F IS A CATEGORY EqUIVALENCE.

EXAMPLE A $\mathbb{R}$-ALGEBRA

$$
A \text {-Bimod } \stackrel{\otimes}{=} \text { End (A-Mod) }
$$

WITH $\otimes=\otimes_{A}$
WITH $\otimes=$ COMPOSITION
$\mathcal{L}=A_{A}\left(\right.$ Arg $_{A}$ $L=I_{d_{A-\mu_{0 d}}}$
VIA $\rho: A$-Bimod $\longrightarrow \operatorname{End}(A-M o d)$

$$
V \longmapsto V_{\oplus_{A}}-
$$

$$
\begin{aligned}
& p_{V, w-1}^{(2)}: p(V) \rho p(w) \longrightarrow p\left(V \otimes_{A} W\right) \\
& \left(a_{V, w z}^{A}\right)^{-1} \stackrel{\text { oar }}{=} p_{V, N}^{(2)}(z): V \otimes_{A}\left(W \otimes_{A} z\right) \rightarrow\left(V \otimes_{A} W\right) \otimes_{A} z
\end{aligned}
$$

I.ISOM. AND Equiv. of MONOIDAL categories

Gand $\theta$ are monoidally 150 morphic $\zeta \stackrel{\otimes}{\approx}$ A IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$\rightarrow F$ IS A CATEGORY ISOMORPHISM.
$\zeta$ Gand $\theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{\approx}$ A IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$\therefore F$ is a category equivalence.
I. Isom. And equiv. of monoidal categories
$\zeta$ and $\theta$ are monoidally 150 morphic $\zeta \stackrel{\otimes}{\approx}$ A IF $\exists$ STRONG MON. FUNCTOR $\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta$
$\therefore F$ IS A CATEGORY ISOMORPHISM.
$\zeta a n d \theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{\approx}$ A IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$\therefore F$ is a category equivalence.
I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

Gand $\theta$ are monoidally isomorphic $\zeta \stackrel{\otimes}{\cong} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

.. F is A CATEGORY ISOMORPHISM.
$\zeta$ Gand $\theta$ are monoidally equivalent $C \stackrel{\otimes}{=} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

.. F is A Category equivalence.

CATEGORY 180 ORPHISM $\zeta \cong \theta$ :
Ffunctors $F: \zeta \rightarrow \theta, G: \theta \rightarrow \zeta . \rightarrow$.

$$
G F=I d_{\xi} \neq F G=I d \theta
$$

category Equivalence $\zeta \simeq \theta$ :
F functor $F: \zeta \rightarrow \theta, G: \theta \rightarrow \zeta .$.

$$
G F \cong I d_{G} \neq F G \cong I d \theta
$$

FIRST TANG FULLY FAITHFUL, $\xrightarrow{\text { THAT APPROACH }}$ ESSENTIAL SUR.

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

Gand $\theta$ are monoidally 150 morphic $\zeta \stackrel{\otimes}{\cong} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$\therefore$. F IS A CATEGORY ISOMORPHISM.
and $\theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{=} A \mathbb{F}$ F STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. $\rightarrow$ F is a Category equivalence.

CATEGORY 180 MORPHISM $\mathscr{C} \cong \theta$ :
Ffunctors $F: \zeta \rightarrow \theta, G: \theta \rightarrow \zeta . \ni$.
Now

$$
G F=I d_{\xi} \neq F G=I d_{\theta}
$$

Take category equivalence $\varphi \sim \theta$ :
APPROACH

$$
\left.\begin{array}{rl}
\text { Frunctors } & F: \zeta \longrightarrow \theta, G: \theta \\
G F \cong I d_{\xi} \neq F G \cong I d \theta
\end{array}\right\}
$$ categs the same? FUNCTOR $F: G \rightarrow \theta$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

Gand $\theta$ are monoidally 150 morphic $\zeta \stackrel{\otimes}{\cong} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. $\Rightarrow$ F IS A CATEGORY ISOMORPHISM.

Gand $\theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{\approx}$ D IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. $\rightarrow$ F is a Category equivalence.

CATEGORY ISOMORPHISM $\zeta \cong \theta$ :
Frunctors $F: \zeta \rightarrow \theta, G: \theta \rightarrow \zeta$.

$$
G^{G F}=I d_{G} \neq \underbrace{F G}=I d_{\theta}
$$

category equivalence $\zeta \simeq \theta$ :
TAPIS
APROACH
FFUNCTORS $F: \zeta \rightarrow \theta, G: \theta \rightarrow \varphi$$\rightarrow$
Now
LETS
take !
NEED TO SAY WHEN
TWO MONOIDAL FUNCTORS
are the same

$$
G F \cong I_{\xi} \notin \underbrace{F G \cong I d_{\theta}}
$$



Th HS Approach
Q: WHEN ARE MONODAL APPROACH ESSENTIALLY SUR. categs the same?
I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

Gand $\theta$ are monoidally isomorphic $\zeta \stackrel{\otimes}{\approx} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$\rightarrow$. F IS A CATEGORY ISOMORPHISM.
$\zeta$ Gand $\theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{=}$ A FF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

.. F is A CATEGORy Equivalence.


NEED TO SAY WHEN
TWO MONOIDAL FUNCTORS
are the same
I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

Gand $\theta$ are monoidally 150 morphic $\zeta \stackrel{\otimes}{\cong} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$\rightarrow$ F IS A CATEGORY ISOMORPHISM.

Gand $\theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{\approx}$ D IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. $\rightarrow$ F is a category equivalence.

a monoidal natural transformation
is a natural trans formation $\phi: F \Longrightarrow F^{\prime}$

NEED TO SAY WHEN
TWO MONOIDAL FUNCTORS are the same

compatible WITH

$$
F^{(0)} \not \ddagger F^{(0)}
$$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

Gand $\theta$ are monoidally 150 morphic $\zeta \stackrel{\otimes}{\cong} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$\Rightarrow$. $F$ IS A CATEGORY ISOMORPHISM.

Gand $\theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{\approx}$ D IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

.. F is a category equivalence.

a monoidal natural transformation
IS A NATURAL TRANSFORMATION $\phi: F \Longrightarrow F^{\prime}$. $\rightarrow$.

WRITE $F \stackrel{\otimes}{\Rightarrow} F^{\prime}$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

Gand $\theta$ are monoidally 150 morphic $\zeta \stackrel{\otimes}{\cong} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$\Rightarrow$. $F$ IS A CATEGORY ISOMORPHISM.

Gand $\theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{\approx}$ D IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. $\rightarrow$ F is a category equivalence.


IS A NATURAL TRANSFORMATION $\phi: F \Longrightarrow F^{\prime}$. $\rightarrow$.

WRITE $F^{\otimes} \stackrel{F^{\prime}}{\Rightarrow}$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

Gand $\theta$ are monoidally 150 morphic $\zeta \stackrel{\otimes}{\cong} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$\Rightarrow$. $F$ IS A CATEGORY ISOMORPHISM.

Gand $\theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{\approx}$ D IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. $\Rightarrow$ F is a Category equivalence.


ISOMORPHISM
IS A NATURAL TRANSFORMATION $\phi: F \stackrel{\sim}{\rightleftharpoons} F^{\prime}$.

WRITE

$$
F \stackrel{\otimes}{\Rightarrow} F^{\prime}
$$


I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

Gand $\theta$ are monoidally 150 morphic $\zeta \stackrel{\otimes}{\cong} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. $\Rightarrow$ F IS A CATEGORY ISOMORPHISM.

Gand $\theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{\approx}$ D IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. $\Rightarrow$ F is a Category equivalence.


ISOMORPHISM
IS A NATURAL TRANSFORMATION $\phi: F \stackrel{\sim}{\rightleftharpoons} F^{\prime}$.

WRITE

$$
\begin{aligned}
& \mathrm{F}^{\otimes} \stackrel{F^{\prime}}{ } \\
& F^{\otimes} \stackrel{\otimes}{\cong} F^{\prime}
\end{aligned}
$$


I. Isom. And equiv. of monoidal categories
$\zeta$ Gand $\theta$ are monoidally 150 morphic $\zeta \stackrel{\otimes}{=} A$ IF $\exists$ STRONG MON. FUNCTOR $\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow D$
$\therefore F$ IS A CATEGORY ISOMORPHISM.
$\zeta a n d \theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{\otimes} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow D
$$

$\therefore F$ is a category equivalence.

ISOMORPHISM
A MONOIDAL NATURAL TRANSFORMATION_BTW (F, $\left.F^{(2)}, F^{(0)}\right) \notin\left(F^{\prime}, F^{\prime(2)}, F^{\prime(0)}\right)$ IS A NATURAL TRANSFORMATION $\phi: F \stackrel{\text { INOPPHISM }}{\Longrightarrow} F^{\prime} \rightarrow$.

WRITE

$$
\begin{aligned}
& F_{\stackrel{\otimes}{\leftrightarrows} F^{\prime}}^{F \stackrel{\otimes}{\cong} F^{\prime}}
\end{aligned}
$$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

Gand $\theta$ are monoidally isomorphic $\zeta \stackrel{\otimes}{\approx} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. F. F IS A CATEGORY ISOMORPHISM.
$\zeta$ Gand $\theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{\approx} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

.. F is A CATEGORy Equivalence.
$\Uparrow$
EXR 3.5 介
WITH, $\underset{\sim}{\text { GMONOIDAL FUNCTOR }} \underset{(F, F(2), F(0)}{\longrightarrow}$


ISOMORPHISM
A MONOIDAL NATURAL TRANSFORMATION—BTW ( $\left.F, F^{(2)}, F^{(0)}\right) \notin\left(F^{\prime}, F^{\prime(2)}, F^{\prime(0)}\right)$
IS A NATURAL TRANSFORMATION $\phi: F \stackrel{\text { ISMORPHIIM }}{\Longrightarrow} F^{\prime}$.
WRITE

$$
\begin{aligned}
& F_{\stackrel{\otimes}{\leftrightarrows} F^{\prime}} \\
& F \stackrel{\otimes}{\cong} F^{\prime}
\end{aligned}
$$

$$
\begin{aligned}
& F(x) \otimes_{D}^{D} F(y) \xrightarrow{F_{x, y}^{(2)}} F(x \otimes y)
\end{aligned}
$$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

Gand $\theta$ are monoidally 150 morphic $\zeta \stackrel{\otimes}{\cong} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. $\Rightarrow$ F IS A CATEGORY ISOMORPHISM.

Gand $\theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{\approx}$ D IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. $\rightarrow$ F is a Category equivalence.

$$
\Uparrow
$$

 $\neq$

$$
F G \stackrel{\otimes}{=} I_{\left(\theta, \otimes^{0}\right)}
$$

EXR 3.5

§

ISOMORPHISM
A MONOIDAL NATURAL FRANSFORMATION—BTW ( $\left.F, F^{(2)}, F^{(0)}\right) \notin\left(F_{1}^{\prime} F^{\prime(2)}, F^{(0)}\right)$
IS A NATURAL TRANS FORMATION L $\phi: F \stackrel{\text { ISOMPMA }}{\Longrightarrow} F^{\prime} \rightarrow$.
WRITE

$$
\begin{aligned}
& F^{\otimes} F^{\prime} \\
& F \stackrel{\otimes}{\cong} F^{\prime}
\end{aligned}
$$

$$
\begin{aligned}
& F(x) \otimes_{\Delta}^{D} F(y) \xrightarrow{F_{x y}^{(x)}} F\left(x \otimes^{i} y\right)
\end{aligned}
$$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

Gand $\theta$ are monoidally 150 morphic $\zeta \stackrel{\otimes}{\cong} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. $\Rightarrow$ F IS A CATEGORY ISOMORPHISM.

Gand $\theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{\approx}$ D IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. $\rightarrow$ F is a Category equivalence.
$\Uparrow$

$$
\begin{aligned}
& \ddagger \\
& F G \stackrel{\otimes}{=} I d_{\left(\theta, \theta^{8}\right)} \\
& \text { \& } \\
& F G \stackrel{\otimes}{=} I d_{\left(\theta, \otimes^{8}\right)}
\end{aligned}
$$

介
EXR 3.5

ISOMORPHISM
A MONOIDAL NATURAL TRANSFORMATTON—BTW ( $\left.F, F^{(2)}, F^{(0)}\right) \neq\left(F_{1}^{\prime} F^{\prime(2)}, F^{(0)}\right)$
IS A NATURAL TRANS FORMATION L $\phi: F \stackrel{\text { ISORPHATM }}{\Longrightarrow} F^{\prime} \rightarrow$.
WRITE

$$
\begin{aligned}
& F^{\otimes} F^{\prime} \\
& F \stackrel{\otimes}{\cong} F^{\prime}
\end{aligned}
$$

$$
\begin{aligned}
& F(x) \otimes_{D}^{D}(y) \xrightarrow{F_{x,( }^{(2)}} F(x \otimes y)
\end{aligned}
$$

I. ISOM. AND EquIV. OF MONOIDAL CATEGORIES

Gand $\theta$ are monoidally 150 morphic $\zeta \stackrel{\otimes}{\cong} D$ IF $\exists$ STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

$\rightarrow$ F IS A CATEGORY ISOMORPHISM.

Gand $\theta$ are monoidally equivalent $\zeta \stackrel{\otimes}{\sim}$ A $\mathcal{F}$ Э STRONG MON. FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

. $\rightarrow$ F is a Category equivalence.


Ex

$$
G-M_{o d} \stackrel{\otimes}{\cong} H_{k G-M o d}
$$

FOR $G$ A GROUP

Ex
$A-B i \operatorname{Mod} \stackrel{\otimes}{\sim} \operatorname{End}(A-M o d)$
FOR A a lk-Algebra
II. MODULE CATEGORIES

MONOIDAL CATEGORY

$$
(\varphi, \otimes, \mu, a, l, r)
$$

CONSISTS OF:
(a) CATEGORY Cl
(b) BIFUNCTOR

$$
\otimes: 6 \times 6 \rightarrow \zeta
$$

(c) OBJECT $\mathbb{1} \in \zeta$
( $d, e, f$ ) NATURAL (tOms:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & : \\
& (x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
x, y, z \in \zeta
$$

$$
l=\left\{l_{x}: \mathcal{L} \otimes x \leadsto x\right\}_{x \in G}
$$

$$
r=\left\{r_{x}: x \otimes \mathbb{L} \xrightarrow{\sim} x\right\}_{x \in G}
$$

SATISFying The

WANT "MODULES" OVER THESE (MIMICKING MODULES OVER MONOIDS)
II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF:
(a) CATEGORY C
(b) BIFUNCTOR

$$
\otimes: 6 \times 6 \rightarrow \zeta
$$

(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) NATURAL (sIMS:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & : \\
& (x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
x, y, z \in h_{1}
$$

$$
l=\left\{l_{x}: \mathcal{L} \otimes x \leadsto x\right\}_{X \in G}
$$

$$
r=\left\{r_{x}: x \otimes \mathbb{L} \xrightarrow{\sim} x\right\}_{x \in \zeta}
$$

SATISFying The
pentagon axiom
\& Triangle axiom

A LEFT G-MODULE category consists of:
(a) A CATEGORY M
(b) A BIFUNCTOR $D: \varphi \times 2 \rightarrow m$ (ACTION BIFUNCTOR)

WANT "MODULES" over these (MIMICKING MODULES OVER MONOIDS)
II. Module categories

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF:
(a) CATEGORY Cl
(b) BIFUNCTOR

$$
\otimes: 6 \times 6 \rightarrow \zeta
$$

(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) NATURAL (toms:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a, y, y & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
x, y, z \in \zeta
$$

$$
l=\left\{l_{x}: \mathcal{L} \otimes x \leadsto x\right\}_{x \in G}
$$

$$
r=\left\{r_{x}: x \otimes \mathbb{L} \xrightarrow{\sim} x\right\}_{x \in G}
$$

SATISFying The
pentagon axiom
\& Triangle axiom
a left G-Module category consists of:
(a) A CATEGORY M
(b) A BIFUNCTOR $D: \varphi \times m \rightarrow q$
(ACTION BIFUNCTOR)
(c) A NATURAL ISOMORPHISM

$$
\begin{aligned}
& M:=\left\{M_{x, y, \mu}:(X \otimes y) \triangleright M \xrightarrow{\sim} X D(Y \triangleright M)\right\}_{x, y \in \ell, M \in M}
\end{aligned}
$$

(d) A NATURAL ISOMORPHISM
II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF:
(a) CATEGORY C
(b) BIFUNCTOR

$$
\otimes: 6 \times 6 \rightarrow \zeta
$$

(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) NATURAL (tOms:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & : \\
& (x \otimes y) \otimes z \\
& \sim x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
x, y, z \in G_{h}
$$

$$
l=\left\{l_{x}: \mathcal{L} \otimes x \leadsto x\right\}_{x \in G}
$$

$$
r=\left\{r_{x}: x \otimes \mathbb{L} \xrightarrow{\sim} x\right\}_{x \in G}
$$

Satisfying The
pentagon axiom
\& Triangle axiom

A LEFT G-MODULE category consists of:
(a) A CATEGORY M
(b) A BIFUNCTOR $D: C \times \eta \rightarrow \eta\binom{$ (aCTION }{ (IFUNCTOR) }
(c) A NATURAL ISOMORPHISM
$\binom{$ LeFTMOD ASSOC }{ constrains }

$$
M:=\left\{M_{x, y, \mu}:(X \otimes y) \triangleright M \xrightarrow{\longrightarrow} X D(y \triangleright M) \int_{x, y \in,\left(\mu_{e} \eta\right.}\right.
$$

(d) A NATURAL isOmORPHISM

$$
p:=\left\{p_{M}: \mathbb{1}^{e} \triangleright M \xrightarrow{\rightarrow} M\right\}_{M \in M}
$$

$$
\binom{\text { LEFTMOD UNIT }}{\text { CONSTANT }}
$$

II. MODULE categories

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF:
(a) CATEGORY C
(b) BIFUNCTOR

$$
\otimes: 6 \times 6 \rightarrow \zeta
$$

(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) NATURAL (toms:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
x, y, y \in h_{i}
$$

$l=\left\{l_{x}: \mathcal{L} \otimes x \leadsto x\right\}_{x \in E}$

$$
r=\left\{r_{x}: x \otimes \mathbb{L} \xrightarrow{\sim} x\right\}_{x \in \zeta}
$$

Satisfying the pentagon axiom
\& Triangle axiom

A LEFT G-MODULE category consists of:
(a) A CATEGORY M
(b) A BIFUNCTOR $D: C \times \eta \rightarrow \eta\binom{$ (BATON }{ BIFUNCTOR) }
(c) A NATURAL 150 MORPHISM

$$
M:=\left\{M_{x, y, \mu}:(X \otimes y) \triangleright M \xrightarrow{\longrightarrow} X D(Y \triangleright M) \int_{x, y \in e, \mu_{e}}\right. \text { contrA }
$$

(d) A natural isomorphism

$$
p:=\left\{p_{M}: \mathbb{1}^{e} D M \xrightarrow{\leftrightharpoons} M\right\}_{M \in \eta}
$$

$$
\binom{\text { LEFTMOD UNIT }}{\text { CONSTRAiNT }}
$$

$$
((x \otimes y) \otimes z) \triangleright M
$$

II. MODULE categories

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF:
(a) CATEGORY C
(b) BIFUNCTOR

$$
\otimes: 6 \times 6 \rightarrow \zeta
$$

(c) OBJECT $\mathbb{1} \in \zeta$
( $d, e, f$ ) NATURAL (toms:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
l=\left\{l_{x}: 1 \otimes x \leadsto x\right\}_{x \in \xi}
$$

$$
r=\left\{r_{x}: x \otimes \mathscr{U} \leadsto x\right\}_{x \in \mu}
$$

SATISFying The pentagon axiom
\& Triangle axiom

A LEFT G-MODULE category consists of:
(a) A CATEGORY M
(b) A BIFUNCTOR $D: C \times \eta \rightarrow \eta\binom{$ (BATON }{ BIFUNCTOR) }
(c) A NATURAL ISOMORPHISM

$$
M:=\left\{M_{x, y, \mu}:(X \otimes y) \triangleright M \xrightarrow{\longrightarrow} X D(Y \triangleright M) \int_{x, y \in e, \mu_{e}}\right. \text { contrail }
$$

(d) A natural isomorphism

$$
p:=\left\{p_{M}: \mathbb{1}^{e} \triangleright M \xrightarrow{\rightarrow} M\right\}_{M \in M}
$$

$$
\binom{\text { LEFTMOD UNIT }}{\text { CONDTRANT }}
$$

.

$$
\left.a_{x, y, z>i d} /(x \otimes y) \otimes z\right) \triangleright M
$$

$(x \otimes(y \otimes z)) \triangleright M$
II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\varphi, \otimes, u, a, l, r)$
CONSISTS OF:
(a) CATEGORY C
(b) BIFUNCTOR

$$
\otimes: \zeta \times \zeta \rightarrow \zeta
$$

(c) OBJECT $\mathbb{U} \in \zeta$
( $d, e, f$ ) NATURAL (toMS:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
l=\left\{l_{x}: 1 \otimes x \simeq x\right\}_{x \in \varepsilon}
$$

$$
r=\left\{r_{x}: x \otimes \mathscr{H} \simeq x\right\}_{x \in \varepsilon}
$$

SATISFYING THE
PENTAGON AXIOM
\& Triangle axiom

A left C-Module category consists of:
(a) A CATEGORY m
(b) A BIFUNCTOR $D: \varphi_{6} \times m \rightarrow q \underset{\left(\begin{array}{l}\text { (BIFUNCTOR) }\end{array}\right)}{\left(\begin{array}{ll}\text { ACTON }\end{array}\right)}$
(c) A NATURAL ISOMORPHISM

$$
M:=\left\{M_{x, y, M}:(X \otimes y) \triangleright M \xrightarrow{\sim} X D(y \triangleright M)\right\}_{x, y \in \varphi, M \in M}
$$

(d) A NATURAL ISOMORPHISM

$$
p:=\left\{p_{M}: \mathbb{1}^{e} \triangleright M \xrightarrow{\longrightarrow} M\right\}_{M \in \eta}
$$

$$
\binom{\text { LEFT MOD UNIT }}{\text { CONSTANT T }}
$$

.

$$
a_{x, y z \triangleright i d}((x \otimes y) \otimes z) \triangleright M
$$

$(x \otimes(y \otimes z)) D M$
$\mu_{x, y \otimes z, \mu} \downarrow$

$$
X \triangleright((y \otimes z) \triangleright M)
$$

$\forall x, y, z \in \boldsymbol{U}$, $M \in M$
II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\varphi, \otimes, u, a, l, r)$
CONSISTS OF:
(a) CATEGORY G
(b) BIFUNCTOR

$$
\otimes: \zeta \times \zeta \rightarrow \zeta
$$

(c) OBJECT $\mathbb{U} \in \zeta$
( $d, e, f$ ) NATURAL (8 0MS:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
l=\left\{l_{x}: 1 \otimes x \simeq x\right\}_{x \in \varepsilon}
$$

$$
r=\left\{r_{x}: x \otimes \mathscr{H} \simeq x\right\}_{x \in \varepsilon}
$$

SATISFYING THE
PENTAGON AXIOM
\& Triangle axiom

A left C-Module category consists of:
(a) A CATEGORY m
(b) A BIFUNCTOR $D: \varphi_{6} \times m \rightarrow q \underset{\left(\begin{array}{l}\text { (BIFUNCTOR) }\end{array}\right)}{\left(\begin{array}{ll}\text { ACTON }\end{array}\right)}$
(c) A NATURAL ISOMORPHISM

$$
M:=\left\{M_{x, y, M}:(X \otimes y) \triangleright M \xrightarrow{\sim} X D(y \triangleright M)\right\}_{x, y \in \varphi, M \in M}
$$

(d) A NATURAL ISOMORPHISM

$$
p:=\left\{p_{M}: \mathbb{1}^{b} \triangleright M \xrightarrow{\longrightarrow} M\right\}_{M \in \eta}
$$

$$
\binom{\text { LEFT MOD UNIT }}{\text { CONSTANT T }}
$$

$$
a_{x, y, z \triangleright i d}((x \otimes y) \otimes z) \triangleright M
$$

$$
(x \otimes(y \otimes z)) \triangleright M
$$

$\mu_{x, y \otimes z, \mu} \downarrow$

$$
X D((y \otimes z) \triangleright M) \xrightarrow{i d \triangleright D y, z, M} X D(Y \triangleright(z \triangleright M))
$$

II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\varphi, \otimes, u, a, l, r)$
CONSISTS OF:
(a) CATEGORY G
(b) BIFUNCTOR

$$
\otimes: \zeta \times \zeta \rightarrow \zeta
$$

(c) OBJECT $\mathbb{U} \in \zeta$
( $d, e, f$ ) NATURAL (8 0MS:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
l=\left\{l_{x}: \mathbb{1} \otimes x \stackrel{\sim}{\leftrightharpoons} x j_{x \in \varepsilon}\right.
$$

$$
r=\left\{r_{x}: x \otimes \mathbb{1} \leadsto x\right\}_{x \in \varepsilon}
$$

Satisfying the
pentagon axiom
\& Triangle axiom

A left C-Module category consists of:
(a) A CATEGORY m
(b) A BIFUNCTOR $D: \varphi_{6} \times m \rightarrow q \underset{\left(\begin{array}{l}\text { (BIFUNCTOR) }\end{array}\right)}{\left(\begin{array}{ll}\text { ACTON }\end{array}\right)}$
(c) A NATURAL ISOMORPHISM
$\left.\begin{array}{c}(\text { LeFT MOD ASSOC } \\ \text { CONSTRAINT }\end{array}\right)$

$$
M:=\left\{M_{x, y, M}:(X \otimes y) \triangleright M \xrightarrow{\sim} X D(y \triangleright M)\right\}_{x, y \in \varphi, M \in M}
$$

(d) A NATURAL ISOMORPHISM

$$
p:=\left\{p_{M}: \mathbb{1}^{b} \triangleright M \xrightarrow{\longrightarrow} M\right\}_{M \in \eta}
$$

$$
\binom{\text { LEFT MOD UNIT }}{\text { CONSTANT T }}
$$


II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\varphi, \otimes, u, a, l, r)$
CONSISTS OF:
(a) CATEGORY C
(b) BIFUNCTOR

$$
\otimes: \zeta \times \zeta \rightarrow \zeta
$$

(c) OBJECT $\mathbb{U} \in \zeta$
( $d, e, f$ ) NATURAL (toMS:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
l=\left\{l_{x}: \mathbb{1} \otimes x \simeq x\right\}_{x \in E}
$$

$$
r=\left\{r_{x}: x \otimes \mathbb{1} \leadsto x\right\}_{x \in E}
$$

SATISFying the
PENTAGON AXIOM
\& Triangle axiom

A left C-Module category consists of:
(a) A CATEGORY $\quad$ m
(b) A BIFUNCTOR $D: \varphi_{6} \times m \rightarrow q \underset{\left(\begin{array}{l}\text { (BIFUNCTOR) }\end{array}\right)}{\left(\begin{array}{ll}\text { ACTON }\end{array}\right)}$
(c) A NATURAL ISOMORPHISM
$\left.\begin{array}{c}\text { (LEA TMOD ASSOC } \\ \text { CONSTRAINT }\end{array}\right)$

$$
M:=\left\{M_{x, y, M}:(X \otimes y) \triangleright M \xrightarrow{\sim} X D(y \triangleright M)\right\}_{x, y \in \varphi, M \in M}
$$

(d) A NATURAL ISOMORPHISM

$$
p:=\left\{p_{M}: \mathbb{1}^{b} \triangleright M \xrightarrow{\longrightarrow} M\right\}_{M \in \eta}
$$

$$
\binom{\text { LEFT MOD UNIT }}{\text { CONSTANT T }}
$$

$\rightarrow$

(PENTAGON AXIOM)
$\forall x, y, z \in C$, $M \in M$
II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\varphi, \otimes, u, a, l, r)$
CONSISTS OF:
(a) CATEGORY C
(b) BIFUNCTOR

$$
\otimes: \zeta \times \zeta \rightarrow \zeta
$$

(c) OBJECT $\mathbb{U} \in \zeta$
( $d, e, f$ ) NATURAL (toMS:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
l=\left\{l_{x}: 1 \otimes x \simeq x\right\}_{x \in \varepsilon}
$$

$$
r=\left\{r_{x}: x \otimes \mathbb{1} \leadsto x\right\}_{x \in E}
$$

SATISFYING THE
PENTAGON AXIOM
\& Triangle axiom

A left C-Module category consists of:
(a) A CATEGORY $\quad$ m
(b) A BIFUNCTOR $D: \varphi_{6} \times m \rightarrow q \underset{\left(\begin{array}{l}\text { (BIFUNCTOR) }\end{array}\right)}{\left(\begin{array}{ll}\text { ACTON }\end{array}\right)}$
(c) A NATURAL ISOMORPHISM
$\left.\begin{array}{c}(\text { LeFT MOD ASSOC } \\ \text { CONSTRAINT }\end{array}\right)$

$$
M:=\left\{M_{x, y, M}:(X \otimes y) \triangleright M \xrightarrow{\sim} X D(y \triangleright M)\right\}_{x, y \in \varphi, M \in M}
$$

(d) A NATURAL ISOMORPHISM $\binom{$ LEFT MOD UNIT }{ CONSTRAINT } $p:=\left\{p_{M}: \mathbb{1}^{b} \triangleright M \xrightarrow{\leftrightharpoons} M\right\}_{M \in \eta}$
$\rightarrow$

II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\varphi, \otimes, \Psi, a, l, r)$
CONSISTS OF:
(a) CATEGORY C
(b) BIFUNCTOR

$$
\otimes: 6 \times 6 \rightarrow \zeta
$$

(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) NATURAL (SIMS:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
l=\left\{l_{x}: \psi \otimes X \leadsto x\right\}_{x \in \xi}
$$

$$
r=\left\{r_{x}: x \otimes \mathbb{L} \leadsto x\right\}_{x \in \zeta}
$$

Satisfying. The
pentagon axiom
\& Triangle axiom

A LEFT G-MODULE category consists of:
(a) A CATEGORY $\sim$
(b) A BIFUNCTOR $D: \varphi \times \eta \rightarrow \eta\left(\begin{array}{c}\text { (BATON } \\ \text { (IFUNCTOR) }\end{array}\right.$
(c) A NATURAL 150 MORPHISM
$\binom{$ LeFT MOD ASSOC }{ CONSTRAiNT }

$$
M:=\left\{M_{x, y, \mu}:(X \otimes y) \triangleright M \xrightarrow{\longrightarrow} X D(y \triangleright M)\right\}_{x, y \in e, \mu_{e}} \text { cont }
$$

(d) A natural isomorphism $\binom{$ LEFTMOD UNIT }{ CONSTRANT } $p:=\left\{p_{M}: \mathbb{1}^{i} \triangleright M \xrightarrow{\rightarrow} M\right\}_{M \in M}$

(PENTAGON AXIOM)

$$
\begin{aligned}
& (X \otimes 1) \triangleright M \\
& r_{x} \triangleright i d{ }_{V}
\end{aligned}
$$

$$
X>M
$$

II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\varphi, \otimes, \mu, a, l, r)$
CONSISTS OF:
(a) CATEGORY G
(b) BIFUNCTOR

$$
\otimes: \zeta \times \zeta \rightarrow \zeta
$$

(c) OBJECT $\mathbb{U} \in \zeta$
( $d, e, f$ ) NATURAL (toMS:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
l=\left\{l_{x}: 1 \otimes x \simeq x\right\}_{x \in \varepsilon}
$$

$$
r=\left\{r_{x}: x \otimes \mathscr{L} \leadsto x\right\}_{x \in \varepsilon}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom

A left C-Module category consists of:
(a) A CATEGORY $M$
(b) A BIFUNCTOR $D: \varphi_{6} \times m \rightarrow q \underset{\left(\begin{array}{l}\text { (BIFUNCTOR) }\end{array}\right)}{\left(\begin{array}{ll}\text { ACTON }\end{array}\right)}$
(c) A NATURAL ISOMORPHISM
$\left.\begin{array}{c}\text { (LEE TMOD ASSOC } \\ \text { CONSTRAINT }\end{array}\right)$

$$
M:=\left\{M_{x, y, M}:(X \otimes y) \triangleright M \xrightarrow{\sim} X D(y \triangleright M)\right\}_{x, y \in \varphi, M \in M}
$$

(d) A NATURAL ISOMORPHISM $\binom{$ LeFT MOD UNIT }{ CONSTANT T }

(PENTAGON AXIOM)

(Triangle axiom)
$\forall x, y, z \in C$, $M \in M$
II. MODULE CATEGORIES ... Modules/repins over structure s should REFLECT THE FEATURES OF 8

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF:
(a) CATEGORy C
(b) BIFUNCTOR

$$
\otimes: \zeta \times \zeta \rightarrow \zeta
$$

(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) NATURAL (tOMS:

$$
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \simeq x \otimes(y \otimes z)
\end{aligned}\right\}
$$

$$
l=\left\{l_{x}: 1 \otimes x \xrightarrow{\sim} x\right\}_{x \in \zeta}
$$

$$
r=\left\{r_{x}: x \otimes \mathbb{L} \simeq x\right\}_{x \in \zeta}
$$

SATISFyING THE
PENTAGON AXIOM
\& Triangle axiom
a left C-Module category consists of:
(a) A CATEGORY M
(b) A BIFUNCTOR $D: C \times 7 \eta \rightarrow \eta{ }^{2} \rightarrow\left(\begin{array}{c}\text { ACTION } \\ \text { (BIFUNCTOR) }\end{array}\right.$
(c) A NATURAL 150 MORPHISM
$\binom{$ LeFT MOD ASSOC }{ CONSTRAiNT }

$$
M:=\left\{M_{x, y, M}:(X \otimes y) \triangleright M \xrightarrow{\sim} X D(y \triangleright M)\right\}_{x, y \in e, \mu_{e}} \text { ConTRA }
$$

(d) A NATURAL isOmORPHISM

$$
\begin{aligned}
& p:=\left\{p_{M}: \mathbb{1}^{b} \triangleright M \xrightarrow{\simeq} M\right\}_{M \in M}
\end{aligned}
$$

$$
\begin{aligned}
& (x \otimes(y \otimes z)) \triangleright M \quad 2 \quad(x \otimes y) \triangleright(z \triangleright M)
\end{aligned}
$$

$$
\begin{aligned}
& X D M \\
& \text { (TRIANGLE AXIOM) } \\
& \forall x, y, z \in \zeta \text {, } \\
& M \in M
\end{aligned}
$$

$\binom{$ LEFT MOD UNIT }{ CONSTRAINT }
II. MODULE categories

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF: ASSUME:
(a) CATEGORY Ce
(b) BIFUNCTOR

$$
\otimes: 6 \times 6 \rightarrow \zeta
$$

(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) Natural (toms:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
l=\left\{l_{x}: 1 \otimes x \leadsto x\right\}_{x \in G}
$$

$$
r=\left\{r_{x}: x \otimes \mathbb{L} \xrightarrow{\sim} x\right\}_{x \in G}
$$

SATISFying The
pentagon axiom
\& Triangle axiom

A LEFT G-MODULE CATEGORY CONSISTS OF:
(a) A CATEGORY M
(b) A BIFUNCTOR $D: C x^{2} \eta \rightarrow{ }^{2} \rightarrow\binom{$ (ACTION }{ BIFUNCTOR) }
(c) A NATURAL 150 MORPHISM
$\binom{$ LeFTMOD ASSOC }{ CONSTRAiNT }

$$
M:=\left\{M_{x, y, M}:(X \otimes y) \triangleright M \xrightarrow{\sim} X D(y \triangleright M)\right\}_{x, y \in e, \mu_{e} M}
$$

(d) A NATURAL isOmORPHISM

$$
p:=\left\{p_{M}: \mathbb{1}^{b} \triangleright M \xrightarrow{\Rightarrow} M\right\}_{M \in M}
$$


(Triangle axiom)

$$
\begin{gathered}
\forall x, y, z \in \zeta, \\
M \in M
\end{gathered}
$$

II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\varphi, \otimes, \mu, a, l, r)$
CONSISTS OF: ASSUME:
(a) CATEGORY
(b) BIFUNCTOR

$$
\otimes: \zeta \times \zeta \rightarrow \zeta
$$

(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) Natural (toms:

$$
\begin{aligned}
& \begin{aligned}
a=\left\{\begin{array}{ll}
a x, y, z: & (x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{array}\right\}
\end{aligned} \\
& l=\left\{e_{x}: \mathbb{1} \otimes x \leadsto \vec{\sim} x j_{x \in \varepsilon}\right. \\
& r=\left\{r_{x}: x \otimes 1 \Perp \leadsto x\right\}_{x \in E}
\end{aligned}
$$

SATISFying The
PENTAGON AXIOM
\& Triangle axiom

A left C-MOdULE category consists of:
(a) A CATEGORY m
(b) A BIFUNCTOR $D: \varphi \times m \rightarrow \psi \underset{\substack{m \\ \text { (BIFUNCTOR) }}}{\substack{\text { ACTION }}}$
(c) A NATURAL ISOMORPHISM
$\left.\begin{array}{c}\text { LEFT MOD ASSOC } \\ \text { CONSTRAINT }\end{array}\right)$

$$
M:=\left\{M_{X, y, M}:(X \otimes Y) \triangleright M \xrightarrow{\sim} X D(Y \triangleright M)\right\}_{X, y \in \varphi, M \in M} \text { ConsTRAI }
$$

(d) A NATURAL ISOMORPHISM

$$
p:=\left\{p_{M}: \mathbb{1}^{b} \triangleright M \xrightarrow{\leftrightharpoons} M\right\}_{M \in M}
$$


(PENTAGON AXIOM)

$$
\binom{\text { LEFT MOD UNIT }}{\text { CONSTRAiNT }}
$$


II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF: ASSUME:
(a) CATEGORY Ce
(b) BIFUNCTOR

$$
\otimes: 6 \times 6 \rightarrow \zeta
$$

(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) Natural (toms:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
l=\left\{l_{x}: 1 \otimes x \leadsto x\right\}_{x \in \xi}
$$

$$
r=\left\{r_{x}: x \otimes \mathbb{L} \xrightarrow{\sim} x\right\}_{x \in G}
$$

SATISFying The
PENTAGON AXIOM
\& Triangle axiom

A left G-MODULE category consists of:
(a) an additive
(a) A CATEGORY $\quad$ a
(b) A BIFUNCTOR $D: C \times m \rightarrow \psi y\binom{$ ACTION }{ BIFUNCTOR) }
(c) A NATURAL ISOMORPHISM
$\left.\begin{array}{c}\text { (LEA TROD ASSOC } \\ \text { CONSTRAINT }\end{array}\right)$

$$
M:=\left\{M_{x, y, \mu}:(X \otimes y) \triangleright M \xrightarrow{\sim} X D(y \triangleright M)\right\}_{x, y \in e, \mu \in m}
$$

(d) A NATURAL ISOMORPHISM

$$
p:=\left\{p_{M}: \mathbb{1}^{b} \triangleright M \xrightarrow{\leftrightharpoons} M\right\}_{M \in M}
$$


(PENTAGON AXIOM)

$$
\binom{\text { LEFT MOD UNIT }}{\text { CONSTRAiNT }}
$$


II. MODULE categories

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF: ASSUME:
(a) ADATEGORY E
(a) CATEGORY C
(b) BIFUNCTOR

$$
\otimes: 6 \times 6 \rightarrow \zeta
$$

(c) OBJECT $\mathbb{L} \in \zeta$
( $d, e, f$ ) NATURAL (toms:

$$
\begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned}
$$

$$
l=\left\{l_{x}: \mathscr{L} \otimes x \leadsto x\right\}_{x \in \varepsilon}
$$

$$
r=\left\{r_{x}: x \otimes \mathbb{L} \xrightarrow{\sim} x\right\}_{x \in G}
$$

Satisfying the
pentagon axiom
\& Triangle axiom

A LEFT €-MODULE CATEGORY CONSISTS OF:
(a) An additive
(a) A CATEGORY m
(b) A BIFUNCTOR D: $C_{6} \times m \rightarrow q$ (ACTION $\rightarrow$ )
$\Rightarrow(X D-): \eta \rightarrow \eta$ 末 $(-\triangleright M): \zeta \rightarrow \eta$ are additive $\forall x, M$
(c) A NATURAL ISOMORPHISM
$\left.\begin{array}{c}\text { (LEE TROD ASSOC } \\ \text { CONSTRAINT }\end{array}\right)$

$$
M:=\left\{M_{x, y, M}:(X \otimes y) \triangleright M \xrightarrow{\sim} X D(y \triangleright M)\right\}_{x, y \in \varphi, M_{e} m}
$$

(d) A NATURAL ISOMORPHISM

$$
p:=\left\{p_{M}: \mathbb{1}^{b} \triangleright M \xrightarrow{\leftrightharpoons} M\right\}_{M \in M}
$$


(triangle axiom)

$$
\forall x, y_{1} z \in \boldsymbol{G},
$$

$$
M \in M
$$

II. Module categories
MONOIDAL CATEGORY

$$
(\varphi, \otimes, \mu, a, l, r)
$$

CONSISTS OF: ASSUME:
(a) CATEGORY Ge
(b) BIFUNCTOR

SATISFying the
PENTAGON AXIOM
\& Triangle axiom

A LEFT C-MODULE CATEGORy CONSISTS OF:
(a) An additive
(a) A CATEGORY $\quad$ M
(b) A BIFUNCTOR $D: C_{6} \times m \rightarrow q$ (ACTION $\rightarrow$ (FUNCTOR)
$\Rightarrow(X D-): \eta \rightarrow \eta \ddagger(-\triangleright M): \zeta \rightarrow \eta$ are AdDITIVE $\forall x, M$


$$
(X \otimes \mathbb{L}) \triangleright M^{\mu_{X, L} M} \operatorname{XD(\mathbb {L}}
$$

$$
r_{x} \nabla i d \int^{2} \text { idDpm }
$$

$$
X D M
$$

(Triangle axiom)

$$
\begin{gathered}
\forall x, y_{1} z \in \boldsymbol{M}, \\
M \in M
\end{gathered}
$$

II. MODULE categories

MONOIDAL CATEGORY $(\varphi, \otimes, \Psi, a, l, r)$
CONSISTS OF: ASSUME:
(a) CATEGORY G
(b) BIFUNCTOR

$$
\otimes: 6 \times 6 \rightarrow \zeta
$$

(c) OBJECT $\mathbb{1} \in \zeta$
( $d, e, f$ ) NATURAL ( ( 0 ms :

$$
\begin{aligned}
& a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& =x \otimes(y \otimes z)
\end{aligned}\right\} \\
& l=\left\{l_{x}: \mathbb{L} \otimes x \stackrel{\sim}{\leadsto} x\right\}_{x \in \xi} \\
& r=\left\{r_{x}: x \otimes \mathbb{L} \xrightarrow{\longrightarrow} x\right\}_{x \in G}
\end{aligned}
$$

Satisfying The
PENTAGON AXIOM
\& Triangle axiom

A LEFT C-MODULE CATEGORY CONSISTS OF:
(a) AN ABEUAN
(a) A CATEGORY m
(b) A BIFUNCTOR D: $C_{6} \times m \rightarrow q$ (DICTION $\rightarrow$ (FUNCTOR)
$\Rightarrow(X D-): \eta \rightarrow \eta$ 末 $(-\triangleright M): \zeta \rightarrow \eta$ are AdDITIVE $\forall x, M$
K c) A NATURAL ISOMORPHISM (LEFTMOD ASSOC)

$$
\frac{E}{E} \quad M:=\left\{M_{x, y, M}:(X \otimes y) \triangleright M \xrightarrow{\sim} X D(Y \triangleright M)\right\}_{X, y \in \varepsilon, M \in M}
$$

(d) a natural 150 morphism $\binom{$ LEFTMOD UNIT }{ CONSTRAINT }

$$
\left\lvert\, \begin{aligned}
& T \\
& T_{H .}
\end{aligned} \quad p\right.:=\left\{p_{M}: \mathbb{1}^{b} \triangleright M \xrightarrow{\longrightarrow} M\right\}_{M \in M}
$$

$$
\underbrace{R_{x, y, 1 z} \triangleright i d}_{R^{H .}} /((x \otimes y) \otimes z) \triangleright M
$$

$$
(X \otimes \mathbb{1}) \triangleright M^{\mu_{X, \mu}, \mu} X \triangleright(\mathbb{L} \triangleright M)
$$

$$
r_{x} p i d \sum^{2} / i d D P_{M}
$$

$$
X D M
$$

(Triangle axiom)

$$
\begin{gathered}
\forall x, y, z \in r, \\
M \in Y
\end{gathered}
$$

II. MODULE categories

MONOIDAL CATEGORY $(\varphi, \otimes, \psi, a, l, r)$
CONSISTS OF: ASSUME: anear
(a) CATEGORY G
(b) BIFUNCTOR

$$
\otimes: \zeta \times 6 \rightarrow \zeta
$$

(c) OBJECT $\mathbb{1} \in \zeta$
( $d, e, f$ ) NATURAL ( ( 0 mS :

$$
\begin{aligned}
& \begin{aligned}
a=\left\{\begin{aligned}
a x, y, z & :(x \otimes y) \otimes z \\
& \Rightarrow x \otimes(y \otimes z)
\end{aligned}\right\}
\end{aligned} \\
& l=\left\{l_{x}: 1 \otimes x \leadsto x\right\}_{x \in G} \\
& r=\left\{r_{x}: x \otimes \mathbb{L} \xrightarrow{\sim} x\right\}_{x \in G}
\end{aligned}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom

A LEFT そ-MODULE CATEGORY CONSISTS OF:
(a) linear
(a) A CATEGORy m
(b) A BIFUNCTOR D: C $\times{ }^{2} \rightarrow \rightarrow$ m (BIFUNCTOR)
$\rightarrow(X D-): \eta \rightarrow \eta \neq(-\triangleright M): \zeta \rightarrow \eta$ are near $\forall x, M$
(k) A NATURAL 150 MORPHISM (LEFTMODASSOC)

$$
\frac{E}{E} \quad M:=\left\{M_{x, y, M}:(X \otimes y) \triangleright M \xrightarrow{\sim} X D(Y \triangleright M)\right\}_{X, y \in \varnothing, M \in M}
$$

Pd) A NATURAL (somorphism $\binom{$ LeFT MOD UNIT }{ CONSTRAiNT }

$$
\begin{aligned}
& T \quad p:=\left\{p_{M}: \mathbb{1}^{b} \triangleright M \xrightarrow{\sim} M\right\}_{M \in M} \\
& \text { H. }
\end{aligned}
$$

$$
(X \otimes \mathbb{1}) \triangleright M^{\mu_{X, \mu, \mu}} X \triangleright(\mathbb{L} \triangleright M)
$$

$$
r_{x}>i d \sum^{2} \text { idDpm }
$$

$$
X D M
$$

(triangle axiom)

$$
\begin{gathered}
\forall x, y, z \in r, \\
M \in M,
\end{gathered}
$$

II. MODULE CATEGORIES

GIVEN $\varphi:=(\varphi, \otimes, \psi, a, l, r)$
LEFT G-MODULE CATEGORy
CONSISTS OF:
(a) CATEGORY -
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

$$
(c, d) \text { NATURAL (tOMS: }
$$

$$
\begin{aligned}
M=\left\{\begin{aligned}
M x, y, M & :(x \otimes y) \triangleright M \\
& \simeq x \triangleright(y \triangleright M)
\end{aligned}\right\}
\end{aligned}
$$

$$
x, y \in \zeta, M \in \eta
$$

$$
p=\left\{p_{M}: \mathbb{L} \triangleright M \stackrel{\sim}{\rightarrow} M\right\}_{M \in \mathcal{M}}
$$

Satisfying the
PENTAGON AXIOM
\& TrIANGLE AXIOM
II. MODULE CATEGORIES

GIVEN $\varphi:=(\varphi, \otimes, \mu, a, l, r)$
LEFT G-MODULE CATEGORy
CONSISTS OF:
(a) CATEGORY -
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

$$
(c, d) \text { NATURAL (tOMS: }
$$

$$
\left.\begin{array}{rl}
M=\left\{\begin{aligned}
M_{x, y, M}: & (x \otimes y) \triangleright M \\
& \simeq
\end{aligned} \quad x \triangleright(y \triangleright M)\right.
\end{array}\right\}
$$

$$
x, y \in \zeta, M \in \eta
$$

$$
p=\left\{p_{M}: \mathbb{L} \triangleright M \stackrel{\sim}{\rightarrow} M\right\}_{M \in \mathcal{M}}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT $\varphi_{-}$-MODULE CATE.

$$
(m, 4, n, q)
$$

DEFINED LIKEWISE
II. MODULE CATEGORIES

GIVEN $\boldsymbol{\zeta}:=(\boldsymbol{\varphi}, \otimes, \boldsymbol{u}, a, l, r)$
LEFT $\zeta$-MODULE CATEGORY
CONSISTS OF:
(a) CATEGORY - m
(b) BIFUNCTOR

$$
\triangleright: C \times m \rightarrow m
$$

$$
(c, d) \text { NATURAL (tOMS: }
$$

$$
\left.\begin{array}{rl}
M=\left\{\begin{aligned}
M_{x, y, M}: & (x \otimes y) \triangleright M \\
& \simeq
\end{aligned} \quad x \triangleright(y \triangleright M)\right.
\end{array}\right\}
$$

$$
x, y \in \zeta, M \in \eta
$$

$$
p=\left\{p_{M}: \mathbb{L} \triangleright M \stackrel{\sim}{\rightarrow} M\right\}_{M \in \mathcal{M}}
$$

Satisfying the
pentagon axiom
\& Triangle axiom
RIGHT $\varphi$-MODULE CATE.

$$
(m, 4, n, q)
$$

DEFINED LIKEWISE

EXAMPLES
II. MODULE CATEGORIES

GIVEN $\varphi:=(\varphi, \otimes, L, a, l, r)$
LEFT $\zeta$-MODULE CATEGORY
CONSISTS OF:
(a) CATEGORY - m
(b) BIFUNCTOR

$$
\nabla: 6 \times m \rightarrow m
$$

(cad) NATURAL (80MS:

$$
M=\left\{\begin{aligned}
M_{x, y, M} & :(x \otimes y) \triangleright M \\
& \sim x \triangleright(y \triangleright M)
\end{aligned}\right\}
$$

$$
x, y \in \zeta \quad M \in \eta
$$

$$
p=\left\{p_{M}: \Perp \Perp M \stackrel{\sim}{\longrightarrow} M\right\}_{M \in \mathcal{M}}
$$

SATISFyING THE
PENTAGON AXIOM
\& TRIANGLE AXIOM
RIGHT $\zeta$-MODULE CATE.

$$
(m, 4, n, q)
$$

DEFINED LIKEWISE

EXAMPLES
???

$$
\equiv \text { REGULAR LEFT } \zeta \text {-MODULE CATEGORY } \equiv
$$

II. MODULE CATEGORIES
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathbb{L}, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY $-m$
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(c,d) Natural (som:

$$
\begin{aligned}
& M=\left\{\begin{aligned}
M_{x, y, M} & :(X \otimes y) \triangleright M \\
& \Rightarrow x \triangleright(y \triangleright M)
\end{aligned}\right\} \\
& x, y \in H_{M} \boldsymbol{m} \\
& p=\left\{p_{M}: \Delta \triangleright M \simeq \leadsto \Delta\right\}_{M \in M}
\end{aligned}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-module cat eg.

$$
(4, \triangleleft, n, q)
$$

DEFINED LIKEWISE

EXAMPLES
GIVEN $\left(b, \otimes^{e}, 1^{e}, a^{e}, l^{e}, r^{e}\right)$,
GET $\quad M:=\zeta \notin D:=\theta^{h}$
FORMS A LEFT $\zeta$-MODULE CATEGORY =REGULAR LEFT C-MODULE CATEGORY $\equiv$
II. MODULE CATEGORIES

EXAMPLES

LIKE FORIR-ALGEBRA
A
regular left a-module
III

$$
\text { (Avs, } D=M_{A} \text { ) }
$$

GIVEN $\left(\varphi, \otimes^{e}, 1^{e}, a^{e}, l^{e}, r^{e}\right)$,
GET $\quad 4:=\zeta \neq D:=\theta^{h}$
FORMS A LEFT G-MODULE CATEGORY =REGULAR LEFT C -MODULE CATEGORY $\equiv$
II. MODULE CATEGORIES

EXAMPLES

LIKE FORIR-ALGEBRA
A
regular left a-module
III

$$
\left(A v s, D=M_{A}\right)
$$

GIVEN $\left(\varphi, \otimes^{e}, 1^{e}, a^{e}, l^{e}, r^{e}\right)$,
GET $\quad 4:=\zeta \notin D:=\theta^{k}$
FORMS A LEFT G-MODULE CATEGORY = REGULAR LEFT 6 -MODULE CATEGORY $\equiv$

GIVEN A STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

Take a left $\theta$-module category ( $\sim /$, $D$ ).
GET A LEFT C-MODULE CATEGORY ( $M, \perp$ ) VIA ???
II. MODULE CATEGORIES

EXAMPLES

LIKE FORIR-ALGEBRA
A
regular left a-module
III

$$
\left(A v s, D=M_{A}\right)
$$

GIVEN $\left(\varphi, \otimes^{e}, 1^{e}, a^{e}, l^{e}, r^{e}\right)$,
GET $\quad 4:=C \notin D:=\theta^{h}$
FORMS A LEFT G-MODULE CATEGORY = REGULAR LEFT 6 -MODULE CATEGORY $\equiv$

GIVEN A STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

Take a left $\theta$-module category ( $\sim /$, $D$ ).
GET A LEFT $C$-MODULE CATEGORY $(M, \rightharpoonup)$

$$
V I A \quad X>M:=F(X) \triangleright M \quad \forall x \in C, M \in \eta
$$

II. MODULE CATEGORIES

EXAMPLES
LIKE FORIR-ALGEBRAS
$A, B$
regular left a-module
III

$$
\text { (Avs, } D=m_{A} \text { ) }
$$

FOR AN alg map

$$
\phi: A \rightarrow B
$$

The restriction of

$$
\begin{gathered}
\left({ }_{B} V, \triangleright\right) \\
\text { To A ALONG } \phi \\
\text { III } \\
(V, a>v:=\phi(a) \triangleright v) \\
\forall a \in A, v \in V
\end{gathered}
$$

GIVEN $\left(\varphi, \otimes^{e}, L^{e}, a^{e}, l^{e}, r^{e}\right)$,
GET $\quad 4:=\zeta \notin D:=\theta^{k}$
FORMS A LEFT G-MODULE CATEGORY = REGULAR LEFT 6 -MODULE CATEGORY $\equiv$

GIVEN A STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

Take a left $\theta$-module category ( $M, \mathrm{D}$ ).
GET A LEFT C-MODULE CATEGORY ( $M, \nabla$ )

$$
V I A \quad X>M:=F(X) \triangleright M \quad \forall x \in C, M \in \eta
$$

II. MODULE CATEGORIES

EXAMPLES

LIKE FORIR-ALGEBRAS
$A, B$
regular left a-module
III

$$
\left(A v s, D=M_{A}\right)
$$

FOR AN ALG MAP

$$
\phi: A \rightarrow B
$$

THE RESTRICTION OF

$$
\begin{gathered}
\left(B_{B} V, \triangleright\right) \\
\text { To A ALonG } \phi \\
\text { III } \\
(V, a \triangleright v:=\phi(a) \triangleright v) \\
\forall a \in A, v \in V
\end{gathered}
$$

GIVEN $\left(b, \otimes^{\varepsilon}, \mathbb{L}^{\varepsilon}, a^{\varepsilon}, e^{e}, r^{\varepsilon}\right)$,
GET $\quad M_{1}:=C \neq D:=\theta^{h}$
FORMS A LEFT G-MODULE CATEGORY =REGULAR LEFT C-MODULE CATEGORY $\equiv$

GIVEN A STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

Take a left $\theta$-module category ( $M, \mathrm{D}$ ).
GET A LEFT C-MODULE CATEGORY ( $\mathcal{M}, \triangleright$ )

$$
V I A \quad X \triangleright M:=F(X) \triangleright M \quad \forall x \in \mathscr{C}, M \in \eta
$$

$\equiv$ RESTRICTION OF $(m, D) \in \theta-M o d$ To C ALONG $F \equiv$
II. MODULE CATEGORIES

EXAMPLES
LIKE FORIR-ALGEBRAS
$A, B$
regular left a-module
III

$$
\text { (Avs, } D=M_{A} \text { ) }
$$

FOR AN alg map

$$
\phi: A \rightarrow B
$$

The restriction of

$$
\begin{gathered}
\left({ }_{B} V, \triangleright\right) \\
\text { To A ALoNG } \phi \\
\text { III } \\
(V, a \triangleright v:=\phi(a)>v) \\
\quad \forall a \in A, v \in V
\end{gathered}
$$

GIVEN $\left(\varphi, \otimes^{e}, L^{e}, a^{e}, l^{e}, r^{e}\right)$,
GET $\quad 4:=C \notin D:=\theta^{h}$
FORMS A LEFT G-MODULE CATEGORY = REGULAR LEFT 6 -MODULE CATEGORY $\equiv$

GIVEN A STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

Take a left $\theta$-module category ( $M, \mathrm{D}$ ).
GET A LEFT C-MODULE CATEGORY ( $M, \nabla$ )

$$
V I A \quad X>M:=F(X) \triangleright M \quad \forall x \in C, M \in \eta
$$

CATEGORY OF $\theta$-MOD CATS DEFINED LATER
$\equiv$ RESTRICTION OF $(m, D) \in \Delta-M o d$, To C $A L O N G F \equiv$
II. MODULE categories

EXER.3.12: COMPLETING THE DETAILS
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \psi, a, l, r)$ LEFT G-MODULE CATEGORy CONSISTS OF:
(a) CATEGORY $-m$
(b) BIFUNCTOR

$$
\Delta: 6 \times m \rightarrow m
$$

(cad) Natural (sIms:
$M=\left\{\begin{aligned} M_{x, y, M} & :(x \otimes y) \triangleright M \\ & \Rightarrow x \triangleright(y \triangleright M)\end{aligned}\right\}$


$$
p=\left\{p_{M}: \mathbb{U} D \sim \sim M\right\}_{M \in M}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-module cat eg. $(-m, 4, n, q)$ DEFINED LIKEWISE

EXAMPLES
GIVEN $\left(b, \otimes^{e}, L^{e}, a^{e}, l^{e}, r^{e}\right)$,
GET $\quad 4:=C \not \& D:=\theta^{h}$
FORMS A LEFT G-MODULE CATEGORY REGULAR LEFT C -MODULE CATEGORY $\equiv$

GIVEN A STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

Take a left $\theta$-module category ( $\mathcal{M}, \mathrm{D}$ ).
GET A LEFT - MODULE CATEGORY ( $M, \nabla$ )

$$
\text { VIA } X \triangleright M:=F(X) \triangleright M \quad \forall x \in C, M \in \eta
$$

CATEGORY OF $\theta$-MOD CATS DEFINED LATER
$\equiv$ RESTRICTION OF $(m, D) \in \Delta-M o d$, To Ce ALONG $F \equiv$
II. MODULE CATEGORIES

EXER.3.12: COMPLETING THE DETAILS
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, u, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY TM
(b) BIFUNCTOR

$$
\nabla: 6 \times m \rightarrow m
$$

(c,d) NATURAL (8OMS:

$$
\left.\begin{array}{rl}
M=\left\{\begin{array}{c}
\mu_{x, y, 1}: \\
\\
\\
\end{array}(x \otimes y) \triangleright M(y \triangleright M)\right.
\end{array}\right\}
$$

SATISFYING THE
PENTAGON AXIOM
\& Triangle axiom
RIGHT C-MODULE CATE. $(-m, 4, n, q)$ DEFINED LIKEWISE

EXAMPLES
$\equiv$ LETS TRY THIS NOW (ON THE BOARD) $\left.\begin{array}{c}\text { INCCLASS } \\ \text { IN }\end{array}\right) \equiv$
GIVEN $\left(b, e^{e}, L^{e}, a^{e}, l^{e}, r^{\varepsilon}\right)$,
GET $\quad 4:=C \neq D:=\theta^{h}$
FORMS A LEFT G-MODULE CATEGORY REGULAR LEFT C -MODULE CATEGORY $\equiv$

GIVEN A STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

Take a left $\theta$-module category ( $M, \mathrm{D}$ ).
GET A LEFT - MODULE CATEGORY ( $M, \nabla$ )

$$
\text { VIA } X \triangleright M:=F(X) \triangleright M \quad \forall x \in C, M \in \eta
$$

CATEGORY OF $\theta$-MOD CATS DEFINED LATER
$\equiv$ RESTRICTION OF $(m, D) \in \Delta-M o d, ~ T O G A L O N G F \equiv$
II. MODULE CATEGORIES

EXER.3.12: COMPLETING THE DETAILS
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \psi, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(ce) Natural (80MS:

$$
\begin{aligned}
M=\left\{\begin{aligned}
M x, y, y: & :(x \otimes y) \triangleright M \\
& \Rightarrow x \triangleright(y \triangleright M)
\end{aligned}\right\}
\end{aligned}
$$

$$
x, y \in \boldsymbol{G}, M_{i} \eta
$$

$$
p=\left\{p_{M}: \mathbb{U} D M \check{\leftrightharpoons} M\right\}_{M \in M}
$$

Satisfying The
pentagon axiom
\& Triangle axiom
RIGHT Є-module cat eg. $(-m, 4, n, q)$ DEFINED LIKEWISE

EXAMPLES
GIVEN A GROUP G WITH SUBGROUP H CAN FORM A STRONG MONOIDAL FUNCTOR

$$
\operatorname{Res}_{H}^{G}: G-\operatorname{Mod} \longrightarrow H-M o d
$$

GIVEN A STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

Take a left $\theta$-module category ( $M, D$ ).
GET A LEFT C -MODULE CATEGORY ( $~ M, ~ \triangleright$ )

$$
V I A \quad X \triangleright M:=F(X) \triangleright M \quad \forall x \in \mathscr{C}, M \in \eta
$$

$\equiv$ RESTRICTION OF $(m, D) \in \Delta$-Mod To G ALONG $F \equiv$
II. MODULE CATEGORIES

EXER.3.12: COMPLETING THE DETAILS
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \psi, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(ce) Natural (80MS:

$$
\begin{aligned}
M=\left\{\begin{aligned}
M x, y, y: & :(x \otimes y) \triangleright M \\
& \Rightarrow x \triangleright(y \triangleright M)
\end{aligned}\right\}
\end{aligned}
$$

$$
x, y \in \boldsymbol{G}, M_{i} \eta
$$

$$
p=\left\{p_{M}:\lfloor\triangleright M \simeq \leadsto M\}_{M \in M}\right.
$$

Satisfying The
pentagon axiom
\& Triangle axiom
RIGHT Є-module cat eg. $(-m, 4, n, q)$ DEFINED LIKEWISE

EXAMPLES
GIVEN A GROUP G WITH SUBGROUP H CAN FORM A STRONG MONOIDAL FUNCTOR

$$
\operatorname{Res}_{H}^{G}: G-\operatorname{Mod} \longrightarrow H-M o d
$$

(GENERALIZING Ford: $G$-Mod $\rightarrow$ Voc For $H=\langle e\rangle$ )

GIVEN A STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

Take a left $\theta$-module category ( $M, D$ ).
GET A LEFT C -MODULE CATEGORY ( $~ M, ~ \triangleright$ )

$$
V I A \quad X \triangleright M:=F(X) \triangleright M \quad \forall x \in C, M \in T
$$

$\equiv$ RESTRICTION OF $(m, D) \in \Delta$-Mod To G ALONG $F \equiv$
II. MODULE CATEGORIES

EXER.3.12: COMPLETING THE DETAILS
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \psi, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(c,d) Natural (toms:

$$
x, y \in \boldsymbol{G}, M_{i} \eta
$$

$$
p=\left\{p_{M}:\lfloor\triangleright M \simeq \leadsto M\}_{M \in M}\right.
$$

Satisfying The
pentagon axiom
\& Triangle axiom
RIGHT $\zeta$-module cat eg. $(-m, 4, n, q)$ DEFINED LIKEWISE

EXAMPLES
GIVEN A GROUP G WITH SUBGROUP H CAN FORM A STRONG MONOIDAL FUNCTOR

$$
\operatorname{Res}_{H}^{G}: G-\operatorname{Mod} \longrightarrow H-M o d
$$

(GENERALIZING Ford: $G$-Mod $\rightarrow$ Voc For $H=\langle e\rangle$ )
$\rightarrow H$-Mod (regular mod.categ.) (egg. Vex) is a Left module category over G-Mod

GIVEN A STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

Take a left $D$-module category ( $\mathcal{M}, \mathrm{D}$ ).
GET A LEFT C -MODULE CATEGORY ( $~ M, ~ \triangleright$ )

$$
V I A \quad X \triangleright M:=F(X) \triangleright M \quad \forall x \in C, M \in T
$$

$\equiv$ RESTRICTION OF $(m, D) \in \Delta$-Mod To G ALONG $F \equiv$
II. Module categories
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \psi, a, l, r)$
LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(cad) Natural (sIms:

$$
\begin{aligned}
M=\left\{\begin{aligned}
M, y, y: & :(x \otimes y) \triangleright M \\
& \Rightarrow x \triangleright(y \triangleright M)
\end{aligned}\right\}
\end{aligned}
$$

$$
x, y \in \boldsymbol{G}, M_{i} \eta
$$

$$
p=\left\{p_{M}:\lfloor\triangleright M \simeq \leadsto M\}_{M \in M}\right.
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-module cat eg. $(-m, \Delta, n, q)$ DEFINED LIKEWISE

EXER.3.12: COMPLETING THE DETAILS
$\left.\begin{array}{r}\text { GET STRONG MONOIDAL FUNCTOR } \\ \text { Braid } \longrightarrow \text { PerM }\end{array}\right\} \begin{gathered}\text { Perm IS A } \\ \text { LEFTMODCAT. } \\ \text { OVER Braid }\end{gathered}$

GIVEN A STRONG MONOIDAC FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

Take a left $\theta$-module category ( $M, D$ ).
GET A LEFT C-MODULE CATEGORY ( $M, D$ )

$$
V I A \quad X \triangleright M:=F(X) \triangleright M \quad \forall X \in \mathscr{C}, M \in \mathcal{M}
$$

$\equiv$ RESTRICTION OF $(m, D) \in \Delta-M o d T o$ Te ALONG $F \equiv$
II. MODULE CATEGORIES

$$
\begin{aligned}
& \begin{array}{l}
\text { GIVEN } \varphi:=(\varphi, \otimes, u, a, l, r) \\
\text { LEFT } \zeta \text {-MODULE CATEGORY }
\end{array} \\
& \text { CONSISTS OF: } \\
& \text { (a) CATEGORY In } \\
& \text { (b) BIFUNCTOR } \\
& \Delta: 6 \times 7 \rightarrow m \\
& \text { (cad) NATURAL (8OMS: } \\
& M=\left\{\begin{aligned}
M x, y, M & :(x \otimes y) \triangleright M \\
& \simeq x \triangleright(y \triangleright M)
\end{aligned}\right\} \\
& x, y \in r_{\mu}, \eta \\
& p=\left\{p_{M}: \Delta \square M \simeq \leadsto \leadsto\right\}_{M \in M} \\
& \text { satisfying The } \\
& \text { pentagon axiom } \\
& \text { \& Triangle axiom }
\end{aligned}
$$

Right $\zeta$-MODULE CATE. $(m, 4, n, q)$ DEFINED LIKEWISE

EXAMPLES
\(\left.\begin{array}{cc}GET STRONG MONOIDAL FUNCTOR \\

Braid \longrightarrow \operatorname{PerM}\end{array}\right\}\)| PerMISA |
| :--- |
| OBJECTS: $n \in \mathbb{N}$ |
| $n \in \mathbb{N}$ |



GIVEN A STRONG MONOIDAL FUNCTOR

$$
\left(F, F^{(2)}, F^{(0)}\right): \zeta \longrightarrow \theta
$$

Take a left $\theta$-module category ( $(\mu, D)$.
GET A LEFT C -MODULE CATEGORY ( $~(~) ~ D) ~$

$$
V I A \quad X \triangleright M:=F(X) \triangleright M \quad \forall x \in C, M \in T
$$

$\equiv$ RESTRICTION OF $(m, D) \in \Delta$-Mod To G ALONG $F \equiv$
II. MODULE CATEGORIES

EXER.3.12: COMPLETING THE DETAILS

$$
\begin{aligned}
& \text { GIVEN } \varphi:=(\varphi, \otimes, u, a, l, r) \\
& \text { LEFT } \zeta \text {-MODULE CATEGORY }
\end{aligned}
$$ CONSISTS OF:

(a) CATEGORY M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(c,d) Natural (som:

$$
x, y \in \xi \in M \in \eta
$$

$$
p=\left\{p_{M}: \mathbb{U} D M \simeq M \leftrightharpoons M\right\}_{M \in M}
$$

Satisfying. The
pentagon axiom
\& Triangle axiom
RIGHT $\varphi$-MODULE CATE.

$$
(y, \triangleleft, n, q)
$$

DEFINED LIKEWISE

EXAMPLES
GET STRONG MONOIDAL FUNCTOR $\left.\begin{array}{c}\text { Braid } \longrightarrow \text { PerM }\end{array}\right\} \begin{gathered}\text { Perm } 1 / A \\ \Rightarrow \begin{array}{l}\text { LeFT MOD CAT. } \\ \text { OVER Braid }\end{array}\end{gathered}$
OBJECTS: $n \in \mathbb{N}$
$n \in \mathbb{N}$
$\begin{array}{rr}\text { MORPHISMS: } & \begin{array}{l}\text { BRAID } \\ \complement^{\text {GROUP }}\end{array} \\ \operatorname{HOM}(n, \mu) & =\left\{\begin{array}{ll}B_{n} & n=\mu \\ \varnothing & n \neq \mu\end{array}\right)\end{array}$

VISUALIZED AS BRAIDS

II. MODULE CATEGORIES


RIGHT Є-MODULE CATE.

$$
(m, 4, n, q)
$$

DEFINED LIKEWISE

EXAMPLES
\(\left.\begin{array}{c}GET STRONG MONOIDAL FUNCTOR \\

Braid \longrightarrow \operatorname{PerM}\end{array}\right\}\)| Perm IS A |
| :--- |
| OBJECTS: $n \in \mathbb{N}$ |
| LEFTMODCAT. |
| OVER Braid |

 visualized as braids visualized as permutations


$$
\underset{\tau_{1}}{X \mid} \underset{\tau_{\tau_{1}^{-1}}}{ } X \underset{\tau_{2}}{ } \mid X
$$

II. MODULE CATEGORIES

$$
\begin{aligned}
& \begin{array}{l}
\text { GIVEN } \varphi:=(\varphi, \otimes, \psi, a, l, r) \\
\text { LEFT } \zeta \text {-MODULE CATEGORY }
\end{array} \\
& \text { CONSISTS OF: } \\
& \text { (a) CATEGORY } m \\
& \text { (b) BIFUNCTOR }
\end{aligned}
$$

RIGHT $\zeta$-MODULE CATE.

$$
(m, 4, n, q)
$$

DEFINED LIKEWISE

EXAMPLES
GET STRONG MONOIDAL FUNCTOR $\left.\begin{array}{rl}\theta: \text { Braid } \longrightarrow \text { PerM }\end{array}\right\} \begin{gathered}\text { PerM IS A } \\ \text { LEFT MOD CAT. } \\ \text { OVER Braid }\end{gathered}$
OBJECTS: $n \in \mathbb{N}$
$n \in \mathbb{N}$
 visualized as braids visualized as permutations

$\theta: n \longmapsto n$ ON OBJECTS
II. MODULE categories


RIGHT $\zeta$-module cat eg.

$$
(4, \triangleleft, n, q)
$$

DEFINED LIKEWISE

EXAMPLES
GET STRONG MONOIDAL FUNCTOR $\left.\quad \begin{array}{rl} \\ \theta: \text { Braid } \longrightarrow \text { PerM }\end{array}\right\} \begin{gathered}\text { PerM IS A } \\ \text { LEFT MOD CAT. } \\ \text { OVER Braid }\end{gathered}$
OBJECTS: $n \in \mathbb{N}$
$n \in \mathbb{N}$
 visualized as braids visualized as permutations

$\theta: n \longmapsto n$ ON OBJECTS

$$
S_{n} \equiv \text { Quot. GP }
$$

\(\left.\begin{array}{c}GET STRONG MONOIDAL FUNCTOR \\

\theta: Braid \longrightarrow PerM\end{array}\right\}\)| Perm IS A |
| :--- |
| OBJECTS: $n \in \mathbb{N}$ |
| LEFT MOD CAT. |
| OVER Braid |



EXER. 3.12: COMPLETING THE DETAILS
$\mathrm{Bn}_{n} \longrightarrow \mathrm{~S}_{n}$ ON MORPHISMS ...
II. MODULE CATEGORIES
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, u, a, l, r)$
LEFT G-MODULE CATEGORy
CONSISTS OF:
(a) CATEGORY $m$
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(cad) Natural (sIms:

$$
\begin{aligned}
& M=\left\{\begin{aligned}
M_{x, y, M} & :(x \otimes y) \triangleright M \\
& \Rightarrow x \triangleright(y \triangleright M)
\end{aligned}\right\} \\
& x, y \in H_{M} \boldsymbol{m} \\
& p=\left\{p_{M}: \Delta \triangleright M \simeq \leadsto \Delta\right\}_{M \in M}
\end{aligned}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
Right $\zeta$-MODULE CATE.

$$
(m, 4, n, q)
$$

DEFINED LIKEWISE
$\longleftarrow$ NOW WE STUDY HOW
To move from
one to another
II. MODULE categories
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORy CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(cad) Natural (som:

$$
\begin{aligned}
M=\left\{\begin{aligned}
M, y, y, M & :(x \otimes y) \triangleright M \\
& \Rightarrow x \triangleright(y \triangleright M)
\end{aligned}\right\}
\end{aligned}
$$

$$
x, y \in \boldsymbol{G}, M_{i} \eta
$$

$$
p=\left\{p_{M}: \mathbb{U} D \sim \sim M\right\}_{M \in M}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT $\zeta$-MODULE CATE.

$$
(m, 4, n, q)
$$

DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM $(m, D, \mu, p)$ To $\left(\mu^{\prime}, D^{\prime}, \mu^{\prime}, p^{\prime}\right)$ CONSISTS OF:
II. MODULE categories
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY $m$
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(cad) Natural (som:

$$
\begin{aligned}
M=\left\{\begin{aligned}
M, y, y, M & :(x \otimes y) \triangleright M \\
& \Rightarrow x \triangleright(y \triangleright M)
\end{aligned}\right\}
\end{aligned}
$$

$$
x, y \in \boldsymbol{G}, M_{i} \eta
$$

$$
p=\left\{p_{M}: \mathbb{U} D \sim \sim M\right\}_{M \in M}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-MODULE CATE.

$$
(m, 4, n, q)
$$

DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM $(m, D, \mu, p)$ To $\left(m^{\prime}, D^{\prime}, \mu^{\prime}, p^{\prime}\right)$ CONSISTS OF:
(a) A FUNCTOR $F:=m \rightarrow \eta^{\prime}$
II. MODULE categories
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(c,d) Natural (sIms:

$$
x, y \in G_{1}, m \in \eta
$$

$$
p=\left\{p_{M}: \mathbb{U} D \sim \sim M\right\}_{M \in M}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT G-MODULE CATE.

$$
(m, 4, n, q)
$$

DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM $(m, D, \mu, p)$ TO $\left(m^{\prime}, D^{\prime}, \mu^{\prime}, p^{\prime}\right)$ CONSISTS OF:
(a) A FUNCTOR $F: m \rightarrow m^{\prime}$
(b) A NATURAL ISOMORPHISM

$$
\begin{aligned}
& \zeta \times \overbrace{\nabla_{0} \circ\left(\tau_{\tau} \times F\right)}^{F \sim \| s} m^{\prime} \\
& \left.\begin{array}{c}
\text { LEFT MODULE FuNCTOR } \\
\text { CONSTANT }
\end{array}\right) \\
& s:=\left\{S_{X, M}: F(X \triangleright M) \xrightarrow{\longrightarrow} X D^{\prime} F(M)\right\}_{X \in e, M \in m}
\end{aligned}
$$

II. MODULE categories
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(cad) Natural (sIms:

$$
x, y \in \boldsymbol{G}, M_{i} \eta
$$

$$
p=\left\{p_{M}: \mathbb{U} D M \check{\leftrightharpoons} M\right\}_{M \in M}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-MODULE CATE.

$$
(-m, 4, n, q)
$$

DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM $(m, D, \mu, p)$ TO $\left(m^{\prime}, D^{\prime}, \mu^{\prime}, p^{\prime}\right)$ CONSISTS OF:
(a) A FUNCTOR $F: M \rightarrow \eta^{\prime}$
(b) A NATURAL ISOMORPHISM
.

$$
\begin{aligned}
& \zeta \times \overbrace{\nabla^{\prime} \circ(\operatorname{Id} \times F)}^{\prod_{0} \sim \downarrow s} m^{\prime} \\
& \left.\begin{array}{c}
\text { (LEFT MODULE FWUCTOR } \\
\text { CONSTANT }
\end{array}\right) \\
& S:=\left\{S_{X, M}: F(X \triangleright M) \xrightarrow{\sim} X \triangleright^{\prime} F(M)\right\}_{X \in e, M \in m} \\
& F((x \otimes y) D M) \\
& X \nabla(Y \delta F(M)) \\
& \text { (PENTAGON AXIOM) } \quad \forall x, y \in \varphi, \mu \in m
\end{aligned}
$$

II. MODULE categories
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(cad) Natural (sIms:

$$
x, y \in \boldsymbol{G}, M_{i} \eta
$$

$$
p=\left\{p_{M}: \mathbb{U} D M \check{\leftrightharpoons} M\right\}_{M \in \mathcal{M}}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-module cater.

$$
(-m, 4, n, q)
$$

DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM $(m, D, \mu, p)$ To ( $\left(\mu^{\prime}, D^{\prime}, \mu^{\prime}, p^{\prime}\right)$ CONSISTS OF:
(a) A FUNCTOR $F:=T \rightarrow \eta^{\prime}$
(b) A NATURAL ISOMORPHISM

$S:=\left\{S_{X, M}: F(X \triangleright M) \xrightarrow{\sim} X \triangleright^{\prime} F(M)\right\}_{X \in e, M \in m}$
. 7.

$$
\begin{aligned}
& F(M x, y, \mu) \quad F((x \otimes y) \triangleright M) \\
& F(X \triangleright(y \triangleright M))
\end{aligned}
$$

$$
X \nabla^{\prime}\left(Y \delta^{\prime} F(M)\right)
$$

(PENTAGON AXIOM) $\quad \forall x, y \in \boldsymbol{\zeta}, \mu \in M$
II. MODULE categories
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(cad) Natural (sIms:

$$
x, y \in \zeta, M \in \eta
$$

$$
p=\left\{p_{M}: \mathbb{U} D M \check{\leftrightharpoons} M\right\}_{M \in \mathcal{M}}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-module cat eg.

$$
(-m, 4, n, q)
$$

DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM ( $\mu, D, \mu, p$ ) TO ( $m^{\prime}, D^{\prime}, \mu^{\prime}, p^{\prime}$ ) CONSISTS OF:
(a) A FUNCTOR $F:=T \rightarrow \eta^{\prime}$
(b) A NATURAL ISOMORPHISM
$S:=\left\{S_{X, M}: F(X \triangleright M) \xrightarrow{\sim} X \triangleright^{\prime} F(M)\right\}_{X \in e, M \in m}$
. 7.

$$
F(M x, y, M) / F((x \triangleright y) \triangleright M)
$$

$$
F(X D(Y \nabla M))
$$

$S_{x, y} \triangleright \downarrow$

$$
X \nabla^{\prime} F(Y \triangleright M) \quad X \nabla^{\prime}(Y \triangleright F(M))
$$

(PENTAGON AXIOM) $\quad \forall x, y \in \zeta, M \in-m$

$$
\begin{aligned}
& \zeta \times \overbrace{\sigma_{0} \circ\left(\tau_{\tau} \times F\right)}^{F \sim \| s} m^{\prime} \\
& \left.\begin{array}{c}
\text { LEFT MODULE FuNCTOR } \\
\text { CONSTANT }
\end{array}\right)
\end{aligned}
$$

II. MODULE categories
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(cad) Natural (sIms:

$$
x, y \in \zeta, M \in \eta
$$

$$
p=\left\{p_{M}: \mathbb{U} D M \check{\leftrightharpoons} M\right\}_{M \in M}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-module cat eg. $(-m, 4, n, q)$ DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM ( $\mu, D, \mu, p$ ) TO ( $m^{\prime}, D^{\prime}, \mu^{\prime}, p^{\prime}$ ) CONSISTS OF:
(a) A FUNCTOR $F:=T \rightarrow \eta^{\prime}$
(b) A NATURAL ISOMORPHISM
$S:=\left\{S_{X, M}: F(X \triangleright M) \xrightarrow{\sim} X \triangleright^{\prime} F(M)\right\}_{X \in e, M \in m}$
. 7.

$$
F(M x, y, M) / F((x \triangleright y) \triangleright M)
$$

$$
F(X D(Y \nabla M))
$$

$S_{x, y} \triangleright \downarrow$

$$
\begin{aligned}
X \triangleright^{\prime} & F(Y \triangleright M) \xrightarrow{\text { id } D^{\prime} S Y, M} X \nabla^{\prime}\left(Y \triangleright^{\prime} F(M)\right) \\
& (P E N T A G O N \text { AXIOM) }
\end{aligned} \quad \forall x, y \in \zeta, M \in-m
$$

$$
\begin{aligned}
& \zeta \times \overbrace{\sigma_{0} \circ\left(\tau_{\tau} \times F\right)}^{F \sim \| s} m^{\prime} \\
& \left.\begin{array}{c}
\text { LEFT MODULE FuNCTOR } \\
\text { CONSTANT }
\end{array}\right)
\end{aligned}
$$

II. MODULE categories
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(cad) Natural (som:

$$
x, y \in \boldsymbol{G}, M_{i} \eta
$$

$$
p=\left\{p_{M}: \mathbb{U} D M \check{\leftrightharpoons} M\right\}_{M \in \mathcal{M}}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-module cat eg. $(4,4, n, q)$ DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM ( $\mu, D, \mu, p$ ) TO ( $m^{\prime}, D^{\prime}, \mu^{\prime}, p^{\prime}$ ) CONSISTS OF:
(a) A FUNCTOR $F:=T \rightarrow \eta^{\prime}$
(b) A NATURAL ISOMORPHISM
$S:=\left\{S_{X, M}: F(X \triangleright M) \xrightarrow{\sim} X \triangleright^{\prime} F(M)\right\}_{X \in e, M \in m}$
..

$$
F(M x, y, M) / F((x \otimes y) \triangleright M) \searrow_{S_{X \otimes Y, M}}
$$

$$
F(X \triangleright(Y \triangleright M)) \quad(X \otimes Y) \triangleright^{\prime} F(M)
$$

$S x, y>M \downarrow$

$$
X \nabla^{\prime} F(Y \triangleright M) \xrightarrow{\downarrow} X \nabla^{\text {id } \nabla^{\prime}}(Y, M \text { (Y' } F(M))
$$

(PENTAGON AXIOM) $\quad \forall x, y \in \zeta, M \in-m$

$$
\begin{aligned}
& \zeta \times \overbrace{\sigma_{0} \circ\left(\tau_{\tau} \times F\right)}^{F \sim \| s} m^{\prime} \\
& \left.\begin{array}{c}
\text { LEFT MODULE FuNCTOR } \\
\text { CONSTANT }
\end{array}\right)
\end{aligned}
$$

II. MODULE categories
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(cad) Natural (sIms:

$$
\begin{aligned}
M=\left\{\begin{aligned}
M x, y, M & :(x \otimes y) \triangleright M \\
& \Rightarrow x \triangleright(y \triangleright M)
\end{aligned}\right\}
\end{aligned}
$$

$$
x, y \in \boldsymbol{G}, M_{i} \eta
$$

$$
p=\left\{p_{M}: \mathbb{U} D M \check{\leftrightharpoons} M\right\}_{M \in \mathcal{M}}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-module cat eg. $(-m, 4, n, q)$ DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM ( $\mu, D, \mu, p$ ) TO ( $m^{\prime}, D^{\prime}, \mu^{\prime}, p^{\prime}$ ) CONSISTS OF:
(a) A FUNCTOR $F:=T \rightarrow \eta^{\prime}$
(b) A NATURAL ISOMORPHISM
$S:=\left\{S_{X, M}: F(X \triangleright M) \xrightarrow{\sim} X \triangleright^{\prime} F(M)\right\}_{X \in e, M \in m}$
..

$$
F(M x, y, M) / F((X \otimes y) \triangleright M) \searrow_{S_{X \otimes Y, M}}
$$

$$
F(X \triangleright(Y \triangleright M)) \quad 2 \quad(X \otimes Y) \nabla^{\prime} F(M)
$$

$S_{x, Y \Delta M} \downarrow$
$X \nabla^{\prime} F(Y \triangleright M) \xrightarrow{i d D^{\prime} S y, M} X \nabla^{\prime}\left(Y M^{\prime} x, y, F(M)\right)$
(PENTAGON AXIOM) $\quad \forall x, y \in \zeta, M \in T$

$$
\begin{aligned}
& \zeta \times \overbrace{\sigma_{0} \circ\left(\tau_{\tau} \times F\right)}^{F \sim \| s} m^{\prime} \\
& \left.\begin{array}{c}
\text { LEFT MODULE FuNCTOR } \\
\text { CONSTANT }
\end{array}\right)
\end{aligned}
$$

II. MODULE categories
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\triangleright: 6 \times m \rightarrow m
$$

(cad) Natural (sIms:

$$
x, y \in r_{i} M \in \eta
$$

$$
p=\left\{p_{M}: \mathbb{L D M} \leadsto \leadsto M\right\}_{M \in M}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-module cat eg. $(4,4, n, q)$ DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM $(m, D, \mu, p)$ TO $\left(m^{\prime}, D^{\prime}, \mu^{\prime}, p^{\prime}\right)$ CONSISTS OF:
(a) A FUNCTOR $F:=T \rightarrow \eta^{\prime}$
(b) A NATURAL ISOMORPHISM

$S:=\left\{S_{X, M}: F(X \triangleright M) \xrightarrow{\sim} X \triangleright^{\prime} F(M)\right\}_{X \in e, M \in m}$
..
$F\left(M_{x, y, M} / F((x \otimes y) \triangleright M)_{S^{S x \otimes y, M}}\right.$

$$
\begin{gathered}
F(\mathbb{L} D M) \xrightarrow{S_{\mu M}} \mathbb{U} \nabla^{\prime} F(M) \\
F\left(p_{n}\right) \searrow^{2} \downarrow P^{\prime} F(M)
\end{gathered}
$$

$$
\begin{aligned}
& F(X D(Y \triangleright M) 1 \\
& S x, y \triangleright M \downarrow \\
& X \triangleright^{\prime} F(Y \triangleright M) \xrightarrow{i d \nabla^{\prime} S_{y} M} x \nabla^{\prime}\left(y D^{\prime} F(M)\right)
\end{aligned}
$$

(PENTAGON AXIOM)
II. MODULE CATEGORIES
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORy CONSISTS OF:
(a) CATEGORY M
(b) BIFUNCTOR

$$
\nabla: 6 \times m \rightarrow m
$$

(c,d) NATURAL (8OMS:

$$
\begin{aligned}
& M=\left\{\begin{aligned}
M_{x, y, M} & :(x \otimes y) \triangleright M \\
& \Rightarrow x \triangleright(y \triangleright M)
\end{aligned}\right\} \\
& x, y \in H_{M} \boldsymbol{m} \\
& p=\left\{p_{M}: \Delta \triangleright M \simeq \leadsto \Delta\right\}_{M \in M}
\end{aligned}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-module cat eg.

$$
(m, 4, n, q)
$$

DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM $(m, D, \mu, p)$ TO $\left(m^{\prime}, D^{\prime}, \mu^{\prime}, p^{\prime}\right)$ CONSISTS OF:
(a) A FUNCTOR $F: r \rightarrow m^{\prime}$
(b) A NATURAL ISOMORPHISM

$$
S:=\left\{S_{X, M}: F(X \triangleright M) \xrightarrow{\sim} X \nabla^{\prime} F(M)\right\}_{X \in e, M \in m}
$$

SATISFying the pentagon axiom \& TrIANGLE axiom
II. MODULE CATEGORIES
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORy CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\nabla: 6 \times m \rightarrow m
$$

(cad) Natural (som:
$M=\left\{\begin{aligned} M_{x, y, M}: & (x \otimes y) \triangleright M \\ & \simeq X \triangleright(y \triangleright M)\end{aligned}\right\}$ $x, y \in \boldsymbol{H}_{\boldsymbol{M}} \mathrm{Men}^{\boldsymbol{m}}$

$$
p=\left\{p_{M}: \mathbb{U} D M \check{\leftrightharpoons} M\right\}_{M \in M}
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-module cat eg.

$$
(-m, 4, n, q)
$$

DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM $(m, D, \mu, p)$ TO $\left(m^{\prime}, D^{\prime}, m^{\prime}, p^{\prime}\right)$ CONSISTS OF:
(a) A FUNCTOR $F: r \rightarrow m^{\prime}$
(b) A NATURAL ISOMORPHISM

$$
S:=\left\{S_{X, M}: F(X \triangleright M) \xrightarrow{\sim} X \triangleright^{\prime} F(M)\right\}_{X \in \mathscr{C}, M \in m}
$$

Satisfying the pentagon axiom \& Triangle axiom
$\leadsto$ FORMS A CATEGORY C-Mod

- objects: LEFT ५-module categories
- MORPHISMS: LEFT G-MODULE FUNCTORS
II. MODULE CATEGORIES
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORy CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\nabla: 6 \times m \rightarrow m
$$

(ce) Natural (80ms:
$M=\left\{\begin{aligned} M_{x, y, M}: & (x \otimes y) \triangleright M \\ & \simeq X \triangleright(y \triangleright M)\end{aligned}\right\}$ $x, y \in \boldsymbol{H}_{\boldsymbol{M}} \mathrm{Men}^{\boldsymbol{m}}$

$$
p=\left\{p_{M}: \mathbb{L} \Delta M \sim \leadsto M\right\}_{M \in M}
$$

Satisfying The
PENTAGON AXIOM
\& Triangle axiom
RIGHT $\varphi$-module cate. $(m, \triangleleft, n, q)$ DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM $(m, D, \mu, p)$ TO $\left(m^{\prime}, D^{\prime}, m^{\prime}, p^{\prime}\right)$ CONSISTS OF:
(a) A FUNCTOR $F: r \rightarrow m^{\prime}$
(b) A NATURAL ISOMORPHISM

$$
S:=\left\{S_{X, M}: F(X \triangleright M) \xrightarrow{\sim} X \triangleright^{\prime} F(M)\right\}_{X \in \mathscr{e}, M \in m}
$$

SATISFyING THE PENTAGON AXIOM \& TRIANGLE AXIOM
$\leadsto$ FORMS A CATEGORY C-Mod

- objects: left ל-module categories
- MORPHISMS: LEFT ל-MODULE FUNCTORS

LIKEWISE,
CAN DEFINE Mod-6
II. MODULE CATEGORIES
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORy CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\nabla: 6 \times m \rightarrow m
$$

(ce) Natural (80ms:
$M=\left\{\begin{aligned} M_{x, y, M}: & (x \otimes y) \triangleright M \\ & \simeq X \triangleright(y \triangleright M)\end{aligned}\right\}$

$$
p=\left\{p_{M}:\lfloor\triangleright M \simeq \leadsto M\}_{M \in M}\right.
$$

satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT $\varphi$-module cate. $(m, \Delta, n, q)$ DEFINED LIKEWISE

A LEFT C-MODULE FUNCTOR FROM $(m, D, \mu, p)$ TO $\left(m^{\prime}, D^{\prime}, m^{\prime}, p^{\prime}\right)$ CONSISTS OF:
(a) A FUNCTOR $F: r \rightarrow m^{\prime}$
(b) A NATURAL ISOMORPHISM

$$
S:=\left\{S_{X, M}: F(X \triangleright M) \xrightarrow{\sim} X \triangleright^{\prime} F(M)\right\}_{X \in e, M \in M}
$$

Satisfying the pentagon axiom \& Triangle axiom
$\leadsto$ FORMS A CATEGORY C-Mod

- object : left そ-module categories
- MORPHISMS: LEFT ५-MODULE FUNCTORS

III. BIMODULE CATEGORIES
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$
LEFT G-MODULE CATEGORY
CONSISTS OF:
(a) CATEGORY - M
(b) BIFUNCTOR

$$
\begin{aligned}
& D: 6 \times m \rightarrow 2 \\
& \text { (cad) NATURAL (8OMS: } \\
& M=\left\{\begin{aligned}
M_{x, y, M} & :(x \otimes y) \triangleright M \\
& \Rightarrow x \triangleright(y \triangleright M)
\end{aligned}\right\} \\
& x, y \in G M \in \eta \\
& p=\left\{p_{M}: \| \square M \stackrel{\Delta}{\leftrightharpoons} M\right\}_{M \in M}
\end{aligned}
$$

satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT $\varphi$-MODULE CATE.

$$
(m, 4, n, q)
$$

DEFINED LIKEWISE
III. BIMODULE CATEGORIES
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORY CONSISTS OF:
(a) CATEGORY in
(b) BIFUNCTOR

$$
\nabla: 6 \times m \rightarrow m
$$

(cad) Natural (sIms:

$$
\begin{aligned}
M=\left\{\begin{aligned}
M, y, y, M & :(x \otimes y) \triangleright M \\
& \Rightarrow x \triangleright(y \triangleright M)
\end{aligned}\right\}
\end{aligned}
$$

$$
x, y \in \varphi, M \in \eta
$$

$$
p=\left\{p_{M}:\lfloor\triangleright M \simeq \leadsto M\}_{M \in M}\right.
$$

Satisfying the
PENTAGON AXIOM
\& Triangle axiom
RIGHT $\zeta$-MODULE CATE.

$$
(m, 4, n, q)
$$

DEFINED LIKEWISE

Take: $(\eta, D, \mu, p) \in \zeta$-Mod

$$
\ddagger(q, \Delta, n, q) \in \operatorname{Mod}-\theta
$$

GET ...

$$
\in(\zeta, \theta)-\operatorname{Bimod}
$$

III. BIMODULE CATEGORIES
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, \mathcal{L}, a, l, r)$ LEFT G-MODULE CATEGORy CONSISTS OF:
(a) CATEGORY m
(b) BIFUNCTOR

$$
\nabla: 6 \times m \rightarrow m
$$

(cad) Natural (sIms:

$$
\left.\begin{array}{rl}
M=\left\{\begin{aligned}
M x, y, M: & (x \otimes y) \triangleright M \\
& \Rightarrow
\end{aligned}\right) \\
& x \triangleright(y \triangleright M)
\end{array}\right\}
$$

Satisfying. the
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-module cat eg.

$$
(m, 4, n, q)
$$

DEFINED LIKEWISE

Take: $(\eta, D, \mu, p) \in \zeta$ - $\operatorname{Mod}$

$$
\ddagger(t y, 4, n, q) \in \operatorname{Mod}-\theta
$$

GET $(m, p, 4, M, n, p, q, b)$

$$
\epsilon(\zeta, \theta)-\text { Bimod }
$$

here, b is a natural isomorphism: ( middle assoc. $\left.\begin{array}{c}\text { constraint. }\end{array}\right)$

$$
\left\{b_{x, M, y}:(X D M) \triangleleft y \xrightarrow{\sim} X D(M \triangleleft y)\right\}_{\substack{X \in \varphi, y \in \theta \\ M \in M}}
$$

SATISFYING COMPATIBILITY CONDITIONS.
III. BIMODULE CATEGORIES
$G \mid V \in N \quad \varphi:=(\varphi, \otimes, u, a, l, r)$ LEFT G-MODULE CATEGORy CONSISTS OF:
(a) CATEGORY $m$
(b) BIFUNCTOR

$$
\nabla: 6 \times m \rightarrow m
$$

(c,d) NATURAL (8OMS:

$$
\begin{aligned}
& M=\left\{\begin{aligned}
M_{x, y, M} & :(x \otimes y) \triangleright M \\
& \Rightarrow x \triangleright(y \triangleright M)
\end{aligned}\right\} \\
& x, y \in H_{M} \boldsymbol{m} \\
& p=\left\{p_{M}: \| \triangleright M \stackrel{\sim}{\rightarrow} M\right\}_{M \in M}
\end{aligned}
$$

satisfying The
PENTAGON AXIOM
\& Triangle axiom
RIGHT Є-module cat eg.

$$
(-m, 4, n, q)
$$

DEFINED LIKEWISE

TAKE: $(m, D, m, p) \in \zeta$ - Mod

$$
\ddagger(t y, 4, n, q) \in \operatorname{Mod}-\theta
$$

GET $(m, p, 4, M, n, p, q, b)$

$$
\epsilon(\zeta, \theta)-\text { Bimod }
$$

HERE, b is a natural isomorphism: ( $\left.\begin{array}{c}\text { middle assoc. } \\ \text { constraint. }\end{array}\right)$

$$
\left\{b_{x, M, y}:(X D M) \triangleleft y \xrightarrow{\sim} X D(M \triangleleft y)\right\}_{\substack{x \in \zeta, y \in \theta \\ M \in M}}
$$

SATISFYING COMPATIBILITY CONDITIONS.

$$
M \in M
$$

Ex. $\zeta=\theta$. GET Greg $\in \zeta$-Bind

$$
\equiv \text { REGULAR G-BIMOD.CAT. } \equiv \text { WITH } D=4=\theta^{\zeta}
$$

LECTURE \#13

TopIcs:
F. Isomorphisms and equivalence of monoidal categories

IF. MODULE categories ( $\{\{3.3 .1,3.3 .2,3.3 .4$ )
II. bimodule categories ( $\$ 3.3 .3$ )

LECTURE \#13
NEXT TIME
WILl USE THIS FRAMEWORK TO "STRICTIFY" MONOIDAL CATS
$\left(\begin{array}{l}1 \\ \text { IV. } \\ \text { IV. }\end{array}\right.$
I. ISOMORPHISMS AND Equivalence of Monoidal categories
IV. MODULE CATEGORIES (\{S3.3.1,3.3.2, 3.3.4)
IV. Bimodule categories

LECTURE \#13
NEXT TIME
WILl USE THIS FRAMEWORK TO "STRICTIFY" MONOIDAL CATS
$\left(\begin{array}{l}1 \\ \text { IV. } \\ \text { IV. }\end{array}\right.$
IV. BImOdule categories
$(\{53.3 .1,3.3 .2,3.3 .4)$
( $\$ 3.3 .3$ )
... AND THEN WEILL DRAW SOME COOL PICTURES!

## Enjoy this lecture? You'll enjoy the textbook! <br> C. Walton's "Symmetries of Algebras, Volume 1" (2024)



Available for purchase at :

619 Wreath (at a discount)
https://www.619wreath.com/

Also on Amazon<br>\&<br>Google Play

Lecture \#13 keywords: bimodule category, module category, module functor, monoidal equivalence, monoidal isomorphism, monoidal natural transformation, strong monoidal functor

