MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #14

LAST TIME

- ISOM. AND EQUIV. OF MONOIDAL CATEGORIES
- MODULE CATEGORIES
- BIMODULE CATEGORIES

TOPICS:

I. STRICTIFICATION (§3.4.1)

II. COHERENCE (§3.4.3)

III. GRAPHICAL CALCULUS (§3.5)

IV. RIGID CATEGORIES (§3.6)

I. STRICTIFICATION

≡ RECALL ≡

A **MONOIDAL CATEGORY** CONSISTS OF:

(a) A CATEGORY $\mathcal{C}$

(b) A BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
(MONOIDAL PRODUCT)

(c) AN OBJECT $\mathbb{1} \in \mathcal{C}$
(MONOIDAL UNIT)

I. STRICTIFICATION

≡ RECALL ≡

A **MONOIDAL CATEGORY** CONSISTS OF:

SATISFYING THE COMPATIBILITY CONDITIONS BELOW:

(a) A CATEGORY $\mathcal{C}$

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(MONOIDAL PRODUCT)

(c) AN OBJECT $\mathbb{1} \in \mathcal{C}$
(MONOIDAL UNIT)

(d, e, f) NATURAL ISOMORPHISMS:

$$a = \left\{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \right\}_{X,Y,Z \in \mathcal{C}} \quad \text{NATURAL IN } X, Y, Z$$

(ASSOCIATIVITY CONSTRAINT)

$$l = \{ l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}} \quad \text{NATURAL IN } X$$

(LEFT UNITALITY CONSTRAINT)

$$r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}} \quad \text{NATURAL IN } X$$

(RIGHT UNITALITY CONSTRAINT)

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(RIGHT UNITALITY CONSTRAINT)

SATISFYING THE COMPATIBILITY CONDITIONS BELOW:

$\forall W, X, Y, Z \in \mathcal{C}$:

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes id_Z"'] \arrow[dr, "a_{W \otimes X, Y, Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W, X, Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "id_W \otimes a_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

(PENTAGON AXIOM)

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(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_W \otimes a_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

(PENTAGON AXIOM)

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & \arrow[d, phantom, "\circlearrowleft"] & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y"] \\
 & X \otimes Y &
\end{tikzcd}$$

(TRIANGLE AXIOM)

I. STRICTIFICATION

A (STRICT) MONOIDAL CATEGORY CONSISTS OF:

(a) A CATEGORY $\mathcal{C}$

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(c) AN OBJECT $\mathbb{1} \in \mathcal{C}$
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(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"'] \arrow[rr, phantom, "\wr"] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W, X, Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_W \otimes a_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$
(PENTAGON AXIOM)

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] \arrow[rr, phantom, bend right, "\wr"] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y"] \\
 & X \otimes Y &
\end{tikzcd}$$
(TRIANGLE AXIOM)

I. STRICTIFICATION

STRICT
A $\wedge$ MONOIDAL CATEGORY CONSISTS OF:

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(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"'] \arrow[rr, phantom, "\circlearrowleft"] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}"] \\
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(PENTAGON AXIOM)

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y"] \\
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\end{tikzcd}$$
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(d, e, f) ~~NATURAL ISOMORPHISMS~~ IDENTITIES:

$$a = \left\{ a_{X,Y,Z} \overset{=\,id_{X,Y,Z}}{} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \right\}_{X,Y,Z \in \mathcal{C}} \text{ NATURAL IN } X, Y, Z$$

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$$l = \{ l_X \overset{=\,id_X}{} : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}} \text{ NATURAL IN } X$$

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(RIGHT UNITALITY CONSTRAINT)

$$X = \mathbb{1} \otimes X = X \otimes \mathbb{1}$$

SATISFYING THE COMPATIBILITY CONDITIONS BELOW:

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(RIGHT UNITALITY CONSTRAINT)

$$X \;::\; \mathbb{1} \otimes X \;=\; X \otimes \mathbb{1}$$

SATISFYING THE COMPATIBILITY CONDITIONS BELOW:

$\forall W, X, Y, Z \in \mathcal{C}$:

$$\begin{tikzcd}
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\end{tikzcd}$$

(PENTAGON AXIOM) VACUOUS

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(RIGHT UNITALITY CONSTRAINT)

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(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"', "="] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W, X, Y \otimes Z}", "="'] \\
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\end{tikzcd}$$

(PENTAGON AXIOM) VACUOUS

→ NO PARENTHESIS NEEDED

$$\begin{tikzcd}
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(d,e,f) ~~NATURAL ISOMORPHISMS~~ IDENTITIES:

$$\underset{= \mathrm{id}_{X,Y,Z}}{a} = \left\{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \right\}_{X,Y,Z \in \mathcal{C}} \quad \text{NATURAL IN } X,Y,Z$$

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(LEFT UNITALITY CONSTRAINT)

$$X = \mathbb{1} \otimes X$$

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(RIGHT UNITALITY CONSTRAINT)

$$\mathbb{1} \otimes X = X \otimes \mathbb{1}$$

SATISFYING THE COMPATIBILITY CONDITIONS BELOW:

$\forall W, X, Y, Z \in \mathcal{C}$:

$$\begin{tikzcd}
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(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"', "="] & \cong & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}", "="'] \\
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(PENTAGON AXIOM) VACUOUS

$\rightsquigarrow$ NO PARENTHESIS NEEDED

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}", "="'] \arrow[dr, "r_X \otimes \mathrm{id}_Y"', "="] & \cong & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y", "="'] \\
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(TRIANGLE AXIOM) VACUOUS

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(RIGHT UNITALITY CONSTRAINT)

$$X = \mathbb{1} \otimes X = X \otimes \mathbb{1}$$

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(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"', "="] \arrow[rr, phantom, "\cong"] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}", "="'] \\
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(PENTAGON AXIOM) VACUOUS

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$$\begin{tikzcd}
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\end{tikzcd}$$

(TRIANGLE AXIOM) VACUOUS

NO INSERTING $\mathbb{1}$S NEEDED

I. STRICTIFICATION

A STRICT MONOIDAL CATEGORY CONSISTS OF:

COMPUTATIONS ARE SIMPLIFIED

(a) A CATEGORY $\mathcal{C}$

(b) A BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
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(MONOIDAL UNIT)

(d, e, f) ~~NATURAL ISOMORPHISMS~~: IDENTITIES

$$a = \left\{ \underbrace{a_{X,Y,Z}}_{= \, id_{X,Y,Z}} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \right\}_{X,Y,Z \in \mathcal{C}} \quad \text{NATURAL IN } X, Y, Z$$

$$X \otimes Y \otimes Z := (X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$$

$$l = \{ \underbrace{l_X}_{= \, id_X} : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}} \quad \text{NATURAL IN } X$$

(LEFT UNITALITY CONSTRAINT)

$$r = \{ \underbrace{r_X}_{= \, id_X} : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}} \quad \text{NATURAL IN } X$$

(RIGHT UNITALITY CONSTRAINT)

$$X = \mathbb{1} \otimes X = X \otimes \mathbb{1}$$

SATISFYING THE COMPATIBILITY CONDITIONS BELOW:

$\forall W, X, Y, Z \in \mathcal{C}$:

$$\begin{tikzcd}
& ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes id_Z"', "="] \arrow[dr, "a_{W \otimes X, Y, Z}", "="'] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W, X \otimes Y, Z}"', "="] & \circlearrowright & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}", "="'] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "id_W \otimes a_{X,Y,Z}", "="'] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

(PENTAGON AXIOM) VACUOUS

→ NO PARENTHESIS NEEDED

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}", "="'] \arrow[dr, "r_X \otimes id_Y"', "="] & \circlearrowright & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "id_X \otimes l_Y", "="'] \\
& X \otimes Y &
\end{tikzcd}$$

(TRIANGLE AXIOM) VACUOUS

NO INSERTING $\mathbb{1}$s NEEDED

I. STRICTIFICATION

COMPUTATIONS ARE SIMPLIFIED

⋮

FOR INSTANCE

RECALL

TAKE MONOIDAL CATEGORIES
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$$F^{(2)} = \{F^{(2)}_{X,Y} : F(X) \otimes^{\mathcal{D}} F(Y) \longrightarrow F(X \otimes^{\mathcal{C}} Y)\}_{X,Y \in \mathcal{C}}.$$

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(F(X) \otimes^{\mathcal{D}} F(Y)) \otimes^{\mathcal{D}} F(Z) \arrow[r, "F^{(2)}_{X,Y} \otimes id"] \arrow[d, "a^{\mathcal{D}}_{F(X),F(Y),F(Z)}"'] & F(X \otimes^{\mathcal{C}} Y) \otimes^{\mathcal{D}} F(Z) \arrow[r, "F^{(2)}_{X \otimes Y, Z}"] \arrow[d, phantom, "="] & F((X \otimes^{\mathcal{C}} Y) \otimes^{\mathcal{C}} Z) \arrow[d, "F(a^{\mathcal{C}}_{X,Y,Z})"] \\
F(X) \otimes^{\mathcal{D}} (F(Y) \otimes^{\mathcal{D}} F(Z)) \arrow[r, "id \otimes F^{(2)}_{Y,Z}"] & F(X) \otimes^{\mathcal{D}} F(Y \otimes^{\mathcal{C}} Z) \arrow[r, "F^{(2)}_{X, Y \otimes Z}"] & F(X \otimes^{\mathcal{C}} (Y \otimes^{\mathcal{C}} Z))
\end{tikzcd}$$

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\mathbb{1}^{\mathcal{D}} \otimes^{\mathcal{D}} F(X) \arrow[r, "l^{\mathcal{D}}_{F(X)}"] \arrow[d, "F^{(0)} \otimes id"'] \arrow[dr, phantom, "="] & F(X) \\
F(\mathbb{1}^{\mathcal{C}}) \otimes^{\mathcal{D}} F(X) \arrow[r, "F^{(2)}_{\mathbb{1},X}"] & F(\mathbb{1} \otimes^{\mathcal{C}} X) \arrow[u, "F(l^{\mathcal{C}}_X)"']
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BECOMES A SQUARE

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\qquad
\begin{tikzcd}
F(X) \arrow[r, "r^{\mathcal{D}}_{F(X)}", "="'] \arrow[d, "id \otimes F^{(0)}"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X) \\
F(X) \otimes^{\mathcal{D}} F(\mathbb{1}^{\mathcal{C}}) \arrow[r, "F^{(2)}_{X,\mathbb{1}}"] & F(X) \arrow[u, "= \; F(r^{\mathcal{C}}_X)"']
\end{tikzcd}$$

I. STRICTIFICATION

WHEN STRICT

COMPUTATIONS ARE SIMPLIFIED

TAKE MONOIDAL CATEGORIES

$\mathcal{C} := (\mathcal{C}, \otimes^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}, \underbrace{a^{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}}}) \;\;\&\;\; \mathcal{D} := (\mathcal{D}, \otimes^{\mathcal{D}}, \mathbb{1}^{\mathcal{D}}, \underbrace{a^{\mathcal{D}}, \ell^{\mathcal{D}}, r^{\mathcal{D}}}).$

IDENTITIES

A MONOIDAL FUNCTOR FROM $\mathcal{C}$ TO $\mathcal{D}$ CONSISTS OF:

(a) A FUNCTOR BTW CATEGORIES $F : \mathcal{C} \longrightarrow \mathcal{D}$.

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(c) A MORPHISM $F^{(0)} : \mathbb{1}^{\mathcal{D}} \longrightarrow F(\mathbb{1}^{\mathcal{C}})$ IN $\mathcal{D}$.

SATISFYING:

$$\begin{tikzcd}
F(X) \otimes^{\mathcal{D}} F(Y) \otimes^{\mathcal{D}} F(Z) \arrow[r, "F^{(2)}_{X,Y} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes F^{(2)}_{Y,Z}"'] \arrow[dr, phantom, "2"] & F(X \otimes^{\mathcal{C}} Y) \otimes^{\mathcal{D}} F(Z) \arrow[d, "F^{(2)}_{X \otimes Y, Z}"] \\
F(X) \otimes^{\mathcal{D}} F(Y \otimes^{\mathcal{C}} Z) \arrow[r, "F^{(2)}_{X, Y \otimes Z}"] & F(X \otimes^{\mathcal{C}} Y \otimes^{\mathcal{C}} Z)
\end{tikzcd}$$

LIKEWISE THESE ARE SIMPLIFIED

$$\begin{tikzcd}
F(X) \arrow[d, "F^{(0)} \otimes \mathrm{id}"'] \arrow[dr, equal, "\mathrm{id}"] \arrow[dr, phantom, "2", near start] & \\
F(\mathbb{1}^{\mathcal{C}}) \otimes^{\mathcal{D}} F(X) \arrow[r, "F^{(2)}_{\mathbb{1},X}"] & F(X)
\end{tikzcd}
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F(X) \arrow[d, "\mathrm{id} \otimes F^{(0)}"'] \arrow[dr, equal, "\mathrm{id}"] \arrow[dr, phantom, "2", near start] & \\
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GET:

$$F^{(2)}_{\mathbb{1},X} \circ (F^{(0)} \otimes \mathrm{id}_{F(X)}) = \mathrm{id}_{F(X)}$$

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F(\mathbb{1}^{\mathcal{C}}) \otimes^{\mathcal{D}} F(X) \arrow[r, "F^{(2)}_{\mathbb{1},X}"'] & F(X) \arrow[ul, bend right, "id"']
\end{tikzcd}
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F(X) \otimes^{\mathcal{D}} F(Y \otimes^{\mathcal{C}} Z) \arrow[r, "F^{(2)}_{X, Y \otimes Z}"] & F(X \otimes^{\mathcal{C}} Y \otimes^{\mathcal{C}} Z)
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GET:

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$$\begin{tikzcd}
& F(X) \arrow[dl, bend right, "F^{(0)} \otimes id"'] \arrow[d, "F^{(0)} \otimes id"] \arrow[dr, phantom, "2"] & \\
F(\mathbb{1}^{\mathcal{C}}) \otimes^{\mathcal{D}} F(X) & F(\mathbb{1}^{\mathcal{C}}) \otimes^{\mathcal{D}} F(X) \arrow[l, "id"'] \arrow[r, "F^{(2)}_{\mathbb{1},X}"] & F(X) \arrow[ul, bend right, "id"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[d, "id \otimes F^{(0)}"'] \arrow[dr, equal, "id"] \arrow[dr, phantom, "2"'] & \\
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$$\begin{tikzcd}
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$$\begin{tikzcd}
(F(X) \otimes^{\mathcal{D}} F(Y)) \otimes^{\mathcal{D}} F(Z) \arrow[r, "F^{(2)}_{X,Y} \otimes \mathrm{id}"] \arrow[d, "a^{\mathcal{D}}_{F(X),F(Y),F(Z)}"'] & F(X \otimes^{\mathcal{C}} Y) \otimes^{\mathcal{D}} F(Z) \arrow[r, "F^{(2)}_{X \otimes Y, Z}"] \arrow[d, phantom, "\circlearrowleft"] & F((X \otimes^{\mathcal{C}} Y) \otimes^{\mathcal{C}} Z) \arrow[d, "F(a^{\mathcal{C}}_{X,Y,Z})"] \\
F(X) \otimes^{\mathcal{D}} (F(Y) \otimes^{\mathcal{D}} F(Z)) \arrow[r, "\mathrm{id} \otimes F^{(2)}_{Y,Z}"] & F(X) \otimes^{\mathcal{D}} F(Y \otimes^{\mathcal{C}} Z) \arrow[r, "F^{(2)}_{X, Y \otimes Z}"] & F(X \otimes^{\mathcal{C}} (Y \otimes^{\mathcal{C}} Z))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1}^{\mathcal{D}} \otimes^{\mathcal{D}} F(X) \arrow[r, "\ell^{\mathcal{D}}_{F(X)}"] \arrow[d, "F^{(0)} \otimes \mathrm{id}"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X) \\
F(\mathbb{1}^{\mathcal{C}}) \otimes^{\mathcal{D}} F(X) \arrow[r, "F^{(2)}_{\mathbb{1},X}"] & F(\mathbb{1} \otimes^{\mathcal{C}} X) \arrow[u, "F(\ell^{\mathcal{C}}_X)"']
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WOULD BE NICE IF WE WERE ALWAYS ABLE TO REDUCE TO THE STRICT CASE

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<u>STRICTIFICATION THEOREM</u>: ANY MONOIDAL CATEGORY $\mathcal{C}$ IS MONOIDALLY EQUIVALENT TO A STRICT MONOIDAL CATEGORY.

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STRICTIFICATION THEOREM: ANY MONOIDAL CATEGORY $\mathcal{C}$ IS MONOIDALLY EQUIVALENT TO A STRICT MONOIDAL CATEGORY.

STRATEGY

- RECALL $(\mathrm{End}(\mathcal{A}),\ \otimes = \circ,\ \mathbb{1} = \mathrm{Id}_{\mathcal{A}})$ IS STRICT.
- DEFINE STRICT MONOIDAL CATEGORY $\mathcal{C}^{str}$ MODELLED ON $\mathrm{End}_{\text{Mod-}\mathcal{C}}(\mathcal{C}_{reg})$.

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- BUILD FULLY FAITHFUL, STRONG MONOIDAL FUNCTOR:
$$\rho : \mathcal{C} \longrightarrow \mathcal{C}^{str}$$

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- BUILD FULLY FAITHFUL, STRONG MONOIDAL FUNCTOR:
$$\rho : \mathcal{C} \longrightarrow \mathcal{C}^{str}$$
- GET $\mathcal{C} \overset{\otimes}{\simeq}$ "ESSENTIAL IMAGE OF ρ" WHICH IS STRICT. ///

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OBJECTS: $\Big(\overset{\text{ENDOFUNCTOR}}{F \in \mathrm{End}(\mathcal{C})},\ \overset{\text{NAT'L ISOM}}{u := \{u_{M,X} : F(M) \otimes X \xrightarrow{\sim} F(M \otimes X)\}_{M,X \in \mathcal{C}}}\Big)$

s.t.

$$\begin{tikzcd}
 & (F(M) \otimes X) \otimes Y \arrow[dl, "u_{M,X} \otimes \mathrm{id}_Y"'] \arrow[dr, "a_{F(M),X,Y}"] & \\
F(M \otimes X) \otimes Y \arrow[d, "u_{M \otimes X, Y}"'] \arrow[rr, phantom, "\circlearrowleft"] & & F(M) \otimes (X \otimes Y) \arrow[d, "u_{M,X,Y}"] \\
F((M \otimes X) \otimes Y) \arrow[rr, "F(a_{M,X,Y})"'] & & F(M \otimes (X \otimes Y))
\end{tikzcd}$$

$\forall M, X, Y \in \mathcal{C}$

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STRICTIFICATION THEOREM: ANY MONOIDAL CATEGORY $\mathcal{C}$ IS MONOIDALLY EQUIVALENT TO A STRICT MONOIDAL CATEGORY.

- DEFINE STRICT MONOIDAL CATEGORY $\mathcal{C}^{str}$ MODELLED ON $\mathrm{End}_{\text{Mod-}\mathcal{C}}(\mathcal{C}_{reg})$.

OBJECTS: $\Big(\overset{\text{ENDOFUNCTOR}}{F \in \mathrm{End}(\mathcal{C})}, \quad \overset{\text{NAT'L ISOM}}{u := \{u_{M,X} : F(M) \otimes X \xrightarrow{\sim} F(M \otimes X)\}_{M,X \in \mathcal{C}}} \Big)$

+ PENTAGON AXIOM

MORPHISMS: $(F,u) \longrightarrow (F',u')$ $\equiv$ NAT'L TRANSF $\theta : F \Longrightarrow F'$ $\ni$.

$$\begin{tikzcd}
F(M) \otimes X \arrow[r, "u_{M,X}"] \arrow[d, "\theta_M \otimes \mathrm{id}_X"'] \arrow[dr, phantom, "\circlearrowleft"] & F(M \otimes X) \arrow[d, "\theta_{M \otimes X}"] \\
F'(M) \otimes X \arrow[r, "u'_{M,X}"] & F'(M \otimes X)
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$$:= \left(\begin{array}{l} Z \otimes - \; : \mathcal{C} \to \mathcal{C} \\ u^Z := \{ \underset{\substack{\| \text{ def} \\ a_{Z,M,X}}}{u^Z_{M,X}} : (Z \otimes M) \otimes X \xrightarrow{\sim} Z \otimes (M \otimes X) \}_{M,X \in \mathcal{C}} \end{array} \right)$$

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BE SURE TO CHECK ... PROPERLY DEFINES ρ

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EXER3.16. MONOIDAL

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EXER3.16

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AN EXPRESSION $w(X_1, X_2, \ldots, X_n)$ IS AN OBJECT IN $\mathcal{C}$ FORMED BY INSERTING PARENTHESES AND $\mathbb{1}$s IN $X_1 \otimes X_2 \otimes \cdots \otimes X_n$

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E.G. EXPRESSIONS IN $X \otimes Y \otimes Z$:

$$(X \otimes Y) \otimes (\mathbb{1} \otimes Z) \qquad\qquad \mathbb{1} \otimes \Big(X \otimes \big((Y \otimes \mathbb{1}) \otimes Z\big)\Big)$$

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$\exists$ WAY* OF MOVING FROM ONE EXPRESSION IN 3 OBJECTS TO ANOTHER

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$$(X \otimes Y) \otimes (\mathbb{1} \otimes Z) \qquad \mathbb{1} \otimes (X \otimes ((Y \otimes \mathbb{1}) \otimes Z)) \xrightarrow{l_{X \otimes ((Y \otimes \mathbb{1}) \otimes Z)}} X \otimes ((Y \otimes \mathbb{1}) \otimes Z) \xrightarrow{id_X \otimes r_Y \otimes id_Z} X \otimes (Y \otimes Z)$$

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$$\begin{tikzcd}
 & & X\otimes(Y\otimes Z) \arrow[dl, "a^{-1}_{X,Y,Z}"'] & & \\
 & (X\otimes Y)\otimes Z & & X\otimes((Y\otimes \mathbb{1})\otimes Z) \arrow[ul, "id_X\otimes r_Y \otimes id_Z"'] & \\
(X\otimes Y)\otimes(\mathbb{1}\otimes Z) & & & & \mathbb{1}\otimes(X\otimes((Y\otimes\mathbb{1})\otimes Z)) \arrow[ul, "l_{X\otimes((Y\otimes\mathbb{1})\otimes Z)}"']
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$$\mathbb{1}\otimes\big(X\otimes((Y\otimes\mathbb{1})\otimes Z)\big) \xrightarrow{l_{X\otimes((Y\otimes\mathbb{1})\otimes Z)}} X\otimes((Y\otimes\mathbb{1})\otimes Z) \xrightarrow{id_X\otimes r_Y\otimes id_Z} X\otimes(Y\otimes Z) \xrightarrow{a^{-1}_{X,Y,Z}} (X\otimes Y)\otimes Z \xrightarrow{id_{X\otimes Y}\otimes l_Z^{-1}} (X\otimes Y)\otimes(\mathbb{1}\otimes Z)$$

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THE PENTAGON & TRIANGLE AXIOMS IMPLY THAT

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UNIQUENESS

$$\begin{tikzcd}
 & & X\otimes(Y\otimes Z) \arrow[dl, "a^{-1}_{X,Y,Z}"'] & & \\
 & (X\otimes Y)\otimes Z \arrow[dl, "id_{X\otimes Y}\otimes l_Z^{-1}"'] & & X\otimes((Y\otimes\mathbb{1})\otimes Z) \arrow[ul, "id_X\otimes r_Y\otimes id_Z"'] & \\
(X\otimes Y)\otimes(\mathbb{1}\otimes Z) & & & & \mathbb{1}\otimes(X\otimes((Y\otimes\mathbb{1})\otimes Z)) \arrow[ul, "l_{X\otimes((Y\otimes\mathbb{1})\otimes Z)}"']
\end{tikzcd}$$

* VIA ASSOCIATIVITY & UNITALITY MORPHISMS

II. COHERENCE

<u>STRICTIFICATION THEOREM</u>: ANY MONOIDAL CATEGORY $\mathcal{C}$ IS MONOIDALLY EQUIVALENT TO A STRICT MONOIDAL CATEGORY.

TAKE n OBJECTS : $X_1, X_2, \ldots, X_n \in \mathcal{C}$.

AN EXPRESSION $w(X_1, X_2, \ldots, X_n)$ IS AN OBJECT IN $\mathcal{C}$ FORMED BY INSERTING PARENTHESES AND $\mathbb{1}$s IN $X_1 \otimes X_2 \otimes \cdots \otimes X_n$

THE PENTAGON & TRIANGLE AXIOMS IMPLY THAT $\exists!$ WAY* OF MOVING FROM ONE EXPRESSION IN <u>4 OBJECTS</u> TO ANOTHER

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STILL HOLDS ... ?? WHAT ABOUT ≥ 5 OBJECTS ??

COHERENCE THEOREM: LET $f, g : w(X_1, \dots, X_n) \longrightarrow w'(X_1, \dots, X_n) \in \mathcal{C}$ BE ISOS COMPRISED OF ASSOC. & UNITALITY MORPHISMS*. THEN, $f = g$.

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PROOF IDEA: VIA THE STRICTIFICATION THEOREM,

HAVE FAITHFUL, STRONG MONOIDAL FUNCTOR

$$\rho : \mathcal{C} \longrightarrow \mathcal{C}^{str}$$

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$$\rho : \mathcal{C} \longrightarrow \mathcal{C}^{str}$$

ARGUE THAT $\rho(f) = \rho(g)$ USING STRONGNESS.

FAITHFULNESS $\Rightarrow$ $f = g$. ///

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III. GRAPHICAL CALCULUS

STRICTIFICATION THEOREM: ANY MONOIDAL CATEGORY $\mathcal{C}$ IS MONOIDALLY EQUIVALENT TO A STRICT MONOIDAL CATEGORY.

TAKE MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

WLOG VIA STRICTIFICATION THEOREM

CAN ASSUME STRICT $(\mathcal{C}, \otimes, \mathbb{1})$

III. GRAPHICAL CALCULUS

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OBJECTS & MORPHISMS IN
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HERE WITH NICE PICTURES

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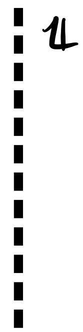

MONOIDAL
UNIT

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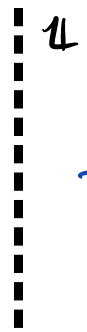

OR SOMETIMES
THE EMPTY STRING
IS USED

MONOIDAL
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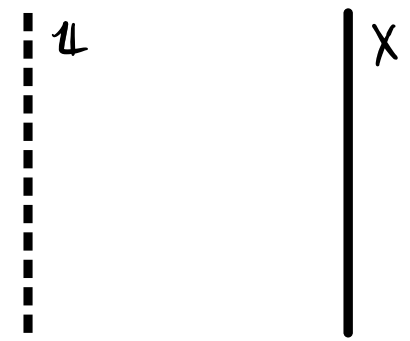

MONOIDAL UNIT

OBJECT $X \in \mathcal{C}$

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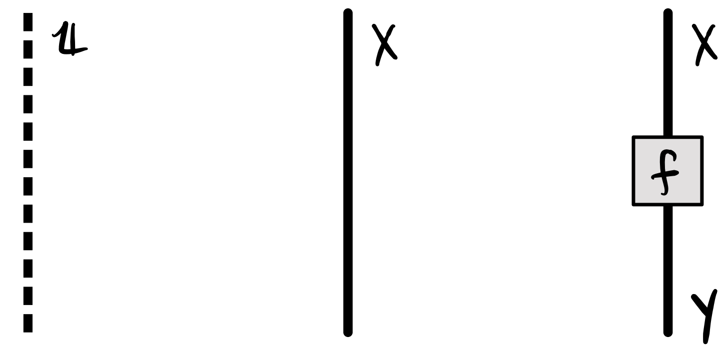

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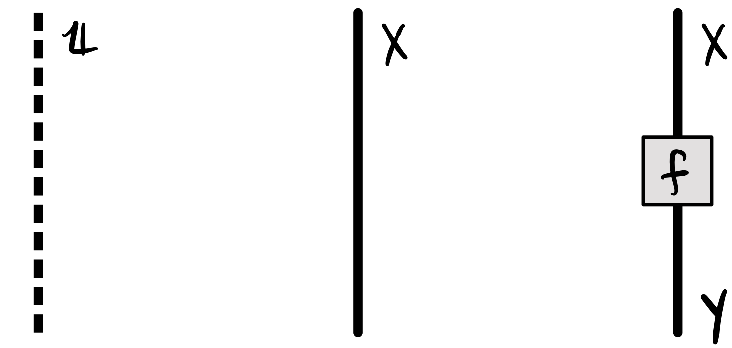

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IN PARTS OF THE LITERATURE:

MORPHISMS $f: X \to Y$

ARE DRAWN <u>UPWARD</u>

CAN ILLUSTRATE OBJECTS & MORPHISMS IN STRICT MONOIDAL CATEGORIES $(\mathcal{C}, \otimes, \mathbb{1})$ HERE WITH NICE PICTURES

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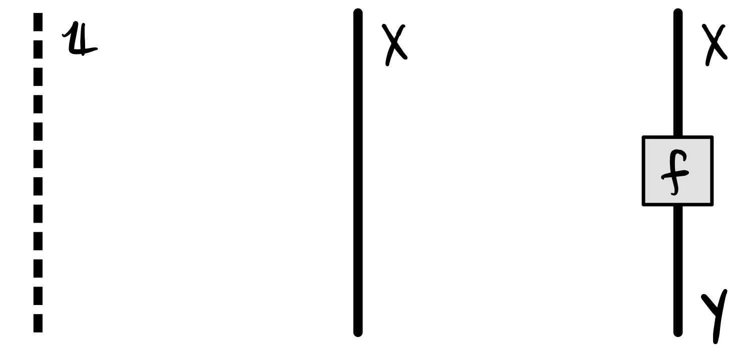

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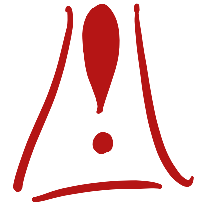

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OPTIMIST vs PESSIMIST CONVENTION

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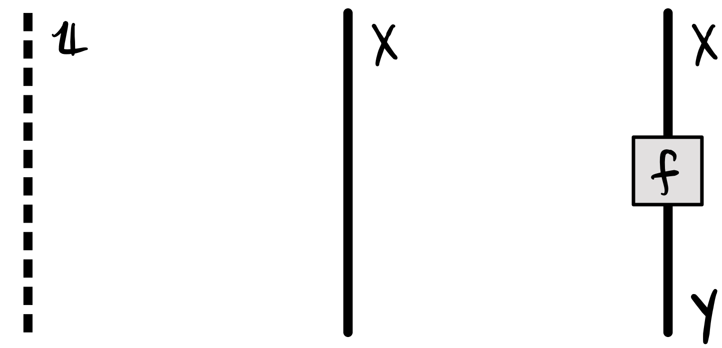

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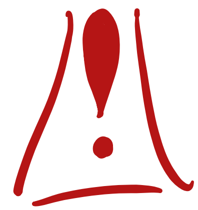

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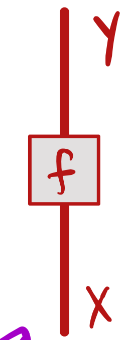

IDEALIST VS REALIST CONVENTION

~~OPTIMIST VS PESSIMIST CONVENTION~~

BOOO

CAN ILLUSTRATE

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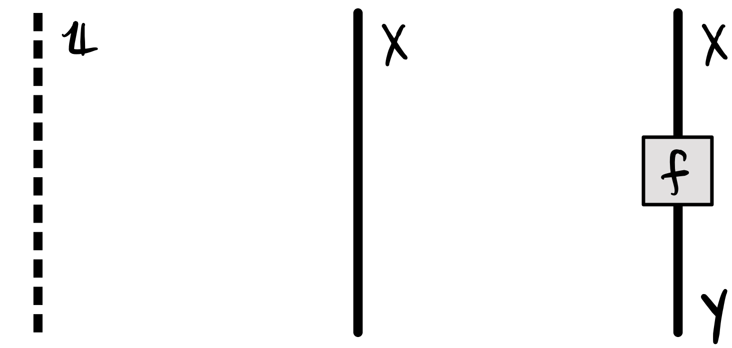

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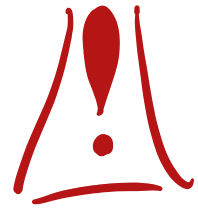

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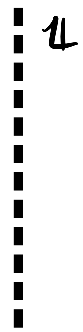

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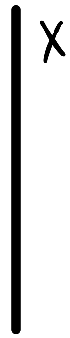
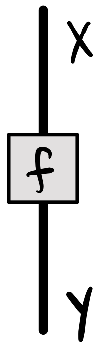


OBJECT

$X \in \mathcal{C}$



MORPHISM

$f : X \to Y \in \mathcal{C}$

IN PARTS OF THE LITERATURE :

MORPHISMS $f : X \to Y$

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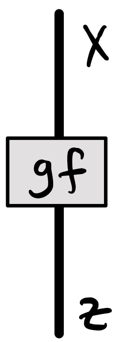
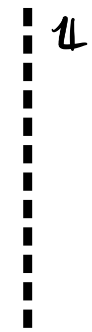


$:=$ GUESS

COMPOSITION
OF MORPHISMS

$f : X \to Y, \; g : Y \to Z \in \mathcal{C}$

III. GRAPHICAL CALCULUS



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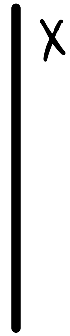
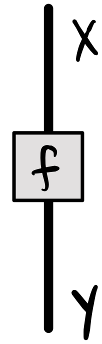


OBJECT $X \in \mathcal{C}$



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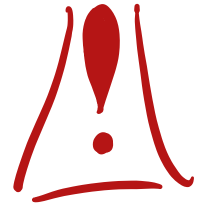

IN PARTS OF THE LITERATURE :

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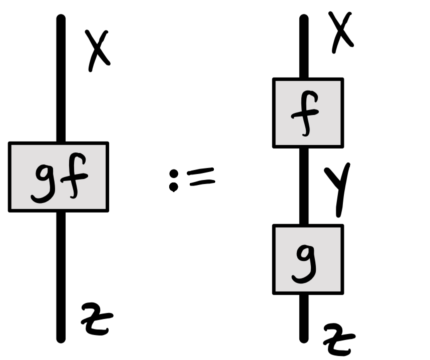

COMPOSITION OF MORPHISMS

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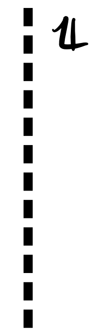
III. GRAPHICAL CALCULUS



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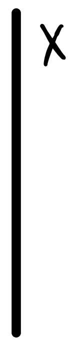

OBJECT
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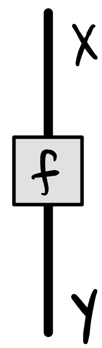

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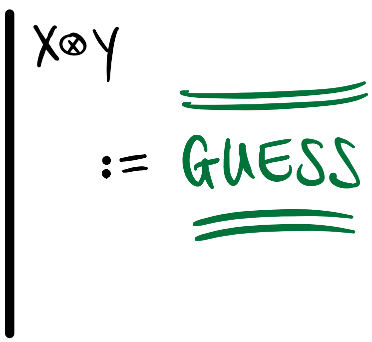
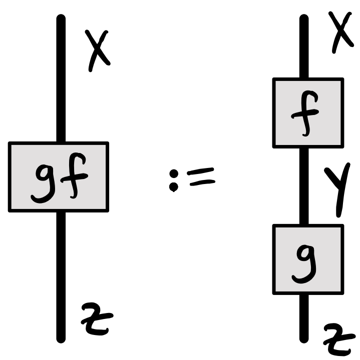


⊗ OF OBJECTS
$X, Y \in \mathcal{C}$



COMPOSITION
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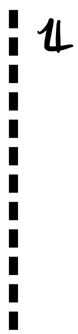
$f: X \to Y,\ g: Y \to Z \in \mathcal{C}$

III. GRAPHICAL CALCULUS



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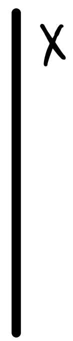
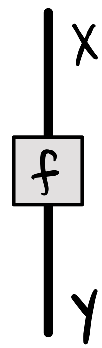


OBJECT

$X \in \mathcal{C}$



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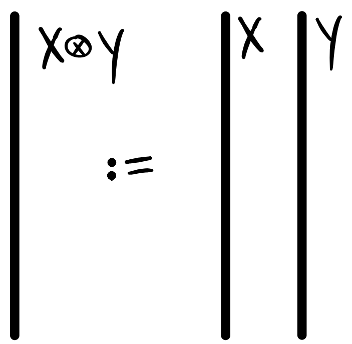

⊗ OF OBJECTS

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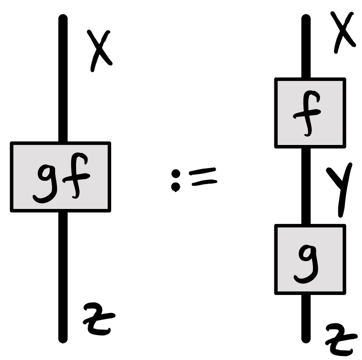
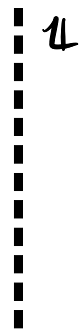


COMPOSITION

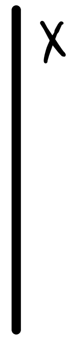
OF MORPHISMS

$f: X \to Y, \ g: Y \to Z \in \mathcal{C}$

III. GRAPHICAL CALCULUS



MONOIDAL UNIT



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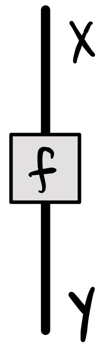

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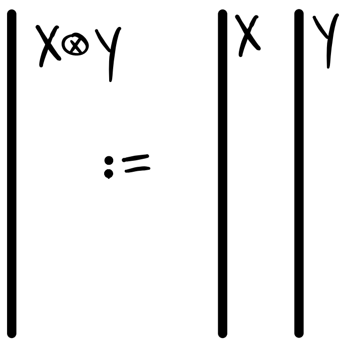
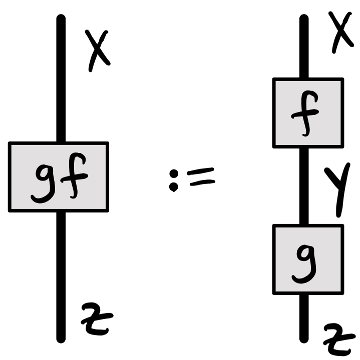


$\otimes$ OF OBJECTS
$X, Y \in \mathcal{C}$

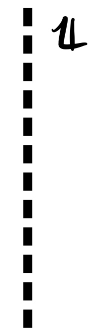


COMPOSITION OF MORPHISMS
$f: X \to Y,\ g: Y \to Z \in \mathcal{C}$

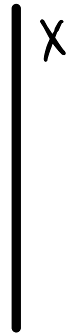
MORPHISM $X \otimes X' \xrightarrow{f \otimes f'} Y \otimes Y'$ CAN BE DRAWN AS:

≡ you TRY ≡

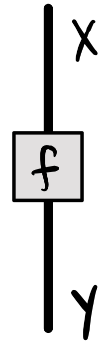
III. GRAPHICAL CALCULUS



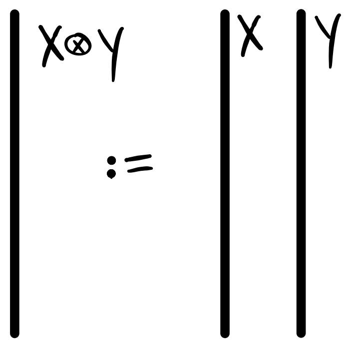
MONOIDAL UNIT



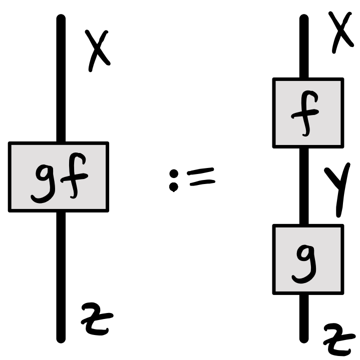
OBJECT $X \in \mathcal{C}$



MORPHISM $f: X \to Y \in \mathcal{C}$



$\otimes$ OF OBJECTS $X, Y \in \mathcal{C}$



COMPOSITION OF MORPHISMS $f: X \to Y,\ g: Y \to Z \in \mathcal{C}$

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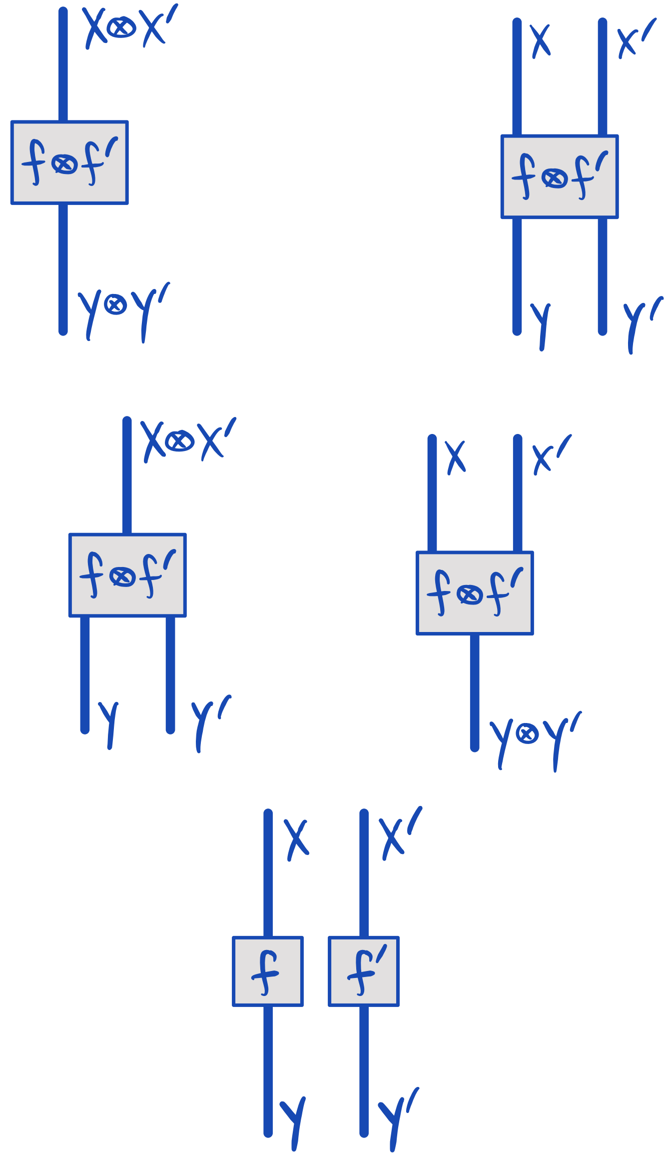
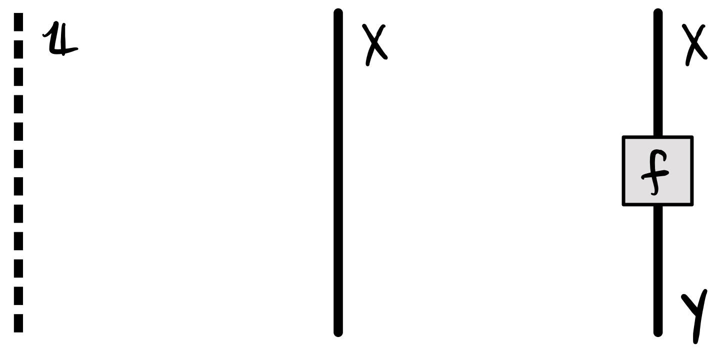


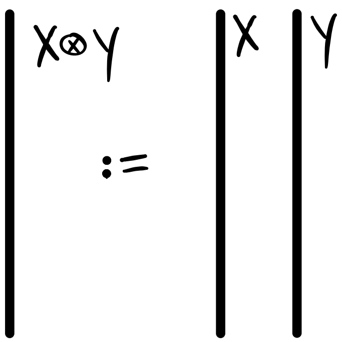
III. GRAPHICAL CALCULUS



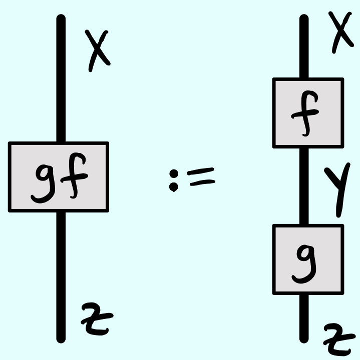
MONOIDAL UNIT

OBJECT $X \in \mathcal{C}$

MORPHISM $f : X \to Y \in \mathcal{C}$



$\otimes$ OF OBJECTS $X, Y \in \mathcal{C}$



COMPOSITION OF MORPHISMS $f : X \to Y$, $g : Y \to Z \in \mathcal{C}$

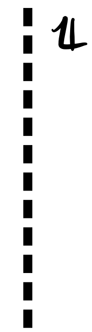
ASSOCIATIVITY:

≡ YOU TRY ≡

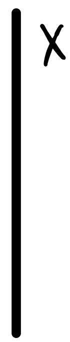
UNITALITY:

≡ YOU TRY ≡

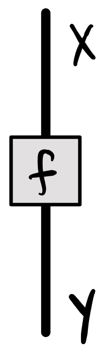
III. GRAPHICAL CALCULUS



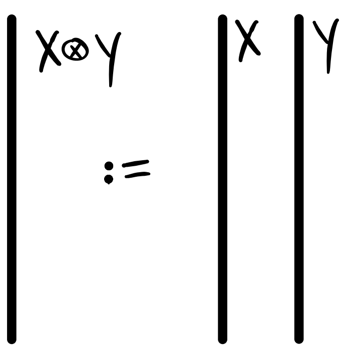
MONOIDAL UNIT



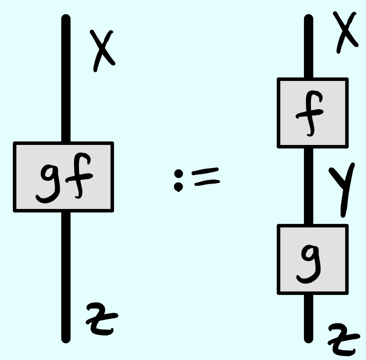
OBJECT
$X \in \mathcal{C}$



MORPHISM
$f : X \to Y \in \mathcal{C}$

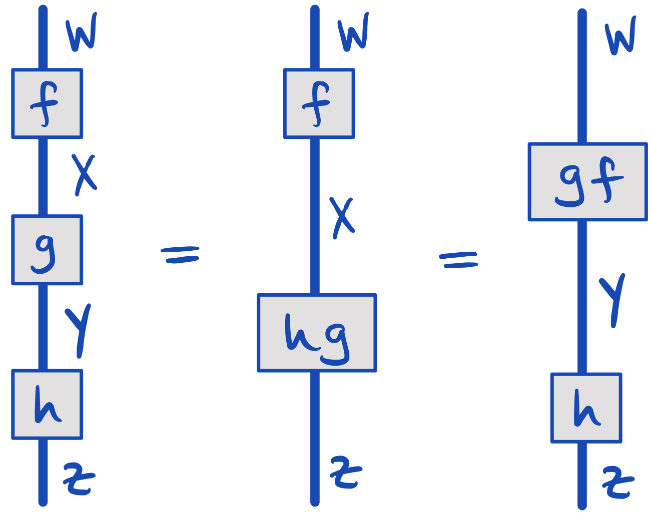


⊗ OF OBJECTS
$X, Y \in \mathcal{C}$

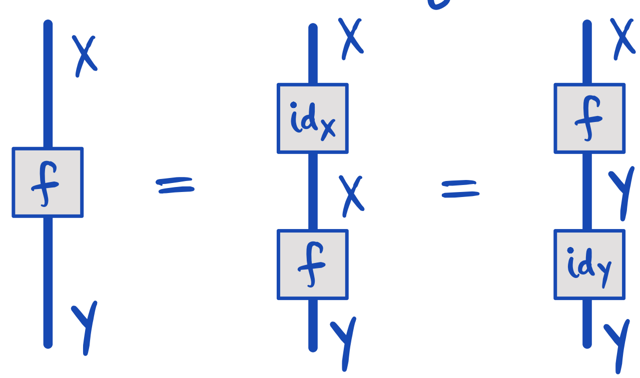


COMPOSITION
OF MORPHISMS
$f : X \to Y,\ g : Y \to Z \in \mathcal{C}$

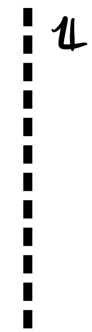
ASSOCIATIVITY:



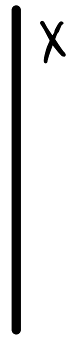
UNITALITY:



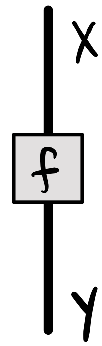
III. GRAPHICAL CALCULUS



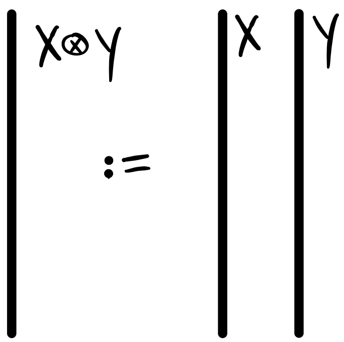
MONOIDAL UNIT



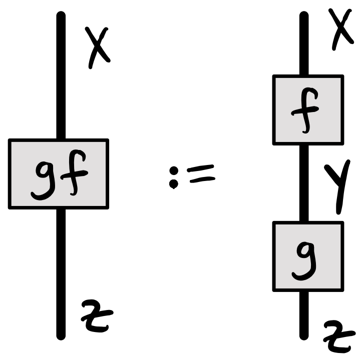
OBJECT $X \in \mathcal{C}$



MORPHISM $f: X \to Y \in \mathcal{C}$



$\otimes$ OF OBJECTS $X, Y \in \mathcal{C}$



COMPOSITION OF MORPHISMS $f: X \to Y,\ g: Y \to Z \in \mathcal{C}$

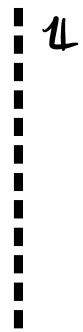
COMMUTATIVE DIAGRAM:

$$\begin{tikzcd}
X \otimes X' \arrow[r, "f \otimes id_{X'}"] \arrow[d, "id_X \otimes f'"'] \arrow[dr, phantom, "2"] & Y \otimes X' \arrow[d, "id_Y \otimes f'"] \\
X \otimes Y' \arrow[r, "f \otimes id_{Y'}"] & Y \otimes Y'
\end{tikzcd}$$

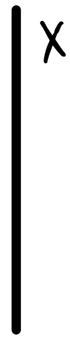
ILLUSTRATED AS:

≡ You Try ≡

III. GRAPHICAL CALCULUS



MONOIDAL UNIT



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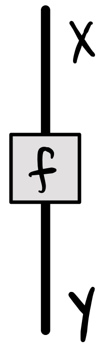
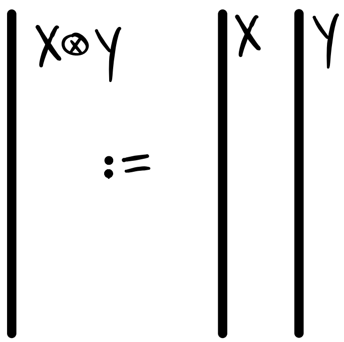
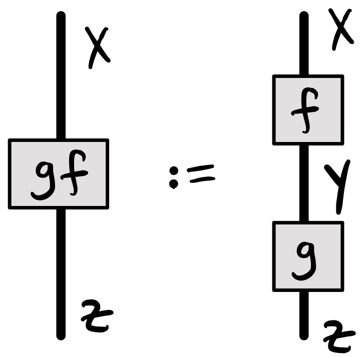
$X \in \mathcal{C}$



MORPHISM
$f: X \to Y \in \mathcal{C}$



⊗ OF OBJECTS
$X, Y \in \mathcal{C}$



COMPOSITION
OF MORPHISMS
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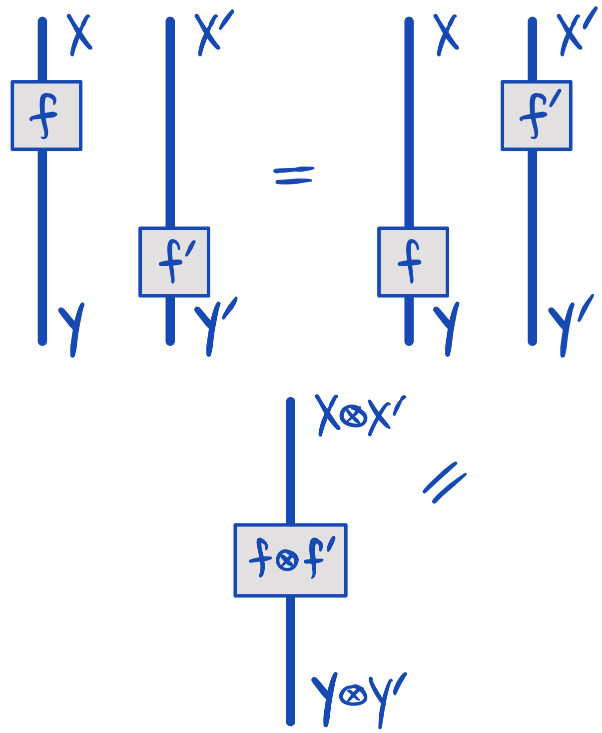
\end{tikzcd}$$

ILLUSTRATED AS:



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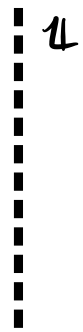
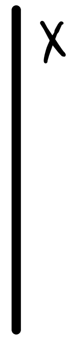
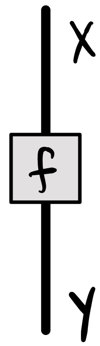


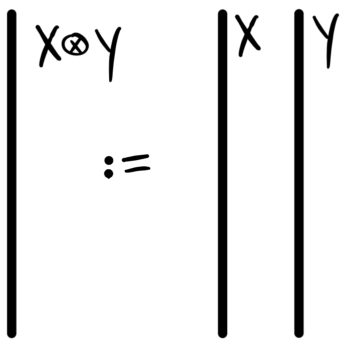
MONOIDAL UNIT



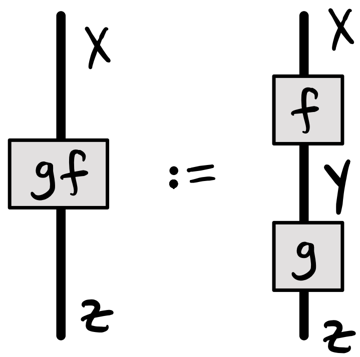
OBJECT $X \in \mathcal{C}$



MORPHISM $f: X \to Y \in \mathcal{C}$



⊗ OF OBJECTS $X, Y \in \mathcal{C}$

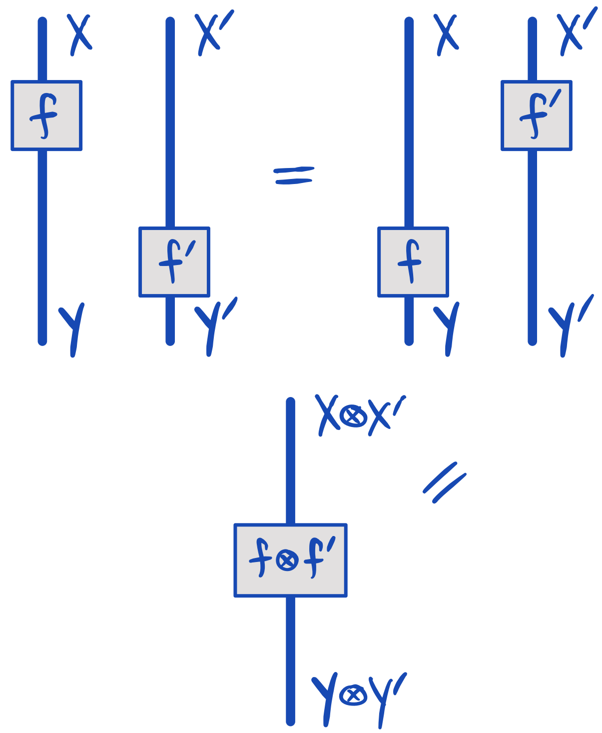


COMPOSITION OF MORPHISMS $f: X \to Y,\ g: Y \to Z \in \mathcal{C}$

LEVEL EXCHANGE

$$\begin{tikzcd} X \otimes X' \arrow[r, "f \otimes id_{X'}"] \arrow[d, "id_X \otimes f'"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \otimes X' \arrow[d, "id_Y \otimes f'"] \\ X \otimes Y' \arrow[r, "f \otimes id_{Y'}"] & Y \otimes Y' \end{tikzcd}$$

ILLUSTRATED AS:



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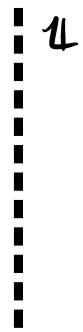
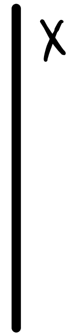


MONOIDAL UNIT



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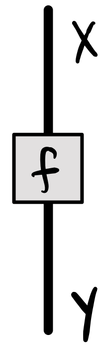
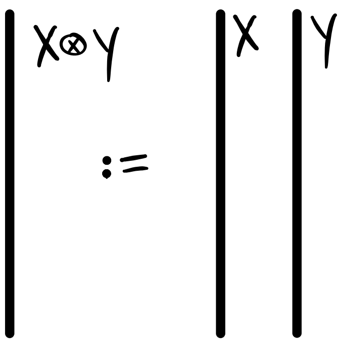
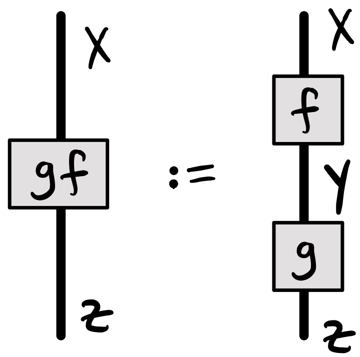


MORPHISM $f : X \to Y \in \mathcal{C}$



$\otimes$ OF OBJECTS $X, Y \in \mathcal{C}$



COMPOSITION OF MORPHISMS $f : X \to Y,\ g : Y \to Z \in \mathcal{C}$

EXERCISE 3.17

FOR $f : X \to Y,\ g : Y \to Z$

$f' : X' \to Y',\ g' : Y' \to Z' \in \mathcal{C}$.

SHOW:

$(gf) \otimes (g'f') = (g \otimes g')(f \otimes f')$.

ILLUSTRATED AS:

≡ You Try ≡

III. GRAPHICAL CALCULUS

EXERCISE 3.17

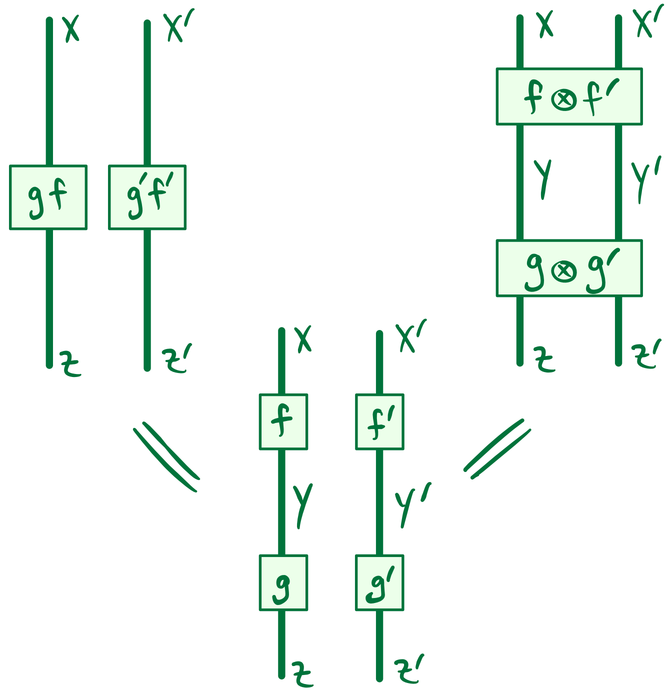
For $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$

$f' : X' \longrightarrow Y'$, $g' : Y' \longrightarrow Z' \in \mathcal{C}$.

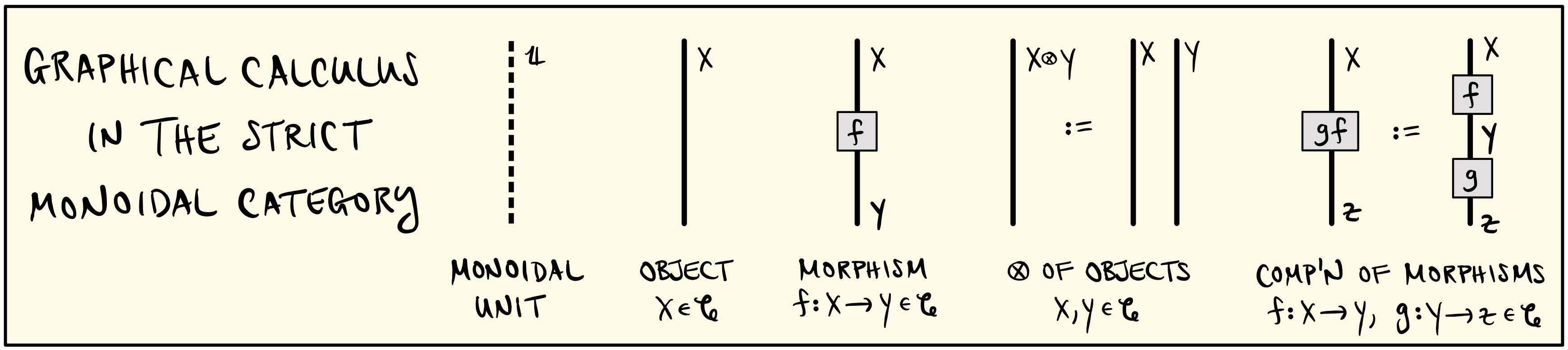
SHOW:

$$(gf) \otimes (g'f') = (g \otimes g')(f \otimes f').$$

ILLUSTRATED AS:



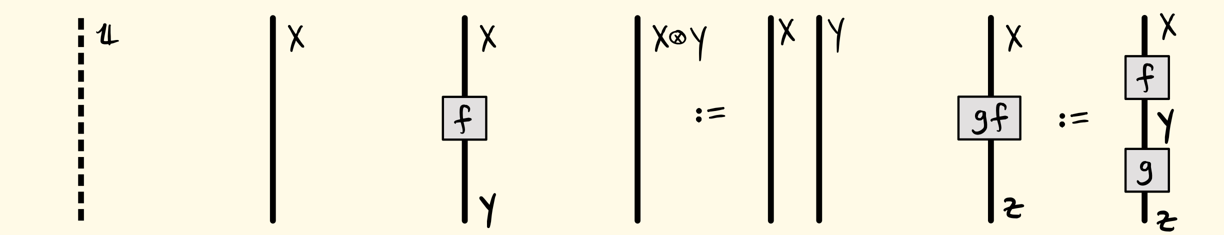
IV. RIGID CATEGORIES : IN STRICT CASE



IV. RIGID CATEGORIES : IN STRICT CASE

THESE ARE MONOIDAL CATEGORIES THAT CONTAIN <u>DUAL</u> STRUCTURES

GRAPHICAL CALCULUS IN THE STRICT MONOIDAL CATEGORY



MONOIDAL UNIT

OBJECT $X \in \mathcal{C}$

MORPHISM $f: X \to Y \in \mathcal{C}$

$\otimes$ OF OBJECTS $X, Y \in \mathcal{C}$

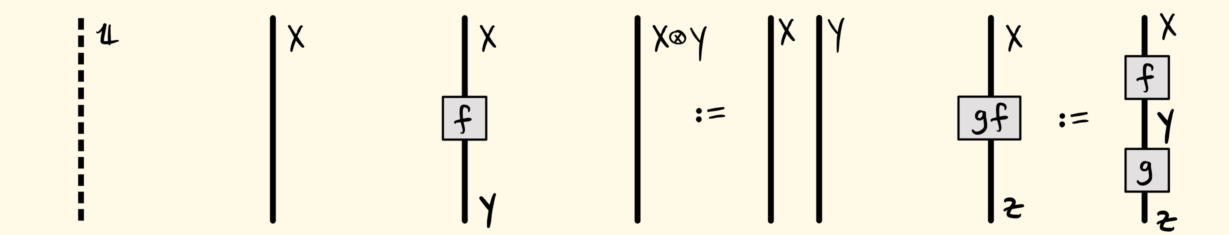
COMP'N OF MORPHISMS $f: X \to Y$, $g: Y \to Z \in \mathcal{C}$

IV. RIGID CATEGORIES : IN STRICT CASE

A LEFT DUAL OF AN OBJECT $X \in \mathcal{C}$ IS AN OBJECT $X^* \in \mathcal{C}$

EQUIPPED WITH MORPHISMS

GRAPHICAL CALCULUS IN THE STRICT MONOIDAL CATEGORY



MONOIDAL UNIT | OBJECT $X \in \mathcal{C}$ | MORPHISM $f: X \to Y \in \mathcal{C}$ | $\otimes$ OF OBJECTS $X, Y \in \mathcal{C}$ | COMP'N OF MORPHISMS $f: X \to Y$, $g: Y \to Z \in \mathcal{C}$

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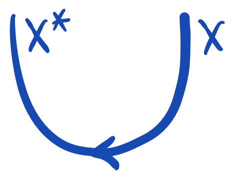
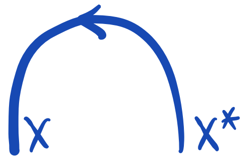




$\ni.$

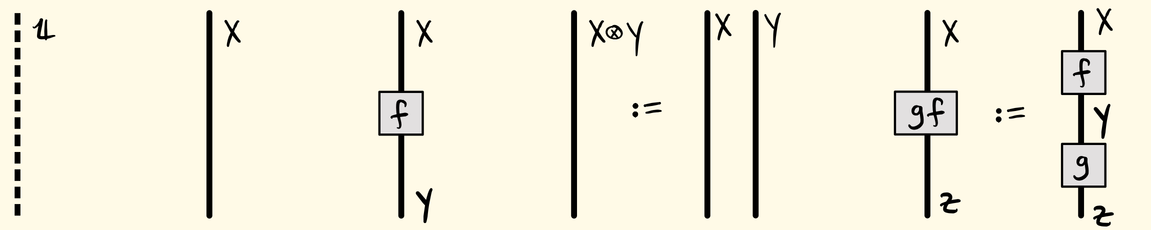
$ev^L_X : X^* \otimes X \longrightarrow \mathbb{1}$

LEFT EVALUATION

$coev^L_X : \mathbb{1} \longrightarrow X \otimes X^*$

LEFT COEVALUATION

GRAPHICAL CALCULUS IN THE STRICT MONOIDAL CATEGORY



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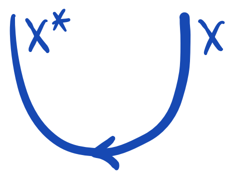

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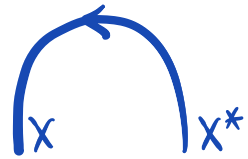

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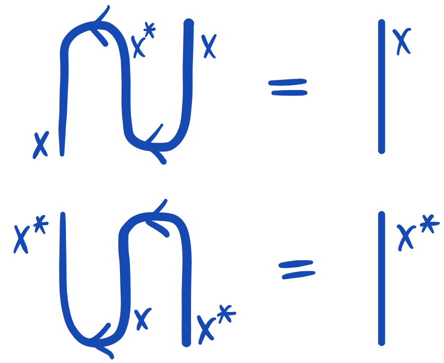
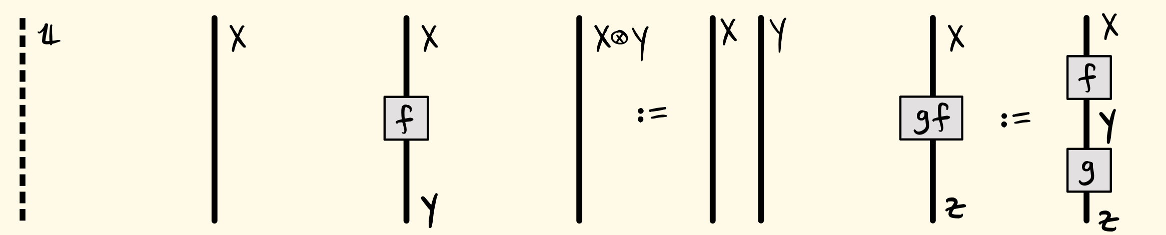


LEFT RIGIDITY AXIOMS

GRAPHICAL CALCULUS IN THE STRICT MONOIDAL CATEGORY



MONOIDAL UNIT

OBJECT $X \in \mathcal{C}$

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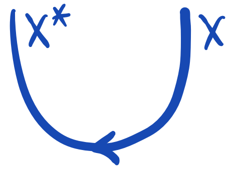
EQUIPPED WITH MORPHISMS



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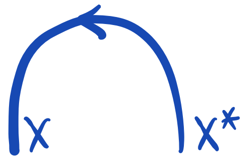

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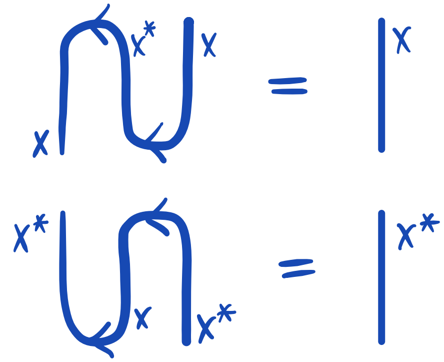


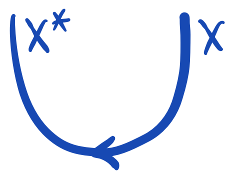
LEFT RIGIDITY AXIOMS

THE ARROWS AREN'T REALLY NECESSARY BUT THEY CAN BE HELPFUL

IV. RIGID CATEGORIES : IN STRICT CASE

A LEFT DUAL OF AN OBJECT $X \in \mathcal{C}$ IS AN OBJECT $X^* \in \mathcal{C}$

EQUIPPED WITH MORPHISMS



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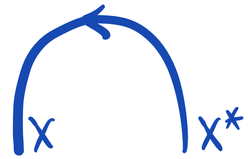

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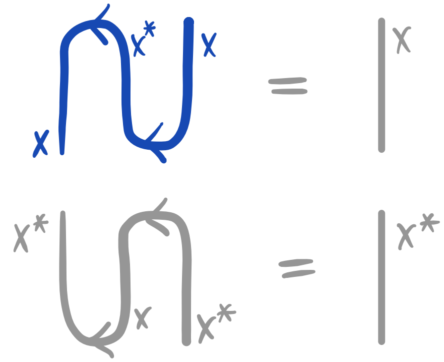
LEFT COEVALUATION

∋.



LEFT RIGIDITY AXIOMS

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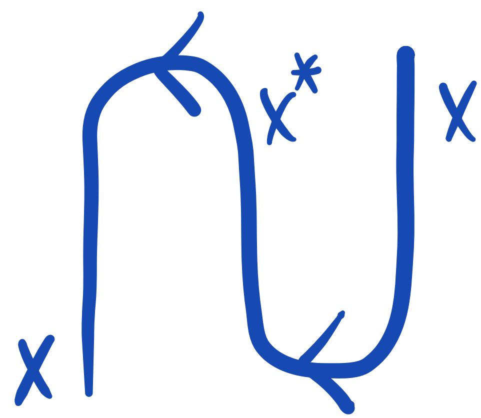

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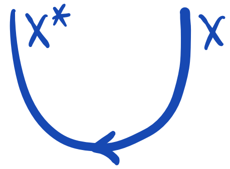

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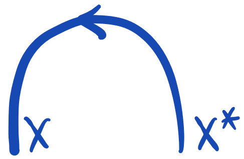

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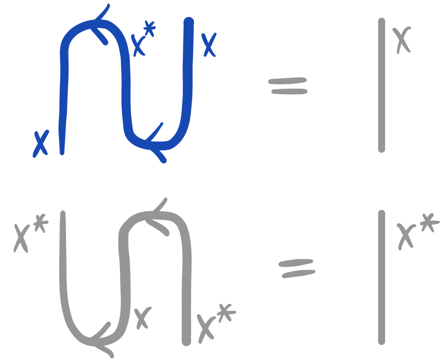

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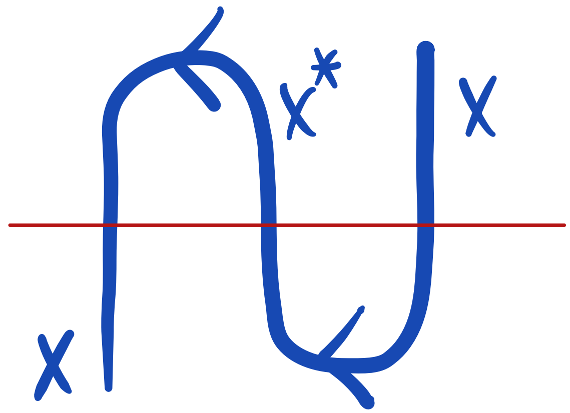

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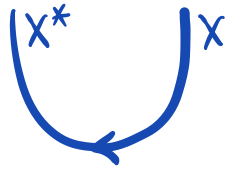
EQUIPPED WITH MORPHISMS



$ev^L_X : X^* \otimes X \longrightarrow \mathbb{1}$

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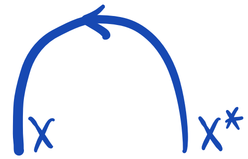

$coev^L_X : \mathbb{1} \longrightarrow X \otimes X^*$

LEFT COEVALUATION

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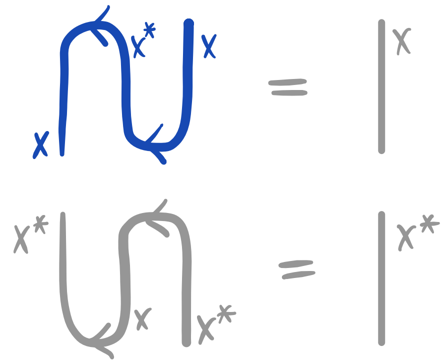
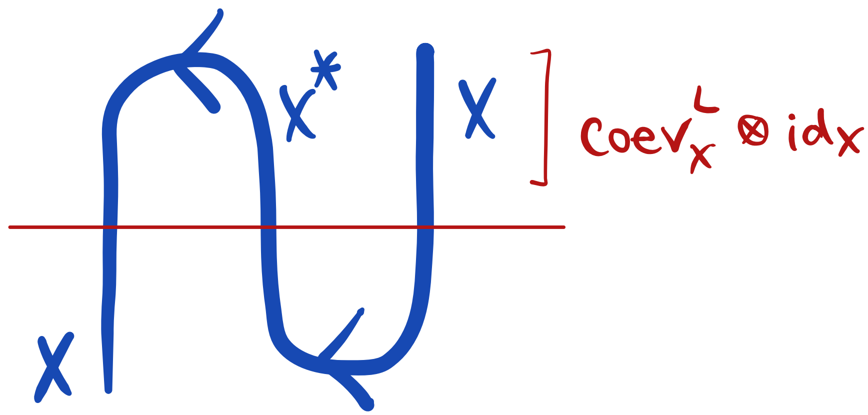


LEFT RIGIDITY AXIOMS

THE ARROWS AREN'T
REALLY NECESSARY
BUT THEY CAN BE HELPFUL



$$X \cong \mathbb{1} \otimes X \xrightarrow{coev^L_X \otimes id_X} X \otimes X^* \otimes X$$

IV. RIGID CATEGORIES : IN STRICT CASE

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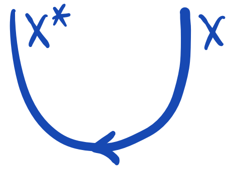

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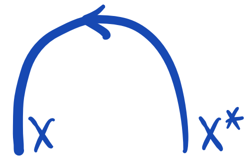

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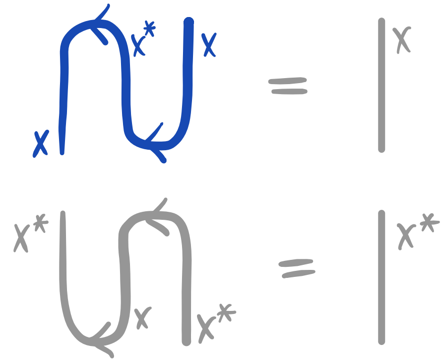

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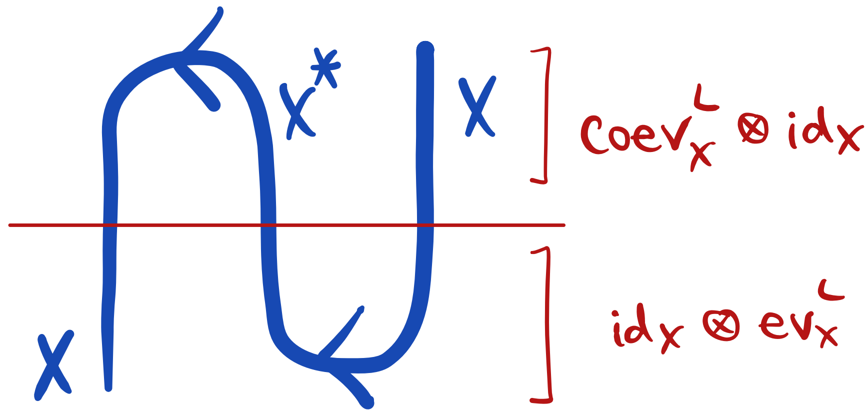

$$X \cong \mathbb{1} \otimes X \xrightarrow{coev^L_X \otimes id_X} X \otimes X^* \otimes X \xrightarrow{id_X \otimes ev^L_X} X \otimes \mathbb{1} \cong X$$

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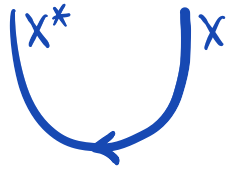

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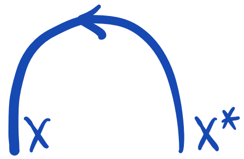

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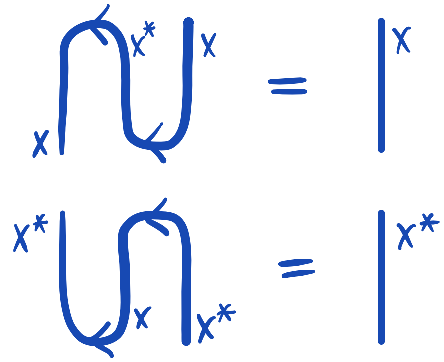

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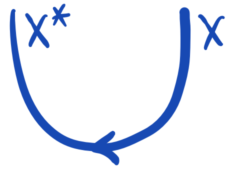

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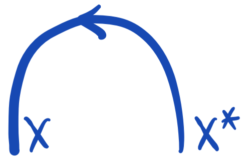

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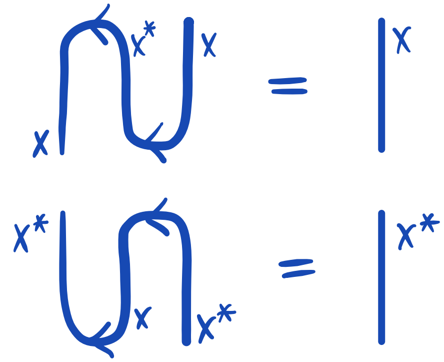

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GET LEFT DUALITY FUNCTOR

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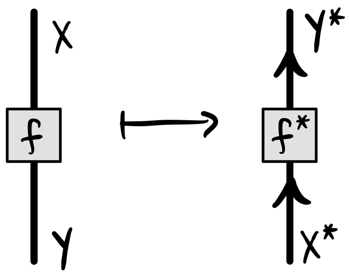

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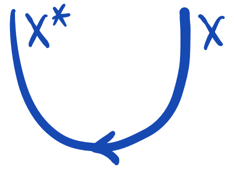

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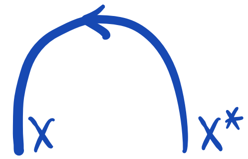

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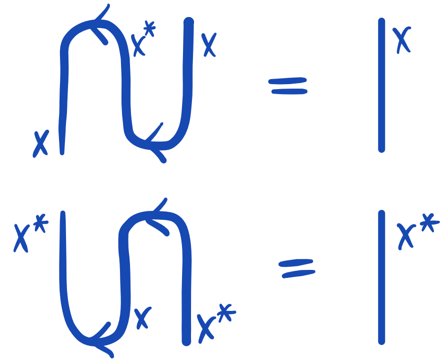

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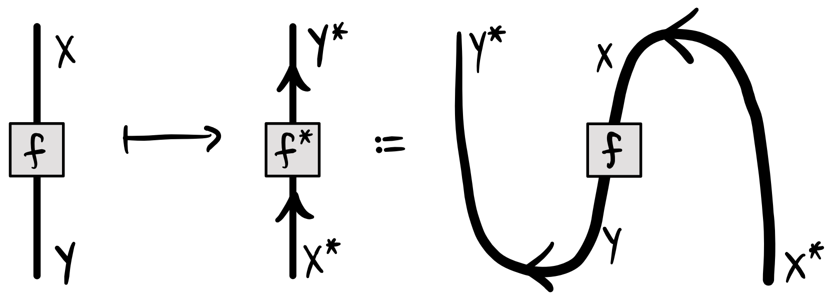

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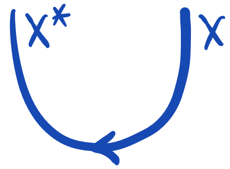

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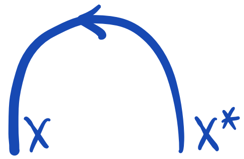

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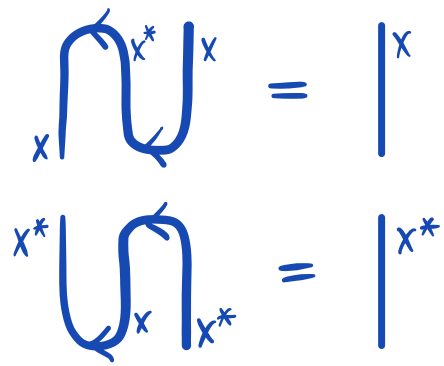

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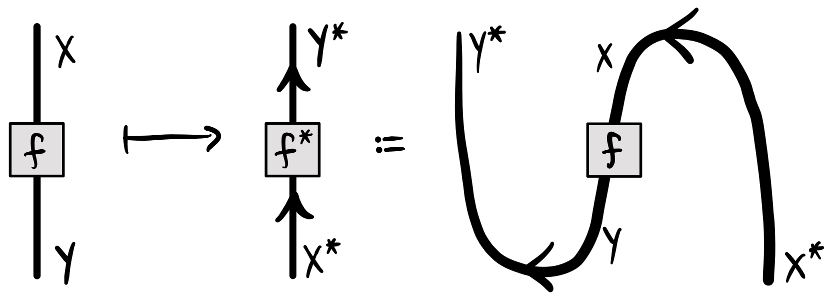

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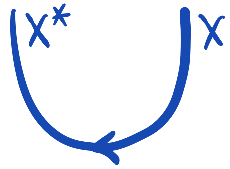

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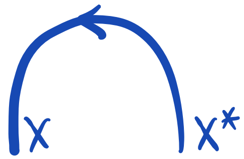

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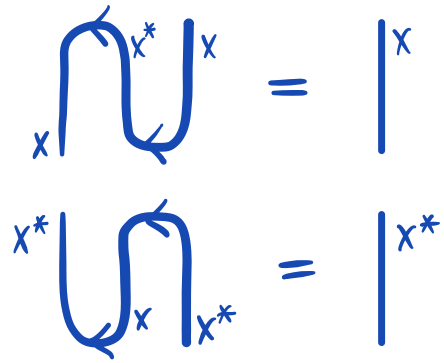

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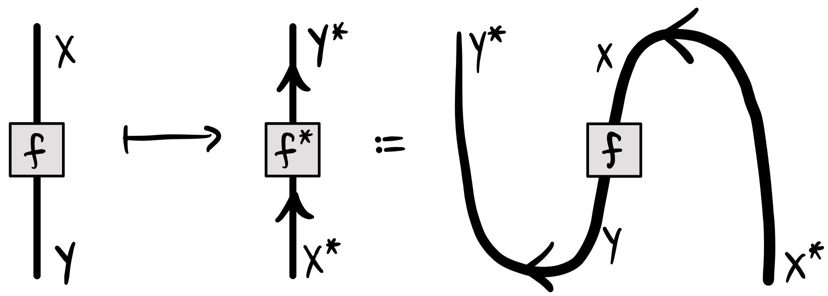

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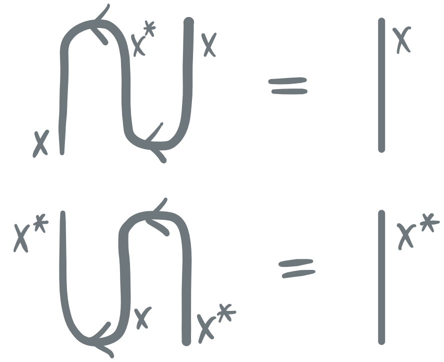

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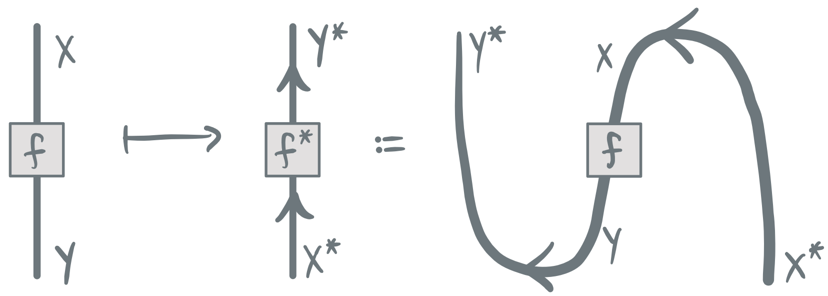

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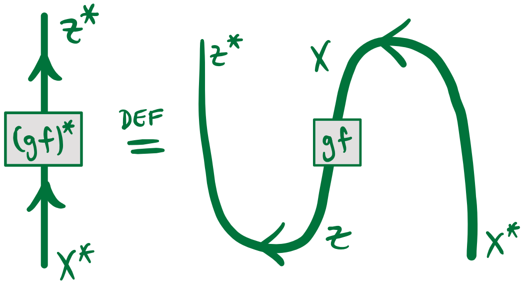

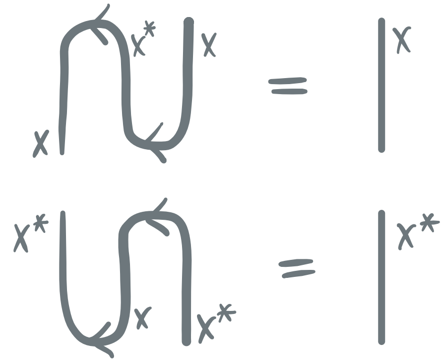

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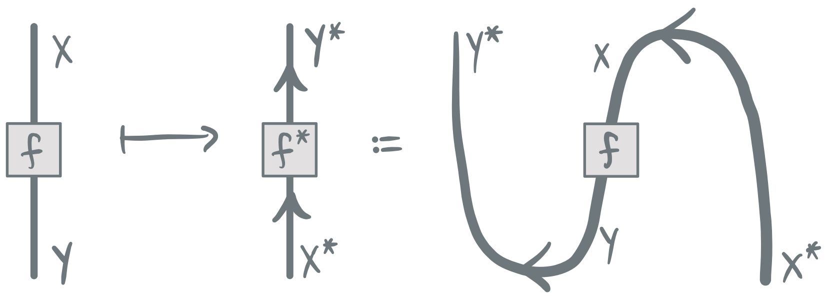

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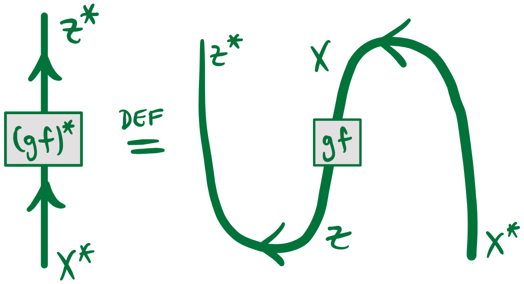

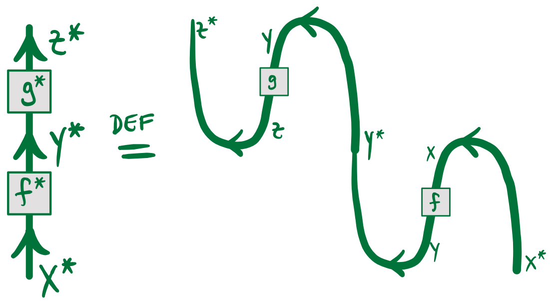

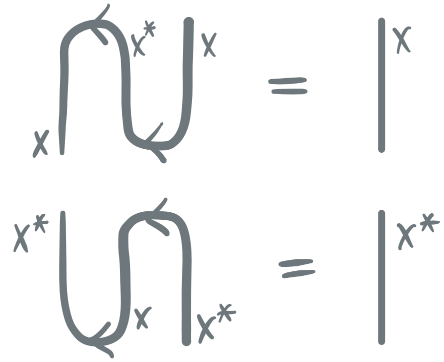

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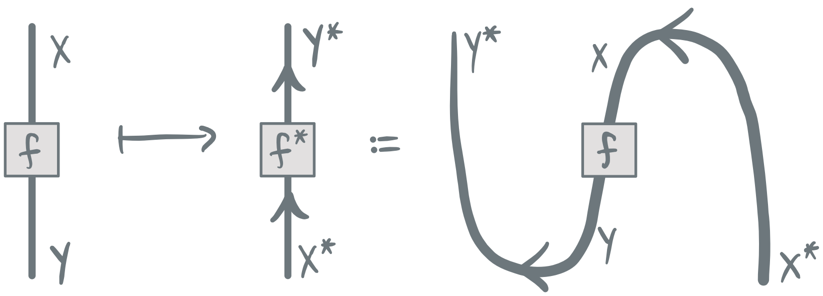

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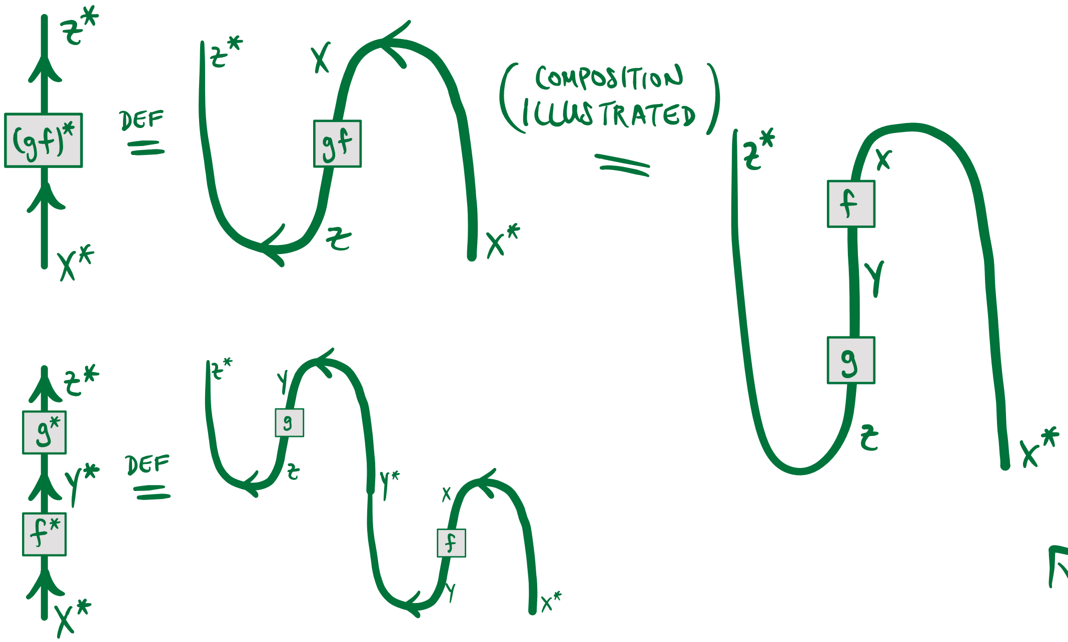

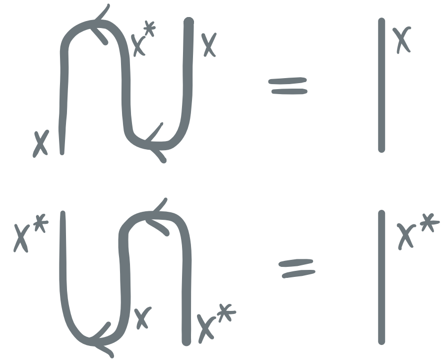

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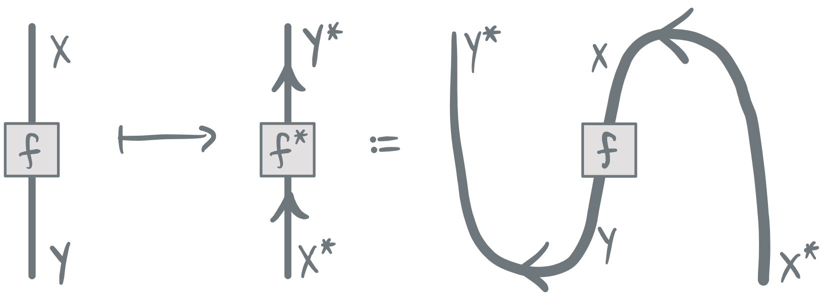

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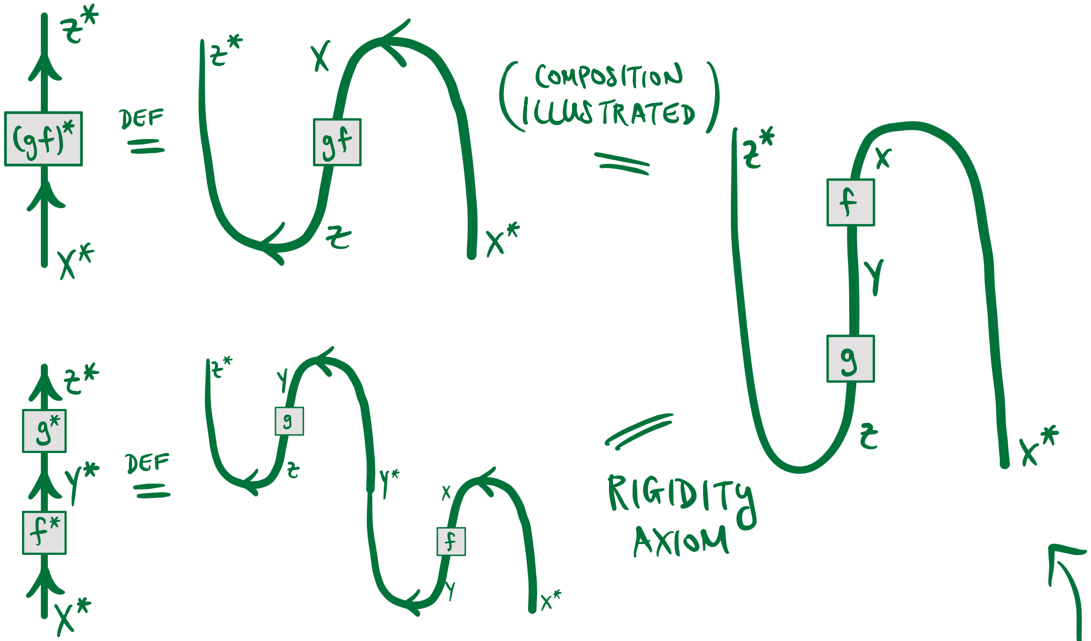

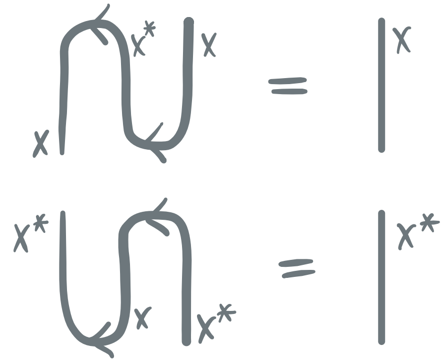

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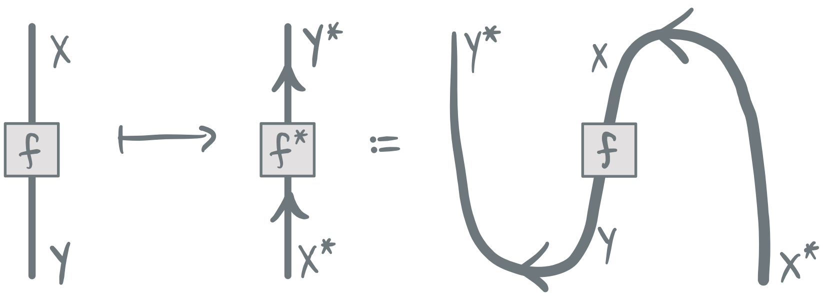

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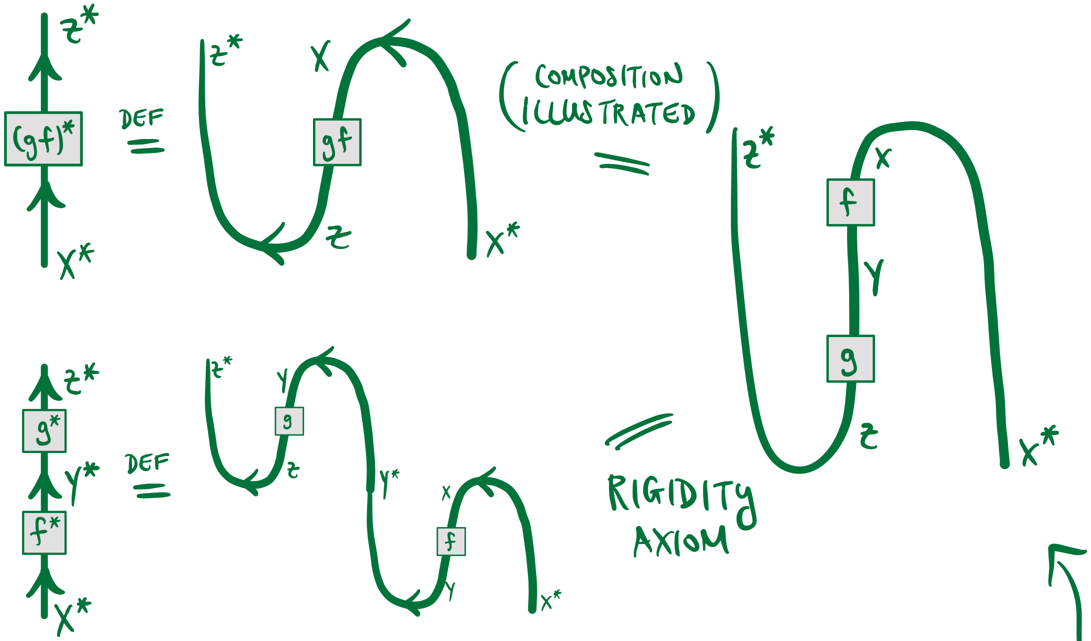

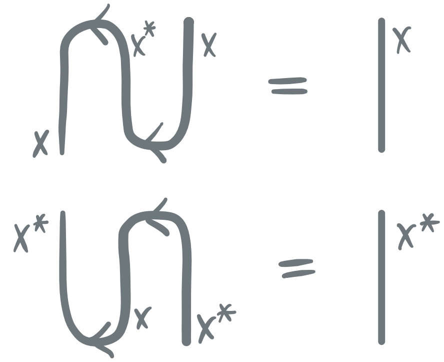

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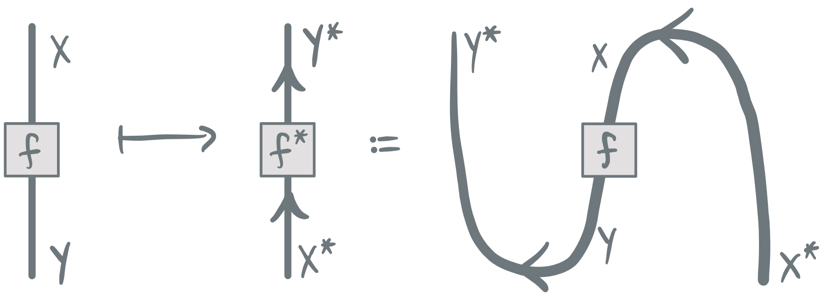

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MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #14

TOPICS:

I. STRICTIFICATION (§3.4.1)

II. COHERENCE (§3.4.3)

III. GRAPHICAL CALCULUS (§3.5)

MORE NEXT TIME

IV. RIGID CATEGORIES (§3.6)

Lecture #14 keywords: Coherence Theorem, dual of an object, graphical calculus, rigidity axioms, rigid category, rigid object, Strictification Theorem