MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #15

LAST TIME

- STRICTIFICATION
- COHERENCE
- GRAPHICAL CALCULUS
- RIGID CATEGORIES
  (JUST A SAMPLE IN THE STRICT CASE)

TOPICS:

I. RIGID CATEGORIES (§3.6)

II. PIVOTAL CATEGORIES (§3.7)

I. RIGID CATEGORIES

MONOIDAL CATEGORIES
CONTAINING
DUAL OBJECTS
& DUAL MORPHISMS

I. RIGID CATEGORIES

MONOIDAL CATEGORIES CONTAINING DUAL OBJECTS & DUAL MORPHISMS

FIX A MONOIDAL CATEGORY $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

AND AN OBJECT $X \in \mathcal{C}$

A LEFT DUAL OF X IS AN OBJECT $X^* \in \mathcal{C}$ EQUIPPED W/ MORPHISMS:

$$ev^L_X : X^* \otimes X \longrightarrow \mathbb{1} \qquad \text{AND} \qquad coev^L_X : \mathbb{1} \longrightarrow X \otimes X^*$$

SUCH THAT:

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X^* \arrow[dr, "r^{-1}_{X^*}"'] \arrow[rrrrr, "id"'] & & & & & X^* \\
 & X^* \otimes \mathbb{1} \arrow[dr, "id \otimes coev^L_X"'] & \circlearrowleft & & \mathbb{1} \otimes X^* \arrow[ur, "l_{X^*}"'] & \\
 & & X^* \otimes (X \otimes X^*) \arrow[r, "a^{-1}_{X^*,X,X^*}"'] & (X^* \otimes X) \otimes X^* \arrow[ur, "ev^L_X \otimes id"'] & &
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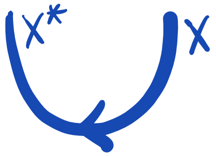
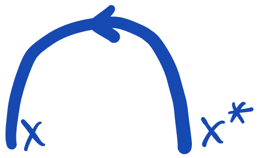
IN STRICT CASE:





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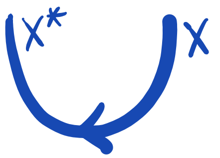

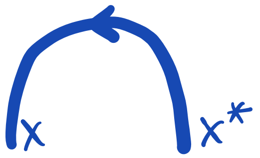

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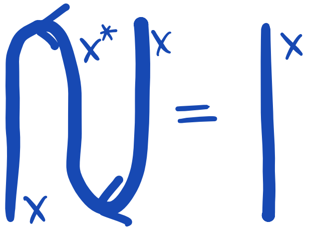

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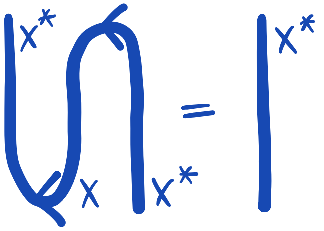

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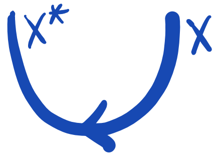
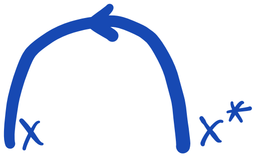


$$coev^L_X : \mathbb{1} \longrightarrow X \otimes X^*$$



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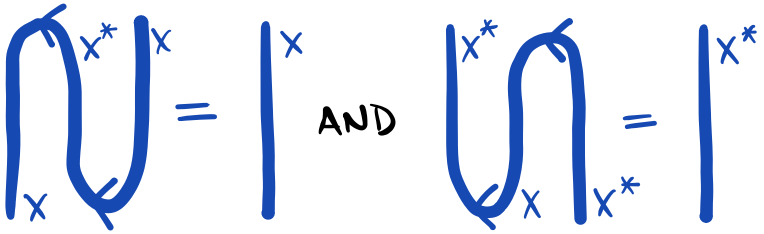

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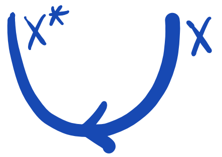

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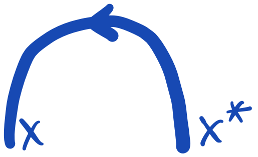

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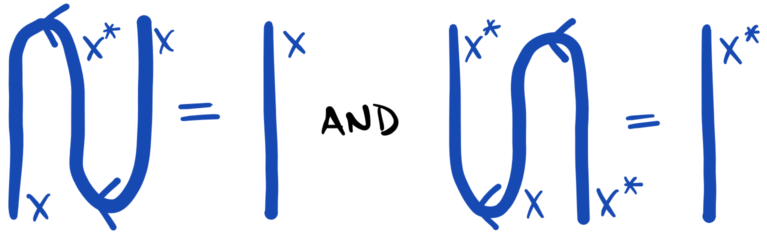

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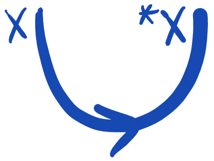

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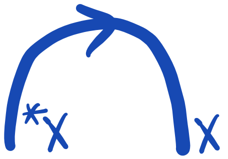

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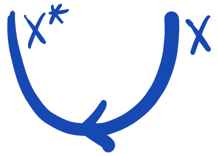

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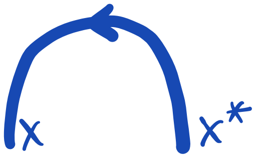

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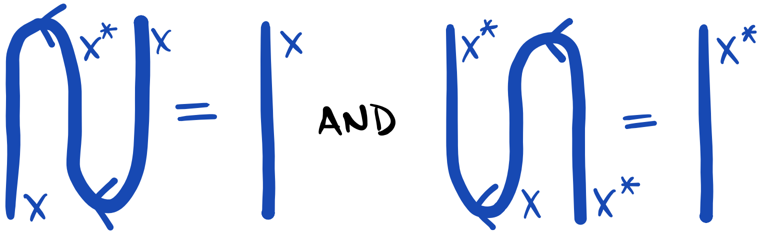

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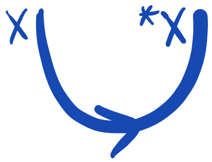

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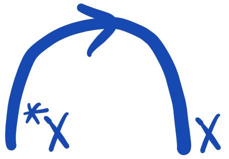

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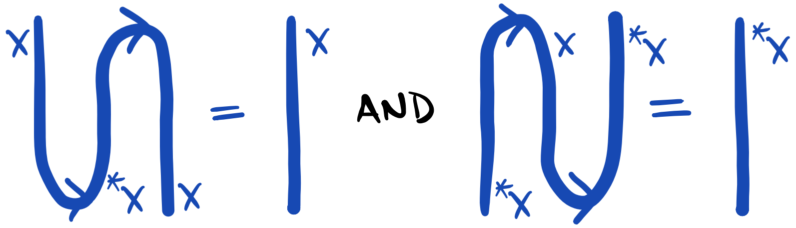

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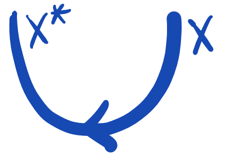

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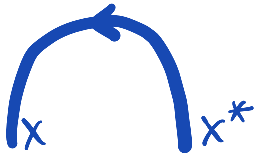

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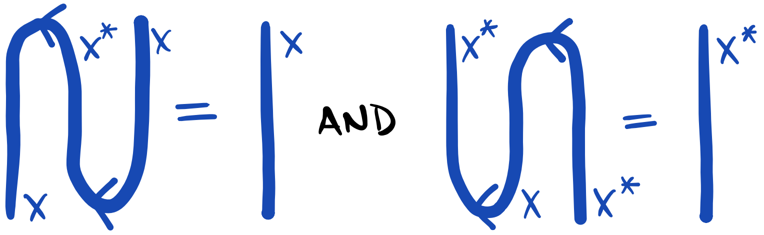

= LEFT RIGIDITY AXIOMS =

$\mathcal{C}$ IS

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IF

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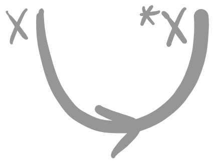

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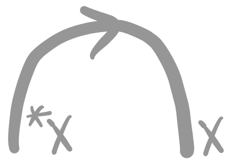

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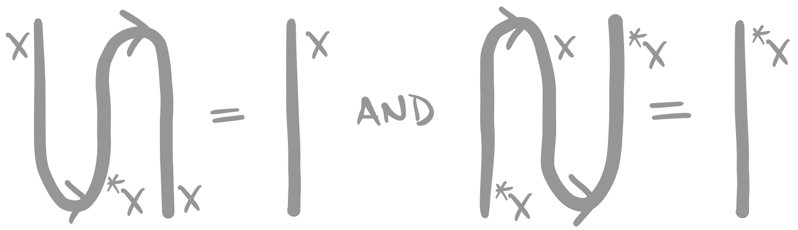

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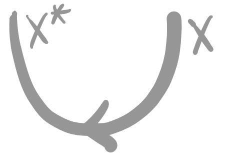

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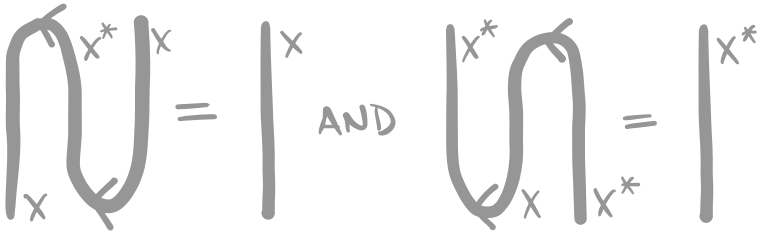

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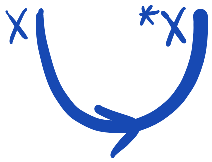

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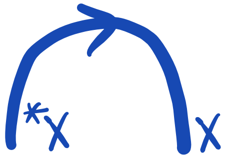

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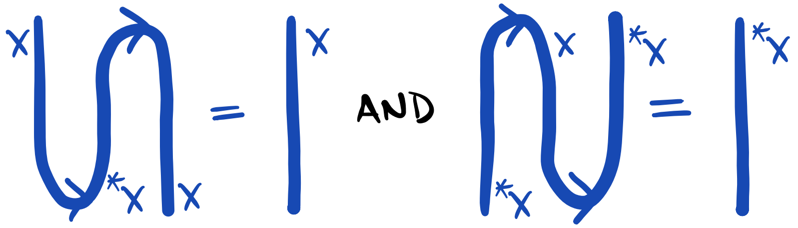

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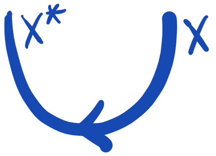

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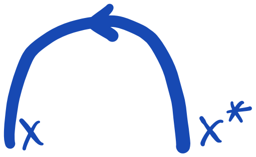

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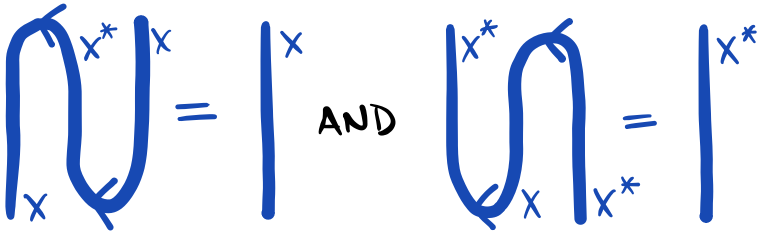

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$\mathcal{C}$ IS RIGID IF

∃ LEFT DUAL X^* & RIGHT DUAL ${}^*X$ $\forall X \in \mathcal{C}$

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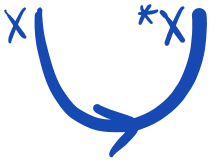
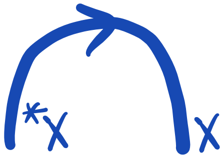
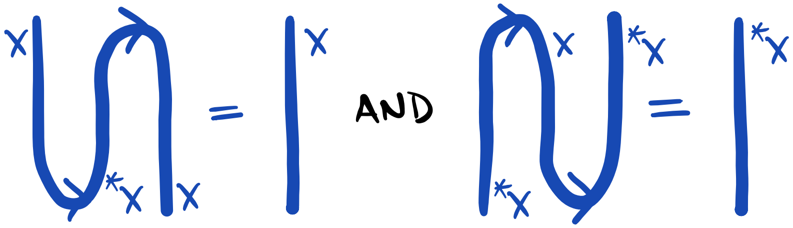


$coev^R_X : \mathbb{1} \longrightarrow {}^*X \otimes X$



SUCH THAT: IN THE STRICT CASE



= RIGHT RIGIDITY AXIOMS =

I. RIGID CATEGORIES

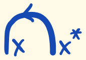
RIGID CATEGORY

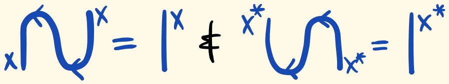
$$\Big(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-)\Big)$$

$\exists\, X^* \in \mathcal{C}$ WITH

$ev^L_X : X^* \otimes X \longrightarrow \mathbb{1}$

$coev^L_X : \mathbb{1} \longrightarrow X \otimes X^*$



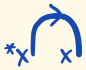


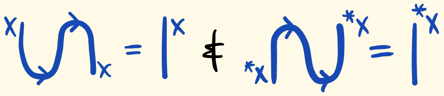
–AND–

$\exists\, {}^*X \in \mathcal{C}$ WITH

$ev^R_X : X \otimes {}^*X \longrightarrow \mathbb{1}$

$coev^R_X : \mathbb{1} \longrightarrow {}^*X \otimes X$





$\forall\, X \in \mathcal{C}$

I. RIGID CATEGORIES

RIGID CATEGORY

$(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-))$

$\exists X^* \in \mathcal{C}$ WITH

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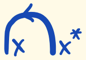
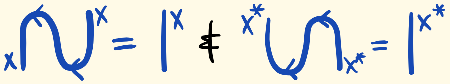
$coev_X^L : \mathbb{1} \longrightarrow X \otimes X^*$ 



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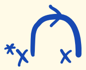
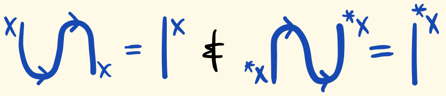
$coev_X^R : \mathbb{1} \longrightarrow {}^*X \otimes X$ 



$\forall X \in \mathcal{C}$

EXERCISE 3.19

LEFT (& RIGHT) DUALS ARE UNIQUE UP TO ISO:

I. RIGID CATEGORIES

RIGID CATEGORY

$$\left(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-)\right)$$

$\exists\, X^* \in \mathcal{C}$ WITH

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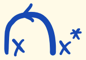

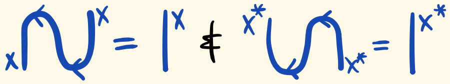

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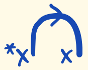
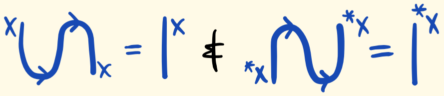
$coev_X^R : \mathbb{1} \longrightarrow {}^*X \otimes X$





$\forall\, X \in \mathcal{C}$

EXERCISE 3.19

LEFT (& RIGHT) DUALS ARE UNIQUE UP TO ISO:

IF $\exists\, Y_1 \quad \in \mathcal{C}$

$e_1 : Y_1 \otimes X \longrightarrow \mathbb{1}$

$c_1 : \mathbb{1} \longrightarrow X \otimes Y_1 \qquad \in \mathcal{C}$

∍. (Y_1, e_1, c_1)

SATISFY LEFT RIGIDITY AXIOMS

I. RIGID CATEGORIES

RIGID CATEGORY

$(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-))$

$\exists X^* \in \mathcal{C}$ WITH

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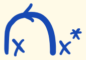

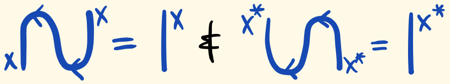

—AND—

$\exists {}^*X \in \mathcal{C}$ WITH

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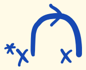

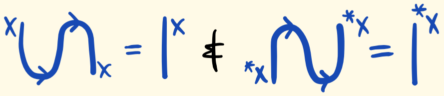

$\forall X \in \mathcal{C}$

EXERCISE 3.19

LEFT (& RIGHT) DUALS ARE UNIQUE UP TO ISO:

IF $\exists Y_1, Y_2 \in \mathcal{C}$ &

$e_1 : Y_1 \otimes X \longrightarrow \mathbb{1}$, $e_2 : Y_2 \otimes X \longrightarrow \mathbb{1}$,

$c_1 : \mathbb{1} \longrightarrow X \otimes Y_1$, $c_2 : \mathbb{1} \longrightarrow X \otimes Y_2$ $\in \mathcal{C}$

∍. (Y_1, e_1, c_1) & (Y_2, e_2, c_2)

SATISFY LEFT RIGIDITY AXIOMS

I. RIGID CATEGORIES

RIGID CATEGORY

$$\Big(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-)\Big)$$

$\exists X^* \in \mathcal{C}$ WITH

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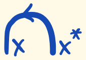
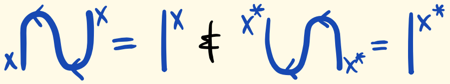
$coev^L_X : \mathbb{1} \to X \otimes X^*$ 



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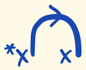
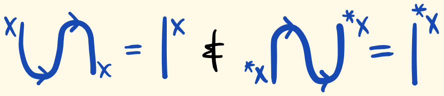
$coev^R_X : \mathbb{1} \to {}^*X \otimes X$ 



$\forall X \in \mathcal{C}$

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$$e_1 : Y_1 \otimes X \to \mathbb{1}, \quad e_2 : Y_2 \otimes X \to \mathbb{1},$$

$$c_1 : \mathbb{1} \to X \otimes Y_1, \quad c_2 : \mathbb{1} \to X \otimes Y_2 \quad \in \mathcal{C}$$

$\ni.$ (Y_1, e_1, c_1) & (Y_2, e_2, c_2)

SATISFY LEFT RIGIDITY AXIOMS

THEN $\exists!$ ISO $f : Y_1 \xrightarrow{\sim} Y_2$ WITH

$$\begin{tikzcd}
Y_1 \otimes X \arrow[dr, "e_1"] \arrow[dd, "f \otimes id_X"'] & \\
\arrow[r, phantom, "\circlearrowright"] & \mathbb{1} \\
Y_2 \otimes X \arrow[ur, "e_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \otimes Y_1 \arrow[dd, "id_X \otimes f"] \\
\mathbb{1} \arrow[ur, "c_1"] \arrow[dr, "c_2"'] \arrow[r, phantom, "\circlearrowright"] & \\
& X \otimes Y_2
\end{tikzcd}$$

I. RIGID CATEGORIES

RIGID CATEGORY

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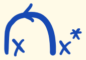

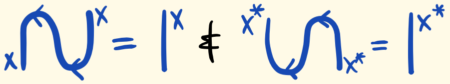

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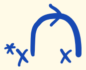

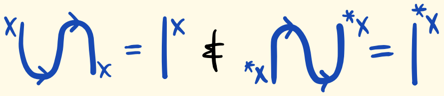

$\forall X \in \mathcal{C}$

EXERCISE 3.20 WE GET $\mathbb{1}^* = \mathbb{1} = {}^*\mathbb{1}$.

I. RIGID CATEGORIES

RIGID CATEGORY

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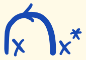

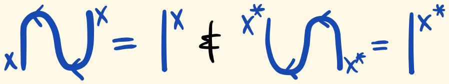

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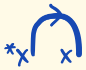

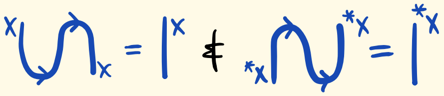

$\forall X \in \mathcal{C}$

EXERCISE 3.20 WE GET $\mathbb{1}^* = \mathbb{1} = {}^*\mathbb{1}$.

≡ LET'S VERIFY THIS ≡
ON THE BOARD

I. RIGID CATEGORIES

RIGID CATEGORY

$$(\underbrace{\mathcal{C},\otimes,\mathbb{1},a,l,r}_{\text{MONOIDAL}},(-)^*,{}^*(-))$$

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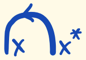

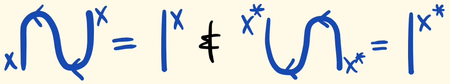

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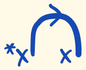

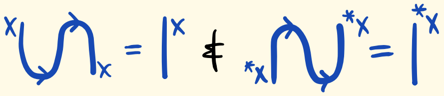

$\forall X \in \mathcal{C}$

EXERCISE 3.20 WE GET $\mathbb{1}^* = \mathbb{1} = {}^*\mathbb{1}$.

NEED $ev_{\mathbb{1}}^L : \overset{\mathbb{1}^*}{\overset{\shortparallel}{\mathbb{1}}} \otimes \mathbb{1} \longrightarrow \mathbb{1}$ $\quad coev_{\mathbb{1}}^L : \mathbb{1} \longrightarrow \mathbb{1}\otimes \overset{\mathbb{1}^*}{\overset{\shortparallel}{\mathbb{1}}}$

SUCH THAT

$$\begin{tikzcd}
& & (X\otimes X^*)\otimes X \arrow[r, "a_{X,X^*,X}"] & X\otimes(X^*\otimes X) \arrow[dr, "id\otimes ev_X^L"] & & \\
& \mathbb{1}\otimes X \arrow[ur, "coev_X^L\otimes id"] & 2 & & X\otimes\mathbb{1} \arrow[dr, "r_X"] & \\
X \arrow[ur, "l_X^{-1}"] \arrow[rrrrr, "id"] & & & & & X
\end{tikzcd}$$

≡ LEFT RIGIDITY AXIOMS HOLD ≡

$$\begin{tikzcd}
X^* \arrow[rrrrr, "id"'] \arrow[dr, "r_{X^*}^{-1}"'] & & & & & X^* \\
& X^*\otimes\mathbb{1} \arrow[dr, "id\otimes coev_X^L"'] & & 2 & \mathbb{1}\otimes X^* \arrow[ur, "l_{X^*}"'] & \\
& & X^*\otimes(X\otimes X^*) \arrow[r, "a^{-1}_{X^*,X,X^*}"'] & (X^*\otimes X)\otimes X^* \arrow[ur, "ev_X^L\otimes id"'] & &
\end{tikzcd}$$

I. RIGID CATEGORIES

RIGID CATEGORY

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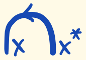

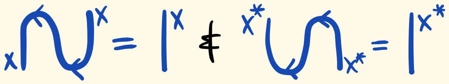

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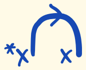

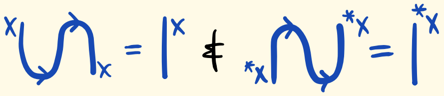

$\forall X \in \mathcal{C}$

EXERCISE 3.20

WE GET $\mathbb{1}^* = \mathbb{1} = {}^*\mathbb{1}$.

NEED

$$\mathrm{ev}^L_{\mathbb{1}} : \overset{\mathbb{1}^*}{\overset{\|}{\mathbb{1}}} \otimes \mathbb{1} \longrightarrow \mathbb{1} \qquad \mathrm{coev}^L_{\mathbb{1}} : \mathbb{1} \longrightarrow \mathbb{1} \otimes \overset{\mathbb{1}^*}{\overset{\|}{\mathbb{1}}}$$

SUCH THAT

$$\begin{tikzcd}
 & & (\mathbb{1} \otimes \mathbb{1}^*) \otimes \mathbb{1} \arrow[r, "a_{\mathbb{1},\mathbb{1}^*,\mathbb{1}}"] & \mathbb{1} \otimes (\mathbb{1}^* \otimes \mathbb{1}) \arrow[dr, "\mathrm{id} \otimes \mathrm{ev}^L_{\mathbb{1}}"] & & \\
 & \mathbb{1} \otimes \mathbb{1} \arrow[ur, "\mathrm{coev}^L_{\mathbb{1}} \otimes \mathrm{id}"] & & \arrow[l, phantom, "\circlearrowright"] & \mathbb{1} \otimes \mathbb{1} \arrow[dr, "r_{\mathbb{1}}"] & \\
\mathbb{1} \arrow[ur, "l^{-1}_{\mathbb{1}}"] \arrow[rrrrr, "\mathrm{id}_{\mathbb{1}}"] & & & & & \mathbb{1}
\end{tikzcd}$$

≡ LEFT RIGIDITY AXIOMS HOLD ≡

$$\begin{tikzcd}
\mathbb{1}^* \arrow[rrrrr, "\mathrm{id}_{\mathbb{1}^*}"'] \arrow[dr, "r^{-1}_{\mathbb{1}^*}"'] & & & & & \mathbb{1}^* \\
 & \mathbb{1}^* \otimes \mathbb{1} \arrow[dr, "\mathrm{id} \otimes \mathrm{coev}^L_{\mathbb{1}}"'] & & \arrow[l, phantom, "\circlearrowright"] & \mathbb{1} \otimes \mathbb{1}^* \arrow[ur, "l_{\mathbb{1}^*}"'] & \\
 & & \mathbb{1}^* \otimes (\mathbb{1} \otimes \mathbb{1}^*) \arrow[r, "a^{-1}_{\mathbb{1}^*,\mathbb{1},\mathbb{1}^*}"'] & (\mathbb{1}^* \otimes \mathbb{1}) \otimes \mathbb{1}^* \arrow[ur, "\mathrm{ev}^L \otimes \mathrm{id}_{\mathbb{1}}"'] & &
\end{tikzcd}$$

I. RIGID CATEGORIES

RIGID CATEGORY

$$\big(\underbrace{\mathcal{C},\otimes,\mathbb{1},a,l,r}_{\text{MONOIDAL}},(-)^*,{}^*(-)\big)$$

$\exists\, X^* \in \mathcal{C}$ WITH

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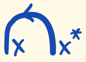
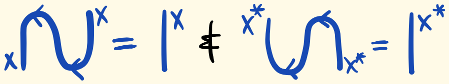


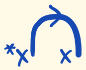


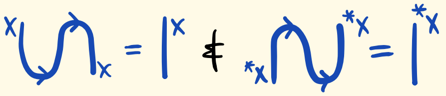
–AND–

$\exists\, {}^*X \in \mathcal{C}$ WITH

$\mathrm{ev}^R_X : X\otimes {}^*X \longrightarrow \mathbb{1}$

$\mathrm{coev}^R_X : \mathbb{1} \longrightarrow {}^*X\otimes X$





$\forall\, X \in \mathcal{C}$

EXERCISE 3.20 WE GET $\mathbb{1}^* = \mathbb{1} = {}^*\mathbb{1}$.

NEED

$$\mathrm{ev}^L_{\mathbb{1}} : \overset{\mathbb{1}^*}{\overset{\shortparallel}{\mathbb{1}}}\otimes \mathbb{1} \longrightarrow \mathbb{1} \qquad \mathrm{coev}^L_{\mathbb{1}} : \mathbb{1} \longrightarrow \mathbb{1}\otimes \overset{\mathbb{1}^*}{\overset{\shortparallel}{\mathbb{1}}}$$

SUCH THAT

$$\begin{tikzcd}
 & (\mathbb{1}\otimes\mathbb{1}^*)\otimes\mathbb{1} \arrow[r, "a_{\mathbb{1},\mathbb{1}^*,\mathbb{1}}"] & \mathbb{1}\otimes(\mathbb{1}^*\otimes\mathbb{1}) \arrow[dr, "\mathrm{id}\otimes\mathrm{ev}^L_{\mathbb{1}}"] & \\
\mathbb{1}\otimes\mathbb{1} \arrow[ur, "\mathrm{coev}^L_{\mathbb{1}}\otimes\mathrm{id}"] \arrow[rrr, phantom, "\circlearrowleft"] & & & \mathbb{1}\otimes\mathbb{1} \arrow[d, "r_{\mathbb{1}}"] \\
\mathbb{1} \arrow[u, "l^{-1}_{\mathbb{1}}"] \arrow[rrr, "\mathrm{id}_{\mathbb{1}}"] & & & \mathbb{1}
\end{tikzcd}$$

≡ LEFT RIGIDITY AXIOMS HOLD ≡

$$\begin{tikzcd}
\mathbb{1}^* \arrow[rrr, "\mathrm{id}_{\mathbb{1}^*}"'] \arrow[d, "r^{-1}_{\mathbb{1}^*}"'] & & & \mathbb{1}^* \\
\mathbb{1}^*\otimes\mathbb{1} \arrow[dr, "\mathrm{id}\otimes\mathrm{coev}^L_{\mathbb{1}}"'] \arrow[rrr, phantom, "\circlearrowleft"] & & & \mathbb{1}\otimes\mathbb{1}^* \arrow[u, "l_{\mathbb{1}^*}"'] \\
 & \mathbb{1}^*\otimes(\mathbb{1}\otimes\mathbb{1}^*) \arrow[r, "a^{-1}_{\mathbb{1}^*,\mathbb{1},\mathbb{1}^*}"'] & (\mathbb{1}^*\otimes\mathbb{1})\otimes\mathbb{1}^* \arrow[ur, "\mathrm{ev}^L\otimes\mathrm{id}_{\mathbb{1}}"'] &
\end{tikzcd}$$

I. RIGID CATEGORIES

RIGID CATEGORY

$$\Big(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-)\Big)$$

$\exists X^* \in \mathcal{C}$ WITH

$ev_X^L : X^* \otimes X \to \mathbb{1}$

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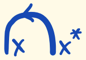

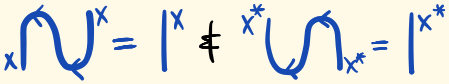

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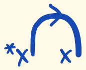

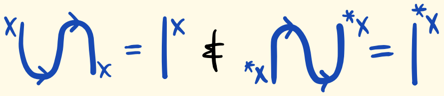

$\forall X \in \mathcal{C}$

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NEED

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 & \mathbb{1}^* \otimes (\mathbb{1} \otimes \mathbb{1}^*) \arrow[r, "a_{\mathbb{1}^*,\mathbb{1},\mathbb{1}^*}^{-1}"'] & (\mathbb{1}^* \otimes \mathbb{1}) \otimes \mathbb{1}^* \arrow[ur, "ev^L \otimes id_{\mathbb{1}}"'] &
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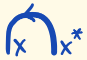
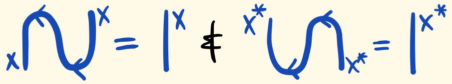
$coev_X^L : \mathbb{1} \longrightarrow X \otimes X^*$ 



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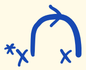
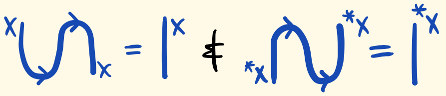
$coev_X^R : \mathbb{1} \longrightarrow {}^*X \otimes X$ 



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\mathbb{1}\otimes\mathbb{1} \arrow[dr, "id\otimes coev^L_{\mathbb{1}}"'] & & & \mathbb{1}\otimes\mathbb{1} \arrow[u, "l_{\mathbb{1}}"'] \\
& \mathbb{1}\otimes(\mathbb{1}\otimes\mathbb{1}) \arrow[r, "a^{-1}_{\mathbb{1},\mathbb{1},\mathbb{1}}"'] & (\mathbb{1}\otimes\mathbb{1})\otimes\mathbb{1} \arrow[ur, "ev^L\otimes id_{\mathbb{1}}"'] &
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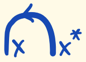

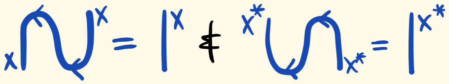

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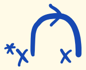

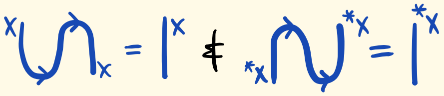

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EXERCISE 3.20 WE GET $\mathbb{1}^* = \mathbb{1} = {}^*\mathbb{1}$.

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 & \mathbb{1}\otimes\mathbb{1} \arrow[ur, "coev^L_{\mathbb{1}}\otimes\mathrm{id}"] & & & \mathbb{1}\otimes\mathbb{1} \arrow[dr, "r_{\mathbb{1}}"] & \\
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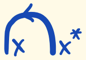

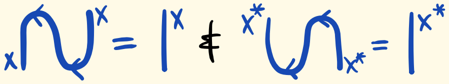

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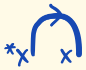

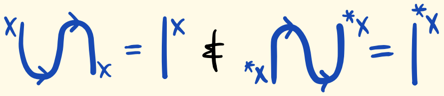

$\forall X \in \mathcal{C}$

<u>EXERCISE 3.20</u> WE GET $\mathbb{1}^* = \mathbb{1} = {}^*\mathbb{1}$.

NEED

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SUCH THAT

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 & \mathbb{1} \otimes \mathbb{1} \arrow[ur, "\overbrace{\mathrm{coev}^L_{\mathbb{1}}}^{r_{\mathbb{1}}^{-1}} \otimes \mathrm{id}"] \arrow[rrr, "\mathrm{id}"] & {} & & \mathbb{1} \otimes \mathbb{1} \arrow[dr, "r_{\mathbb{1}}"] & \\
\mathbb{1} \arrow[ur, "l_{\mathbb{1}}^{-1}"] \arrow[rrrrr, "\mathrm{id}_{\mathbb{1}}"] & & & & & \mathbb{1}
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I. RIGID CATEGORIES

RIGID CATEGORY

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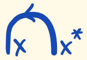
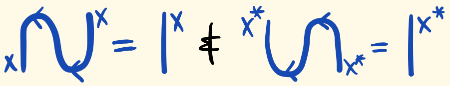
$coev_X^L : \mathbb{1} \to X \otimes X^*$ 



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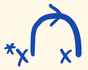
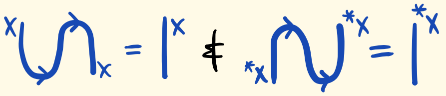
$coev_X^R : \mathbb{1} \to {}^*X \otimes X$ 



$\forall X \in \mathcal{C}$

EXERCISE 3.20 WE GET $\mathbb{1}^* = \mathbb{1} = {}^*\mathbb{1}$.

NEED $\therefore$ DEFINE

$$ev^L_{\mathbb{1}} : \overset{\mathbb{1}^*}{\overset{\|}{\mathbb{1}}} \otimes \mathbb{1} \xrightarrow{l_{\mathbb{1}}} \mathbb{1} \qquad coev^L_{\mathbb{1}} : \mathbb{1} \xrightarrow{r_{\mathbb{1}}^{-1}} \mathbb{1} \otimes \overset{\mathbb{1}^*}{\overset{\|}{\mathbb{1}}}$$

SUCH THAT

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 & (\mathbb{1} \otimes \mathbb{1}) \otimes \mathbb{1} \arrow[r, "a_{\mathbb{1},\mathbb{1},\mathbb{1}}"] \arrow[from=1-2, to=2-3, phantom, "\wr \text{ TRIANGLE AXIOM}"] & \mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1}) \arrow[dr, "id \otimes \overbrace{ev^L_{\mathbb{1}}}^{l_{\mathbb{1}}}"] & \\
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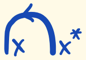

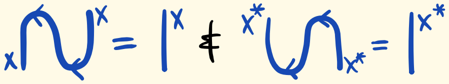

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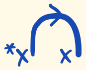

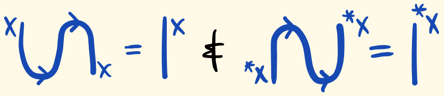

$\forall X\in\mathcal{C}$

EXERCISE 3.20 WE GET $\mathbb{1}^* = \mathbb{1} = {}^*\mathbb{1}$.

∴ DEFINE

NEED $ev^L_{\mathbb{1}} : \overset{\mathbb{1}^*}{\overset{\parallel}{\mathbb{1}}}\otimes\mathbb{1} \xrightarrow{l_{\mathbb{1}}} \mathbb{1}$ $\quad coev^L_{\mathbb{1}} : \mathbb{1} \xrightarrow{r^{-1}_{\mathbb{1}}} \mathbb{1}\otimes\overset{\mathbb{1}^*}{\overset{\parallel}{\mathbb{1}}}$

SUCH THAT ✱ NOW STS $l_{\mathbb{1}} = r_{\mathbb{1}}$

$$\begin{tikzcd}
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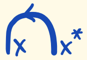
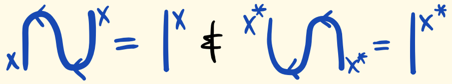
$coev^L_X : \mathbb{1} \to X \otimes X^*$ 



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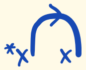
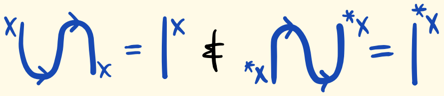
$coev^R_X : \mathbb{1} \to {}^*X \otimes X$ 



$\forall X \in \mathcal{C}$

EXERCISE 3.20

WE GET $\mathbb{1}^* = \mathbb{1} = {}^*\mathbb{1}$.

NEED $\quad \therefore$ DEFINE

$$\overset{\mathbb{1}^* \atop \|}{ev^L_{\mathbb{1}}} : \mathbb{1} \otimes \mathbb{1} \xrightarrow{l_{\mathbb{1}}} \mathbb{1} \qquad coev^L_{\mathbb{1}} : \mathbb{1} \xrightarrow{r_{\mathbb{1}}^{-1}} \mathbb{1} \otimes \overset{\mathbb{1}^* \atop \|}{\mathbb{1}}$$

SUCH THAT

$*$ NOW STS $l_{\mathbb{1}} = r_{\mathbb{1}}$ (EXER. 3.1)

$$\begin{tikzcd}
 & & (\mathbb{1} \otimes \mathbb{1}) \otimes \mathbb{1} \arrow[r, "a_{\mathbb{1},\mathbb{1},\mathbb{1}}"] & \mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1}) \arrow[dr, "id \otimes \overbrace{ev^L_{\mathbb{1}}}^{l_{\mathbb{1}}}"] & & \\
 & \mathbb{1} \otimes \mathbb{1} \arrow[ur, "\overbrace{coev^L_{\mathbb{1}}}^{r_{\mathbb{1}}^{-1}} \otimes id"] \arrow[rrr, "id"'] \arrow[rrr, phantom, bend left=30, "\circlearrowright \ \text{TRIANGLE AXIOM}"] & & & \mathbb{1} \otimes \mathbb{1} \arrow[dr, "r_{\mathbb{1}}"] & \\
\mathbb{1} \arrow[ur, "l_{\mathbb{1}}^{-1}"] \arrow[rrrrr, "id_{\mathbb{1}}"'] \arrow[rrrrr, phantom, bend left=15, "* \qquad \circlearrowright"] & & & & & \mathbb{1}
\end{tikzcd}$$

I. RIGID CATEGORIES

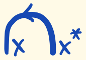
RIGID CATEGORY

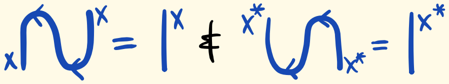
$(\underbrace{\mathcal{C},\otimes,\mathbb{1},a,l,r}_{\text{MONOIDAL}},(-)^*,{}^*(-))$

$\exists\, X^* \in \mathcal{C}$ WITH

$ev^L_X : X^* \otimes X \to \mathbb{1}$

$coev^L_X : \mathbb{1} \to X \otimes X^*$





—AND—

$\exists\, {}^*X \in \mathcal{C}$ WITH

$ev^R_X : X \otimes {}^*X \to \mathbb{1}$

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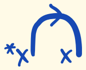

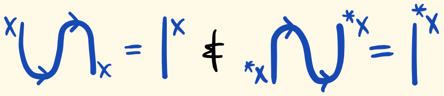

$\forall\, X \in \mathcal{C}$

<u>EXERCISE 3.20</u> WE GET $\mathbb{1}^* = \mathbb{1} = {}^*\mathbb{1}$.

NEED

DEFINE

$$ev^L_{\mathbb{1}} : \overset{\mathbb{1}^*}{\overset{\parallel}{\mathbb{1}}} \otimes \mathbb{1} \xrightarrow{l_{\mathbb{1}}} \mathbb{1} \qquad coev^L_{\mathbb{1}} : \mathbb{1} \xrightarrow{r^{-1}_{\mathbb{1}}} \mathbb{1} \otimes \overset{\mathbb{1}^*}{\overset{\parallel}{\mathbb{1}}}$$

$\boxed{l_{\mathbb{1}} = r_{\mathbb{1}} \text{ (EXER. 3.1)}}$

SUCH THAT

$$\begin{tikzcd}
 & & (\mathbb{1}\otimes\mathbb{1})\otimes\mathbb{1} \arrow[r, "a_{\mathbb{1},\mathbb{1},\mathbb{1}}"] & \mathbb{1}\otimes(\mathbb{1}\otimes\mathbb{1}) \arrow[dr, "id\otimes ev^L_{\mathbb{1}}"] & & \\
 & \mathbb{1}\otimes\mathbb{1} \arrow[ur, "coev^L_{\mathbb{1}}\otimes id"] & & 2 & \mathbb{1}\otimes\mathbb{1} \arrow[dr, "r_{\mathbb{1}}"] & \\
\mathbb{1} \arrow[ur, "l^{-1}_{\mathbb{1}}"] \arrow[rrrrr, "id_{\mathbb{1}}"] & & & & & \mathbb{1}
\end{tikzcd}$$

= LEFT RIGIDITY AXIOMS HOLD =

$$\begin{tikzcd}
\mathbb{1} \arrow[rrrrr, "id_{\mathbb{1}}"] \arrow[dr, "r^{-1}_{\mathbb{1}}"'] & & & & & \mathbb{1} \\
 & \mathbb{1}\otimes\mathbb{1} \arrow[dr, "id\otimes coev^L_{\mathbb{1}}"'] & & & \mathbb{1}\otimes\mathbb{1} \arrow[ur, "l_{\mathbb{1}}"'] & \\
 & & \mathbb{1}\otimes(\mathbb{1}\otimes\mathbb{1}) \arrow[r, "a^{-1}_{\mathbb{1},\mathbb{1},\mathbb{1}}"'] & (\mathbb{1}\otimes\mathbb{1})\otimes\mathbb{1} \arrow[ur, "ev^L\otimes id_{\mathbb{1}}"'] & &
\end{tikzcd}$$

I. RIGID CATEGORIES

RIGID CATEGORY

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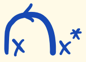

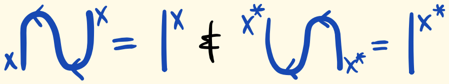

—AND—

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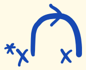

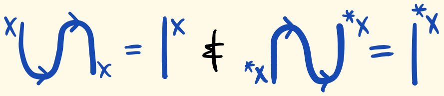

$\forall X \in \mathcal{C}$

EXERCISE 3.20 WE GET $\mathbb{1}^* = \mathbb{1} = {}^*\mathbb{1}$.

NEED $ev^L_{\mathbb{1}} : \overset{\mathbb{1}^*}{\overset{\parallel}{\mathbb{1}}} \otimes \mathbb{1} \xrightarrow{l_{\mathbb{1}}} \mathbb{1}$ DEFINE $coev^L_{\mathbb{1}} : \mathbb{1} \xrightarrow{r^{-1}_{\mathbb{1}}} \mathbb{1} \otimes \overset{\mathbb{1}^*}{\overset{\parallel}{\mathbb{1}}}$

$l_{\mathbb{1}} = r_{\mathbb{1}}$ (EXER. 3.1)

SUCH THAT

$$\begin{tikzcd}
 & & (\mathbb{1} \otimes \mathbb{1}) \otimes \mathbb{1} \arrow[r, "a_{\mathbb{1},\mathbb{1},\mathbb{1}}"] & \mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1}) \arrow[dr, "id \otimes ev^L_{\mathbb{1}}"] & & \\
 & \mathbb{1} \otimes \mathbb{1} \arrow[ur, "coev^L_{\mathbb{1}} \otimes id"] & 2 & & \mathbb{1} \otimes \mathbb{1} \arrow[dr, "r_{\mathbb{1}}"] & \\
\mathbb{1} \arrow[ur, "l^{-1}_{\mathbb{1}}"] \arrow[rrrrr, "id_{\mathbb{1}}"] & & & & & \mathbb{1}
\end{tikzcd}$$

≡ LEFT RIGIDITY AXIOMS HOLD ≡

$$\begin{tikzcd}
\mathbb{1} \arrow[rrrrr, "id_{\mathbb{1}}"] \arrow[dr, "r^{-1}_{\mathbb{1}}"'] & & & & & \mathbb{1} \\
 & \mathbb{1} \otimes \mathbb{1} \arrow[dr, "id \otimes coev^L_{\mathbb{1}}"'] & 2 & & \mathbb{1} \otimes \mathbb{1} \arrow[ur, "l_{\mathbb{1}}"'] & \\
 & & \mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1}) \arrow[r, "a^{-1}_{\mathbb{1},\mathbb{1},\mathbb{1}}"'] & (\mathbb{1} \otimes \mathbb{1}) \otimes \mathbb{1} \arrow[ur, "ev^L \otimes id_{\mathbb{1}}"'] & &
\end{tikzcd}$$

BY SIMILAR ARGUMENT

I. RIGID CATEGORIES

RIGID CATEGORY

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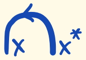

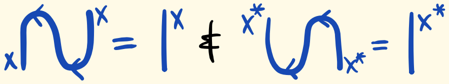

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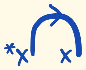

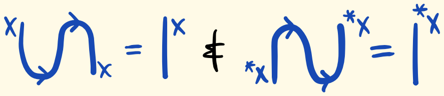

$\forall X \in \mathcal{C}$

EXERCISE 3.20 WE GET $\mathbb{1}^* \overset{\checkmark}{=} \mathbb{1} \overset{\text{you do!}}{=} {}^*\mathbb{1}$.

NEED

DEFINE

$$\mathrm{ev}^L_{\mathbb{1}} : \overset{\mathbb{1}^*}{\overset{\|}{\mathbb{1}}} \otimes \mathbb{1} \xrightarrow{l_{\mathbb{1}}} \mathbb{1} \qquad \mathrm{coev}^L_{\mathbb{1}} : \mathbb{1} \xrightarrow{r^{-1}_{\mathbb{1}}} \mathbb{1} \otimes \overset{\mathbb{1}^*}{\overset{\|}{\mathbb{1}}}$$

$\boxed{l_{\mathbb{1}} = r_{\mathbb{1}} \text{ (EXER. 3.1)}}$

SUCH THAT

$$\begin{tikzcd}
 & & (\mathbb{1} \otimes \mathbb{1}) \otimes \mathbb{1} \arrow[r, "a_{\mathbb{1},\mathbb{1},\mathbb{1}}"] & \mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1}) \arrow[dr, "\mathrm{id} \otimes \mathrm{ev}^L_{\mathbb{1}}"] & & \\
 & \mathbb{1} \otimes \mathbb{1} \arrow[ur, "\mathrm{coev}^L_{\mathbb{1}} \otimes \mathrm{id}"] & & & \mathbb{1} \otimes \mathbb{1} \arrow[dr, "r_{\mathbb{1}}"] & \\
\mathbb{1} \arrow[ur, "l^{-1}_{\mathbb{1}}"] \arrow[rrrrr, "\mathrm{id}_{\mathbb{1}}"] & & & & & \mathbb{1}
\arrow[from=2-2, to=2-5, phantom, "2"]
\end{tikzcd}$$

≡ LEFT RIGIDITY AXIOMS HOLD ≡

$$\begin{tikzcd}
\mathbb{1} \arrow[rrrrr, "\mathrm{id}_{\mathbb{1}}"] \arrow[dr, "r^{-1}_{\mathbb{1}}"'] & & & & & \mathbb{1} \\
 & \mathbb{1} \otimes \mathbb{1} \arrow[dr, "\mathrm{id} \otimes \mathrm{coev}^L_{\mathbb{1}}"'] & & & \mathbb{1} \otimes \mathbb{1} \arrow[ur, "l_{\mathbb{1}}"'] & \\
 & & \mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1}) \arrow[r, "a^{-1}_{\mathbb{1},\mathbb{1},\mathbb{1}}"'] & (\mathbb{1} \otimes \mathbb{1}) \otimes \mathbb{1} \arrow[ur, "\mathrm{ev}^L \otimes \mathrm{id}_{\mathbb{1}}"'] & &
\arrow[from=2-2, to=2-5, phantom, "2"]
\end{tikzcd}$$

BY SIMILAR ARGUMENT

I. RIGID CATEGORIES

RIGID CATEGORY

$(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-))$

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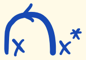

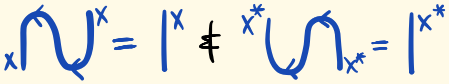

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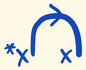

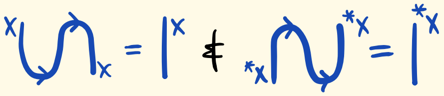

$\forall X \in \mathcal{C}$

you do!

EXERCISE 3.20 WE GET $\mathbb{1}^* \overset{✓}{=} \mathbb{1} = {}^*\mathbb{1}$.

NEED

$$\begin{tikzcd} \mathbb{1}^* \arrow[d, phantom, "\shortparallel"] & & & \mathbb{1}^* \arrow[d, phantom, "\shortparallel"] \\ ev_{\mathbb{1}}^L : \mathbb{1} \otimes \mathbb{1} \arrow[r, "l_{\mathbb{1}}"] & \mathbb{1} & coev_{\mathbb{1}}^L : \mathbb{1} \arrow[r, "r_{\mathbb{1}}^{-1}"] & \mathbb{1} \otimes \mathbb{1} \end{tikzcd}$$

DEFINE

SUCH THAT

$l_{\mathbb{1}} = r_{\mathbb{1}}$ (EXER. 3.1)

ANOTHER COOL EXERCISE :

EXERCISE 3.20 WE GET ${}^*(X^*) \cong X \cong ({}^*X)^*$.

I. RIGID CATEGORIES

RIGID CATEGORY

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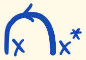

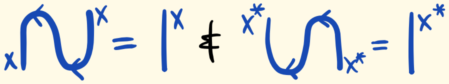

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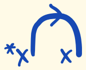

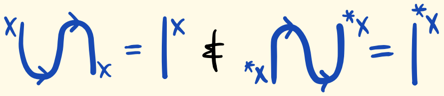

$\forall\, X \in \mathcal{C}$

EXAMPLES INVOLVING $\Bbbk$-VECTOR SPACES

CONSIDER $(\mathrm{FdVec}, \otimes := \otimes_{\Bbbk}, \mathbb{1} = \Bbbk)$

I. RIGID CATEGORIES

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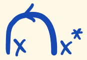

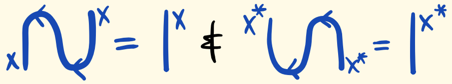

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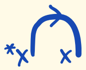

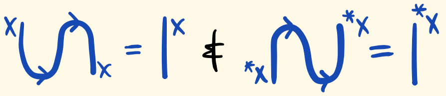

$\forall\, X \in \mathcal{C}$

EXAMPLES INVOLVING $\Bbbk$-VECTOR SPACES

CONSIDER $(\mathrm{FdVec}, \otimes := \otimes_{\Bbbk}, \mathbb{1} = \Bbbk)$

FOR $V \in \mathrm{FdVec}$ WITH BASIS $\{b_i\}_i$,

GET $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ IS THE LEFT DUAL OF V.

I. RIGID CATEGORIES

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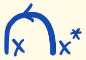
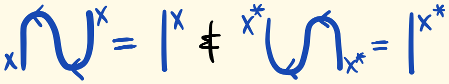
$coev_X^L : \mathbb{1} \longrightarrow X \otimes X^*$ 



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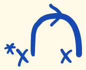
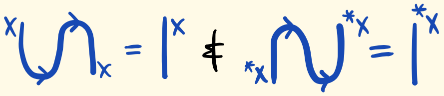
$coev_X^R : \mathbb{1} \longrightarrow {}^*X \otimes X$ 



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HERE: $ev_V^L : V^* \otimes_{\Bbbk} V \longrightarrow \Bbbk$ $\quad$ $coev_V^L : \Bbbk \longrightarrow V \otimes_{\Bbbk} V^*$

$$f \otimes_{\Bbbk} v \longmapsto f(v) \qquad 1_{\Bbbk} \longmapsto \sum_{i=1}^{\dim_{\Bbbk} V} b_i \otimes b_i^*$$

I. RIGID CATEGORIES

RIGID CATEGORY

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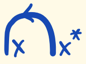

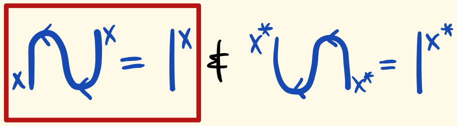

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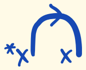

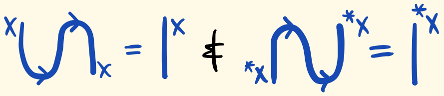

$\forall X \in \mathcal{C}$

EXAMPLES INVOLVING $\Bbbk$-VECTOR SPACES

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FOR $V \in \mathrm{FdVec}$ WITH BASIS $\{b_i\}_i$,

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HERE: $ev^L_V : V^* \otimes_{\Bbbk} V \longrightarrow \Bbbk$

$$f \otimes_{\Bbbk} v \longmapsto f(v)$$

$coev^L_V : \Bbbk \longrightarrow V \otimes_{\Bbbk} V^*$

$$1_{\Bbbk} \longmapsto \sum_{i=1}^{\dim_{\Bbbk} V} b_i \otimes b_i^*$$

$$V \xrightarrow{coev^L_V \otimes id_V} V \otimes V^* \otimes V \xrightarrow{id_V \otimes ev^L_V} V$$

I. RIGID CATEGORIES

RIGID CATEGORY

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$ev_X^L : X^*\otimes X \longrightarrow \mathbb{1}$

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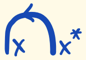

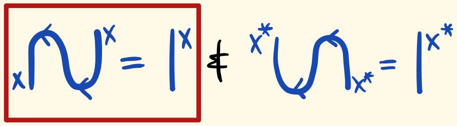

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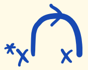

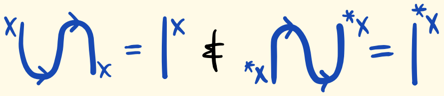

$\forall X\in\mathcal{C}$

EXAMPLES INVOLVING $\Bbbk$-VECTOR SPACES

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I. RIGID CATEGORIES

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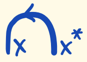

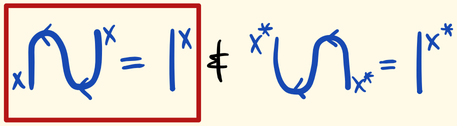

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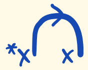

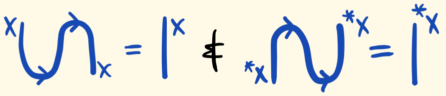

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$$v = 1_{\mathbb{k}} \otimes_{\mathbb{k}} v \mapsto \Sigma_i b_i \otimes b_i^* \otimes v$$

$$\|$$

$$\Sigma_{i,j}\, b_i \otimes b_i^* \otimes \lambda_j b_j$$

$$\|$$

$$\Sigma_{i,j}\, \lambda_j\, b_i \otimes b_i^* \otimes b_j$$

$\Big[\, v = \Sigma_j \lambda_j b_j$ FOR SOME $\lambda_j \in \mathbb{k} \,\Big]$

I. RIGID CATEGORIES

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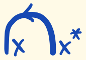

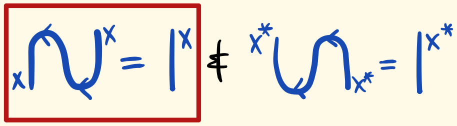

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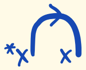

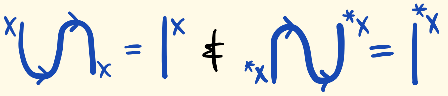

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$$\| $$

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$$\|$$

$$\textstyle\sum_{i,j} \lambda_j b_i \otimes b_i^* \otimes b_j \longmapsto \sum_{i,j} \lambda_j b_i \otimes b_i^*(b_j)$$

[$v = \sum_j \lambda_j b_j$ FOR SOME $\lambda_j \in \Bbbk$]

I. RIGID CATEGORIES

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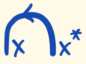

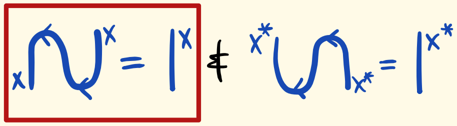

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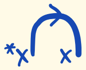

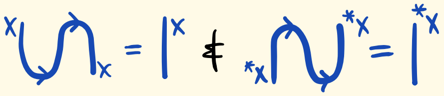

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$$\| $$

$$\textstyle\sum_{i,j} b_i \otimes b_i^* \otimes \lambda_j b_j \qquad\qquad \sum_i \lambda_i b_i \otimes 1_{\Bbbk}$$

$$\| \qquad\qquad\qquad\qquad \| \; j=i$$

$$\textstyle\sum_{i,j} \lambda_j b_i \otimes b_i^* \otimes b_j \longmapsto \sum_{i,j} \lambda_j b_i \otimes \underbrace{b_i^*(b_j)}_{\delta_{i,j} 1_{\Bbbk}}$$

$\Big[\, v = \sum_j \lambda_j b_j$ FOR SOME $\lambda_j \in \Bbbk \,\Big]$

I. RIGID CATEGORIES

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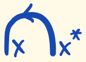

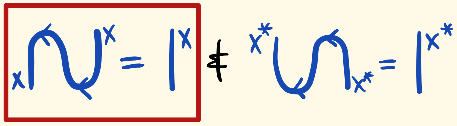

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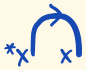

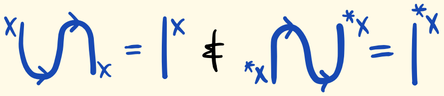

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$$V \xrightarrow{coev_V^L \otimes id_V} V \otimes V^* \otimes V \xrightarrow{id_V \otimes ev_V^L} V \; = id_V$$

$v \equiv 1_{\mathbb{k}} \otimes_{\mathbb{k}} v \longmapsto \sum_i b_i \otimes b_i^* \otimes v$ $\qquad$ $v \otimes_{\mathbb{k}} 1_{\mathbb{k}} \equiv v$ ✓

$\sum_i b_i \otimes b_i^* \otimes v = \sum_{i,j} b_i \otimes b_i^* \otimes \lambda_j b_j = \sum_{i,j} \lambda_j b_i \otimes b_i^* \otimes b_j \longmapsto \sum_{i,j} \lambda_j b_i \otimes \underbrace{b_i^*(b_j)}_{\delta_{i,j} 1_{\mathbb{k}}}$

$\sum_{i,j} \lambda_j b_i \otimes b_i^*(b_j) \overset{j=i}{=} \sum_i \lambda_i b_i \otimes 1_{\mathbb{k}} = v \otimes_{\mathbb{k}} 1_{\mathbb{k}}$

$\left[\begin{array}{l} v = \sum_j \lambda_j b_j \\ \text{FOR SOME } \lambda_j \in \mathbb{k} \end{array} \right]$

I. RIGID CATEGORIES

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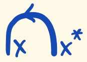

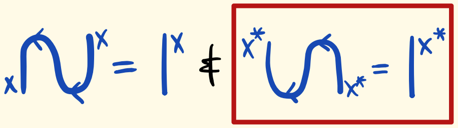

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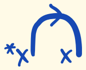

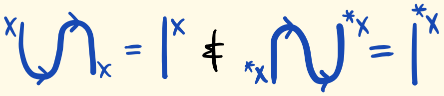

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I. RIGID CATEGORIES

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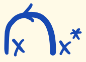

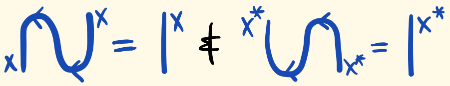

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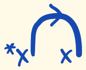

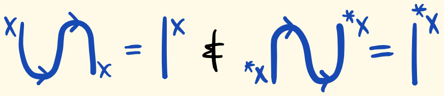

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I. RIGID CATEGORIES

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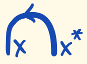

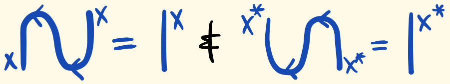

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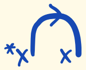

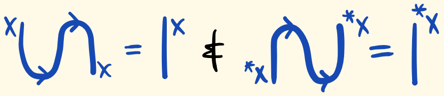

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IS $(\mathsf{Vec}, \otimes_{\Bbbk}, \Bbbk)$ RIGID??

↑ INCLUDING INFINITE DIM'L VECTOR SPACES

|||
THINK
ABOUT
THIS
|||

I. RIGID CATEGORIES

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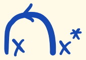

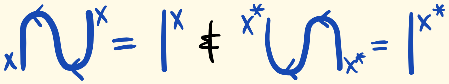

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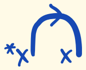

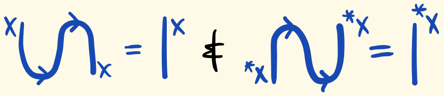

$\forall X \in \mathcal{C}$

EXAMPLES INVOLVING $\Bbbk$-VECTOR SPACES

CONSIDER $(\mathsf{FdVec}, \otimes := \otimes_{\Bbbk}, \mathbb{1} = \Bbbk)$

FOR $V \in \mathsf{FdVec}$ WITH BASIS $\{b_i\}_i$,

GET $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ IS THE LEFT DUAL OF V.

HERE: $ev^L_V : V^* \otimes_{\Bbbk} V \longrightarrow \Bbbk$ $\quad coev^L_V : \Bbbk \longrightarrow V \otimes_{\Bbbk} V^*$

$f \otimes_{\Bbbk} v \longmapsto f(v)$ $\quad 1_{\Bbbk} \longmapsto \sum_{i=1}^{\dim_{\Bbbk} V} b_i \otimes b_i^*$

I. RIGID CATEGORIES

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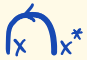

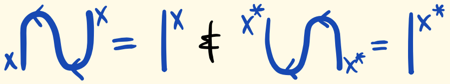

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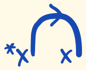

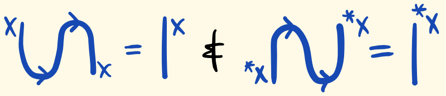

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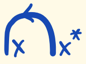

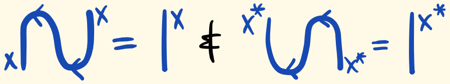

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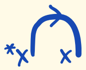

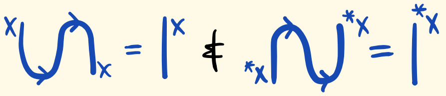

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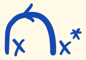

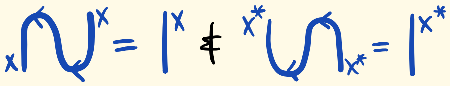

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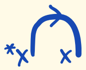

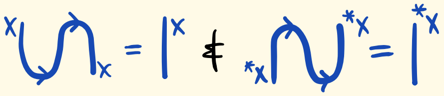

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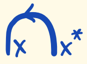

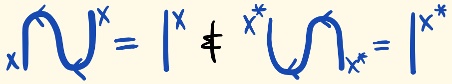

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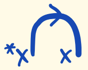

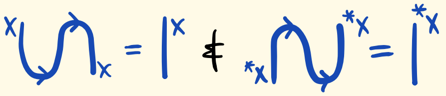

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↑ NOTATION $\qquad$ FOR SOME FINITE n

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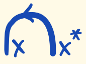

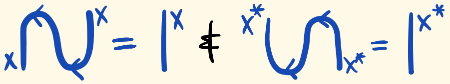

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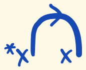

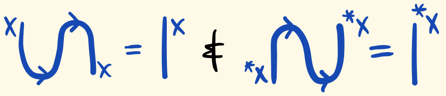

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FOR SOME <u>FINITE</u> n

THEN

$$V \xrightarrow{coev^L_V \otimes id_V} V \otimes_{\Bbbk} W \otimes_{\Bbbk} V \xrightarrow{id_V \otimes ev^L_V} V$$

I. RIGID CATEGORIES

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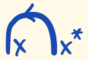
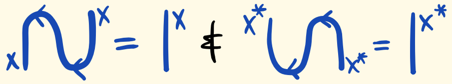
$coev_X^L : \mathbb{1} \to X \otimes X^*$ 



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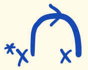
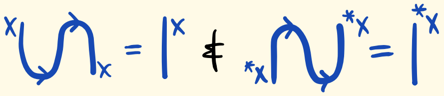
$coev_X^R : \mathbb{1} \to {}^*X \otimes X$ 



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FOR SOME <u>FINITE</u> n

THEN

$$\begin{tikzcd} V \arrow[r, "coev_V^L \otimes id_V"] & V \otimes_{\Bbbk} W \otimes_{\Bbbk} V \arrow[r, "id_V \otimes ev_V^L"] & V \end{tikzcd}$$

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I. RIGID CATEGORIES

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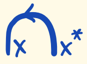
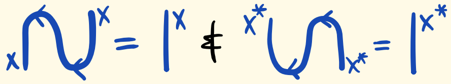
$coev_X^L : \mathbb{1} \to X \otimes X^*$ 



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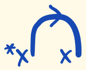
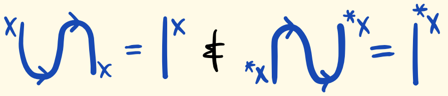
$coev_X^R : \mathbb{1} \to {}^*X \otimes X$ 



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THEN

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$$v = 1_{\Bbbk} \otimes v \longmapsto \sum_{i=1}^{n} b_i \otimes w_i \otimes v$$

$\left[v = \sum_{j=1}^{\overset{\text{FINITE}}{m}} \lambda_j b_j \right.$
FOR SOME $\left. \lambda_j \in \Bbbk \right]$

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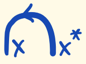

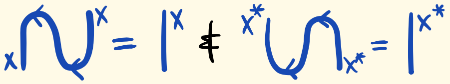

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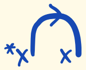

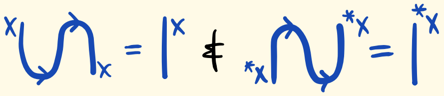

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THEN

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$\left[\, v = \sum_{j=1}^{\overset{\text{FINITE}}{m}} \lambda_j b_j \right.$
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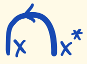

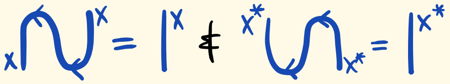

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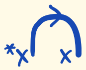
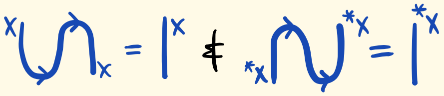
$coev_X^R : \mathbb{1} \to {}^*X \otimes X$





$\forall X \in \mathcal{C}$

EXAMPLES INVOLVING 𝕜-VECTOR SPACES

CONSIDER $(\mathsf{Vec}, \otimes := \otimes_{\Bbbk}, \mathbb{1} = \Bbbk)$

FOR $V \in \mathsf{Vec}$ WITH BASIS $\{b_i\}_i$,

SUPPOSE $W \in \mathsf{Vec}$ IS THE LEFT DUAL OF V.

HERE: $ev_V^L : W \otimes_{\Bbbk} V \longrightarrow \Bbbk$ $\quad coev_V^L : \Bbbk \longrightarrow V \otimes_{\Bbbk} W$

$w \otimes v \longmapsto w(v)$ $\quad 1_{\Bbbk} \longmapsto \sum_{i=1}^{n} b_i \otimes w_i$

↑ NOTATION

FOR SOME FINITE n

THEN

$$V \xrightarrow{coev_V^L \otimes id_V} V \otimes_{\Bbbk} W \otimes_{\Bbbk} V \xrightarrow{id_V \otimes ev_V^L} V$$

$$v \equiv 1_{\Bbbk} \otimes v \longmapsto \sum_{i=1}^{n} b_i \otimes w_i \otimes v \longmapsto \sum_{i=1}^{n} \sum_{j=1}^{m} \underbrace{\lambda_j}_{\in \Bbbk} b_i \underbrace{w_i(b_j)}_{\in \Bbbk}$$

$\left[v = \sum_{j=1}^{\overset{\text{FINITE}}{m}} \lambda_j b_j \right.$

$\left. \text{FOR SOME } \lambda_j \in \Bbbk \right]$

IMAGE IS THE SPAN OF $\{b_i\}_{i=1}^{n}$

I. RIGID CATEGORIES

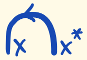
RIGID CATEGORY

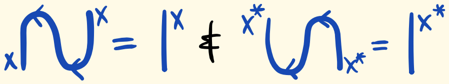
$$\Big(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-)\Big)$$

$\exists\, X^* \in \mathcal{C}$ WITH

$ev^L_X : X^* \otimes X \to \mathbb{1}$

$coev^L_X : \mathbb{1} \to X \otimes X^*$



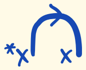


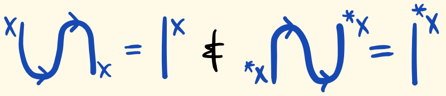
–AND–

$\exists\, {}^*X \in \mathcal{C}$ WITH

$ev^R_X : X \otimes {}^*X \to \mathbb{1}$

$coev^R_X : \mathbb{1} \to {}^*X \otimes X$





$\forall\, X \in \mathcal{C}$

EXAMPLES INVOLVING $\Bbbk$-VECTOR SPACES

CONSIDER $(Vec, \otimes := \otimes_{\Bbbk}, \mathbb{1} = \Bbbk)$

FOR $V \in Vec$ WITH BASIS $\{b_i\}_i$,

SUPPOSE $W \in Vec$ IS <u>THE</u> LEFT DUAL OF V.

HERE: $ev^L_V : W \otimes_{\Bbbk} V \longrightarrow \Bbbk$ $\qquad$ $coev^L_V : \Bbbk \longrightarrow V \otimes_{\Bbbk} W$

$w \otimes v \longmapsto w(v)$ $\qquad$ $1_{\Bbbk} \longmapsto \sum_{i=1}^{n} b_i \otimes w_i$

NOTATION $\qquad$ FOR SOME <u>FINITE</u> n

THEN

$$V \xrightarrow{coev^L_V \otimes id_V} V \otimes_{\Bbbk} W \otimes_{\Bbbk} V \xrightarrow{id_V \otimes ev^L_V} V \neq id_V$$

$$v \equiv 1_{\Bbbk} \otimes v \longmapsto \sum_{i=1}^{n} b_i \otimes w_i \otimes v \longmapsto \sum_{i=1}^{n} \sum_{j=1}^{m} \underbrace{\lambda_j}_{\in \Bbbk} b_i \underbrace{w_i(b_j)}_{\in \Bbbk}$$

$$\left[\begin{array}{l} v = \sum_{j=1}^{m} \lambda_j b_j \quad (m \text{ FINITE}) \\ \text{FOR SOME } \lambda_j \in \Bbbk \end{array}\right]$$

IMAGE IS THE SPAN OF $\{b_i\}_{i=1}^{n}$
(FINITE-DIM'L)

I. RIGID CATEGORIES

RIGID CATEGORY

$$(\underbrace{\mathcal{C},\otimes,\mathbb{1},a,l,r}_{\text{MONOIDAL}},(-)^*,{}^*(-))$$

$\exists X^* \in \mathcal{C}$ WITH

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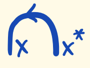
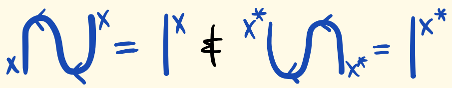
$coev^L_X : \mathbb{1}\to X\otimes X^*$ 



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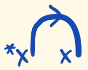
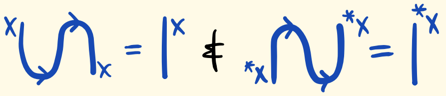
$coev^R_X : \mathbb{1}\to {}^*X\otimes X$ 



$\forall X\in\mathcal{C}$

EXAMPLES INVOLVING $\Bbbk$-VECTOR SPACES

CONSIDER $(\mathsf{Vec}, \otimes := \otimes_{\Bbbk}, \mathbb{1}=\Bbbk)$ NOT RIGID

FOR $V\in\mathsf{Vec}$ WITH BASIS $\{b_i\}_i$,

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↑ NOTATION $\qquad$ FOR SOME FINITE n

THEN

$$V \xrightarrow{coev^L_V\otimes id_V} V\otimes_{\Bbbk} W\otimes_{\Bbbk} V \xrightarrow{id_V\otimes ev^L_V} V \neq id_V$$

$$v\equiv 1_{\Bbbk}\otimes v \longmapsto \sum_{i=1}^{n} b_i\otimes w_i\otimes v \longmapsto \sum_{i=1}^{n}\sum_{j=1}^{m} \underbrace{\lambda_j}_{\in\Bbbk} b_i\, \underbrace{w_i(b_j)}_{\in\Bbbk}$$

$\left[v=\sum_{j=1}^{\overset{\text{FINITE}}{m}}\lambda_j b_j\right.$ FOR SOME $\left.\lambda_j\in\Bbbk\right]$

IMAGE IS THE SPAN OF $\{b_i\}_{i=1}^{n}$
(FINITE-DIM'L)

I. RIGID CATEGORIES

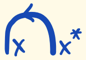
RIGID CATEGORY

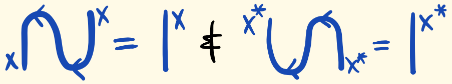
$(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-))$

$\exists X^* \in \mathcal{C}$ WITH

$ev_X^L : X^* \otimes X \longrightarrow \mathbb{1}$

$coev_X^L : \mathbb{1} \longrightarrow X \otimes X^*$





-AND-

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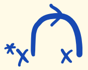
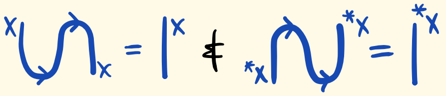
$coev_X^R : \mathbb{1} \longrightarrow {}^*X \otimes X$





$\forall X \in \mathcal{C}$

EXAMPLES INVOLVING 𝕜-VECTOR SPACES

RIGID	NOT RIGID
FdVec	Vec

I. RIGID CATEGORIES

RIGID CATEGORY

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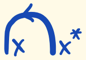

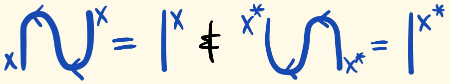

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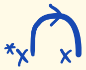

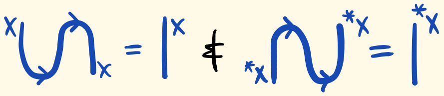

$\forall X\in\mathcal{C}$

EXAMPLES INVOLVING $\Bbbk$-VECTOR SPACES

RIGID	NOT RIGID
FdVec	Vec

FOR A GROUP G

G-FdMod	G-Mod

I. RIGID CATEGORIES

RIGID CATEGORY

$$(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-))$$

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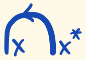

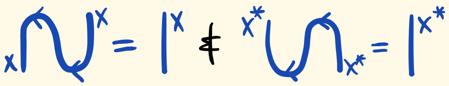

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$\exists {}^*X \in \mathcal{C}$ WITH

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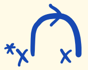

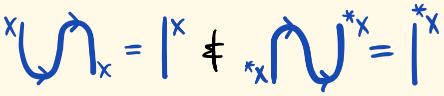

$\forall X \in \mathcal{C}$

EXAMPLES INVOLVING $\mathbb{k}$-VECTOR SPACES

RIGID	NOT RIGID
FdVec	Vec

FOR A GROUP G

G-FdMod	EXER.3.22	G-Mod

I. RIGID CATEGORIES

RIGID CATEGORY

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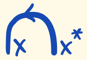
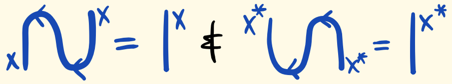


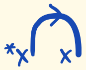


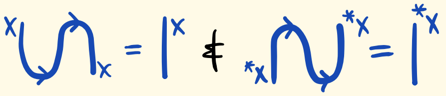
—AND—

$\exists {}^*X \in \mathcal{C}$ WITH

$ev^R_X : X \otimes {}^*X \to \mathbb{1}$

$coev^R_X : \mathbb{1} \to {}^*X \otimes X$





$\forall X \in \mathcal{C}$

EXAMPLES INVOLVING 𝕜-VECTOR SPACES

RIGID		NOT RIGID
FdVec		Vec
	FOR A GROUP G	
G-FdMod	EXER.3.22	G-Mod
	FOR A 𝕜-ALGEBRA A	
		A-FdBimod (MOST OF THE TIME)

I. RIGID CATEGORIES

RIGID CATEGORY

$$(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-))$$

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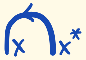

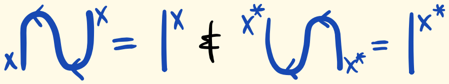

—AND—

$\exists {}^*X \in \mathcal{C}$ WITH

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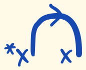

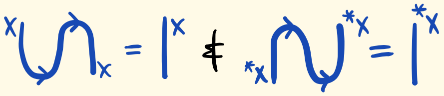

$\forall X \in \mathcal{C}$

EXAMPLES INVOLVING $\Bbbk$-VECTOR SPACES

RIGID	NOT RIGID
FdVec	Vec

FOR A GROUP G

G-FdMod $\boxed{\text{EXER.3.22}}$ G-Mod

FOR A $\Bbbk$-ALGEBRA A

A-FdBimod ↑ (FINITE-DIM'L SEMISIMPLE)

A-FdBimod (MOST OF THE TIME)

I. RIGID CATEGORIES

RIGID CATEGORY

$$(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-))$$

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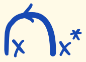

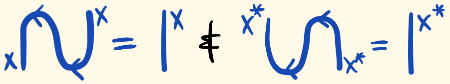

– AND –

$\exists {}^*X \in \mathcal{C}$ WITH

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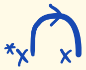
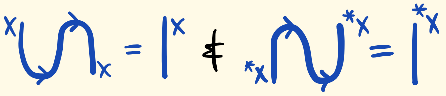




$\forall X \in \mathcal{C}$

EXAMPLES INVOLVING 𝕜-VECTOR SPACES

RIGID	NOT RIGID
FdVec	Vec

FOR A GROUP G

G-FdMod

EXER. 3.22

G-Mod

FOR A 𝕜-ALGEBRA A

A-FdBimod
↑
(FINITE-DIM'L SEMISIMPLE)

A-FdBimod
(MOST OF THE TIME)

EXER. 1.31

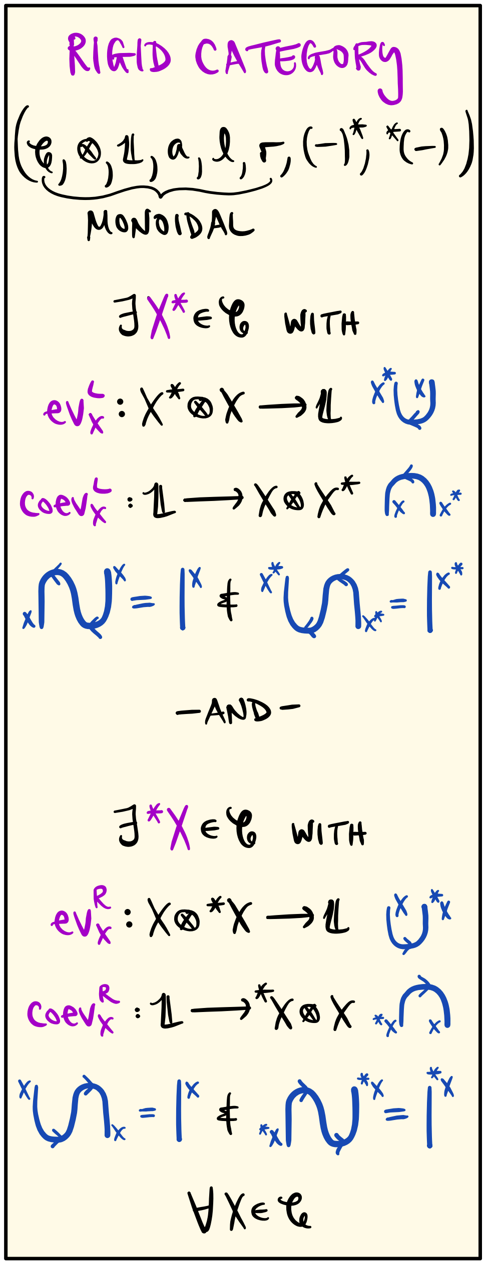
SEMISIMPLE
2-DIM'L

$\frac{\mathbb{k}[x]}{(x^2-1)}$ - FdBimod

NOT S.S., 2-DIM'L

$\frac{\mathbb{k}[x]}{(x^2)}$ - FdBimod

I. RIGID CATEGORIES



RIGID CATEGORY

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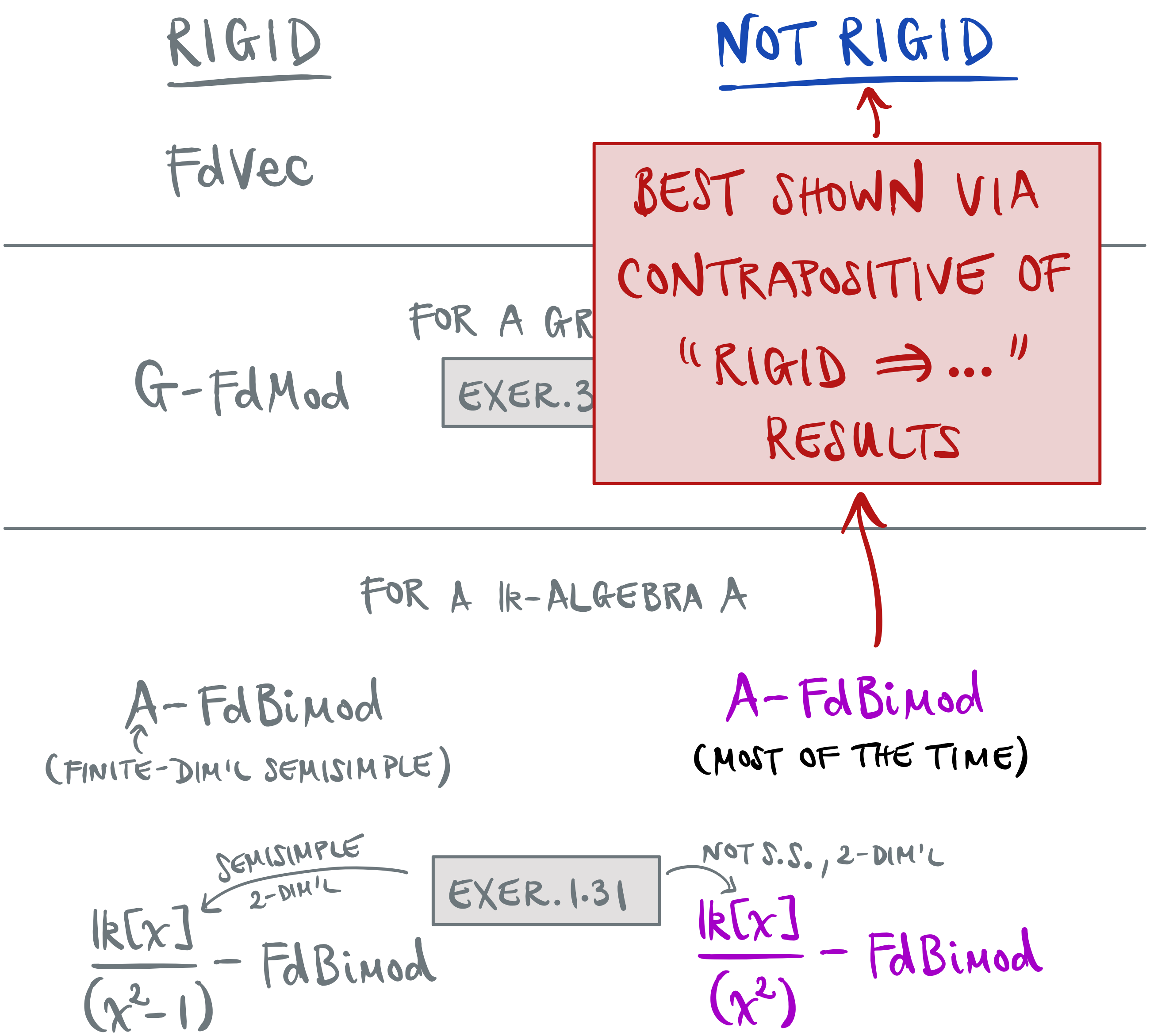
$\forall X \in \mathcal{C}$

EXAMPLES INVOLVING $\mathbb{k}$-VECTOR SPACES



RIGID	NOT RIGID
FdVec	
G-FdMod	
A-FdBimod (FINITE-DIM'L SEMISIMPLE)	A-FdBimod (MOST OF THE TIME)
$\frac{\mathbb{k}[x]}{(x^2-1)}$ - FdBimod	$\frac{\mathbb{k}[x]}{(x^2)}$ - FdBimod

FOR A GR

EXER.3

BEST SHOWN VIA CONTRAPOSITIVE OF "RIGID ⇒ ..." RESULTS

FOR A $\mathbb{k}$-ALGEBRA A

SEMISIMPLE 2-DIM'L

EXER. 1.31

NOT S.S., 2-DIM'L

I. RIGID CATEGORIES

RIGID CATEGORY

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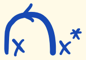

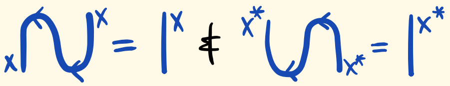

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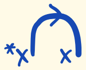
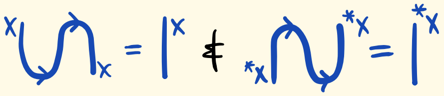




$\forall X \in \mathcal{C}$

LET'S ESTABLISH A

"RIGID $\Rightarrow$ ..."

RESULT

I. RIGID CATEGORIES

(RECALL NOTATION: $L \dashv R$)

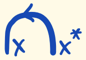
RIGID CATEGORY

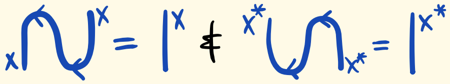
$$\Big(\underbrace{\mathcal{C},\otimes,\mathbb{1},a,l,r}_{\text{MONOIDAL}},(-)^*,{}^*(-)\Big)$$

$\exists X^* \in \mathcal{C}$ WITH

$ev^L_X : X^* \otimes X \longrightarrow \mathbb{1}$

$coev^L_X : \mathbb{1} \longrightarrow X \otimes X^*$





—AND—

$\exists {}^*X \in \mathcal{C}$ WITH

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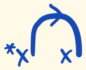

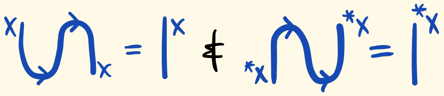

$\forall X \in \mathcal{C}$

PROP: IN A RIGID CATEGORY, WE HAVE THE FOLLOWING ADJUNCTIONS:

(a)

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "X\otimes -"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "{}^*X\otimes -"] \end{tikzcd}$$

(b)

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "X^*\otimes -"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "X\otimes -"] \end{tikzcd}$$

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I. RIGID CATEGORIES

(RECALL NOTATION: $L \dashv R$)

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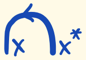
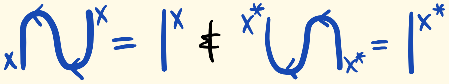
$coev_X^L : \mathbb{1} \longrightarrow X \otimes X^*$ 



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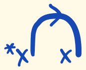
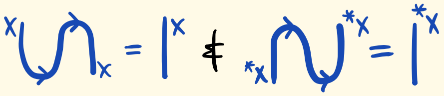
$coev_X^R : \mathbb{1} \longrightarrow {}^*X \otimes X$ 



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$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "- \otimes {}^*Y"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "- \otimes Y"] \end{tikzcd}$$

(d)
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "- \otimes Y"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "- \otimes Y^*"] \end{tikzcd}$$

$\forall X, Y \in \mathcal{C}$

I. RIGID CATEGORIES

(RECALL NOTATION: L ⊣ R)

RIGID CATEGORY

$$\big(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-)\big)$$

$\exists\, X^* \in \mathcal{C}$ WITH

$ev^L_X : X^* \otimes X \to \mathbb{1}$

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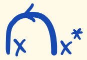

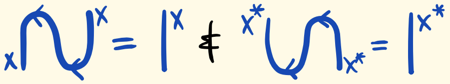

— AND —

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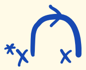
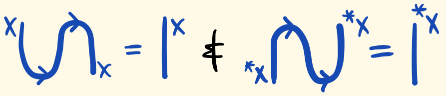
$coev^R_X : \mathbb{1} \to {}^*X \otimes X$





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IMPLIES $(X \otimes -)$ HAS A LEFT & RIGHT ADJOINT

(c)

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "- \otimes {}^*Y"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "- \otimes Y"] \end{tikzcd}$$

(d)

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "- \otimes Y"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "- \otimes Y^*"] \end{tikzcd}$$

$\forall X, Y \in \mathcal{C}$

I. RIGID CATEGORIES

(RECALL NOTATION : $L \dashv R$)

RIGID CATEGORY

$$\left(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-)\right)$$

$\exists X^* \in \mathcal{C}$ WITH

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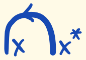
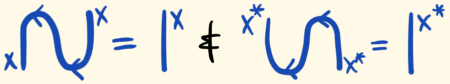
$coev_X^L : \mathbb{1} \to X \otimes X^*$ 



$= |^X$ & $= |^{X^*}$

—AND—

$\exists {}^*X \in \mathcal{C}$ WITH

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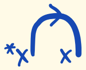
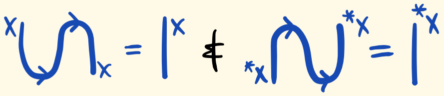
$coev_X^R : \mathbb{1} \to {}^*X \otimes X$ 



$= |^X$ & $= |^{*X}$

$\forall X \in \mathcal{C}$

PROP : IN A RIGID CATEGORY,

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$\therefore$ $(X \otimes -)$ IS LEFT & RIGHT EXACT

(c)

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$\forall X, Y \in \mathcal{C}$

I. RIGID CATEGORIES

(RECALL NOTATION: $L \dashv R$)

RIGID CATEGORY

$$(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^{*}, {}^{*}(-))$$

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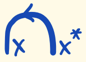

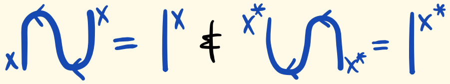

—AND—

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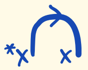

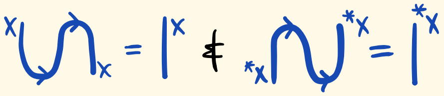

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IMPLIES $(- \otimes Y)$ HAS A LEFT & RIGHT ADJOINT

(c)

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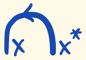

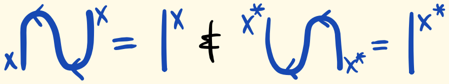

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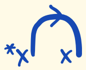

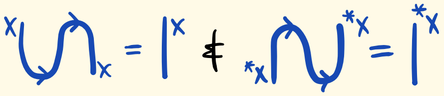

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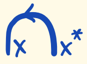

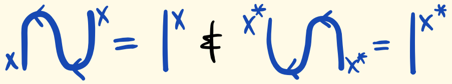

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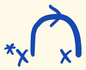

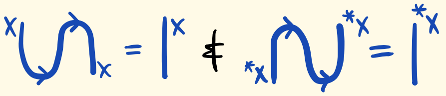

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↑ LET'S SKETCH THE PROOF

$\forall\, X, Y \in \mathcal{C}$

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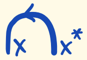

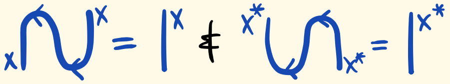

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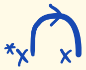

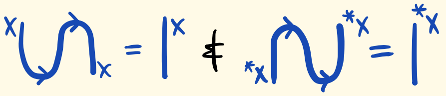

$\forall\, X \in \mathcal{C}$

(d)

LET'S SKETCH THE PROOF

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "- \otimes Y"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "- \otimes Y^*"]
\end{tikzcd}$$

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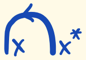

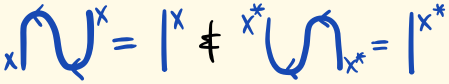

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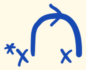

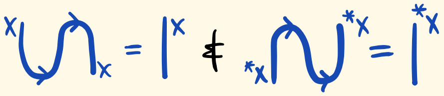

$\forall X \in \mathcal{C}$

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NATURAL IN X, Z

I. RIGID CATEGORIES

RIGID CATEGORY

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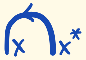

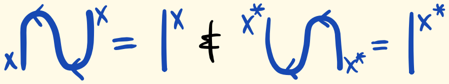

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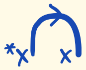

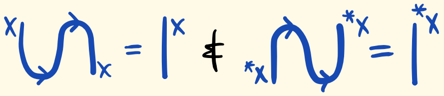

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≡ LET'S TRY THIS ON THE BOARD ≡

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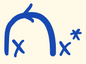

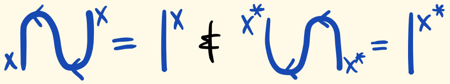

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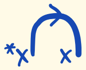

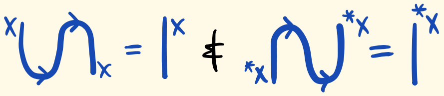

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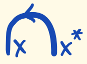

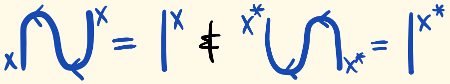

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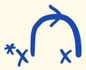

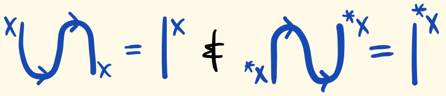

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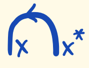

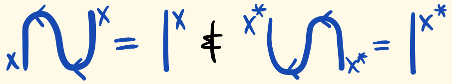

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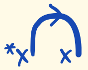

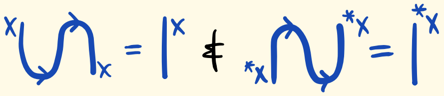

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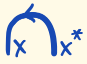

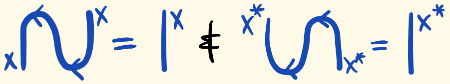

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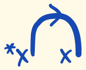

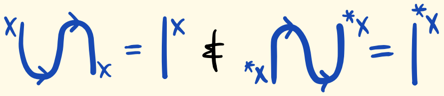

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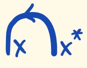
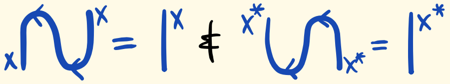
$coev^L_X : \mathbb{1} \longrightarrow X \otimes X^*$ 



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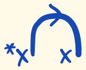
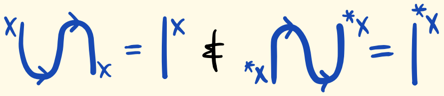
$coev^R_X : \mathbb{1} \longrightarrow {}^*X \otimes X$ 



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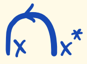
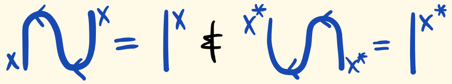
$coev_X^L : \mathbb{1} \to X\otimes X^*$ 



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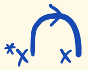
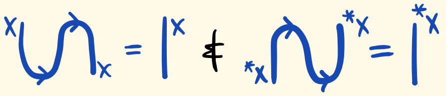
$coev_X^R : \mathbb{1} \to {}^*X\otimes X$ 



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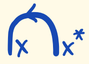

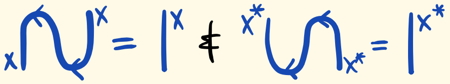

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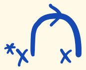

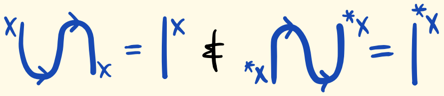

$\forall X \in \mathcal{C}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "-\otimes Y"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "-\otimes Y^*"] \end{tikzcd}$$

$\forall Y \in \mathcal{C}$

NEED BIJECTION:

$$\mathfrak{J}_{X,Z} : \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^*)$$

NATURAL IN X, Z

WLOG PROCEED IN STRICT CASE

$$\mathfrak{J}_{X,Z}(f : X \otimes Y \to Z) := X \xrightarrow{id_X \otimes coev^L_Y} X \otimes Y \otimes Y^* \xrightarrow{f \otimes id_{Y^*}} Z \otimes Y^*$$

$$\mathfrak{J}^{-1}_{X,Z}(g : X \to Z \otimes Y^*) := X \otimes Y \xrightarrow{g \otimes id_Y} Z \otimes Y^* \otimes Y \xrightarrow{id_Z \otimes ev^L_Y} Z$$

$$\begin{tikzcd} & & Z \otimes Y^* \otimes Y \arrow[rrd, "id_Z \otimes ev^L_Y"] & & \\ \mathfrak{J}^{-1}\mathfrak{J}(f) : X \otimes Y \arrow[rru, "\mathfrak{J}(f) \otimes id_Y"] \arrow[rd, "id_X \otimes coev^L_Y \otimes id_Y"'] \arrow[rrr, bend right=60, "id_X \otimes id_Y"'] & \circlearrowleft \text{ DEF} & & \circlearrowleft \text{ LEVEL EXCHANGE} & Z \\ & X \otimes Y \otimes Y^* \otimes Y \arrow[ru, "f \otimes id_{Y^*} \otimes id_Y"] \arrow[rr, "id_X \otimes id_Y \otimes ev^L_Y"] & \circlearrowleft \text{ RIGIDITY AXIOM} & X \otimes Y \arrow[ru, "f"'] & \end{tikzcd}$$

$$✓ \ \big\| \quad f$$

LIKEWISE

$$\underline{\underline{\mathfrak{J}\mathfrak{J}^{-1}(g) = g}}$$

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I. RIGID CATEGORIES

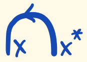
RIGID CATEGORY

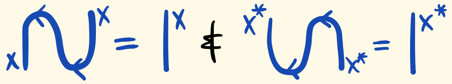
$$\left(\underbrace{\mathcal{C},\otimes,\mathbb{1},a,l,r}_{\text{MONOIDAL}},(-)^*,{}^*(-)\right)$$

$\exists X^* \in \mathcal{C}$ WITH

$ev^L_X : X^*\otimes X \longrightarrow \mathbb{1}$

$coev^L_X : \mathbb{1} \longrightarrow X\otimes X^*$





—AND—

$\exists {}^*X \in \mathcal{C}$ WITH

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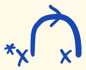

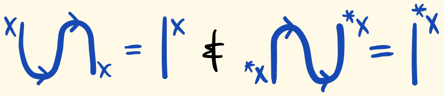

$\forall X \in \mathcal{C}$

PROP: IN A RIGID CATEGORY, WE HAVE THE FOLLOWING ADJUNCTIONS:

(a)

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "X\otimes -"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "{}^*X\otimes -"] \end{tikzcd}$$

(b)

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "X^*\otimes -"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "X\otimes -"] \end{tikzcd}$$

(c)

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "-\otimes {}^*Y"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "-\otimes Y"] \end{tikzcd}$$

(d)

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "-\otimes Y"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "-\otimes Y^*"] \end{tikzcd}$$

$\forall X, Y \in \mathcal{C}$

I. RIGID CATEGORIES

VERY IMPORTANT RESULT!

RIGID CATEGORY

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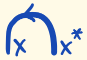

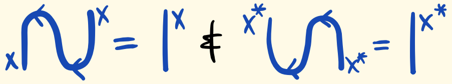

-AND-

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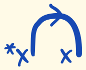

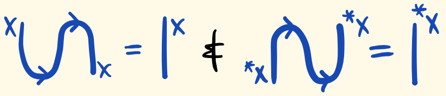

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(b)

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "X^*\otimes -"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "X\otimes -"] \end{tikzcd}$$

ARE EXACT

(c)

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "-\otimes {}^*Y"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "-\otimes Y"] \end{tikzcd}$$

(d)

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "-\otimes Y"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "-\otimes Y^*"] \end{tikzcd}$$

$\forall X, Y \in \mathcal{C}$

I. RIGID CATEGORIES

DROPPING DIRECTIONAL ARROWS FOR BREVITY

RIGID CATEGORY

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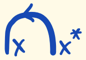

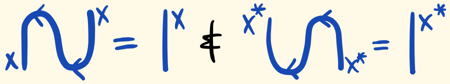

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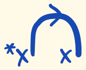

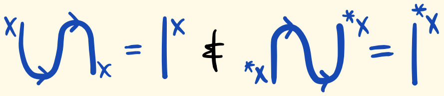

$\forall X \in \mathcal{C}$

COOL FACTS

I. RIGID CATEGORIES

DROPPING DIRECTIONAL ARROWS FOR BREVITY

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$$\big(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-)\big)$$

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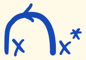

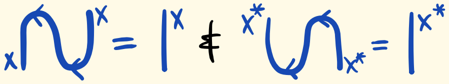

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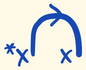

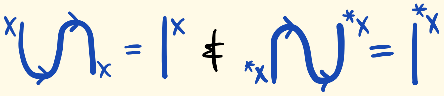

$\forall\, X \in \mathcal{C}$

COOL FACTS

GET CONTRAVARIANT, STRONG MONOIDAL FUNCTORS

$(-)^* : \mathcal{C}^{rev} \longrightarrow \mathcal{C}$ ${}^*(-) : \mathcal{C}^{rev} \longrightarrow \mathcal{C}$

I. RIGID CATEGORIES

DROPPING DIRECTIONAL ARROWS FOR BREVITY

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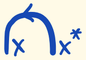

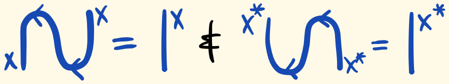

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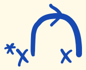

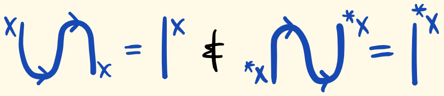

$\forall X \in \mathcal{C}$

COOL FACTS

GET CONTRAVARIANT, STRONG MONOIDAL FUNCTORS

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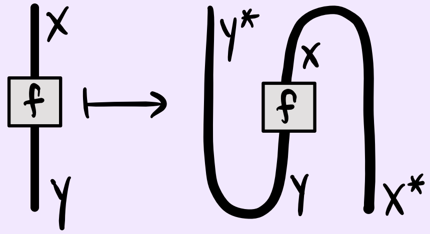

I. RIGID CATEGORIES

DROPPING DIRECTIONAL ARROWS FOR BREVITY

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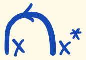

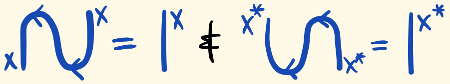

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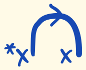

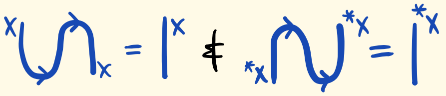

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COOL FACTS

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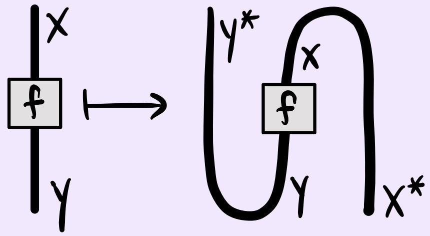

${}^*(-) : \mathcal{C}^{rev} \longrightarrow \mathcal{C}$

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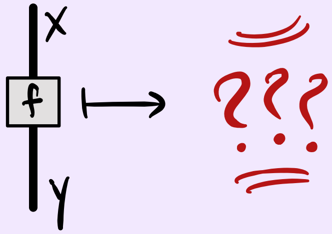

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I. RIGID CATEGORIES

DROPPING DIRECTIONAL ARROWS FOR BREVITY

RIGID CATEGORY

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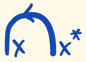

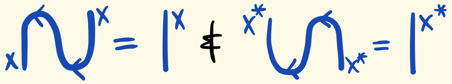

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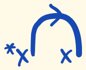

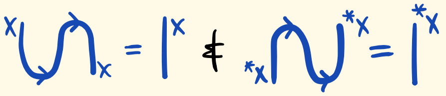

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COOL FACTS

GET CONTRAVARIANT, STRONG MONOIDAL FUNCTORS

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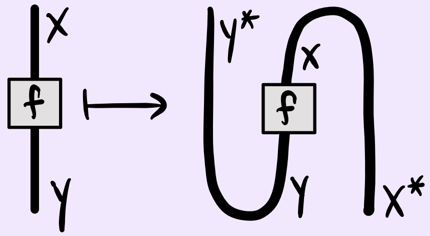

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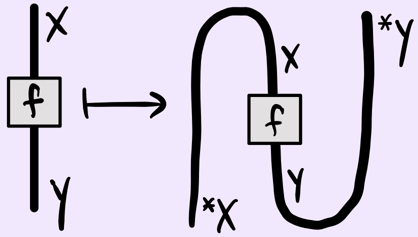

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I. RIGID CATEGORIES

DROPPING DIRECTIONAL ARROWS FOR BREVITY

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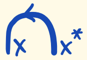

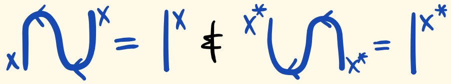

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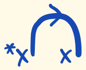

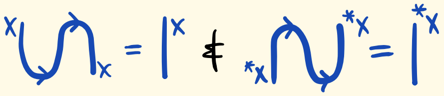

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COOL FACTS

GET CONTRAVARIANT, STRONG MONOIDAL FUNCTORS

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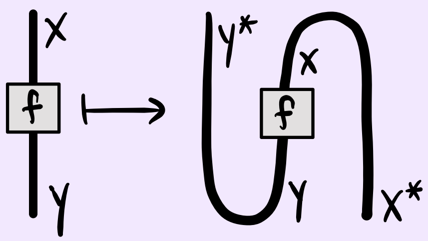

${}^*(-) : \mathcal{C}^{rev} \longrightarrow \mathcal{C}$

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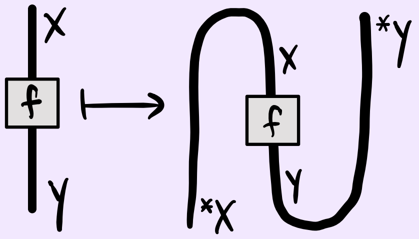
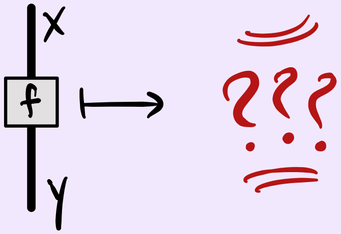


GET COVARIANT, STRONG MONOIDAL FUNCTORS

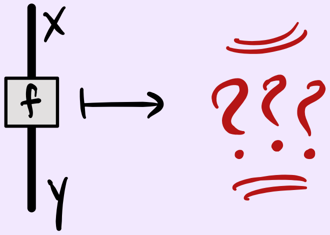
$(-)^{**} : \mathcal{C} \longrightarrow \mathcal{C}$

$X \longmapsto X^{**}$



${}^{**}(-) : \mathcal{C} \longrightarrow \mathcal{C}$

$X \longmapsto {}^{**}X$



I. RIGID CATEGORIES

DROPPING DIRECTIONAL ARROWS FOR BREVITY

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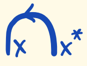

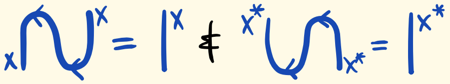

–AND–

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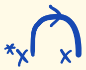
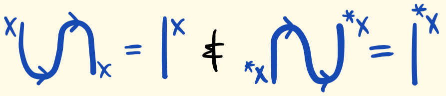




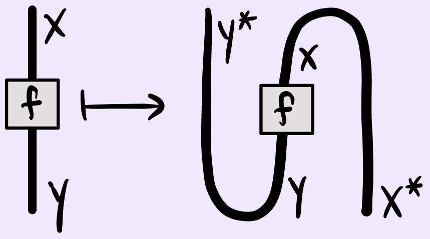
$\forall X \in \mathcal{C}$

COOL FACTS

GET CONTRAVARIANT, STRONG MONOIDAL FUNCTORS

$(-)^* : \mathcal{C}^{rev} \longrightarrow \mathcal{C}$

$X \longmapsto X^*$



${}^*(-) : \mathcal{C}^{rev} \longrightarrow \mathcal{C}$

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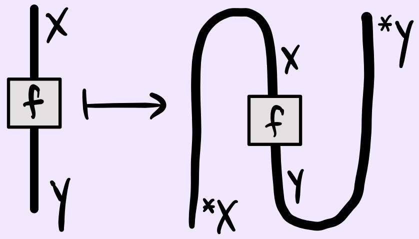
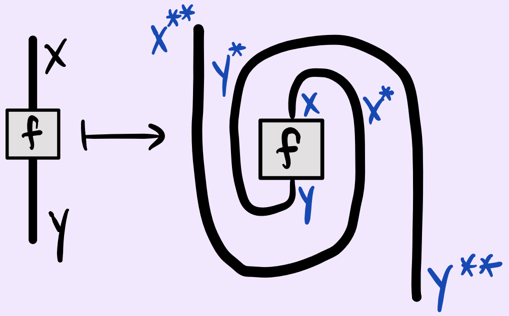


GET COVARIANT, STRONG MONOIDAL FUNCTORS

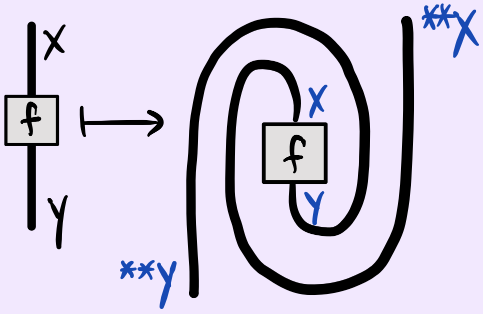
$(-)^{**} : \mathcal{C} \longrightarrow \mathcal{C}$

$X \longmapsto X^{**}$



${}^{**}(-) : \mathcal{C} \longrightarrow \mathcal{C}$

$X \longmapsto {}^{**}X$



I. RIGID CATEGORIES

RIGID CATEGORY

$$\big(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-)\big)$$

$\exists\, X^* \in \mathcal{C}$ WITH

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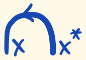

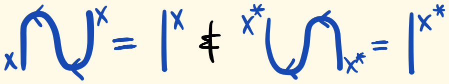

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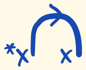

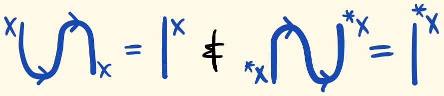

$\forall\, X \in \mathcal{C}$

COOL FACTS

RIGIDITY IS PRESERVED UNDER STRONG MONOIDAL FUNCTORS

I. RIGID CATEGORIES

RIGID CATEGORY

$$\Big(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, l, r}_{\text{MONOIDAL}}, (-)^*, {}^*(-)\Big)$$

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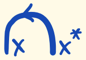

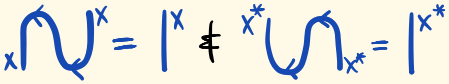

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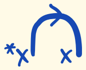

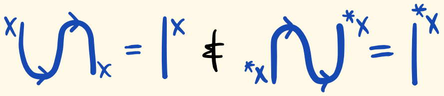

$\forall X \in \mathcal{C}$

COOL FACTS

RIGIDITY IS PRESERVED UNDER STRONG MONOIDAL FUNCTORS

IF $(F, F^{(2)}, F^{(0)}) : \mathcal{C} \longrightarrow \mathcal{D}$ IS STRONG MONOIDAL

THEN $(X^*, ev_X^L, coev_X^L)$ = LEFT DUAL OF $X \in \mathcal{C}$

$\Rightarrow$ $F(X^*)$ = LEFT DUAL OF $F(X) \in \mathcal{D}$ WITH:

$$ev^L_{F(X)} : F(X^*) \otimes^{\mathcal{D}} F(X) \xrightarrow[???]{} \mathbb{1}^{\mathcal{D}}$$

$$coev^L_{F(X)} : \mathbb{1}^{\mathcal{D}} \xrightarrow[???]{} F(X) \otimes^{\mathcal{D}} F(X^*)$$

I. RIGID CATEGORIES

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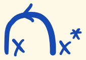
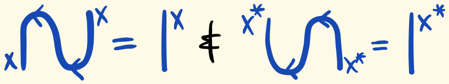
$coev_X^L : \mathbb{1} \to X \otimes X^*$ 



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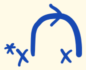
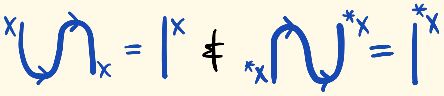
$coev_X^R : \mathbb{1} \to {}^*X \otimes X$ 



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$$ev^L_{F(X)} : \begin{tikzcd} F(X^*) \overset{\mathcal{D}}{\otimes} F(X) \arrow[r, "F^{(2)}_{X^*,X}"] & F(X^* \overset{\mathcal{C}}{\otimes} X) \arrow[r, "F(ev_X^L)"] & F(\mathbb{1}^{\mathcal{C}}) \arrow[r, "F^{(-0)}"] & \mathbb{1}^{\mathcal{D}} \end{tikzcd}$$

$$coev^L_{F(X)} : \begin{tikzcd} \mathbb{1}^{\mathcal{D}} \arrow[r, "F^{(0)}"] & F(\mathbb{1}^{\mathcal{C}}) \arrow[r, "F(coev_X^L)"] & F(X \otimes X^*) \arrow[r, "F^{(-2)}_{X,X^*}"] & F(X) \overset{\mathcal{D}}{\otimes} F(X^*) \end{tikzcd}$$

SIMILAR FOR RIGHT DUALS.

I. RIGID CATEGORIES

RIGID CATEGORY

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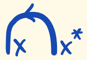
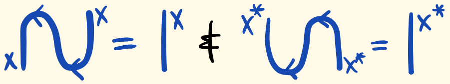
$coev_X^L : \mathbb{1} \to X \otimes X^*$ 



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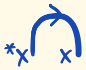
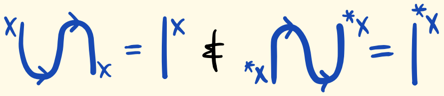
$coev_X^R : \mathbb{1} \to {}^*X \otimes X$ 



$\forall X \in \mathcal{C}$

COOL FACTS

TAKE MONOIDAL TRANSF'N ϕ BELOW

STRONG MONOIDAL

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left=60, "F"] \arrow[rr, bend right=60, "F'"'] \arrow[rr, phantom, "\otimes \ \Downarrow \phi"] & & \mathcal{D}
\end{tikzcd}$$

RIGID

JUST MONOIDAL

STRONG MONOIDAL

I. RIGID CATEGORIES

RIGID CATEGORY

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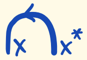

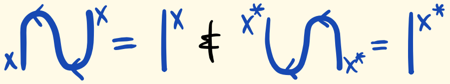

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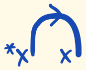

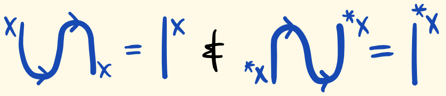

$\forall X \in \mathcal{C}$

COOL FACTS

TAKE MONOIDAL TRANSF'N ϕ BELOW

STRONG MONOIDAL

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left=50, "F"] \arrow[rr, bend right=50, "F'"'] \arrow[rr, phantom, "\overset{\otimes}{\cong}\ \Downarrow \phi"] & & \mathcal{D}
\end{tikzcd}$$

RIGID

JUST MONOIDAL

STRONG MONOIDAL

GET THAT ϕ IS A MONOIDAL ISOM.

I. RIGID CATEGORIES

RIGID CATEGORY

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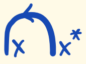

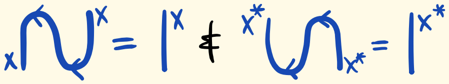

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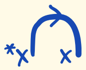

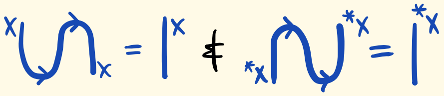

$\forall X \in \mathcal{C}$

VERIFIED IN VARIOUS EXERCISES

HAVE FUN!

COOL FACTS

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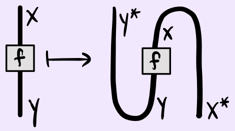
$X \longmapsto X^*$



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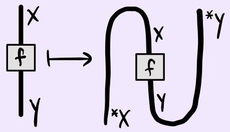

GET COVARIANT, STRONG MONOIDAL FUNCTORS

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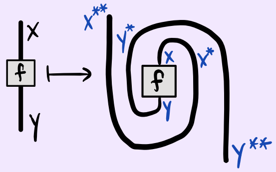

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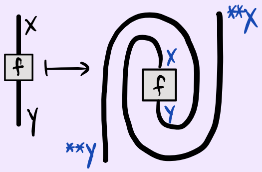

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& F(X^* \overset{\mathcal{C}}{\otimes} X) \arrow[r, "F(ev^L_X)"] & F(\mathbb{1}^{\mathcal{C}}) \arrow[ur, "F^{(-0)}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
coev^L_{F(X)} : \mathbb{1}^{\mathcal{D}} \arrow[dr, "F^{(0)}"'] & & & F(X) \overset{\mathcal{D}}{\otimes} F(X^*) \\
& F(\mathbb{1}^{\mathcal{C}}) \arrow[r, "F(coev^L_X)"] & F(X \otimes X^*) \arrow[ur, "F^{(-2)}_{X,X^*}"'] &
\end{tikzcd}$$

SIMILAR FOR RIGHT DUALS.

COOL FACTS

TAKE MONOIDAL TRANSF'N ϕ BELOW

$$\begin{tikzcd}[column sep=huge]
\underset{\text{RIGID}}{\mathcal{C}} \arrow[r, bend left=50, "\overset{\text{STRONG MONOIDAL}}{F}", ""{name=U, below}] \arrow[r, bend right=50, "\underset{\text{STRONG MONOIDAL}}{F'}"', ""{name=D, above}] & \underset{\text{JUST MONOIDAL}}{\mathcal{D}} \arrow[Rightarrow, from=U, to=D, "\phi", "\overset{\otimes}{\cong}"']
\end{tikzcd}$$

GET THAT ϕ IS A MONOIDAL ISOM.

II. PIVOTAL CATEGORIES

RIGID CATEGORY

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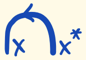

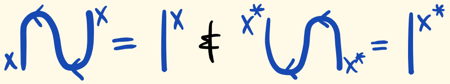

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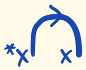

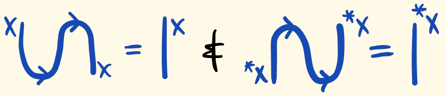

$\forall X \in \mathcal{C}$

GET CONTRAVARIANT, STRONG MONOIDAL FUNCTORS

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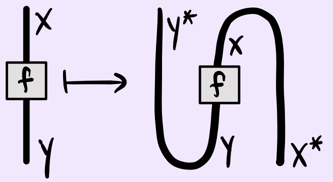

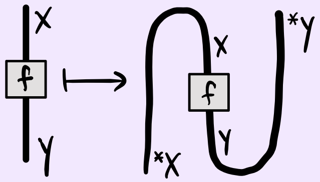

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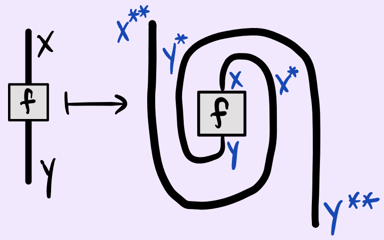

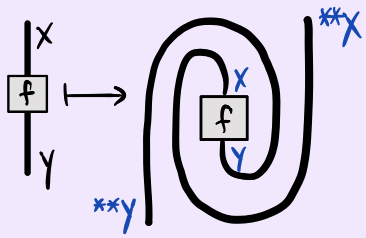

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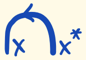
RIGID CATEGORY

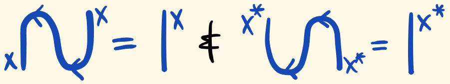
$$\big(\underbrace{\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r,}_{\text{MONOIDAL}} (-)^*, {}^*(-)\big)$$

$\exists\, X^* \in \mathcal{C}$ WITH

$ev^L_X : X^* \otimes X \longrightarrow \mathbb{1}$

$coev^L_X : \mathbb{1} \longrightarrow X \otimes X^*$





–AND–

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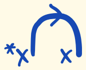
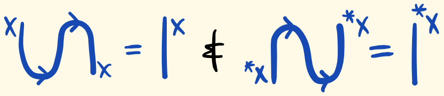




$\forall\, X \in \mathcal{C}$

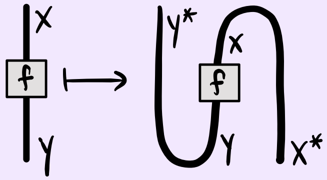
THESE ARE RIGID CATEGORIES WHERE

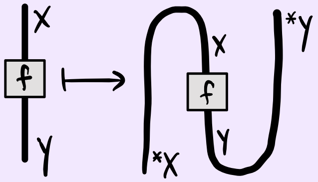
$$(-)^* \overset{\otimes}{\cong} {}^*(-)$$

GET CONTRAVARIANT, STRONG MONOIDAL FUNCTORS

$(-)^* : \mathcal{C}^{rev} \longrightarrow \mathcal{C}$ $\qquad$ ${}^*(-) : \mathcal{C}^{rev} \longrightarrow \mathcal{C}$

$X \longmapsto X^*$ $\qquad$ $X \longmapsto {}^*X$





GET COVARIANT, STRONG MONOIDAL FUNCTORS

$(-)^{**} : \mathcal{C} \longrightarrow \mathcal{C}$ $\qquad$ ${}^{**}(-) : \mathcal{C} \longrightarrow \mathcal{C}$

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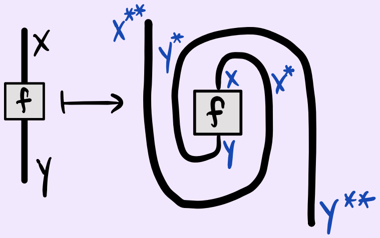

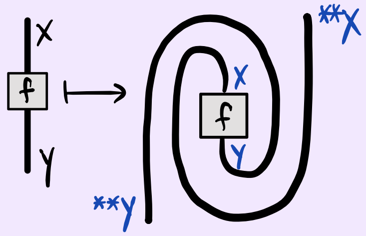

$$(-)^{**} \overset{\otimes}{\cong} \mathrm{Id}_{\mathcal{C}} \overset{\otimes}{\cong} {}^{**}(-)$$

II. PIVOTAL CATEGORIES

RIGID CATEGORY

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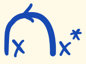

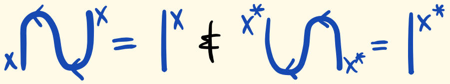

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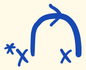

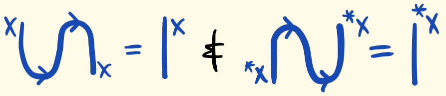

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THIS IS AN EXTRA STRUCTURE

GET CONTRAVARIANT, STRONG MONOIDAL FUNCTORS

$(-)^* : \mathcal{C}^{rev} \longrightarrow \mathcal{C}$

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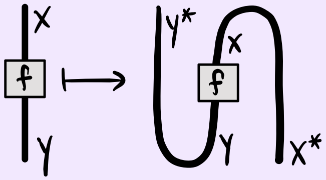

${}^*(-) : \mathcal{C}^{rev} \longrightarrow \mathcal{C}$

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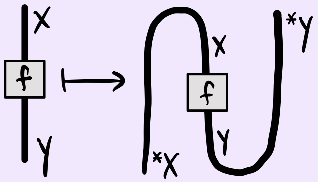

GET COVARIANT, STRONG MONOIDAL FUNCTORS

$(-)^{**} : \mathcal{C} \longrightarrow \mathcal{C}$

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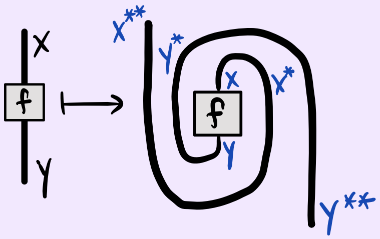

${}^{**}(-) : \mathcal{C} \longrightarrow \mathcal{C}$

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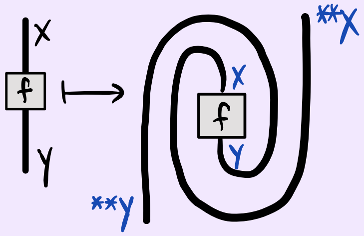

$$(-)^{**} \overset{\otimes}{\cong} \mathrm{Id}_{\mathcal{C}} \overset{\otimes}{\cong} {}^{**}(-)$$

II. PIVOTAL CATEGORIES

RIGID CATEGORY

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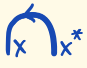

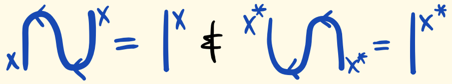

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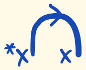

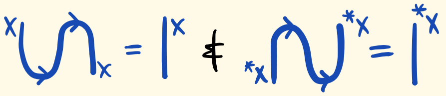

$\forall X \in \mathcal{C}$

THESE ARE RIGID CATEGORIES WHERE

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THIS IS AN EXTRA STRUCTURE

E.G. ...

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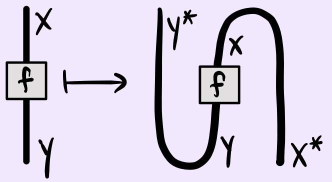

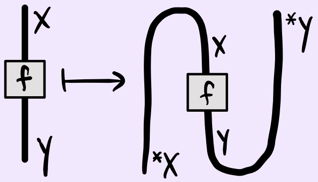

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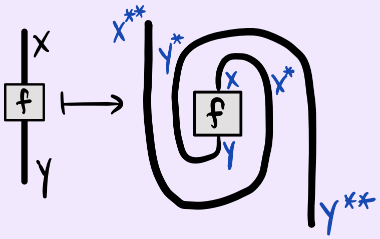

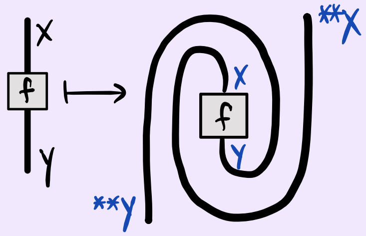

$$\exists\, j : Id_{\mathcal{C}} \xRightarrow[\sim]{\otimes} (-)^{**}$$

II. PIVOTAL CATEGORIES

RIGID CATEGORY

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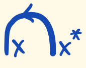

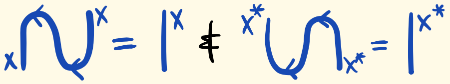

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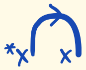

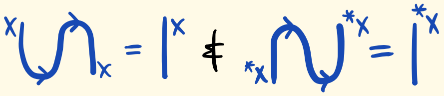

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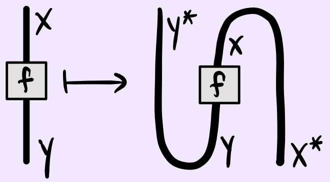

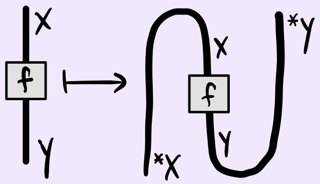

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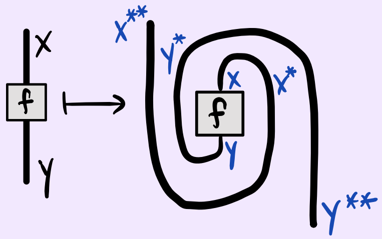

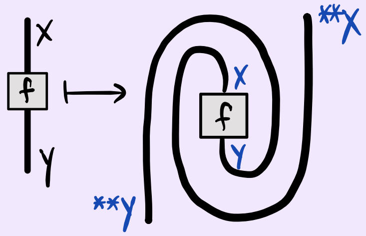

CALLED A PIVOTAL STRUCTURE

$\exists j : \mathrm{Id}_{\mathcal{C}} \overset{\otimes}{\underset{\sim}{\Longrightarrow}} (-)^{**}$

II. PIVOTAL CATEGORIES

RIGID CATEGORY

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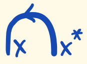

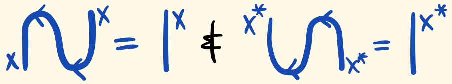

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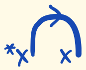

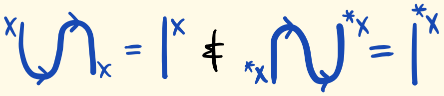

$\forall X \in \mathcal{C}$

THESE ARE RIGID CATEGORIES WHERE

$$\exists \hat{j} : (-)^* \overset{\otimes}{\underset{\sim}{\Longrightarrow}} {}^*(-)$$

THIS IS AN EXTRA STRUCTURE

E.G. . . .

COOL FACT FOR $\mathcal{C}$ RIGID

$\exists$ PIVOTAL STRUCTURE j ON $\mathcal{C}$

$\iff$ $\exists$ NAT'L ISOM. $\hat{j}$ AS ABOVE.

CALLED A PIVOTAL STRUCTURE

$$\exists j : Id_{\mathcal{C}} \overset{\otimes}{\underset{\sim}{\Longrightarrow}} (-)^{**}$$

II. PIVOTAL CATEGORIES

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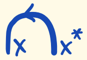
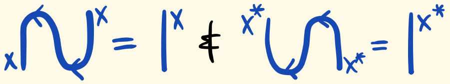
$coev_X^L : \mathbb{1} \longrightarrow X \otimes X^*$ 



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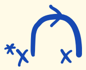
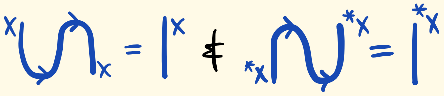
$coev_X^R : \mathbb{1} \longrightarrow {}^*X \otimes X$ 



$\forall X \in \mathcal{C}$

THIS IS AN EXTRA STRUCTURE

E.G...

COOL FACT FOR $\mathcal{C}$ RIGID

$\exists$ PIVOTAL STRUCTURE j ON $\mathcal{C}$

$\Longleftrightarrow$ $\exists$ NAT'L ISOM. $\hat{j}$ AS ABOVE.

EXAMPLES INCLUDE:

FdVec G-FdMod

CALLED A PIVOTAL STRUCTURE

$$\exists\, j : Id_{\mathcal{C}} \xRightarrow{\overset{\otimes}{\sim}} (-)^{**}$$

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SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #15

THERE'S SO MUCH
MORE WE COULD COVER

TOPICS:

I. RIGID CATEGORIES (§3.6)
II. PIVOTAL CATEGORIES (§3.7)

MATH 466/566
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LECTURE #15

THERE'S SO MUCH MORE WE COULD COVER

BUT WE'RE OUT OF TIME ☹ .... PLEASE READ MORE IF you'RE CURIOUS ☺

TOPICS:

✓ I. RIGID CATEGORIES (§3.6)

✓ II. PIVOTAL CATEGORIES (§3.7)

MATH 466/566
SPRING 2024

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LECTURE #15

PLEASE READ MORE
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TOPICS:

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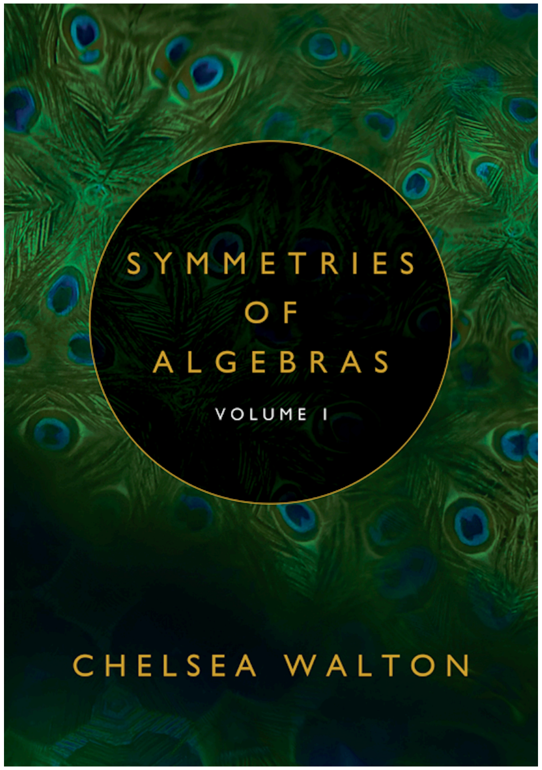
✓ II. PIVOTAL CATEGORIES (§3.7)

NEXT TIME: FUSION CATEGORIES



Lecture #15 keywords: coevaluation, dual of an object, evaluation, pivotal category, rigidity axioms, rigid category, rigid object