Preview for Lecture #13 of C. Walton's Spring 2024 course on "Symmetries of Algebras" at Rice U.
Keywords: bimodule category, module category, module functor, monoidal equivalence, monoidal isomorphism, monoidal natural transformation, strong monoidal functor
https://math.rice.edu/~notlaw/symalgbook.html, https://www.619wreath.com/

II. MODULE CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

(d,e,f) NATURAL ISOMS:

$a = \{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \}_{X,Y,Z \in \mathcal{C}}$

$l = \{ l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$

$r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$

SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

A LEFT $\mathcal{C}$-MODULE CATEGORY CONSISTS OF:

(a) A CATEGORY $\mathcal{M}$

(b) A BIFUNCTOR $\triangleright : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ (ACTION BIFUNCTOR)

(c) A NATURAL ISOMORPHISM (LEFT MOD ASSOC CONSTRAINT)

$$m := \{ m_{X,Y,M} : (X \otimes Y) \triangleright M \xrightarrow{\sim} X \triangleright (Y \triangleright M) \}_{X,Y \in \mathcal{C},\, M \in \mathcal{M}}$$

(d) A NATURAL ISOMORPHISM (LEFT MOD UNIT CONSTRAINT)

$$p := \{ p_M : \mathbb{1}^{\mathcal{C}} \triangleright M \xrightarrow{\sim} M \}_{M \in \mathcal{M}}$$

$\ni$.

$$\begin{tikzcd}
& ((X \otimes Y) \otimes Z) \triangleright M \arrow[dl, "a_{X,Y,Z} \triangleright id"'] \arrow[dr, "m_{X \otimes Y, Z, M}"] & \\
(X \otimes (Y \otimes Z)) \triangleright M \arrow[d, "m_{X, Y \otimes Z, M}"'] \arrow[rr, phantom, "\circlearrowleft"] & & (X \otimes Y) \triangleright (Z \triangleright M) \arrow[d, "m_{X,Y,Z \triangleright M}"] \\
X \triangleright ((Y \otimes Z) \triangleright M) \arrow[rr, "id \triangleright m_{Y,Z,M}"] & & X \triangleright (Y \triangleright (Z \triangleright M))
\end{tikzcd}$$

(PENTAGON AXIOM)

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \triangleright M \arrow[rr, "m_{X,\mathbb{1},M}"] \arrow[dr, "r_X \triangleright id"'] & \arrow[d, phantom, "\circlearrowleft"] & X \triangleright (\mathbb{1} \triangleright M) \arrow[dl, "id \triangleright p_M"] \\
& X \triangleright M &
\end{tikzcd}$$

(TRIANGLE AXIOM)

$\forall X, Y, Z \in \mathcal{C},\ M \in \mathcal{M}$