MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #13

TOPICS:

I. ISOMORPHISMS AND EQUIVALENCE OF MONOIDAL CATEGORIES (§3.2.2)

II. MODULE CATEGORIES (§§3.3.1, 3.3.2, 3.3.4)

III. BIMODULE CATEGORIES (§3.3.3)

I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

(d,e,f) NATURAL ISOMS:

$$a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$$

$$\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

$$r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

SATISFYING THE **PENTAGON AXIOM** & **TRIANGLE AXIOM**

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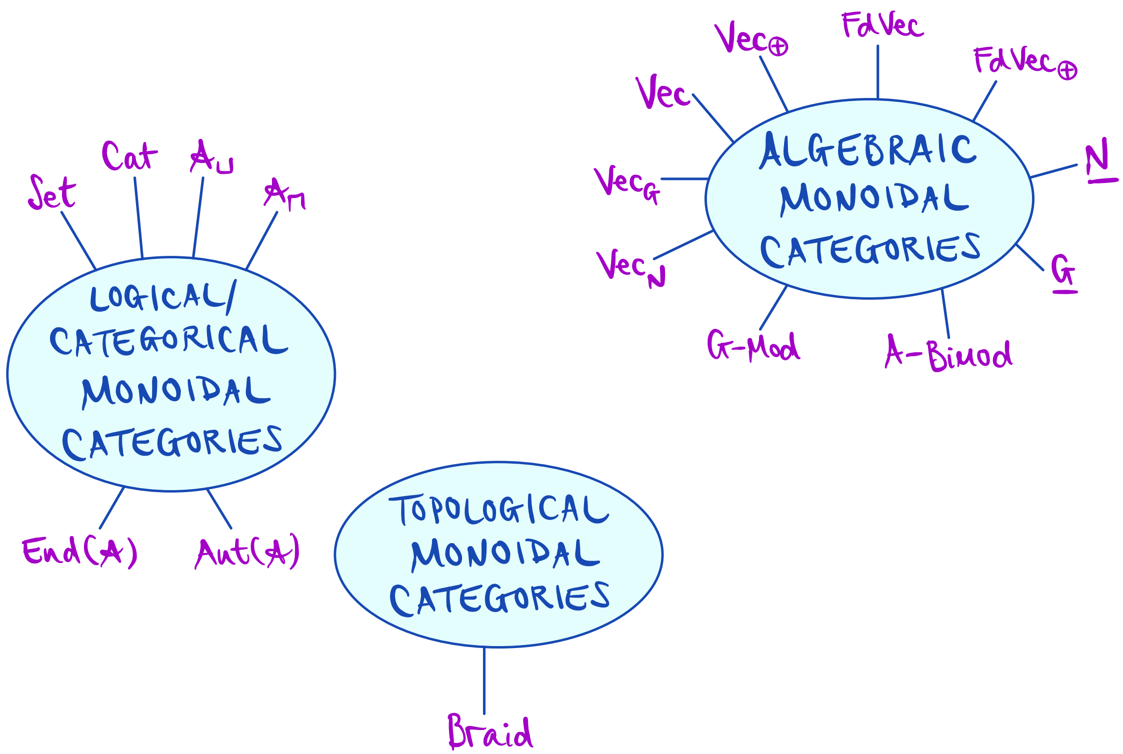
SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM



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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

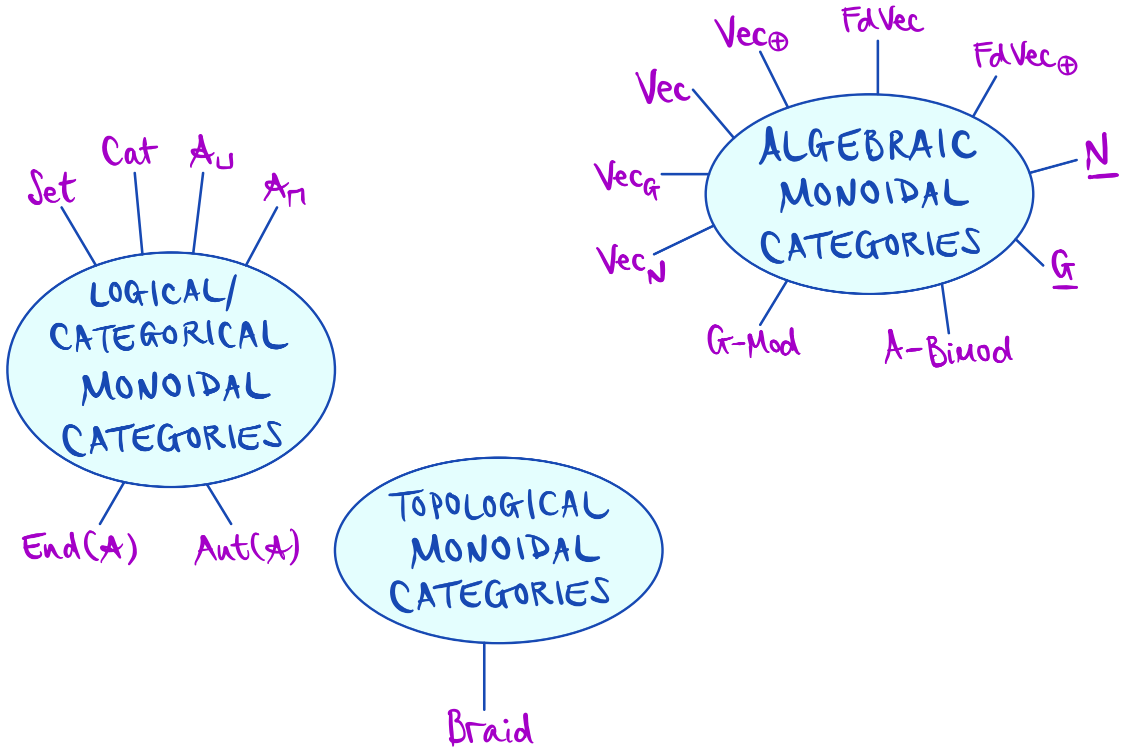
A **MONOIDAL FUNCTOR** FROM $\mathcal{C}$ TO $\mathcal{D}$ CONSISTS OF:

(a) A FUNCTOR BTW CATEGORIES $F : \mathcal{C} \longrightarrow \mathcal{D}$.

(b) A NATURAL TRANSF'N $F^{(2)} = \{ F^{(2)}_{X,Y} : F(X) \otimes^{\mathcal{D}} F(Y) \longrightarrow F(X \otimes^{\mathcal{C}} Y) \}_{X,Y \in \mathcal{C}}$.

(c) A MORPHISM $F^{(0)} : \mathbb{1}^{\mathcal{D}} \longrightarrow F(\mathbb{1}^{\mathcal{C}})$ IN $\mathcal{D}$.

SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS



I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

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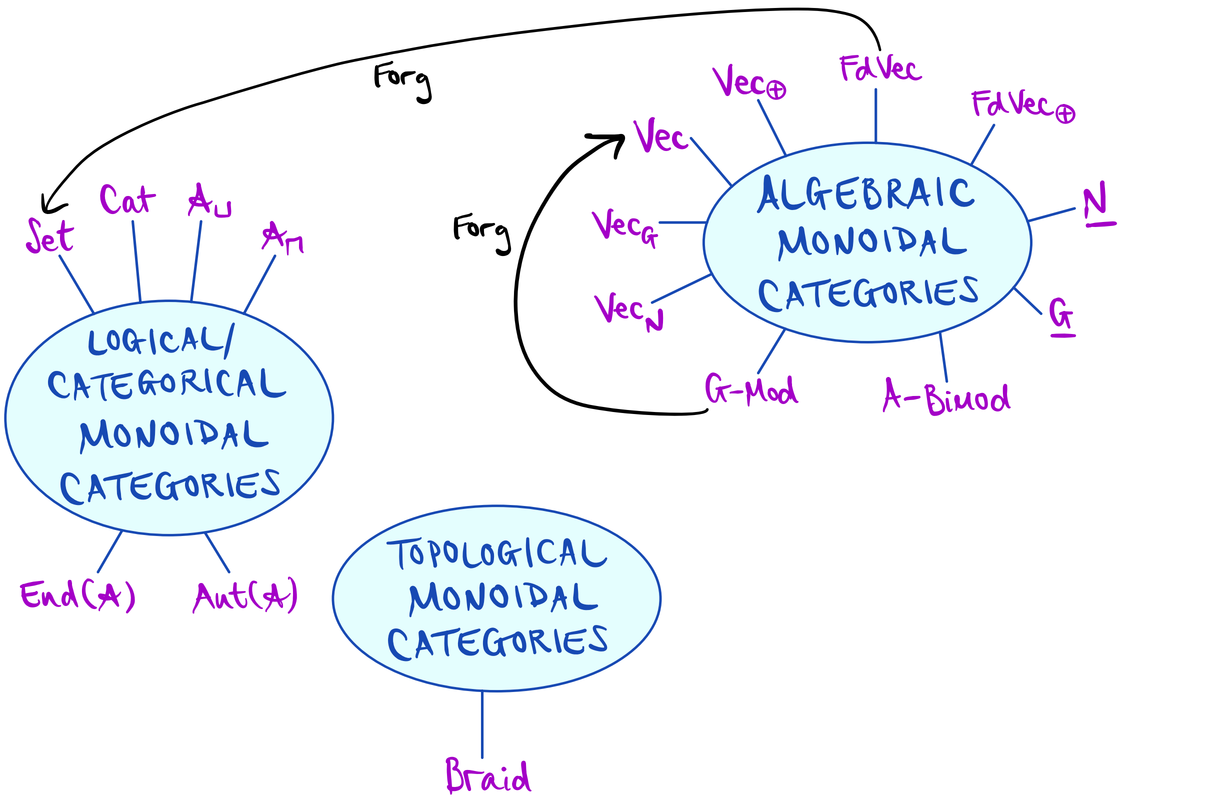

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A **STRONG MONOIDAL FUNCTOR** FROM $\mathcal{C}$ TO $\mathcal{D}$ CONSISTS OF:

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(c) ~~A MORPHISM~~ AN ISO $F^{(0)} : \mathbb{1}^{\mathcal{D}} \to F(\mathbb{1}^{\mathcal{C}})$ IN $\mathcal{D}$.

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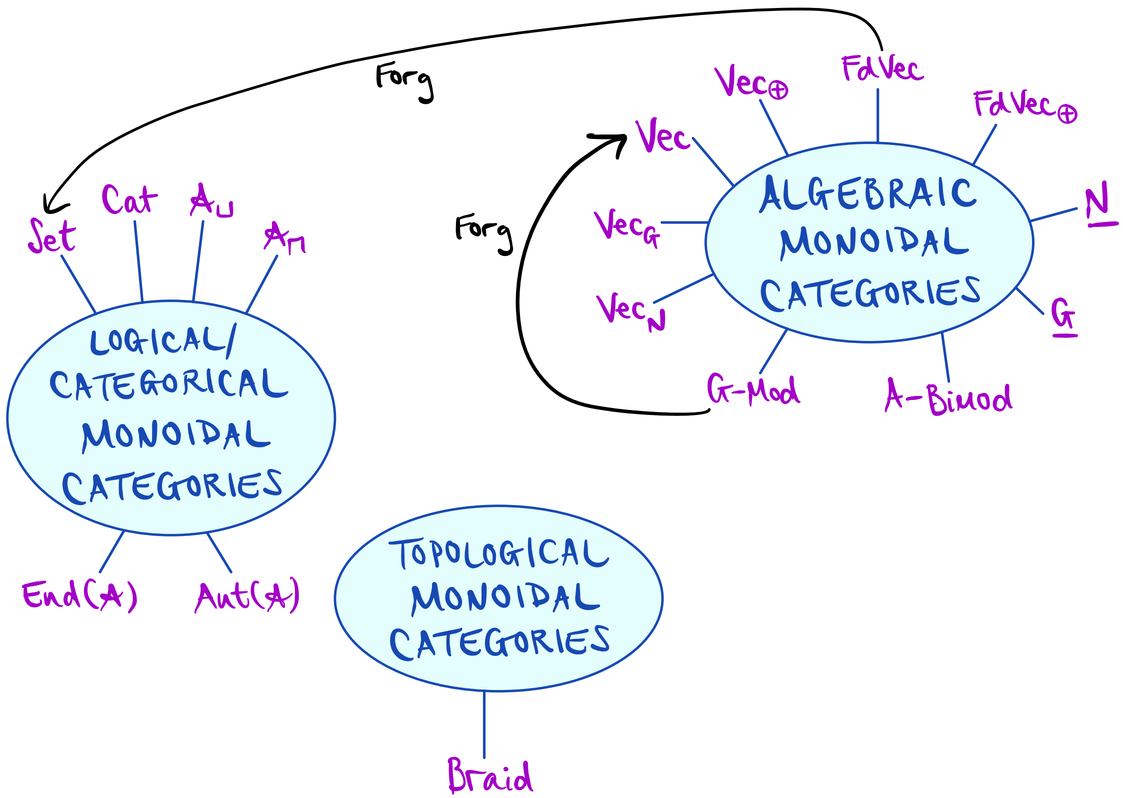

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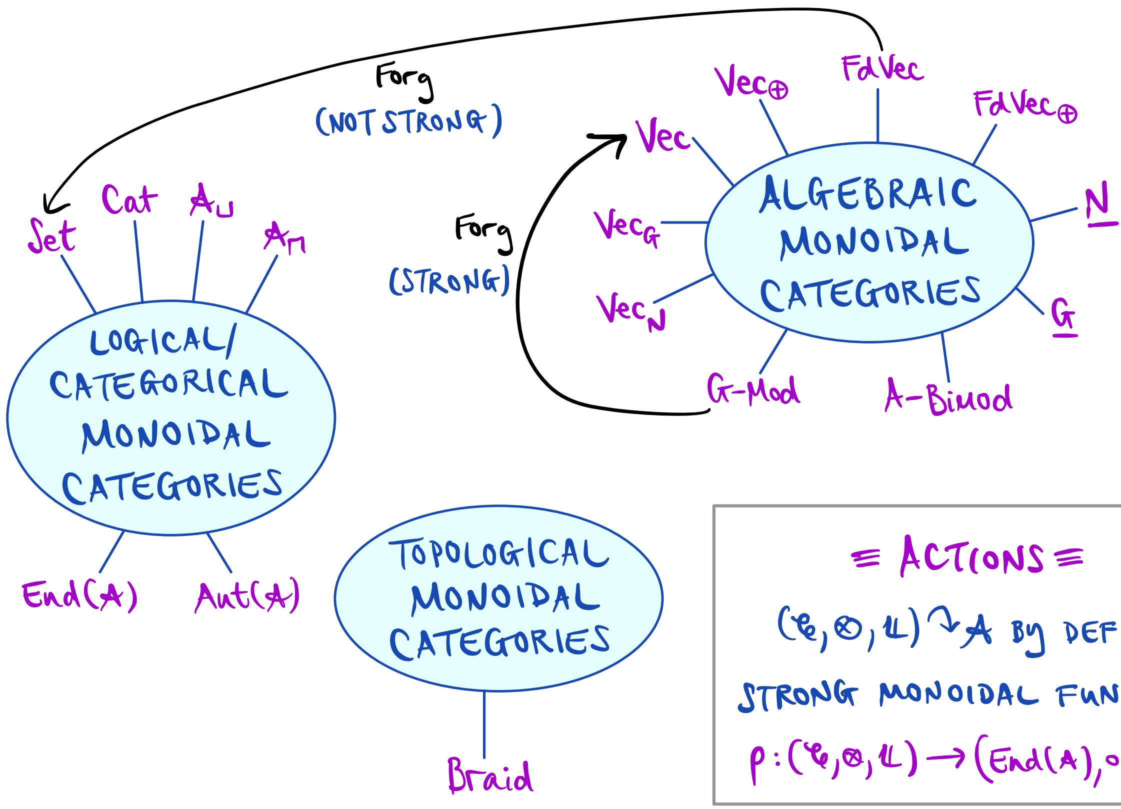

≡ ACTIONS ≡

$(\mathcal{C}, \otimes, \mathbb{1}) \curvearrowright \mathcal{A}$ BY DEF IS:

STRONG MONOIDAL FUNCTOR

$$\rho : (\mathcal{C}, \otimes, \mathbb{1}) \to (\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$$

I. Isom. and Equiv. of Monoidal Categories

Monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

consists of:

(a) category $\mathcal{C}$

(b) bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) object $\mathbb{1} \in \mathcal{C}$

(d,e,f) natural isoms:

$$a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$$

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Satisfying the pentagon axiom & triangle axiom

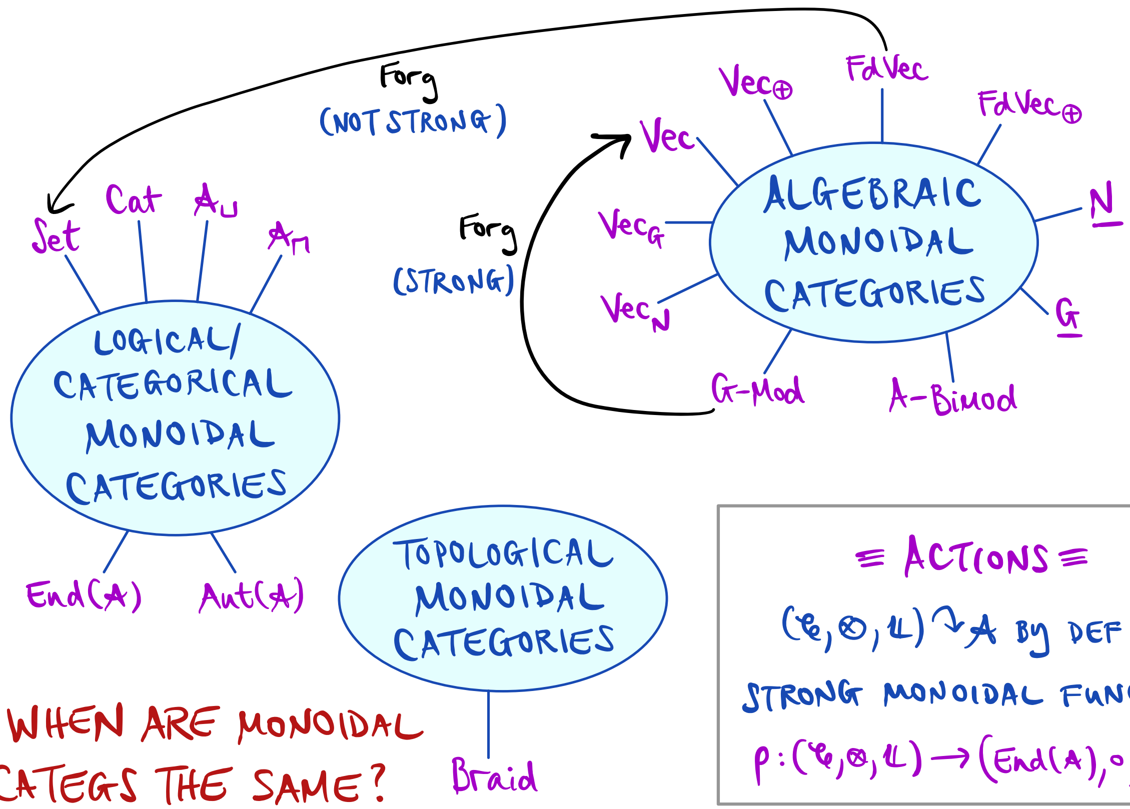
A **strong monoidal functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of:

(a) A functor btw categories $F : \mathcal{C} \longrightarrow \mathcal{D}$.

(b) A natural ~~transf'n~~ isomorphism $F^{(2)} = \{ F^{(2)}_{X,Y} : F(X) \otimes^{\mathcal{D}} F(Y) \longrightarrow F(X \otimes^{\mathcal{C}} Y) \}_{X,Y \in \mathcal{C}}$.

(c) ~~A morphism~~ an iso $F^{(0)} : \mathbb{1}^{\mathcal{D}} \longrightarrow F(\mathbb{1}^{\mathcal{C}})$ in $\mathcal{D}$.

Subject to associativity & unitality axioms



Q: When are monoidal categs the same?

≡ Actions ≡

$(\mathcal{C}, \otimes, \mathbb{1}) \curvearrowright \mathcal{A}$ by def is:

strong monoidal functor

$\rho : (\mathcal{C}, \otimes, \mathbb{1}) \to (\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$

I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

≡ RECALL ≡

TWO CATEGORIES $\mathcal{C}$ AND $\mathcal{D}$ ARE THE SAME VIA

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I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

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CATEGORY ISOMORPHISM $\mathcal{C} \cong \mathcal{D}$:

$\exists$ FUNCTORS $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ s.t.

$$GF = Id_{\mathcal{C}} \quad \& \quad FG = Id_{\mathcal{D}}$$

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CATEGORY EQUIVALENCE $\mathcal{C} \simeq \mathcal{D}$:

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⇕

$\exists$ FULLY FAITHFUL, ESSENTIALLY SURJ. FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

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Q: WHEN ARE MONOIDAL CATEGS THE SAME? $\longrightarrow$ FIRST TAKING THIS APPROACH

I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

TAKE MONOIDAL CATEGS.

$\mathcal{C} := (\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, a^{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}})$

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

$\mathcal{C} \simeq \mathcal{D}$

Q: WHEN ARE MONOIDAL CATEGS THE SAME? → FIRST TAKING THIS APPROACH → $\exists$ FULLY FAITHFUL, ESSENTIALLY SURJ. FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

$\mathcal{C}$ AND $\mathcal{D}$ ARE MONOIDALLY ISOMORPHIC

$$\mathcal{C} \overset{\otimes}{\cong} \mathcal{D}$$

IF $\exists$ STRONG MONOIDAL FUNCTOR

$$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$$

s.t.

F IS A CATEGORY ISOMORPHISM.

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EXERCISE 3.6 G GROUP. GET:

$$G\text{-Mod} \overset{\otimes}{\cong} \Bbbk G\text{-Mod}$$

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SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

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EXERCISE 3.6 G GROUP. GET:

$$G\text{-Mod} \overset{\otimes}{\cong} \Bbbk G\text{-Mod}$$

$(\Bbbk G\text{-Mod}, \otimes, \mathbb{1})$ DEFINED BY:

$$\left(\textstyle\sum_{g \in G} \lambda_g g\right) \blacktriangleright (v \otimes_{\Bbbk} v') := \textstyle\sum_{g \in G} \lambda_g (g \triangleright v) \otimes_{\Bbbk} (g \triangleright' v')$$

FOR $(V, \triangleright), (V', \triangleright') \in \Bbbk G\text{-Mod}$

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FOR $(V, \triangleright), (V', \triangleright') \in \Bbbk G\text{-Mod}$

$$\left(\sum_{g \in G} \lambda_g g\right) \blacktriangleright 1_{\Bbbk} := \sum_{g \in G} \lambda_g (g \triangleright 1_{\Bbbk}) = \sum_{g \in G} \lambda_g$$

FOR $\mathbb{1} = \Bbbk$

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FOR $(V, \triangleright), (V', \triangleright') \in \Bbbk G\text{-Mod}$

$\left(\sum_{g \in G} \lambda_g g\right) \blacktriangleright 1_{\Bbbk} := \sum_{g \in G} \lambda_g (g \triangleright 1_{\Bbbk})$

FOR $\mathbb{1} = \Bbbk$ $\qquad = \sum_{g \in G} \lambda_g$

I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

TAKE MONOIDAL CATEGS.

$\mathcal{C} := (\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, a^{\mathcal{C}}, l^{\mathcal{C}}, r^{\mathcal{C}})$

&

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A **STRONG MONOIDAL FUNCTOR** FROM $\mathcal{C}$ TO $\mathcal{D}$ CONSISTS OF:

(a) A FUNCTOR BTW CATEGORIES $F: \mathcal{C} \longrightarrow \mathcal{D}$.

(b) A NATURAL ISOMORPHISM $F^{(2)} = \{F^{(2)}_{X,Y} : F(X) \otimes^{\mathcal{D}} F(Y) \xrightarrow{\sim} F(X \otimes^{\mathcal{C}} Y)\}_{X,Y \in \mathcal{C}}$.

(c) AN ISO $F^{(0)} : \mathbb{1}^{\mathcal{D}} \xrightarrow{\sim} F(\mathbb{1}^{\mathcal{C}})$ IN $\mathcal{D}$.

SUBJECT TO ASSOCIATIVITY & UNITALITY AXIOMS

$\mathcal{C}$ AND $\mathcal{D}$ ARE **MONOIDALLY EQUIVALENT**

$$\mathcal{C} \overset{\otimes}{\simeq} \mathcal{D}$$

IF $\exists$ STRONG MONOIDAL FUNCTOR

$$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$$

s.t.

F IS A CATEGORY EQUIVALENCE.

EXAMPLE: A $\Bbbk$-ALGEBRA

NOT NECESSARILY MONOIDAL } A-Mod

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EXAMPLE A $\mathbb{k}$-ALGEBRA

IS MONOIDAL → End(A-Mod)

WITH $\otimes$ = COMPOSITION

$\mathbb{1} = \mathrm{Id}_{A\text{-Mod}}$

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$$A\text{-Bimod} \overset{\otimes}{\simeq} \text{End}(A\text{-Mod})$$

WITH $\otimes = \otimes_A$, $\mathbb{1} = {}_A(A_{reg})_A$

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$V \longmapsto V \otimes_A -$

$\rho^{(2)}_{V,W} : \rho(V) \circ \rho(W) \longrightarrow \rho(V \otimes_A W)$

$(a^A_{V,W,Z})^{-1} \overset{DEF}{=} \rho^{(2)}_{V,W}(Z) : V \otimes_A (W \otimes_A Z) \longrightarrow (V \otimes_A W) \otimes_A Z$

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I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE MONOIDALLY ISOMORPHIC

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Q: WHEN ARE MONOIDAL CATEGS THE SAME? —(FIRST TAKING THIS APPROACH)→ $\exists$ FULLY FAITHFUL, ESSENTIALLY SURJ. FUNCTOR $F : \mathcal{C} \to \mathcal{D}$ ✓

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$\mathscr{C}$ AND $\mathscr{D}$ ARE **MONOIDALLY ISOMORPHIC** $\mathscr{C} \overset{\otimes}{\cong} \mathscr{D}$ IF $\exists$ STRONG MON. FUNCTOR $(F, F^{(2)}, F^{(0)}) : \mathscr{C} \to \mathscr{D}$ s.t. F IS A CATEGORY ISOMORPHISM.

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$\exists$ FUNCTORS $F : \mathscr{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathscr{C}$ s.t.

$$GF = Id_{\mathscr{C}} \;\&\; FG = Id_{\mathscr{D}}$$

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I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE MONOIDALLY ISOMORPHIC	$\mathcal{C}$ AND $\mathcal{D}$ ARE MONOIDALLY EQUIVALENT
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NOW LET'S TAKE THIS APPROACH

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⋮

NEED TO SAY WHEN TWO MONOIDAL FUNCTORS ARE THE SAME

Q: WHEN ARE MONOIDAL CATEGS THE SAME? → FIRST TAKING THIS APPROACH

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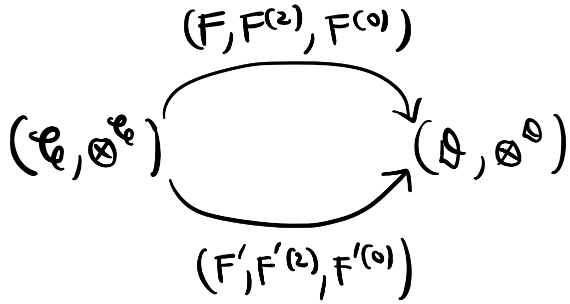
TAKE MONOIDAL FUNCTORS

$$(F, F^{(2)}, F^{(0)}),\ (F', F'^{(2)}, F'^{(0)}) : (\mathcal{C}, \otimes^{\mathcal{C}}) \longrightarrow (\mathcal{D}, \otimes^{\mathcal{D}})$$



NEED TO SAY WHEN TWO MONOIDAL FUNCTORS ARE THE SAME

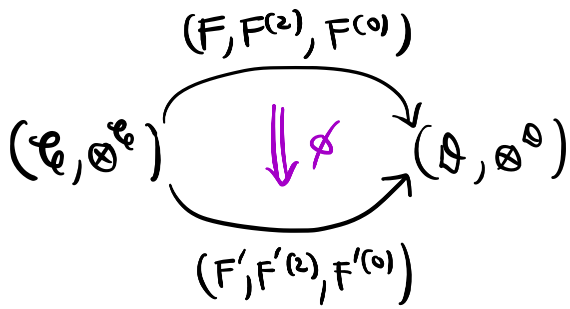
I. Isom. and equiv. of monoidal categories

$\mathcal{C}$ and $\mathcal{D}$ are **monoidally isomorphic** $\mathcal{C} \overset{\otimes}{\cong} \mathcal{D}$ if $\exists$ strong mon. functor $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$ s.t. F is a category isomorphism.

$\mathcal{C}$ and $\mathcal{D}$ are **monoidally equivalent** $\mathcal{C} \overset{\otimes}{\simeq} \mathcal{D}$ if $\exists$ strong mon. functor $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$ s.t. F is a category equivalence.

Take monoidal functors

$$(F, F^{(2)}, F^{(0)}),\ (F', F'^{(2)}, F'^{(0)}) : (\mathcal{C}, \otimes^{\mathcal{C}}) \longrightarrow (\mathcal{D}, \otimes^{\mathcal{D}})$$



A **monoidal natural transformation** is a natural transformation $\phi : F \Rightarrow F'$

Compatible with $F^{(2)}$ & $F'^{(2)}$

Compatible with $F^{(0)}$ & $F'^{(0)}$

Need to say when two monoidal functors are the same

I. Isom. and equiv. of monoidal categories

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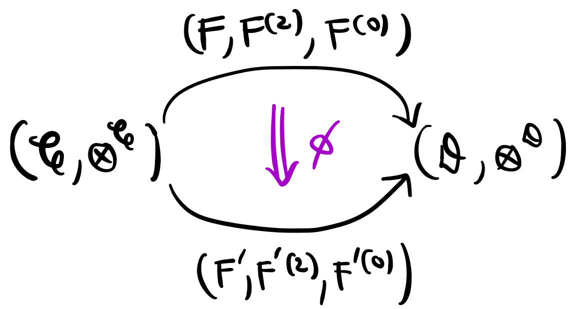

A **monoidal natural transformation**

is a natural transformation $\phi : F \Rightarrow F'$ s.t.

$$\begin{array}{ccc} F(X) \otimes^{\mathcal{D}} F(Y) & \xrightarrow{F^{(2)}_{X,Y}} & F(X \otimes^{\mathcal{C}} Y) \\ \phi_X \otimes^{\mathcal{D}} \phi_Y \big\downarrow & \circlearrowleft & \big\downarrow \phi_{X \otimes^{\mathcal{C}} Y} \\ F'(X) \otimes^{\mathcal{D}} F'(Y) & \xrightarrow{F'^{(2)}_{X,Y}} & F'(X \otimes^{\mathcal{C}} Y) \end{array} \quad \& \quad \begin{array}{ccc} & & F(\mathbb{1}^{\mathcal{C}}) \\ & \overset{F^{(0)}}{\nearrow} & \\ \mathbb{1}^{\mathcal{D}} & \circlearrowleft & \big\downarrow \phi_{\mathbb{1}^{\mathcal{C}}} \\ & \underset{F'^{(0)}}{\searrow} & \\ & & F'(\mathbb{1}^{\mathcal{C}}) \end{array}$$

$\forall X, Y \in \mathcal{C}$

> Write
>
> $F \overset{\otimes}{\Rightarrow} F'$

I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

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TAKE MONOIDAL FUNCTORS

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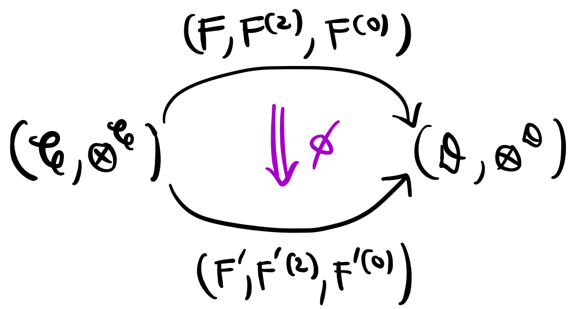

A MONOIDAL NATURAL ~~TRANSFORMATION~~ ISOMORPHISM IS A NATURAL TRANSFORMATION $\phi : F \Rightarrow F'$ s.t.

$$\begin{array}{ccc} F(X) \otimes^{\mathcal{D}} F(Y) & \xrightarrow{F^{(2)}_{X,Y}} & F(X \otimes^{\mathcal{C}} Y) \\ \downarrow {\scriptstyle \phi_X \otimes^{\mathcal{D}} \phi_Y} & \circlearrowleft & \downarrow {\scriptstyle \phi_{X \otimes^{\mathcal{C}} Y}} \\ F'(X) \otimes^{\mathcal{D}} F'(Y) & \xrightarrow{F'^{(2)}_{X,Y}} & F'(X \otimes^{\mathcal{C}} Y) \end{array} \quad \forall X, Y \in \mathcal{C}$$

&

$$\begin{array}{ccc} & \xrightarrow{F^{(0)}} & F(\mathbb{1}^{\mathcal{C}}) \\ \mathbb{1}^{\mathcal{D}} & \circlearrowleft & \downarrow {\scriptstyle \phi_{\mathbb{1}^{\mathcal{C}}}} \\ & \xrightarrow{F'^{(0)}} & F'(\mathbb{1}^{\mathcal{C}}) \end{array}$$

WRITE $F \overset{\otimes}{\Rightarrow} F'$

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Take monoidal functors

$$(F, F^{(2)}, F^{(0)}),\ (F', F'^{(2)}, F'^{(0)}) : (\mathcal{C}, \otimes^{\mathcal{C}}) \to (\mathcal{D}, \otimes^{\mathcal{D}})$$

$$(\mathcal{C}, \otimes^{\mathcal{C}}) \ \underset{(F', F'^{(2)}, F'^{(0)})}{\overset{(F, F^{(2)}, F^{(0)})}{\rightrightarrows}} \ (\mathcal{D}, \otimes^{\mathcal{D}}), \qquad \overset{\sim}{\Downarrow} \phi$$

A monoidal natural ~~transformation~~ isomorphism is a natural ~~transformation~~ isomorphism $\phi : F \overset{\sim}{\Rightarrow} F'$ $\ni$.

$$\begin{array}{ccc} F(X) \otimes^{\mathcal{D}} F(Y) & \xrightarrow{F^{(2)}_{X,Y}} & F(X \otimes^{\mathcal{C}} Y) \\ {\scriptstyle \phi_X \otimes^{\mathcal{D}} \phi_Y} \downarrow & \circlearrowright & \downarrow {\scriptstyle \phi_{X \otimes^{\mathcal{C}} Y}} \\ F'(X) \otimes^{\mathcal{D}} F'(Y) & \xrightarrow{F'^{(2)}_{X,Y}} & F'(X \otimes^{\mathcal{C}} Y) \end{array} \quad \forall X, Y \in \mathcal{C}$$

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Write $F \overset{\otimes}{\Rightarrow} F'$

I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **MONOIDALLY ISOMORPHIC** $\mathcal{C} \overset{\otimes}{\cong} \mathcal{D}$ IF $\exists$ STRONG MON. FUNCTOR $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \to \mathcal{D}$ ∋. F IS A CATEGORY ISOMORPHISM.

$\mathcal{C}$ AND $\mathcal{D}$ ARE **MONOIDALLY EQUIVALENT** $\mathcal{C} \overset{\otimes}{\simeq} \mathcal{D}$ IF $\exists$ STRONG MON. FUNCTOR $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \to \mathcal{D}$ ∋. F IS A CATEGORY EQUIVALENCE.

TAKE MONOIDAL FUNCTORS

$$(F, F^{(2)}, F^{(0)}),\ (F', F'^{(2)}, F'^{(0)}) : (\mathcal{C}, \otimes^{\mathcal{C}}) \longrightarrow (\mathcal{D}, \otimes^{\mathcal{D}})$$

$$\phi : (F, F^{(2)}, F^{(0)}) \overset{\sim}{\Rightarrow} (F', F'^{(2)}, F'^{(0)}) : (\mathcal{C}, \otimes^{\mathcal{C}}) \to (\mathcal{D}, \otimes^{\mathcal{D}})$$

A **MONOIDAL NATURAL** ~~TRANSFORMATION~~ ISOMORPHISM IS A NATURAL ~~TRANSFORMATION~~ ISOMORPHISM $\phi : F \overset{\sim}{\Rightarrow} F'$ ∋.

$$\begin{array}{ccc} F(X) \otimes^{\mathcal{D}} F(Y) & \xrightarrow{F^{(2)}_{X,Y}} & F(X \otimes^{\mathcal{C}} Y) \\ \downarrow \phi_X \otimes^{\mathcal{D}} \phi_Y & \circlearrowleft & \downarrow \phi_{X \otimes^{\mathcal{C}} Y} \\ F'(X) \otimes^{\mathcal{D}} F'(Y) & \xrightarrow{F'^{(2)}_{X,Y}} & F'(X \otimes^{\mathcal{C}} Y) \end{array} \quad \forall X, Y \in \mathcal{C}$$

$\&$

$$\mathbb{1}^{\mathcal{D}} \xrightarrow{F^{(0)}} F(\mathbb{1}^{\mathcal{C}}) \xrightarrow{\phi_{\mathbb{1}^{\mathcal{C}}}} F'(\mathbb{1}^{\mathcal{C}}) \quad = \quad \mathbb{1}^{\mathcal{D}} \xrightarrow{F'^{(0)}} F'(\mathbb{1}^{\mathcal{C}})$$

WRITE

~~$F \overset{\otimes}{\Rightarrow} F'$~~

$F \overset{\otimes}{\cong} F'$

I. Isom. and Equiv. of Monoidal Categories

$\mathcal{C}$ and $\mathcal{D}$ are **monoidally isomorphic** $\mathcal{C} \overset{\otimes}{\cong} \mathcal{D}$ if $\exists$ strong mon. functor $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$ s.t. F is a category isomorphism.

$\mathcal{C}$ and $\mathcal{D}$ are **monoidally equivalent** $\mathcal{C} \overset{\otimes}{\simeq} \mathcal{D}$ if $\exists$ strong mon. functor $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$ s.t. F is a category equivalence.

A **monoidal natural** ~~transformation~~ isomorphism btw $(F, F^{(2)}, F^{(0)})$ & $(F', F'^{(2)}, F'^{(0)})$ is a natural ~~transformation~~ isomorphism $\phi : F \overset{\sim}{\Rightarrow} F'$ s.t.

$$\begin{array}{ccc} F(X) \otimes^{\mathcal{D}} F(Y) & \xrightarrow{F^{(2)}_{X,Y}} & F(X \otimes^{\mathcal{C}} Y) \\ {\scriptstyle \phi_X \otimes^{\mathcal{D}} \phi_Y} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \phi_{X \otimes^{\mathcal{C}} Y}} \\ F'(X) \otimes^{\mathcal{D}} F'(Y) & \xrightarrow{F'^{(2)}_{X,Y}} & F'(X \otimes^{\mathcal{C}} Y) \end{array} \quad \forall X, Y \in \mathcal{C}$$

&

$$\begin{array}{ccc} & \overset{F^{(0)}}{\nearrow} & F(\mathbb{1}^{\mathcal{C}}) \\ \mathbb{1}^{\mathcal{D}} & \circlearrowleft & \downarrow {\scriptstyle \phi_{\mathbb{1}^{\mathcal{C}}}} \\ & \underset{F'^{(0)}}{\searrow} & F'(\mathbb{1}^{\mathcal{C}}) \end{array}$$

Write ~~$F \overset{\otimes}{\Rightarrow} F'$~~ $F \overset{\otimes}{\cong} F'$

I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **MONOIDALLY ISOMORPHIC** $\mathcal{C} \overset{\otimes}{\cong} \mathcal{D}$ IF $\exists$ STRONG MON. FUNCTOR $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$ s.t. F IS A CATEGORY ISOMORPHISM.	$\mathcal{C}$ AND $\mathcal{D}$ ARE **MONOIDALLY EQUIVALENT** $\mathcal{C} \overset{\otimes}{\simeq} \mathcal{D}$ IF $\exists$ STRONG MON. FUNCTOR $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$ s.t. F IS A CATEGORY EQUIVALENCE.

$\Updownarrow$ EXER 3.5 $\Updownarrow$

$\exists$ MONOIDAL FUNCTORS

$$(\mathcal{C}, \otimes^{\mathcal{C}}) \underset{(G, G^{(2)}, G^{(0)})}{\overset{(F, F^{(2)}, F^{(0)})}{\rightleftarrows}} (\mathcal{D}, \otimes^{\mathcal{D}})$$

WITH ... WITH ...

A **MONOIDAL NATURAL** ~~TRANSFORMATION~~ ISOMORPHISM BTW $(F, F^{(2)}, F^{(0)})$ & $(F', F'^{(2)}, F'^{(0)})$ IS A NATURAL ~~TRANSFORMATION~~ ISOMORPHISM $\phi : F \overset{\sim}{\Rightarrow} F'$ s.t.

$$\begin{array}{ccc} F(X) \otimes^{\mathcal{D}} F(Y) & \xrightarrow{F^{(2)}_{X,Y}} & F(X \otimes^{\mathcal{C}} Y) \\ \downarrow{\scriptstyle \phi_X \otimes^{\mathcal{D}} \phi_Y} & \circlearrowleft & \downarrow{\scriptstyle \phi_{X \otimes^{\mathcal{C}} Y}} \\ F'(X) \otimes^{\mathcal{D}} F'(Y) & \xrightarrow{F'^{(2)}_{X,Y}} & F'(X \otimes^{\mathcal{C}} Y) \end{array} \quad \forall X, Y \in \mathcal{C}$$

&

$$\begin{array}{ccc} & & F(\mathbb{1}^{\mathcal{C}}) \\ & \nearrow^{F^{(0)}} & \\ \mathbb{1}^{\mathcal{D}} & \circlearrowleft & \downarrow{\scriptstyle \phi_{\mathbb{1}^{\mathcal{C}}}} \\ & \searrow_{F'^{(0)}} & \\ & & F'(\mathbb{1}^{\mathcal{C}}) \end{array}$$

WRITE ~~$F \overset{\otimes}{\Rightarrow} F'$~~ $F \overset{\otimes}{\cong} F'$

I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **MONOIDALLY ISOMORPHIC**	$\mathcal{C}$ AND $\mathcal{D}$ ARE **MONOIDALLY EQUIVALENT**
$\mathcal{C} \overset{\otimes}{\cong} \mathcal{D}$ IF $\exists$ STRONG MON. FUNCTOR $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$ $\ni.$ F IS A CATEGORY ISOMORPHISM.	$\mathcal{C} \overset{\otimes}{\simeq} \mathcal{D}$ IF $\exists$ STRONG MON. FUNCTOR $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$ $\ni.$ F IS A CATEGORY EQUIVALENCE.

$\Updownarrow$ EXER 3.5 $\Updownarrow$

$\exists$ MONOIDAL FUNCTORS

$$(\mathcal{C}, \otimes^{\mathcal{C}}) \underset{(G, G^{(2)}, G^{(0)})}{\overset{(F, F^{(2)}, F^{(0)})}{\rightleftarrows}} (\mathcal{D}, \otimes^{\mathcal{D}})$$

WITH $GF \overset{\otimes}{\cong} \mathrm{Id}_{(\mathcal{C}, \otimes^{\mathcal{C}})}$ & $FG \overset{\otimes}{\cong} \mathrm{Id}_{(\mathcal{D}, \otimes^{\mathcal{D}})}$ (for monoidally isomorphic)

WITH (for monoidally equivalent)

A **MONOIDAL NATURAL** ~~TRANSFORMATION~~ ISOMORPHISM BTW $(F, F^{(2)}, F^{(0)})$ & $(F', F'^{(2)}, F'^{(0)})$ IS A NATURAL ~~TRANSFORMATION~~ ISOMORPHISM $\phi : F \overset{\sim}{\Rightarrow} F'$ $\ni.$

$$\begin{array}{ccc} F(X) \otimes^{\mathcal{D}} F(Y) & \xrightarrow{F^{(2)}_{X,Y}} & F(X \otimes^{\mathcal{C}} Y) \\ \downarrow{\scriptstyle \phi_X \otimes^{\mathcal{D}} \phi_Y} & \circlearrowleft & \downarrow{\scriptstyle \phi_{X \otimes^{\mathcal{C}} Y}} \\ F'(X) \otimes^{\mathcal{D}} F'(Y) & \xrightarrow{F'^{(2)}_{X,Y}} & F'(X \otimes^{\mathcal{C}} Y) \end{array} \quad \forall X, Y \in \mathcal{C}$$

&

$$\begin{array}{ccc} & \overset{F^{(0)}}{\nearrow} & F(\mathbb{1}^{\mathcal{C}}) \\ \mathbb{1}^{\mathcal{D}} & \circlearrowleft & \downarrow{\scriptstyle \phi_{\mathbb{1}^{\mathcal{C}}}} \\ & \underset{F'^{(0)}}{\searrow} & F'(\mathbb{1}^{\mathcal{C}}) \end{array}$$

WRITE

~~$F \overset{\otimes}{\Rightarrow} F'$~~

$F \overset{\otimes}{\cong} F'$

I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE MONOIDALLY ISOMORPHIC

$\mathcal{C} \overset{\otimes}{\cong} \mathcal{D}$ IF $\exists$ STRONG MON. FUNCTOR

$$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$$

∋. F IS A CATEGORY ISOMORPHISM.

$\mathcal{C}$ AND $\mathcal{D}$ ARE MONOIDALLY EQUIVALENT

$\mathcal{C} \overset{\otimes}{\simeq} \mathcal{D}$ IF $\exists$ STRONG MON. FUNCTOR

$$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$$

∋. F IS A CATEGORY EQUIVALENCE.

$\Updownarrow$ EXER 3.5 $\Updownarrow$

$\exists$ MONOIDAL FUNCTORS

$$(\mathcal{C}, \otimes^{\mathcal{C}}) \underset{(G,\, G^{(2)},\, G^{(0)})}{\overset{(F,\, F^{(2)},\, F^{(0)})}{\rightleftarrows}} (\mathcal{D}, \otimes^{\mathcal{D}})$$

WITH

$$GF \overset{\otimes}{=} \mathrm{Id}_{(\mathcal{C}, \otimes^{\mathcal{C}})} \quad \& \quad FG \overset{\otimes}{=} \mathrm{Id}_{(\mathcal{D}, \otimes^{\mathcal{D}})}$$

WITH

$$GF \overset{\otimes}{\cong} \mathrm{Id}_{(\mathcal{C}, \otimes^{\mathcal{C}})} \quad \& \quad FG \overset{\otimes}{\cong} \mathrm{Id}_{(\mathcal{D}, \otimes^{\mathcal{D}})}$$

A MONOIDAL NATURAL ~~TRANSFORMATION~~ ISOMORPHISM BTW $(F, F^{(2)}, F^{(0)})$ & $(F', F'^{(2)}, F'^{(0)})$ IS A NATURAL ~~TRANSFORMATION~~ ISOMORPHISM $\phi : F \overset{\sim}{\Rightarrow} F'$ ∋.

$$\begin{array}{ccc} F(X) \otimes^{\mathcal{D}} F(Y) & \xrightarrow{F^{(2)}_{X,Y}} & F(X \otimes^{\mathcal{C}} Y) \\ {\scriptstyle \phi_X \otimes^{\mathcal{D}} \phi_Y} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \phi_{X \otimes^{\mathcal{C}} Y}} \\ F'(X) \otimes^{\mathcal{D}} F'(Y) & \xrightarrow{F'^{(2)}_{X,Y}} & F'(X \otimes^{\mathcal{C}} Y) \end{array} \quad \forall X, Y \in \mathcal{C}$$

&

$$\begin{array}{ccc} & & F(\mathbb{1}^{\mathcal{C}}) \\ & \overset{F^{(0)}}{\nearrow} & \\ \mathbb{1}^{\mathcal{D}} & \circlearrowleft & \downarrow {\scriptstyle \phi_{\mathbb{1}^{\mathcal{C}}}} \\ & \underset{F'^{(0)}}{\searrow} & \\ & & F'(\mathbb{1}^{\mathcal{C}}) \end{array}$$

WRITE

~~$F \overset{\otimes}{\Rightarrow} F'$~~

$F \overset{\otimes}{\cong} F'$

I. ISOM. AND EQUIV. OF MONOIDAL CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE MONOIDALLY ISOMORPHIC
$\mathcal{C} \overset{\otimes}{\cong} \mathcal{D}$ IF $\exists$ STRONG MON. FUNCTOR
$$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$$
s.t. F IS A CATEGORY ISOMORPHISM.

$\mathcal{C}$ AND $\mathcal{D}$ ARE MONOIDALLY EQUIVALENT
$\mathcal{C} \overset{\otimes}{\simeq} \mathcal{D}$ IF $\exists$ STRONG MON. FUNCTOR
$$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$$
s.t. F IS A CATEGORY EQUIVALENCE.

EXER 3.5

$\Updownarrow$ WITH $\exists$ MONOIDAL FUNCTORS WITH $\Updownarrow$

$$(F, F^{(2)}, F^{(0)}) : (\mathcal{C}, \otimes^{\mathcal{C}}) \rightleftarrows (\mathcal{D}, \otimes^{\mathcal{D}}) : (G, G^{(2)}, G^{(0)})$$

(isomorphism case:) $GF \overset{\otimes}{=} \mathrm{Id}_{(\mathcal{C}, \otimes^{\mathcal{C}})}$ & $FG \overset{\otimes}{=} \mathrm{Id}_{(\mathcal{D}, \otimes^{\mathcal{D}})}$

(equivalence case:) $GF \overset{\otimes}{\cong} \mathrm{Id}_{(\mathcal{C}, \otimes^{\mathcal{C}})}$ & $FG \overset{\otimes}{\cong} \mathrm{Id}_{(\mathcal{D}, \otimes^{\mathcal{D}})}$

REVISIT

Ex

$$G\text{-Mod} \overset{\otimes}{\cong} \Bbbk G\text{-Mod}$$

FOR G A GROUP

Ex

$$A\text{-Bimod} \overset{\otimes}{\simeq} \mathrm{End}(A\text{-Mod})$$

FOR A A $\Bbbk$-ALGEBRA

II. MODULE CATEGORIES

MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR
$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

(d, e, f) NATURAL ISOMS:

$$a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$$

$$\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

$$r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

SATISFYING THE
PENTAGON AXIOM
& TRIANGLE AXIOM

WANT "MODULES" OVER THESE
(MIMICKING MODULES OVER MONOIDS)

II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

(d, e, f) NATURAL ISOMS:

$$a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$$

$$l = \{ l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

$$r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) A CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ (ACTION BIFUNCTOR)

WANT "MODULES" OVER THESE (MIMICKING MODULES OVER MONOIDS)

II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

(d, e, f) NATURAL ISOMS:

$$a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$$

$$\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

$$r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

SATISFYING THE **PENTAGON AXIOM** & **TRIANGLE AXIOM**

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) A CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$
(**ACTION BIFUNCTOR**)

(c) A NATURAL ISOMORPHISM

$$\mathcal{C} \times \mathcal{C} \times \mathcal{M} \ \overset{\triangleright \circ (\otimes \times \mathrm{Id}_{\mathcal{M}})}{\underset{\triangleright \circ (\mathrm{Id}_{\mathcal{C}} \times \triangleright)}{\rightrightarrows}} \ \mathcal{M}, \qquad \sim \Downarrow m$$

(**LEFT MODULE ASSOCIATIVITY CONSTRAINT**)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM

$$\mathcal{M} \ \overset{\mathbb{1}^{\mathcal{C}} \triangleright -}{\underset{\mathrm{Id}_{\mathcal{M}}}{\rightrightarrows}} \ \mathcal{M}, \qquad \sim \Downarrow p$$

(**LEFT MODULE UNITALITY CONSTRAINT**)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

...

II. MODULE CATEGORIES

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
>
> CONSISTS OF:
>
> (a) CATEGORY $\mathcal{C}$
>
> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
>
> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
>
> (d,e,f) NATURAL ISOMS:
>
> $a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$
>
> $\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> $r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) A CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ (ACTION BIFUNCTOR)

(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

...

II. MODULE CATEGORIES

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
>
> CONSISTS OF:
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> (a) CATEGORY $\mathcal{C}$
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>
> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
>
> (d, e, f) NATURAL ISOMS:
>
> $a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$
>
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>
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> SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

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(a) A CATEGORY $\mathcal{M}$

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(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

∋.

$$((X \otimes Y) \otimes Z) \triangleright M$$

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
> CONSISTS OF:
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>
> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
>
> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
>
> (d,e,f) NATURAL ISOMS:
>
> $a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$
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>
> $r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

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(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

s.t.

$$((X \otimes Y) \otimes Z) \triangleright M$$

$$\downarrow a_{X,Y,Z} \triangleright \mathrm{id}$$

$$(X \otimes (Y \otimes Z)) \triangleright M$$

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

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(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

s.t.

$$((X \otimes Y) \otimes Z) \triangleright M \xrightarrow{a_{X,Y,Z} \triangleright \mathrm{id}} (X \otimes (Y \otimes Z)) \triangleright M \xrightarrow{m_{X,Y \otimes Z,M}} X \triangleright ((Y \otimes Z) \triangleright M)$$

$\forall X, Y, Z \in \mathcal{C}$, $M \in \mathcal{M}$

II. MODULE CATEGORIES

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
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> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
>
> (d,e,f) NATURAL ISOMS:
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> $a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$
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>
> $r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
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(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

s.t.

$$((X \otimes Y) \otimes Z) \triangleright M \xrightarrow{a_{X,Y,Z} \triangleright id} (X \otimes (Y \otimes Z)) \triangleright M \xrightarrow{m_{X,Y \otimes Z, M}} X \triangleright ((Y \otimes Z) \triangleright M) \xrightarrow{id \triangleright m_{Y,Z,M}} X \triangleright (Y \triangleright (Z \triangleright M))$$

$\forall X, Y, Z \in \mathcal{C}$, $M \in \mathcal{M}$

II. MODULE CATEGORIES

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
>
> CONSISTS OF:
>
> (a) CATEGORY $\mathcal{C}$
>
> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
>
> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
>
> (d,e,f) NATURAL ISOMS:
>
> $a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$
>
> $\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> $r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> SATISFYING THE **PENTAGON AXIOM** & **TRIANGLE AXIOM**

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) A CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ (ACTION BIFUNCTOR)

(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

s.t.

$$((X \otimes Y) \otimes Z) \triangleright M \xrightarrow{a_{X,Y,Z} \triangleright \mathrm{id}} (X \otimes (Y \otimes Z)) \triangleright M \xrightarrow{m_{X,Y \otimes Z,M}} X \triangleright ((Y \otimes Z) \triangleright M) \xrightarrow{\mathrm{id} \triangleright m_{Y,Z,M}} X \triangleright (Y \triangleright (Z \triangleright M))$$

$$((X \otimes Y) \otimes Z) \triangleright M \xrightarrow{m_{X \otimes Y, Z, M}} (X \otimes Y) \triangleright (Z \triangleright M)$$

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
> CONSISTS OF:
>
> (a) CATEGORY $\mathcal{C}$
>
> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
>
> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
>
> (d, e, f) NATURAL ISOMS:
>
> $a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$
>
> $\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> $r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) A CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ (ACTION BIFUNCTOR)

(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

∋.

$$\begin{array}{ccc}
 & ((X \otimes Y) \otimes Z) \triangleright M & \\
\swarrow^{a_{X,Y,Z} \triangleright id} & & \searrow^{m_{X \otimes Y, Z, M}} \\
(X \otimes (Y \otimes Z)) \triangleright M & \circlearrowleft & (X \otimes Y) \triangleright (Z \triangleright M) \\
\downarrow m_{X, Y \otimes Z, M} & & \downarrow m_{X,Y,Z \triangleright M} \\
X \triangleright ((Y \otimes Z) \triangleright M) & \xrightarrow{id \triangleright m_{Y,Z,M}} & X \triangleright (Y \triangleright (Z \triangleright M))
\end{array}$$

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

(PENTAGON AXIOM)

II. MODULE CATEGORIES

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
>
> CONSISTS OF:
>
> (a) CATEGORY $\mathcal{C}$
>
> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
>
> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
>
> (d,e,f) NATURAL ISOMS:
>
> $a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$
>
> $\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> $r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) A CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ (ACTION BIFUNCTOR)

(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

s.t.

$$\begin{array}{ccc}
 & ((X \otimes Y) \otimes Z) \triangleright M & \\
{\scriptstyle a_{X,Y,Z} \triangleright id} \swarrow & & \searrow {\scriptstyle m_{X \otimes Y, Z, M}} \\
(X \otimes (Y \otimes Z)) \triangleright M & \circlearrowright & (X \otimes Y) \triangleright (Z \triangleright M) \\
\downarrow {\scriptstyle m_{X, Y \otimes Z, M}} & & \downarrow {\scriptstyle m_{X,Y,Z \triangleright M}} \\
X \triangleright ((Y \otimes Z) \triangleright M) & \xrightarrow{id \triangleright m_{Y,Z,M}} & X \triangleright (Y \triangleright (Z \triangleright M))
\end{array}$$

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

(PENTAGON AXIOM)

$(X \otimes \mathbb{1}) \triangleright M$

II. MODULE CATEGORIES

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$
> CONSISTS OF:
> (a) CATEGORY $\mathcal{C}$
> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
> (d, e, f) NATURAL ISOMS:
> $a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$
> $l = \{ l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
> $r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
> SATISFYING THE **PENTAGON AXIOM** & **TRIANGLE AXIOM**

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) A CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ (ACTION BIFUNCTOR)

(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

s.t.

$$((X \otimes Y) \otimes Z) \triangleright M \xrightarrow{a_{X,Y,Z} \triangleright \mathrm{id}} (X \otimes (Y \otimes Z)) \triangleright M \xrightarrow{m_{X,Y \otimes Z,M}} X \triangleright ((Y \otimes Z) \triangleright M) \xrightarrow{\mathrm{id} \triangleright m_{Y,Z,M}} X \triangleright (Y \triangleright (Z \triangleright M))$$

$$((X \otimes Y) \otimes Z) \triangleright M \xrightarrow{m_{X \otimes Y,Z,M}} (X \otimes Y) \triangleright (Z \triangleright M) \xrightarrow{m_{X,Y,Z \triangleright M}} X \triangleright (Y \triangleright (Z \triangleright M))$$

(the two paths agree: $\circlearrowleft$)

(PENTAGON AXIOM)

$$(X \otimes \mathbb{1}) \triangleright M \xrightarrow{r_X \triangleright \mathrm{id}} X \triangleright M$$

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ CONSISTS OF:
>
> (a) CATEGORY $\mathcal{C}$
>
> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
>
> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
>
> (d,e,f) NATURAL ISOMS:
>
> $a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$
>
> $\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> $r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> SATISFYING THE **PENTAGON AXIOM** & **TRIANGLE AXIOM**

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) A CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ (ACTION BIFUNCTOR)

(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

$\ni$.

$$\begin{array}{ccc}
 & ((X \otimes Y) \otimes Z) \triangleright M & \\
{\scriptstyle a_{X,Y,Z} \triangleright \mathrm{id}} \swarrow & \circlearrowleft & \searrow {\scriptstyle m_{X \otimes Y, Z, M}} \\
(X \otimes (Y \otimes Z)) \triangleright M & & (X \otimes Y) \triangleright (Z \triangleright M) \\
\downarrow {\scriptstyle m_{X, Y \otimes Z, M}} & & \downarrow {\scriptstyle m_{X,Y,Z \triangleright M}} \\
X \triangleright ((Y \otimes Z) \triangleright M) & \xrightarrow{\mathrm{id} \triangleright m_{Y,Z,M}} & X \triangleright (Y \triangleright (Z \triangleright M))
\end{array}$$

(PENTAGON AXIOM)

$$\begin{array}{ccc}
(X \otimes \mathbb{1}) \triangleright M & \xrightarrow{m_{X,\mathbb{1},M}} & X \triangleright (\mathbb{1} \triangleright M) \\
{\scriptstyle r_X \triangleright \mathrm{id}} \searrow & \circlearrowleft & \swarrow {\scriptstyle \mathrm{id} \triangleright p_M} \\
 & X \triangleright M &
\end{array}$$

(TRIANGLE AXIOM)

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

... MODULES/REPNS OVER A STRUCTURE $\mathcal{S}$ SHOULD REFLECT THE FEATURES OF $\mathcal{S}$

> MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
>
> CONSISTS OF:
>
> (a) CATEGORY $\mathcal{C}$
>
> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
>
> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
>
> (d, e, f) NATURAL ISOMS:
>
> $a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$
>
> $\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> $r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

(a) A CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ (ACTION BIFUNCTOR)

(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

$\ni$.

$$\begin{array}{ccc} & ((X \otimes Y) \otimes Z) \triangleright M & \\ {}^{a_{X,Y,Z} \triangleright \mathrm{id}} \swarrow & & \searrow {}^{m_{X \otimes Y, Z, M}} \\ (X \otimes (Y \otimes Z)) \triangleright M & \circlearrowleft & (X \otimes Y) \triangleright (Z \triangleright M) \\ \downarrow {\scriptstyle m_{X, Y \otimes Z, M}} & & \downarrow {\scriptstyle m_{X,Y,Z \triangleright M}} \\ X \triangleright ((Y \otimes Z) \triangleright M) & \xrightarrow{\mathrm{id} \triangleright m_{Y,Z,M}} & X \triangleright (Y \triangleright (Z \triangleright M)) \end{array}$$

(PENTAGON AXIOM)

$$\begin{array}{ccc} (X \otimes \mathbb{1}) \triangleright M & \xrightarrow{m_{X,\mathbb{1},M}} & X \triangleright (\mathbb{1} \triangleright M) \\ {}_{r_X \triangleright \mathrm{id}} \searrow & \circlearrowleft & \swarrow {}_{\mathrm{id} \triangleright p_M} \\ & X \triangleright M & \end{array}$$

(TRIANGLE AXIOM)

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
>
> CONSISTS OF: **ASSUME: ADDITIVE**
>
> (a) CATEGORY $\mathcal{C}$
>
> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
>
> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
>
> (d,e,f) NATURAL ISOMS:
>
> $a = \{a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)\}_{X,Y,Z \in \mathcal{C}}$
>
> $\ell = \{\ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X\}_{X \in \mathcal{C}}$
>
> $r = \{r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X\}_{X \in \mathcal{C}}$
>
> SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

(a) A CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ (ACTION BIFUNCTOR)

(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M)\}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M\}_{M \in \mathcal{M}}$$

s.t.

$$\begin{array}{ccc} & ((X \otimes Y) \otimes Z) \triangleright M & \\ {\scriptstyle a_{X,Y,Z} \triangleright id} \swarrow & & \searrow {\scriptstyle m_{X \otimes Y, Z, M}} \\ (X \otimes (Y \otimes Z)) \triangleright M & \circlearrowleft & (X \otimes Y) \triangleright (Z \triangleright M) \\ \downarrow {\scriptstyle m_{X, Y \otimes Z, M}} & & \downarrow {\scriptstyle m_{X,Y,Z \triangleright M}} \\ X \triangleright ((Y \otimes Z) \triangleright M) & \xrightarrow{id \triangleright m_{Y,Z,M}} & X \triangleright (Y \triangleright (Z \triangleright M)) \end{array}$$

(PENTAGON AXIOM)

$$\begin{array}{ccc} (X \otimes \mathbb{1}) \triangleright M & \xrightarrow{m_{X,\mathbb{1},M}} & X \triangleright (\mathbb{1} \triangleright M) \\ {\scriptstyle r_X \triangleright id} \searrow & \circlearrowleft & \swarrow {\scriptstyle id \triangleright p_M} \\ & X \triangleright M & \end{array}$$

(TRIANGLE AXIOM)

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF: **ASSUME: ADDITIVE**

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

(d,e,f) NATURAL ISOMS:

$a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$

$\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$

$r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) A CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ (ACTION BIFUNCTOR)

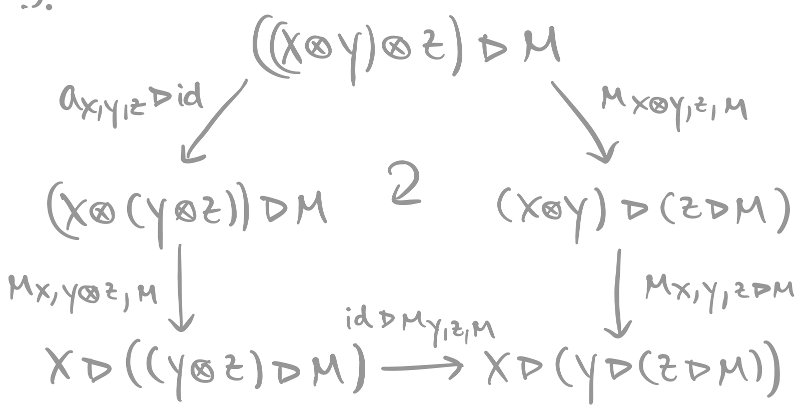
(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$

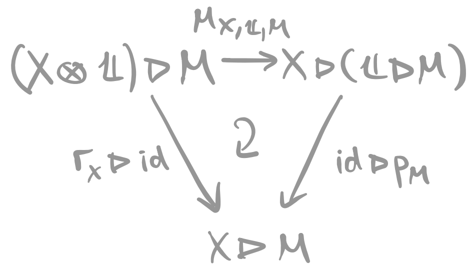
(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$

$\ni$.



(PENTAGON AXIOM)



(TRIANGLE AXIOM)

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF: ASSUME: ADDITIVE

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

(d,e,f) NATURAL ISOMS:

$a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$

$\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$

$r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) ~~A~~ AN ADDITIVE CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ (ACTION BIFUNCTOR)

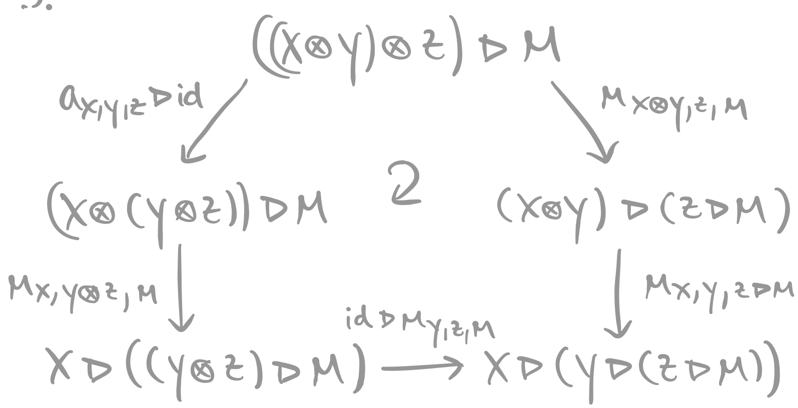
(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$

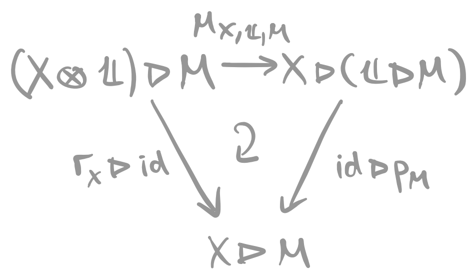
(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$

∋.



(PENTAGON AXIOM)



(TRIANGLE AXIOM)

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF: ~~ASSUME:~~ ASSUME: ADDITIVE

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

(d, e, f) NATURAL ISOMS:

$a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$

$l = \{ l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$

$r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) ~~A~~ AN ADDITIVE CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ (ACTION BIFUNCTOR)

s.t. $(X \triangleright -) : \mathcal{M} \to \mathcal{M}$ & $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ ARE ADDITIVE $\forall X, M$

(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

s.t.

$$\begin{array}{ccc}
 & ((X \otimes Y) \otimes Z) \triangleright M & \\
{}^{a_{X,Y,Z} \triangleright id} \swarrow & & \searrow {}^{m_{X \otimes Y, Z, M}} \\
(X \otimes (Y \otimes Z)) \triangleright M & \circlearrowleft & (X \otimes Y) \triangleright (Z \triangleright M) \\
\downarrow {}^{m_{X, Y \otimes Z, M}} & & \downarrow {}^{m_{X,Y,Z \triangleright M}} \\
X \triangleright ((Y \otimes Z) \triangleright M) & \xrightarrow{id \triangleright m_{Y,Z,M}} & X \triangleright (Y \triangleright (Z \triangleright M))
\end{array}$$

(PENTAGON AXIOM)

$$\begin{array}{ccc}
(X \otimes \mathbb{1}) \triangleright M & \xrightarrow{m_{X,\mathbb{1},M}} & X \triangleright (\mathbb{1} \triangleright M) \\
{}^{r_X \triangleright id} \searrow & \circlearrowleft & \swarrow {}^{id \triangleright p_M} \\
 & X \triangleright M &
\end{array}$$

(TRIANGLE AXIOM)

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
>
> CONSISTS OF: ASSUME: ADDITIVE
>
> (a) CATEGORY $\mathcal{C}$
>
> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
>
> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
>
> (d, e, f) NATURAL ISOMS:
>
> $a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$
>
> $\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> $r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) ~~A~~ AN ADDITIVE CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ (ACTION BIFUNCTOR)

s.t. $(X \triangleright -) : \mathcal{M} \to \mathcal{M}$ & $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ ARE ADDITIVE $\forall X, M$

KEEP THE REST

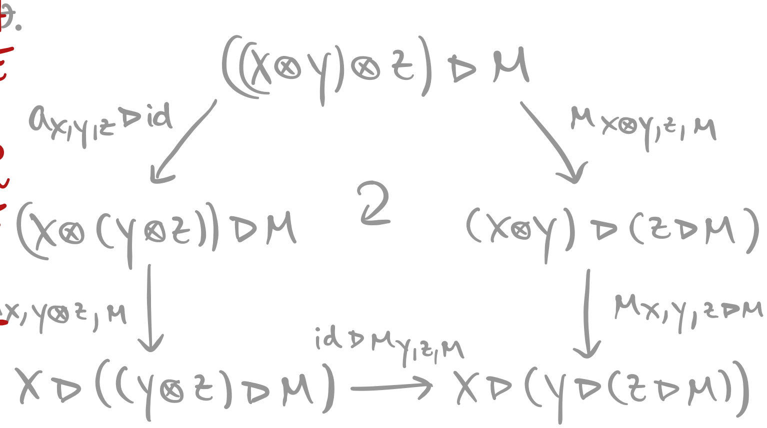
(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$

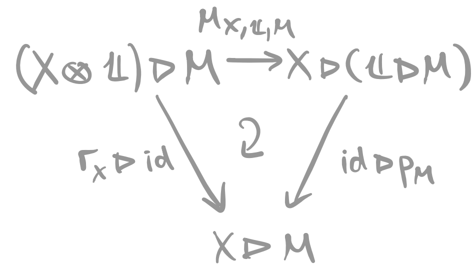
(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$

s.t.



(PENTAGON AXIOM)



(TRIANGLE AXIOM)

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
>
> CONSISTS OF: ASSUME: ABELIAN
>
> (a) CATEGORY $\mathcal{C}$
>
> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
>
> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
>
> (d, e, f) NATURAL ISOMS:
>
> $a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$
>
> $\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> $r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$
>
> SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) ~~A~~ AN ABELIAN CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ (ACTION BIFUNCTOR)

s.t. $(X \triangleright -) : \mathcal{M} \to \mathcal{M}$ & $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ ARE ADDITIVE $\forall X, M$

KEEP THE REST

(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$

s.t.

$$\begin{array}{ccccc}
 & & ((X \otimes Y) \otimes Z) \triangleright M & & \\
 & \swarrow^{a_{X,Y,Z} \triangleright id} & & \searrow^{m_{X \otimes Y, Z, M}} & \\
(X \otimes (Y \otimes Z)) \triangleright M & & \circlearrowleft & & (X \otimes Y) \triangleright (Z \triangleright M) \\
\downarrow^{m_{X, Y \otimes Z, M}} & & & & \downarrow^{m_{X,Y,Z \triangleright M}} \\
X \triangleright ((Y \otimes Z) \triangleright M) & & \xrightarrow{id \triangleright m_{Y,Z,M}} & & X \triangleright (Y \triangleright (Z \triangleright M))
\end{array}$$

(PENTAGON AXIOM)

$$\begin{array}{ccc}
(X \otimes \mathbb{1}) \triangleright M & \xrightarrow{m_{X,\mathbb{1},M}} & X \triangleright (\mathbb{1} \triangleright M) \\
 & \searrow^{r_X \triangleright id} \quad \circlearrowleft \quad \swarrow^{id \triangleright p_M} & \\
 & X \triangleright M &
\end{array}$$

(TRIANGLE AXIOM)

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

> **MONOIDAL CATEGORY** $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
>
> CONSISTS OF: ASSUME: LINEAR
>
> (a) CATEGORY $\mathcal{C}$
>
> (b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
>
> (c) OBJECT $\mathbb{1} \in \mathcal{C}$
>
> (d, e, f) NATURAL ISOMS:
>
> $a = \{a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)\}_{X,Y,Z \in \mathcal{C}}$
>
> $\ell = \{\ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X\}_{X \in \mathcal{C}}$
>
> $r = \{r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X\}_{X \in \mathcal{C}}$
>
> SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A **LEFT $\mathcal{C}$-MODULE CATEGORY** CONSISTS OF:

(a) ~~A~~ A LINEAR CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ (ACTION BIFUNCTOR)

s.t. $(X \triangleright -) : \mathcal{M} \to \mathcal{M}$ & $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ ARE LINEAR $\forall X, M$

KEEP THE REST

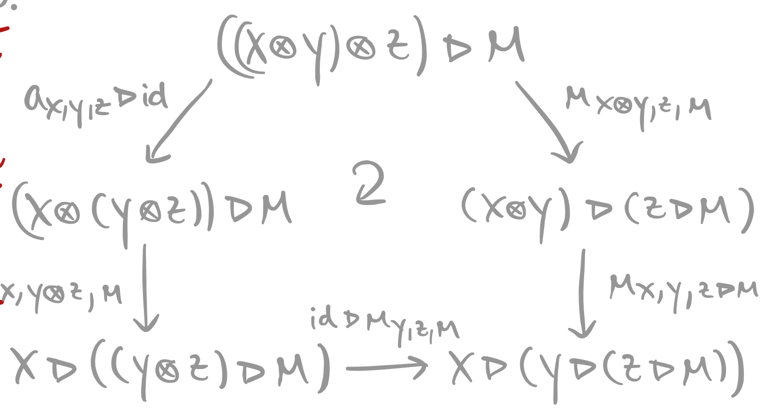
(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$m := \{m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M)\}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$

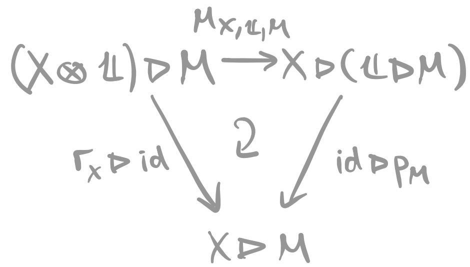
(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$p := \{p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M\}_{M \in \mathcal{M}}$

s.t.



(PENTAGON AXIOM)



(TRIANGLE AXIOM)

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY

CONSISTS OF:

(a) CATEGORY $\mathcal{M}$

(b) BIFUNCTOR

$\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

(c,d) NATURAL ISOMS:

$$m = \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

$$p = \{ p_M : \mathbb{1} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY

CONSISTS OF:

(a) CATEGORY $\mathcal{M}$

(b) BIFUNCTOR

$\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

(c,d) NATURAL ISOMS:

$$m = \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\ M \in \mathcal{M}}$$

$$p = \{ p_M : \mathbb{1} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG.

$(\mathcal{M}, \triangleleft, n, q)$

DEFINED LIKEWISE

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY

CONSISTS OF:

(a) CATEGORY $\mathcal{M}$

(b) BIFUNCTOR

$\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

(c,d) NATURAL ISOMS:

$$m = \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

$$p = \{ p_M : \mathbb{1} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG.

$(\mathcal{M}, \triangleleft, n, q)$

DEFINED LIKEWISE

EXAMPLES

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

(a) CATEGORY $\mathcal{M}$

(b) BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

(c,d) NATURAL ISOMS:

$$m = \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

$$p = \{ p_M : \mathbb{1} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

EXAMPLES

???

≡ REGULAR LEFT $\mathcal{C}$-MODULE CATEGORY ≡

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

(a) CATEGORY $\mathcal{M}$

(b) BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

(c,d) NATURAL ISOMS:

$$m = \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

$$p = \{ p_M : \mathbb{1} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

EXAMPLES

GIVEN $(\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, a^{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}})$,

GET $\mathcal{M} := \mathcal{C}$ & $\triangleright := \otimes^{\mathcal{C}}$

FORMS A LEFT $\mathcal{C}$-MODULE CATEGORY

= REGULAR LEFT $\mathcal{C}$-MODULE CATEGORY =

II. MODULE CATEGORIES

EXAMPLES

LIKE FOR $\Bbbk$-ALGEBRA

A

REGULAR LEFT A-MODULE

$\cong$

$(A_{vs}, \triangleright = m_A)$

GIVEN $(\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, a^{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}})$,

GET $\mathcal{M} := \mathcal{C}$ & $\triangleright := \otimes^{\mathcal{C}}$

FORMS A LEFT $\mathcal{C}$-MODULE CATEGORY

= REGULAR LEFT $\mathcal{C}$-MODULE CATEGORY =

II. MODULE CATEGORIES

EXAMPLES

LIKE FOR $\Bbbk$-ALGEBRA

A

REGULAR LEFT A-MODULE

$\left(A_{Vs},\ \triangleright = m_A\right)$

GIVEN $(\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, a^{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}})$,

GET $\mathcal{M} := \mathcal{C}$ & $\triangleright := \otimes^{\mathcal{C}}$

FORMS A LEFT $\mathcal{C}$-MODULE CATEGORY

= REGULAR LEFT $\mathcal{C}$-MODULE CATEGORY =

GIVEN A STRONG MONOIDAL FUNCTOR

$$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$$

TAKE A LEFT $\mathcal{D}$-MODULE CATEGORY $(\mathcal{M}, \triangleright)$.

GET A LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \blacktriangleright)$

VIA ???

II. MODULE CATEGORIES

EXAMPLES

LIKE FOR $\Bbbk$-ALGEBRA

A

REGULAR LEFT A-MODULE

$\cong$

$(A_{VS}, \triangleright = m_A)$

GIVEN $(\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, a^{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}})$,

GET $\mathcal{M} := \mathcal{C}$ & $\triangleright := \otimes^{\mathcal{C}}$

FORMS A LEFT $\mathcal{C}$-MODULE CATEGORY

= REGULAR LEFT $\mathcal{C}$-MODULE CATEGORY =

GIVEN A STRONG MONOIDAL FUNCTOR

$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$

TAKE A LEFT $\mathcal{D}$-MODULE CATEGORY $(\mathcal{M}, \triangleright)$.

GET A LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \blacktriangleright)$

VIA $X \blacktriangleright M := F(X) \triangleright M \quad \forall X \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

EXAMPLES

LIKE FOR $\Bbbk$-ALGEBRAS A, B

REGULAR LEFT A-MODULE

$$\cong$$

$$(A_{vs},\ \triangleright = m_A)$$

GIVEN $(\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, a^{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}})$,

GET $\mathcal{M} := \mathcal{C}$ & $\triangleright := \otimes^{\mathcal{C}}$

FORMS A LEFT $\mathcal{C}$-MODULE CATEGORY

= REGULAR LEFT $\mathcal{C}$-MODULE CATEGORY =

FOR AN ALG MAP $\phi : A \longrightarrow B$

THE RESTRICTION OF $({}_B V, \triangleright)$ TO A ALONG ϕ

$$\cong$$

$$(V,\ a \blacktriangleright v := \phi(a) \triangleright v) \quad \forall a \in A,\ v \in V$$

GIVEN A STRONG MONOIDAL FUNCTOR

$$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$$

TAKE A LEFT $\mathcal{D}$-MODULE CATEGORY $(\mathcal{M}, \triangleright)$.

GET A LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \blacktriangleright)$

VIA $X \blacktriangleright M := F(X) \triangleright M \quad \forall X \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

EXAMPLES

LIKE FOR $\Bbbk$-ALGEBRAS A, B

REGULAR LEFT A-MODULE
$\parallel$
$(A_{vs}, \triangleright = m_A)$

GIVEN $(\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, a^{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}})$,

GET $\mathcal{M} := \mathcal{C}$ & $\triangleright := \otimes^{\mathcal{C}}$

FORMS A LEFT $\mathcal{C}$-MODULE CATEGORY

≡ REGULAR LEFT $\mathcal{C}$-MODULE CATEGORY ≡

FOR AN ALG MAP $\phi : A \longrightarrow B$

THE RESTRICTION OF $({}_B V, \triangleright)$ TO A ALONG ϕ
$\parallel$
$(V, \; a \blacktriangleright v := \phi(a) \triangleright v)$ $\forall a \in A, v \in V$

GIVEN A STRONG MONOIDAL FUNCTOR

$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$

TAKE A LEFT $\mathcal{D}$-MODULE CATEGORY $(\mathcal{M}, \triangleright)$.

GET A LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \blacktriangleright)$

VIA $X \blacktriangleright M := F(X) \triangleright M$ $\quad \forall X \in \mathcal{C}, M \in \mathcal{M}$

≡ RESTRICTION OF $(\mathcal{M}, \triangleright) \in \mathcal{D}$-Mod TO $\mathcal{C}$ ALONG F ≡

II. MODULE CATEGORIES

EXAMPLES

LIKE FOR $\Bbbk$-ALGEBRAS A, B

REGULAR LEFT A-MODULE $\cong (A_{vs}, \triangleright = m_A)$

GIVEN $(\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, a^{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}})$,

GET $\mathcal{M} := \mathcal{C}$ & $\triangleright := \otimes^{\mathcal{C}}$

FORMS A LEFT $\mathcal{C}$-MODULE CATEGORY

= REGULAR LEFT $\mathcal{C}$-MODULE CATEGORY =

FOR AN ALG MAP $\phi : A \longrightarrow B$

THE RESTRICTION OF $({}_B V, \triangleright)$ TO A ALONG ϕ $\cong$ $(V, \; a \blacktriangleright v := \phi(a) \triangleright v)$ $\forall a \in A, v \in V$

GIVEN A STRONG MONOIDAL FUNCTOR $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$

TAKE A LEFT $\mathcal{D}$-MODULE CATEGORY $(\mathcal{M}, \triangleright)$.

GET A LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \blacktriangleright)$

VIA $X \blacktriangleright M := F(X) \triangleright M \quad \forall X \in \mathcal{C}, M \in \mathcal{M}$

= RESTRICTION OF $(\mathcal{M}, \triangleright) \in \mathcal{D}\text{-Mod}$ TO $\mathcal{C}$ ALONG F =

($\mathcal{D}$-Mod: CATEGORY OF $\mathcal{D}$-MOD CATS DEFINED LATER)

II. MODULE CATEGORIES

EXER. 3.12: COMPLETING THE DETAILS

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

(a) CATEGORY $\mathcal{M}$

(b) BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

(c,d) NATURAL ISOMS:

$m = \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$

$p = \{ p_M : \mathbb{1} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

EXAMPLES

GIVEN $(\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, a^{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}})$,

GET $\mathcal{M} := \mathcal{C}$ & $\triangleright := \otimes^{\mathcal{C}}$

FORMS A LEFT $\mathcal{C}$-MODULE CATEGORY

≡ REGULAR LEFT $\mathcal{C}$-MODULE CATEGORY ≡

GIVEN A STRONG MONOIDAL FUNCTOR

$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$

TAKE A LEFT $\mathcal{D}$-MODULE CATEGORY $(\mathcal{M}, \triangleright)$.

GET A LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \blacktriangleright)$

VIA $X \blacktriangleright M := F(X) \triangleright M \quad \forall X \in \mathcal{C}, M \in \mathcal{M}$

≡ RESTRICTION OF $(\mathcal{M}, \triangleright) \in \mathcal{D}\text{-Mod}$ TO $\mathcal{C}$ ALONG F ≡

(CATEGORY OF $\mathcal{D}$-MOD CATS DEFINED LATER)

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY

CONSISTS OF:

(a) CATEGORY $\mathcal{M}$

(b) BIFUNCTOR

$\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

(c,d) NATURAL ISOMS:

$$m = \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\ M \in \mathcal{M}}$$

$$p = \{ p_M : \mathbb{1} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

EXAMPLES

EXER. 3.12: COMPLETING THE DETAILS

≡ LET'S TRY THIS NOW (ON THE BOARD IN CLASS) ≡

GIVEN $(\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, a^{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}})$,

GET $\mathcal{M} := \mathcal{C}$ & $\triangleright := \otimes^{\mathcal{C}}$

FORMS A LEFT $\mathcal{C}$-MODULE CATEGORY

≡ REGULAR LEFT $\mathcal{C}$-MODULE CATEGORY ≡

GIVEN A STRONG MONOIDAL FUNCTOR

$$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$$

TAKE A LEFT $\mathcal{D}$-MODULE CATEGORY $(\mathcal{M}, \triangleright)$.

GET A LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \blacktriangleright)$

VIA $X \blacktriangleright M := F(X) \triangleright M \quad \forall X \in \mathcal{C},\ M \in \mathcal{M}$

≡ RESTRICTION OF $(\mathcal{M}, \triangleright) \in \mathcal{D}\text{-Mod}$ TO $\mathcal{C}$ ALONG F ≡

($\mathcal{D}$-Mod: CATEGORY OF $\mathcal{D}$-MOD CATS DEFINED LATER)

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

(a) CATEGORY $\mathcal{M}$

(b) BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

(c,d) NATURAL ISOMS:

$$m = \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

$$p = \{ p_M : \mathbb{1} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

EXER. 3.12: COMPLETING THE DETAILS

EXAMPLES

GIVEN A GROUP G WITH SUBGROUP H CAN FORM A STRONG MONOIDAL FUNCTOR

$$\mathrm{Res}^G_H : G\text{-Mod} \longrightarrow H\text{-Mod}$$

GIVEN A STRONG MONOIDAL FUNCTOR $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$

TAKE A LEFT $\mathcal{D}$-MODULE CATEGORY $(\mathcal{M}, \triangleright)$.

GET A LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \blacktriangleright)$ VIA $X \blacktriangleright M := F(X) \triangleright M \quad \forall X \in \mathcal{C},\, M \in \mathcal{M}$

= RESTRICTION OF $(\mathcal{M}, \triangleright) \in \mathcal{D}$-Mod TO $\mathcal{C}$ ALONG F =

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EXER. 3.12: COMPLETING THE DETAILS

EXAMPLES

GIVEN A GROUP G WITH SUBGROUP H CAN FORM A STRONG MONOIDAL FUNCTOR

$$\mathrm{Res}^G_H : G\text{-Mod} \longrightarrow H\text{-Mod}$$

(GENERALIZING $\mathrm{Forg} : G\text{-Mod} \to \mathrm{Vec}$ FOR $H = \langle e \rangle$)

GIVEN A STRONG MONOIDAL FUNCTOR

$$(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$$

TAKE A LEFT $\mathcal{D}$-MODULE CATEGORY $(\mathcal{M}, \triangleright)$.

GET A LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \blacktriangleright)$ VIA $X \blacktriangleright M := F(X) \triangleright M \quad \forall X \in \mathcal{C}, M \in \mathcal{M}$

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

EXER. 3.12: COMPLETING THE DETAILS

EXAMPLES

GIVEN A GROUP G WITH SUBGROUP H CAN FORM A STRONG MONOIDAL FUNCTOR

$$\mathrm{Res}^G_H : G\text{-Mod} \longrightarrow H\text{-Mod}$$

(GENERALIZING $\mathrm{Forg} : G\text{-Mod} \to \mathrm{Vec}$ FOR $H = \langle e \rangle$)

$\rightsquigarrow$ H-Mod (REGULAR MOD. CATEG.) (E.G. Vec) IS A LEFT MODULE CATEGORY OVER G-Mod

GIVEN A STRONG MONOIDAL FUNCTOR $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$

TAKE A LEFT $\mathcal{D}$-MODULE CATEGORY $(\mathcal{M}, \triangleright)$.

GET A LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \blacktriangleright)$ VIA $X \blacktriangleright M := F(X) \triangleright M \quad \forall X \in \mathcal{C},\ M \in \mathcal{M}$

≡ RESTRICTION OF $(\mathcal{M}, \triangleright) \in \mathcal{D}$-Mod TO $\mathcal{C}$ ALONG F ≡

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

EXER. 3.12: COMPLETING THE DETAILS

EXAMPLES

GET STRONG MONOIDAL FUNCTOR

$$\text{Braid} \longrightarrow \text{Perm}$$

$\Rightarrow$ Perm IS A LEFT MOD CAT. OVER Braid

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

EXER. 3.12: COMPLETING THE DETAILS

EXAMPLES

GET STRONG MONOIDAL FUNCTOR

$$\text{Braid} \longrightarrow \text{Perm}$$

$\Rightarrow$ Perm IS A LEFT MOD CAT. OVER Braid

OBJECTS: $n \in \mathbb{N}$ (Braid); $n \in \mathbb{N}$ (Perm)

MORPHISMS:

$\operatorname{Hom}(n,m) = \begin{cases} B_n & n = m \\ \emptyset & n \neq m \end{cases}$ (B_n: BRAID GROUP)

$\operatorname{Hom}(n,m) = \begin{cases} S_n & n = m \\ \emptyset & n \neq m \end{cases}$ (S_n: SYMMETRIC GROUP)

GIVEN A STRONG MONOIDAL FUNCTOR $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$

TAKE A LEFT $\mathcal{D}$-MODULE CATEGORY $(\mathcal{M}, \triangleright)$.

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VIA $X \blacktriangleright M := F(X) \triangleright M \quad \forall X \in \mathcal{C},\ M \in \mathcal{M}$

= RESTRICTION OF $(\mathcal{M}, \triangleright) \in \mathcal{D}$-Mod TO $\mathcal{C}$ ALONG F =

II. MODULE CATEGORIES

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CONSISTS OF:

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SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG.

$(\mathcal{M}, \triangleleft, n, q)$

DEFINED LIKEWISE

EXER. 3.12: COMPLETING THE DETAILS

EXAMPLES

GET STRONG MONOIDAL FUNCTOR

$$\text{Braid} \longrightarrow \text{Perm}$$

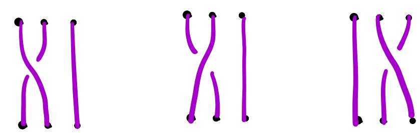
$\Rightarrow$ Perm IS A LEFT MOD CAT. OVER Braid

OBJECTS: $n \in \mathbb{N}$ $\qquad$ $n \in \mathbb{N}$

MORPHISMS:

$\mathrm{Hom}(n,m) = \begin{cases} B_n & n = m \\ \emptyset & n \neq m \end{cases}$ (B_n: BRAID GROUP) $\qquad$ $\mathrm{Hom}(n,m) = \begin{cases} S_n & n = m \\ \emptyset & n \neq m \end{cases}$ (S_n: SYMMETRIC GROUP)

VISUALIZED AS BRAIDS



$\sigma_1 \qquad \sigma_1^{-1} \qquad \sigma_2$

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

EXER. 3.12: COMPLETING THE DETAILS

EXAMPLES

GET STRONG MONOIDAL FUNCTOR

$$\text{Braid} \longrightarrow \text{Perm}$$

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OBJECTS: $n \in \mathbb{N}$; $n \in \mathbb{N}$

MORPHISMS:

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(B_n: BRAID GROUP)

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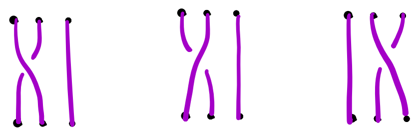
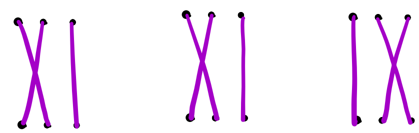
(S_n: SYMMETRIC GROUP)

VISUALIZED AS BRAIDS



σ_1 σ_1^{-1} σ_2

VISUALIZED AS PERMUTATIONS



τ_1 τ_1^{-1} τ_2

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

EXER. 3.12: COMPLETING THE DETAILS

EXAMPLES

GET STRONG MONOIDAL FUNCTOR

$\Theta : \text{Braid} \longrightarrow \text{Perm}$

$\Rightarrow$ Perm IS A LEFT MOD CAT. OVER Braid

OBJECTS: $n \in \mathbb{N}$ | $n \in \mathbb{N}$

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$\text{Hom}(n,m) = \begin{cases} B_n & n = m \\ \emptyset & n \neq m \end{cases}$ (BRAID GROUP)

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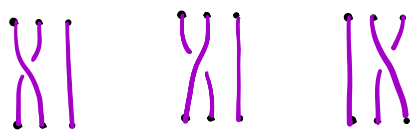
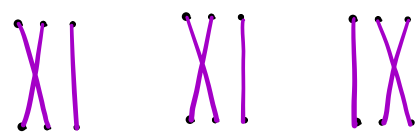
VISUALIZED AS BRAIDS



σ_1 $\quad \sigma_1^{-1}$ $\quad \sigma_2$

VISUALIZED AS PERMUTATIONS



τ_1 $\quad \tau_1^{-1}$ $\quad \tau_2$

$\Theta : n \longmapsto n$ ON OBJECTS

II. MODULE CATEGORIES

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

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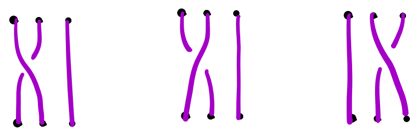
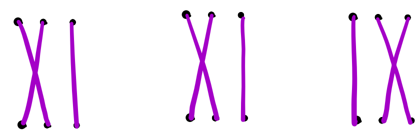


σ_1 $\quad \sigma_1^{-1}$ $\quad \sigma_2$

VISUALIZED AS PERMUTATIONS



τ_1 $\quad \tau_1^{-1}$ $\quad \tau_2$

$S_n \equiv$ QUOT. GP OF B_n

$\Theta : n \longmapsto n$ ON OBJECTS

$B_n \twoheadrightarrow S_n$ ON MORPHISMS ...

II. MODULE CATEGORIES

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SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

← NOW WE STUDY HOW TO MOVE FROM ONE TO ANOTHER

RIGHT $\mathcal{C}$-MODULE CATEG.

$(\mathcal{M}, \triangleleft, n, q)$

DEFINED LIKEWISE

II. MODULE CATEGORIES

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

A LEFT $\mathcal{C}$-MODULE FUNCTOR FROM $(\mathcal{M}, \triangleright, m, p)$ TO $(\mathcal{M}', \triangleright', m', p')$ CONSISTS OF:

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

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SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

A LEFT $\mathcal{C}$-MODULE FUNCTOR FROM $(\mathcal{M}, \triangleright, m, p)$ TO $(\mathcal{M}', \triangleright', m', p')$ CONSISTS OF:

(a) A FUNCTOR $F : \mathcal{M} \to \mathcal{M}'$

II. MODULE CATEGORIES

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SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG.

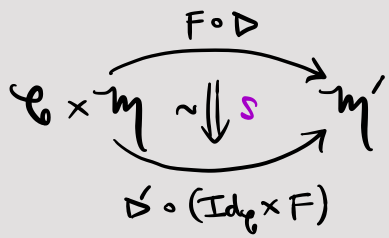
$(\mathcal{M}, \triangleleft, n, q)$

DEFINED LIKEWISE

A **LEFT $\mathcal{C}$-MODULE FUNCTOR** FROM $(\mathcal{M}, \triangleright, m, p)$ TO $(\mathcal{M}', \triangleright', m', p')$ CONSISTS OF:

(a) A FUNCTOR $F : \mathcal{M} \to \mathcal{M}'$

(b) A NATURAL ISOMORPHISM



(LEFT MODULE FUNCTOR CONSTRAINT)

$s := \{ s_{X,M} : F(X \triangleright M) \xrightarrow{\sim} X \triangleright' F(M) \}_{X \in \mathcal{C},\, M \in \mathcal{M}}$

II. MODULE CATEGORIES

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SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG.

$(\mathcal{M}, \triangleleft, n, q)$

DEFINED LIKEWISE

A LEFT $\mathcal{C}$-MODULE FUNCTOR FROM

$(\mathcal{M}, \triangleright, m, p)$ TO $(\mathcal{M}', \triangleright', m', p')$ CONSISTS OF:

(a) A FUNCTOR $F : \mathcal{M} \to \mathcal{M}'$

(b) A NATURAL ISOMORPHISM

$$\mathcal{C} \times \mathcal{M} \;\underset{\triangleright' \circ (\mathrm{Id}_{\mathcal{C}} \times F)}{\overset{F \circ \triangleright}{\rightrightarrows}}\; \mathcal{M}', \qquad s : F \circ \triangleright \overset{\sim}{\Rightarrow} \triangleright' \circ (\mathrm{Id}_{\mathcal{C}} \times F)$$

(LEFT MODULE FUNCTOR CONSTRAINT)

$s := \{ s_{X,M} : F(X \triangleright M) \xrightarrow{\sim} X \triangleright' F(M) \}_{X \in \mathcal{C},\, M \in \mathcal{M}}$

s.t.

$F((X \otimes Y) \triangleright M)$

$X \triangleright' (Y \triangleright' F(M))$

(PENTAGON AXIOM) $\forall X, Y \in \mathcal{C},\ M \in \mathcal{M}$

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(b) BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

(c,d) NATURAL ISOMS:

$$m = \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

$$p = \{ p_M : \mathbb{1} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

A LEFT $\mathcal{C}$-MODULE FUNCTOR FROM $(\mathcal{M}, \triangleright, m, p)$ TO $(\mathcal{M}', \triangleright', m', p')$ CONSISTS OF:

(a) A FUNCTOR $F : \mathcal{M} \to \mathcal{M}'$

(b) A NATURAL ISOMORPHISM

$$s : F \circ \triangleright \overset{\sim}{\Rightarrow} \triangleright' \circ (\mathrm{Id}_{\mathcal{C}} \times F) \quad : \mathcal{C} \times \mathcal{M} \to \mathcal{M}'$$

(LEFT MODULE FUNCTOR CONSTRAINT)

$$s := \{ s_{X,M} : F(X \triangleright M) \xrightarrow{\sim} X \triangleright' F(M) \}_{X \in \mathcal{C},\, M \in \mathcal{M}}$$

s.t.

$F((X \otimes Y) \triangleright M)$

$F(m_{X,Y,M}) \swarrow$

$F(X \triangleright (Y \triangleright M))$

$X \triangleright' (Y \triangleright' F(M))$

(PENTAGON AXIOM) $\quad \forall X, Y \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

(a) CATEGORY $\mathcal{M}$

(b) BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

(c,d) NATURAL ISOMS:

$$m = \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\ M \in \mathcal{M}}$$

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SATISFYING THE **PENTAGON AXIOM** & **TRIANGLE AXIOM**

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

A **LEFT $\mathcal{C}$-MODULE FUNCTOR** FROM $(\mathcal{M}, \triangleright, m, p)$ TO $(\mathcal{M}', \triangleright', m', p')$ CONSISTS OF:

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$$s := \{ s_{X,M} : F(X \triangleright M) \xrightarrow{\sim} X \triangleright' F(M) \}_{X \in \mathcal{C},\ M \in \mathcal{M}}$$

s.t.

$$F((X \otimes Y) \triangleright M) \xrightarrow{F(m_{X,Y,M})} F(X \triangleright (Y \triangleright M)) \xrightarrow{s_{X, Y \triangleright M}} X \triangleright' F(Y \triangleright M) \qquad X \triangleright' (Y \triangleright' F(M))$$

(PENTAGON AXIOM) $\forall X, Y \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

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(PENTAGON AXIOM) $\forall X, Y \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

A LEFT $\mathcal{C}$-MODULE FUNCTOR FROM $(\mathcal{M}, \triangleright, m, p)$ TO $(\mathcal{M}', \triangleright', m', p')$ CONSISTS OF:

(a) A FUNCTOR $F : \mathcal{M} \to \mathcal{M}'$

(b) A NATURAL ISOMORPHISM

$$\mathcal{C} \times \mathcal{M} \; \overset{F \circ \triangleright}{\underset{\triangleright' \circ (\mathrm{Id}_{\mathcal{C}} \times F)}{\rightrightarrows}} \; \mathcal{M}', \qquad \sim \Downarrow s$$

(LEFT MODULE FUNCTOR CONSTRAINT)

$$s := \{ s_{X,M} : F(X \triangleright M) \xrightarrow{\sim} X \triangleright' F(M) \}_{X \in \mathcal{C},\, M \in \mathcal{M}}$$

s.t.

$$F((X \otimes Y) \triangleright M) \xrightarrow{F(m_{X,Y,M})} F(X \triangleright (Y \triangleright M)) \xrightarrow{s_{X, Y \triangleright M}} X \triangleright' F(Y \triangleright M) \xrightarrow{\mathrm{id} \triangleright' s_{Y,M}} X \triangleright' (Y \triangleright' F(M))$$

$$F((X \otimes Y) \triangleright M) \xrightarrow{s_{X \otimes Y, M}} (X \otimes Y) \triangleright' F(M)$$

(PENTAGON AXIOM) $\forall X, Y \in \mathcal{C},\ M \in \mathcal{M}$

II. Module Categories

Given $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

Left $\mathcal{C}$-module category consists of:

(a) category $\mathcal{M}$

(b) bifunctor
$\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

(c,d) natural isoms:

$$m = \left\{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \right\}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

$$p = \left\{ p_M : \mathbb{1} \triangleright M \xrightarrow{\sim} M \right\}_{M \in \mathcal{M}}$$

Satisfying the

pentagon axiom

& **triangle axiom**

Right $\mathcal{C}$-module categ.
$(\mathcal{M}, \triangleleft, n, q)$
defined likewise

A **left $\mathcal{C}$-module functor** from $(\mathcal{M}, \triangleright, m, p)$ to $(\mathcal{M}', \triangleright', m', p')$ consists of:

(a) A functor $F : \mathcal{M} \to \mathcal{M}'$

(b) A natural isomorphism

$$s : F \circ \triangleright \overset{\sim}{\Longrightarrow} \triangleright' \circ (\mathrm{Id}_{\mathcal{C}} \times F) : \mathcal{C} \times \mathcal{M} \to \mathcal{M}'$$

(**left module functor constraint**)

$$s := \left\{ s_{X,M} : F(X \triangleright M) \xrightarrow{\sim} X \triangleright' F(M) \right\}_{X \in \mathcal{C},\, M \in \mathcal{M}}$$

$\ni$.

$$\left(\mathrm{id}_X \triangleright' s_{Y,M}\right) \circ s_{X, Y \triangleright M} \circ F(m_{X,Y,M}) = m'_{X,Y,F(M)} \circ s_{X \otimes Y, M} : F((X \otimes Y) \triangleright M) \to X \triangleright' (Y \triangleright' F(M))$$

(**pentagon axiom**) $\quad \forall X, Y \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

(a) CATEGORY $\mathcal{M}$

(b) BIFUNCTOR
$\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$

(c,d) NATURAL ISOMS:

$$m = \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

$$p = \{ p_M : \mathbb{1} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

A LEFT $\mathcal{C}$-MODULE FUNCTOR FROM $(\mathcal{M}, \triangleright, m, p)$ TO $(\mathcal{M}', \triangleright', m', p')$ CONSISTS OF:

(a) A FUNCTOR $F : \mathcal{M} \to \mathcal{M}'$

(b) A NATURAL ISOMORPHISM

$$\mathcal{C} \times \mathcal{M} \; \underset{\triangleright' \circ (\mathrm{Id}_{\mathcal{C}} \times F)}{\overset{F \circ \triangleright}{\rightrightarrows}} \; \mathcal{M}', \qquad s : F \circ \triangleright \overset{\sim}{\Rightarrow} \triangleright' \circ (\mathrm{Id}_{\mathcal{C}} \times F)$$

(LEFT MODULE FUNCTOR CONSTRAINT)

$$s := \{ s_{X,M} : F(X \triangleright M) \xrightarrow{\sim} X \triangleright' F(M) \}_{X \in \mathcal{C},\, M \in \mathcal{M}}$$

s.t.

$$\begin{array}{ccccc} & & F((X \otimes Y) \triangleright M) & & \\ & \swarrow^{F(m_{X,Y,M})} & & \searrow^{s_{X \otimes Y, M}} & \\ F(X \triangleright (Y \triangleright M)) & & \circlearrowleft & & (X \otimes Y) \triangleright' F(M) \\ \downarrow{\scriptstyle s_{X, Y \triangleright M}} & & & & \downarrow{\scriptstyle m'_{X,Y,F(M)}} \\ X \triangleright' F(Y \triangleright M) & & \xrightarrow{\mathrm{id} \triangleright' s_{Y,M}} & & X \triangleright' (Y \triangleright' F(M)) \end{array}$$

(PENTAGON AXIOM)

$$\begin{array}{ccc} F(\mathbb{1} \triangleright M) & \xrightarrow{s_{\mathbb{1},M}} & \mathbb{1} \triangleright' F(M) \\ & {\scriptstyle F(p_M)} \searrow \;\circlearrowleft\; \swarrow {\scriptstyle p'_{F(M)}} & \\ & F(M) & \end{array}$$

(TRIANGLE AXIOM)

$\forall X, Y \in \mathcal{C},\ M \in \mathcal{M}$

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

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A LEFT $\mathcal{C}$-MODULE FUNCTOR FROM $(\mathcal{M}, \triangleright, m, p)$ TO $(\mathcal{M}', \triangleright', m', p')$ CONSISTS OF:

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(b) A NATURAL ISOMORPHISM

$$s := \{ s_{X,M} : F(X \triangleright M) \xrightarrow{\sim} X \triangleright' F(M) \}_{X \in \mathcal{C},\, M \in \mathcal{M}}$$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

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(a) CATEGORY $\mathcal{M}$

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

→ FORMS A CATEGORY $\mathcal{C}$-Mod

- OBJECTS: LEFT $\mathcal{C}$-MODULE CATEGORIES
- MORPHISMS: LEFT $\mathcal{C}$-MODULE FUNCTORS

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

(a) CATEGORY $\mathcal{M}$

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A LEFT $\mathcal{C}$-MODULE FUNCTOR FROM $(\mathcal{M}, \triangleright, m, p)$ TO $(\mathcal{M}', \triangleright', m', p')$ CONSISTS OF:

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

→ FORMS A CATEGORY $\mathcal{C}$-Mod

- OBJECTS: LEFT $\mathcal{C}$-MODULE CATEGORIES
- MORPHISMS: LEFT $\mathcal{C}$-MODULE FUNCTORS

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE → LIKEWISE, CAN DEFINE Mod-$\mathcal{C}$

II. MODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

FORMS A CATEGORY $\mathcal{C}$-Mod

- OBJECTS: LEFT $\mathcal{C}$-MODULE CATEGORIES
- MORPHISMS: LEFT $\mathcal{C}$-MODULE FUNCTORS

LIKEWISE, CAN DEFINE Mod-$\mathcal{C}$

CAN ALSO GET MONOIDAL CATS LIKE ENDOMORPHISM ALGS

$\mathrm{End}_{\mathcal{C}\text{-Mod}}(\mathcal{M})$

$\otimes$ = COMPOSITION

$\mathbb{1} = \mathrm{Id}_{\mathcal{M}}$

III. BIMODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

LEFT $\mathcal{C}$-MODULE CATEGORY

CONSISTS OF:

(a) CATEGORY $\mathcal{M}$

(b) BIFUNCTOR

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SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG.

$(\mathcal{M}, \triangleleft, n, q)$

DEFINED LIKEWISE

STICKING THESE TOGETHER

III. BIMODULE CATEGORIES

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

TAKE: $(\mathcal{M}, \triangleright, m, p) \in \mathcal{C}\text{-Mod}$

& $(\mathcal{M}, \triangleleft, n, q) \in \text{Mod-}\mathcal{D}$

GET ...

$\in (\mathcal{C}, \mathcal{D})\text{-Bimod}$

III. BIMODULE CATEGORIES

> GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
>
> LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:
>
> (a) CATEGORY $\mathcal{M}$
>
> (b) BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$
>
> (c,d) NATURAL ISOMS:
>
> $$m = \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$
>
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>
> SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

TAKE: $(\mathcal{M}, \triangleright, m, p) \in \mathcal{C}\text{-Mod}$ & $(\mathcal{M}, \triangleleft, n, q) \in \text{Mod-}\mathcal{D}$

GET $(\mathcal{M}, \triangleright, \triangleleft, m, n, p, q, b) \in (\mathcal{C}, \mathcal{D})\text{-Bimod}$

HERE, b IS A NATURAL ISOMORPHISM: (MIDDLE ASSOC. CONSTRAINT)

$$\{ b_{X,M,Y} : (X \triangleright M) \triangleleft Y \xrightarrow{\sim} X \triangleright (M \triangleleft Y) \}_{X \in \mathcal{C},\, Y \in \mathcal{D},\, M \in \mathcal{M}}$$

SATISFYING COMPATIBILITY CONDITIONS.

III. BIMODULE CATEGORIES

GIVEN $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

RIGHT $\mathcal{C}$-MODULE CATEG. $(\mathcal{M}, \triangleleft, n, q)$ DEFINED LIKEWISE

TAKE : $(\mathcal{M}, \triangleright, m, p) \in \mathcal{C}\text{-Mod}$ & $(\mathcal{M}, \triangleleft, n, q) \in \text{Mod-}\mathcal{D}$

GET $(\mathcal{M}, \triangleright, \triangleleft, m, n, p, q, b) \in (\mathcal{C}, \mathcal{D})\text{-Bimod}$

HERE, b IS A NATURAL ISOMORPHISM: (MIDDLE ASSOC. CONSTRAINT)

$$\{ b_{X,M,Y} : (X \triangleright M) \triangleleft Y \xrightarrow{\sim} X \triangleright (M \triangleleft Y) \}_{X \in \mathcal{C},\, Y \in \mathcal{D},\, M \in \mathcal{M}}$$

SATISFYING COMPATIBILITY CONDITIONS.

Ex. $\mathcal{C} = \mathcal{D}$. GET $\mathcal{C}_{reg} \in \mathcal{C}\text{-Bimod}$

= REGULAR $\mathcal{C}$-BIMOD. CAT. = WITH $\triangleright = \triangleleft = \otimes^{\mathcal{C}}$

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #13

TOPICS:

I. ISOMORPHISMS AND EQUIVALENCE OF MONOIDAL CATEGORIES (§ 3.2.2)

II. MODULE CATEGORIES (§§ 3.3.1, 3.3.2, 3.3.4)

III. BIMODULE CATEGORIES (§ 3.3.3)

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LECTURE #13

NEXT TIME
WILL USE THIS FRAMEWORK
TO "STRICTIFY" MONOIDAL CATS

TOPICS:

I. ISOMORPHISMS AND EQUIVALENCE OF MONOIDAL CATEGORIES (§3.2.2) ✓

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WILL USE THIS FRAMEWORK TO "STRICTIFY" MONOIDAL CATS

TOPICS:

✓ I. ISOMORPHISMS AND EQUIVALENCE OF MONOIDAL CATEGORIES (§3.2.2)

✓ II. MODULE CATEGORIES (§§3.3.1, 3.3.2, 3.3.4)

✓ III. BIMODULE CATEGORIES (§3.3.3)

... AND THEN WE'LL DRAW SOME COOL PICTURES!

Lecture #13 keywords: bimodule category, module category, module functor, monoidal equivalence, monoidal isomorphism, monoidal natural transformation, strong monoidal functor