MATH $466 / 566$
SPRING 2024

LAST TM ME

- rigid categories
- pivotal categories
chelsea walton RICE U.

TOPICS:
I. Fusion categories
(853.9.1, 3.9.3)
II. FUSION RULES \& RANK (\$\$3.9.1, 3.9.3)
III. FROBENIUS-PERRON DIMENSION (\$3.9.2)
I. Fusion categories

1
SUPER NICE monoidal categories
I. Fusion categories

A MONOIDAL CATEGORY

$$
\zeta:=(\zeta, \otimes, \mathbb{L}, a, l, r)
$$

IS FUSION IF:
(a) Cis abelian
(b) Cis (in-) LINEAR
(c) G is locally finite
(d) $\mathbb{L}$ IS ABSOLUTELY SIMPLE
(e) $C$ is RIGID
(f) G is semisimple
(g) G is finite
I. Fusion categories

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I. Fusion categories

A MONOIDAL CATEGORY

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\mathfrak{b}:=(\mathfrak{l}, \otimes, \mathbb{u}, a, l, r)
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IS FUSION IF:
$\operatorname{HOM}_{b}(x, y) \in A b$ $\forall x, y \in \zeta$


PREADDITIVE CATEGORY
(a) C is abelian
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A MONOIDAL CATEGORY $b:=(\zeta, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
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SUPER NICE


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SUPER NICE

are additive $\forall x \in \zeta$
AbELIAN MONOIDAL:

$$
X \otimes-1-\otimes X
$$

ARE ADDITIVE $\forall x \in G$

I. FUSION categories

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SUPER NICE

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I. Fusion categories

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SUPER NICE

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IS FUSION IF:
$H_{0} M_{C}(x, y) \in V e c$ $\forall x, y \in \zeta$
$\downarrow$ DEF
LINEAR CATEGORY
(a) Cis abelian
(b) Cis (in-) LINEAR
(c) C IS LOcAlly FINITE
(d) $\mathbb{L}$ is Absolutely simple
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SUPER NICE
monoidal categories

LINEAR MONOIDAL:

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x \otimes-1-\otimes x
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are linear $\forall x \in \zeta$
I. Fusion categories

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SUPER NICE
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I. Fusion categories

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SUPER NICE
Monoidal categories

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(g) G is finite
I. Fusion categories
a monoidal category $\zeta:=(\zeta, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
(a) Cis abelian

$$
X^{\not{ }^{0} \in \zeta \text { IS SEMISIMPLE } \mathbb{F} X \cong \bigcup_{i \in I} X_{i}}
$$

For simple objects $X_{i}$.
$\xi$ is semisimple if all objects are ss
(b) Cis (in-) LINEAR
(c) I is locally finite
(d) $\mathbb{L}$ is Absolutely simple
(e) C IS RIGID
(f) 6 Is SEmisimple
(g) G is finite
I. Fusion categories

A MONOIDAL CATEGORY $b:=(l, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
(a) C is abelian
(b) Cis (in-) LINEAR
(c) I Is LOCALLY FINITE
(d) $\mathbb{L}$ is Absolutely simple
(e) C IS RIGID
(f) G is semisimple
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MONOIDAL CATEGORIES
$X^{\not{ }^{0} \in \zeta \text { is semisimple }} \mathfrak{F} X \cong \mathbb{U}_{\text {ie }} X_{i}$
For simple objects $X_{i}$.
$G$ is semisimple if all objects are ss
simple
OBJECTS
$x^{\neq 0} e$ is simple
IF THE ONLY SUBBOETS OF $X$ Are $X \neq 0$
indecomposable OBJECTS
$X^{\not{ }^{0} \in G \text { IS INDECOMPOSABLE }}$
IF $X \neq X_{1} \cup X_{2}$
$\forall$ Nonzero subObJ. $X_{1}, X_{2}$ of $X$
I. Fusion categories

A MONOIDAL CATEGORY $b:=(l, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
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MONOIDAL CATEGORIES

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X^{\neq 0} \in \zeta \text { is semisimple } \mathbb{F} X \cong \mathbb{U}_{i \in I} X_{i}
$$

For simple objects $X_{i}$.
$G$ is semisimple if all objects are ss


IF THE ONLY SUBOBJS OF $X$ are $X \neq 0$
indecomposable OBJECTS
$X^{+0}$ +e IS In decomposable
IF $X \neq X_{1} \cup X_{2}$
$\forall$ Nonzero dubobs. $X_{1}$, $X_{2}$ of $X$
I. Fusion categories

A MONOIDAL CATEGORY $l \cdot=(l, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
(a) C is abelian
(b) Cis (in-) LINEAR
(c) $\mathbb{Z}$ is locally finite
(d) $\mathbb{L}$ is absolutely simple
(e) C IS RIGID
(f) G is semisimple
(g) $G$ IS FINITE

MONOIDAL CATEGORIES
$X^{\neq 0} \in \zeta$ is semisimple $\mathbb{F} X \cong \mathbb{U}_{\text {LeI }} X_{i}$
For simple objects $X_{i}$.
$G$ is semisimple if all objects are ss
 OBJECTS

PROP:
in semisimple categories, INDECOMPOSABLE OBJECTS OF FINITE LENGTH are simple.

Semisimple CATEGORIES
I. Fusion categories
a MONOIDAL CATEGORY

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\mathfrak{l}:=(\boldsymbol{l}, \otimes, \mathbb{u}, a, l, r)
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IS FUSION IF:
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monoidal categories

$$
\forall x \in \xi_{e}:
$$

$$
\begin{aligned}
& \exists X^{*} \in \zeta \text { with } \\
& \mathrm{ev}_{x}^{L}: X^{*} \otimes X \rightarrow \mathbb{L} X^{*}\left({ }^{x}\right) \\
& \left.\operatorname{cosv}_{x}^{L}: \mathbb{1} \rightarrow X \otimes X^{*} \widehat{X}_{x}\right)_{x^{*}} \\
& x^{n} \bigcup^{x}=\left.\right|^{x} \neq x^{*}\left(\Omega_{x^{*}}=\left.\right|^{x^{*}}\right. \\
& \text {-AND - }
\end{aligned}
$$

I. Fusion categories

A MONOIDAL CATEGORY $l:=(l, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
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$$
\forall x \in \zeta:
$$

-AND -

$$
\begin{gathered}
\exists^{*} X \in \mathscr{C} \text { wITH } \\
\operatorname{ev}_{x}^{R}: X \otimes^{*} X \rightarrow \mathbb{U} \bigcup^{X} J^{* x} \\
\operatorname{coev}_{x}^{R}: \mathbb{U} \rightarrow{ }^{*} X \otimes X{ }_{* x} \Gamma_{x} \\
{ }^{x} \prod_{x}=\left.\right|^{x} \neq{ }_{*_{x}} \|_{U^{*}}^{*}=\left.\right|^{*}
\end{gathered}
$$

I. Fusion categories

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End $(\mathbb{U}) \cong \mathbb{k}$
I. Fusion categories

A MONOIDAL CATEGORY $b:=(l, \otimes, \mathbb{L}, a, l, r)$

IS FUSION IF:

End ec $(\mathbb{1}) \cong \mathbb{k}$

1 Is A SIMPLE OBJECT OF $\zeta$
(a) G is abelian *
(b) Cis (kkk-) LINEAR $*$
(c) I is locally finite
(d) $\mathbb{L}$ IS AbSOLUTELY SIMPLE
(e) $C$ is RIGID *
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(d) $\mathbb{L}$ Is Absolutely simple
(e) $C$ is RIGID *
(f) G is semisimple
(g) G is finite

Ende $(\mathbb{1}) \cong \mathbb{I} k$
$\mathscr{L}$ is a simple object of 6
PF/Take a subobject of $\mathbb{L}$ :
$(X,($ cactic $: X \rightarrow \mathbb{4})$ Wlor monero, SIMPLE
GET S.E.S. $0 \rightarrow X \hookrightarrow \mathbb{H} \rightarrow \operatorname{coker}(1) \rightarrow 0$.
I. Fusion categories

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Endec $(\mathbb{1}) \cong \mathbb{k}$

PF/TAKE A sUbobJect of $\mathbb{L}$ :
$(X, \stackrel{(\text { nowos) }}{i}: X \rightarrow \mathbb{U})$ WLOG MNEERO, SIMPLE GET S.E.S. $0 \rightarrow X \hookrightarrow \backsim \longrightarrow \operatorname{coker}(\mathrm{l}) \rightarrow 0$. $(-)^{*}$ IS EXACT $\Rightarrow$
GET S.E.S. $0 \rightarrow \operatorname{arker}(1)^{*} \rightarrow \mathbb{H} \xrightarrow{*} x^{*} \rightarrow 0$.
$(X \otimes-)$ EXACT $\Rightarrow$ GET S.E.S.:
$0 \rightarrow X \otimes \operatorname{coker}()^{*} \longrightarrow X \xrightarrow{\text { id } \otimes \varepsilon^{*}} X \otimes X^{*} \rightarrow 0$
I. Fusion categories

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End $(\mathbb{1}) \cong \mathbb{k}$
$\Downarrow$ Assume *
1 Is A SIMPLE OBJECT OF $\zeta$
PF/ Take a subobject of $\mathbb{L}$ :

GET S.E.S. $0 \rightarrow X \xrightarrow{\imath} \mathbb{\longrightarrow} \operatorname{coker}(1) \rightarrow 0$.
$(-)^{*}$ IS EXACT $\Rightarrow$
GET S.E.S. $0 \rightarrow \operatorname{arker}(1)^{*} \rightarrow \mathbb{H} \xrightarrow{*} x^{*} \rightarrow 0$.
$(X \otimes-)$ EXACT $\Rightarrow$ GET S.E.S.:
$0 \rightarrow X \otimes \operatorname{coker}(1)^{*} \longrightarrow X \xrightarrow{\text { id ss. }} X \otimes X^{*} \rightarrow 0$
$x$ sImple $\Rightarrow$ id $x \otimes i^{*}$ manic
I. Fusion categories

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$(X \otimes-)$ EXACT $\Rightarrow$ GET S.E.S.:

$$
0 \rightarrow X \otimes \operatorname{coker}\left((1)^{*} \longrightarrow X \xrightarrow{\text { id } 8 c^{*}} X \otimes X^{*} \rightarrow 0\right.
$$

$X$ sImple $\Rightarrow i d x \otimes i^{*}$ manic
$\Rightarrow i d x \otimes c^{*} 150$.
I. Fusion categories

A MONOIDAL CATEGORY $b:=(l, \otimes, \mathbb{L}, a, l, r)$

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Ende $(\mathbb{1}) \cong \mathbb{k}$
$\mathcal{L}$ IS A SIMPLE OBJECT OF 6
PF/TAKE A sUbObJECT OF $\mathbb{L}$ :
$(X, \stackrel{(\text { nouro }}{i}: X \rightarrow \mathbb{U})$ WLOG MNUERR, SIMPLE
GET $i d_{X} \otimes \&^{*}: X \rightarrow X \otimes X^{*}$ AN $1 \delta 0$.
I. Fusion categories

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(g) G is finite

End ec $(\mathbb{1}) \cong \mathbb{k}$

11 is a simple object of $\zeta$
PF/ TAKE A subobject of $\mathbb{L}$ :

GET $i d_{X} \otimes i^{*}: X \rightarrow X \otimes X^{*}$ an 180.
Now $\phi: \mathbb{U} \xrightarrow{\operatorname{coev}_{x}} X \otimes X^{*} \xrightarrow{\left(i d \otimes \mathcal{C}^{*}\right)^{-1}} X$
Is A NONZERO MORPHISM.
So, $\mathbb{U} \xrightarrow{\phi} X \xrightarrow{\longrightarrow} \mathbb{L}$ is NONZERO
1 Absolutely simple $\Rightarrow \angle \phi=\lambda^{e^{\| k^{x}} \text { id }}$
I. Fusion categories

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(g) G IS FINITE

End $(\mathbb{1}) \cong \mathbb{k}$
$\mathcal{L}$ IS A SIMPLE OBJECT OF $\zeta$
PF/ Take a subobject of $\mathbb{1}$ :

GET $i d_{X} \otimes i^{*}: X \rightarrow X \otimes X^{*}$ an 180.
Now $\phi: \mathbb{U} \xrightarrow{\operatorname{coev}_{x}^{\iota}} X \otimes X^{*} \xrightarrow{\left(i d \otimes B^{*}\right)^{-1}} X$
Is A NONZERO MORPHISM.
So, $\mathbb{U} \xrightarrow{\phi} X \xrightarrow{\longrightarrow} \mathbb{L}$ IS NONZERO
11 Absolutely simple $\Rightarrow L \phi=\lambda$ id ${ }^{\text {ul k }}$
$\therefore 1 \varnothing$ IS AN $150 \Rightarrow$ L $\varnothing$ EPIC
I. Fusion categories

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(e) $\zeta$ is RIGID *
(f) G is semisimple
(g) G IS FINITE

End ec $(\mathbb{1}) \cong \mathbb{k}$
$\Downarrow$ Assume *
1 Is A SIMPLE OBJECT OF $G$
PF/ Take a subobject of $\mathbb{L}$ :
$(X, \stackrel{\text { naos) }}{i}: X \rightarrow \mathbb{U})$ WOG MNUERR, SIMPLE
GET $i d_{X} \otimes i^{*}: X \rightarrow X \otimes X^{*}$ an 180.
Now $\phi: \mathbb{U} \xrightarrow{\operatorname{coev}_{x}^{\iota}} X \otimes X^{*} \xrightarrow{\left(i d \otimes B^{*}\right)^{-1}} X$
Is A NONZERO MORPHISM.
So, $\mathbb{U} \xrightarrow{\phi} X \xrightarrow{\longrightarrow}$ is NONZERO
11 Absolutely simple $\Rightarrow 1 \phi=\lambda^{\text {di }^{\mu x^{x}}}$ id
$\therefore 1 \varnothing$ IS AN $150 \Rightarrow 1 \varnothing$ EPIC

$$
\Rightarrow \mathbb{I E P I C} \quad \therefore X \cong \mathbb{U} \mathbb{Z}
$$

I. Fusion categories

A MONOIDAL CATEGORY $b:=(l, \otimes, \mathbb{L}, a, l, r)$

IS FUSION IF:
(a) C is abelian *
(b) Cis (ik-)LINEAR $*$
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I. Fusion categories

A MONOIDAL CATEGORY $b:=(l, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
(a) C is abelian *
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(c) $\zeta$ is LOCALLy FINITE**,
(d) $\mathbb{L}$ IS AbSOLUTELY SIMPLE
(e) $C$ is RIGID *
(f) G is semisimple
(g) G is finite

End $(\mathbb{1}) \cong \mathbb{\cong}$
介 Assume , $_{\text {** }}$
$\mathscr{L}$ is a simple object of 6
PF/
IL simple

$$
\sqrt{\Downarrow} \text { [SCHuR's LemmA }]
$$

Ende(4)
DIVISION ALGEBRA/ ${ }^{\mathbb{R}}$
I. Fusion categories

A MONOIDAL CATEGORY $b:=(l, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
(a) C is abelian *
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(c) $\zeta$ is LOCALLy FINITE**,
(d) $\mathbb{L}$ IS AbSOLUTELY SIMPLE
(e) $C$ is RIGID *
(f) G is semisimple
(g) G is finite

End (ex) $\cong$ \|k
P ASsume *, **
$\mathcal{L}$ IS A SIMPLE OBJECT OF $G$
PF/
4 simple

$$
\sqrt{ }\left[*^{*}, \text { schur's Lemma }\right]
$$

Ende(14) FINITE-DIMENSIONAL DIVISION ALGEBRA/ IR
I. Fusion categories

A MONOIDAL CATEGORY $b:=(l, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
(a) C is abelian *
(b) Cis (ik-)LINEAR $*$
(c) $\zeta$ is locally finite **,
(d) $\mathbb{L}$ IS AbSOLUTELY SIMPLE
(e) $\zeta$ is RIGID *
(f) G is semisimple
(g) $G$ IS FINITE

End $(\mathbb{1}) \cong \mathbb{\cong}$
M ASSUME *)**
$\mathcal{L}$ IS A SIMPLE OBJECT OF 6
PF/
4 simple

$$
\sqrt{ }\left[*^{*}, \text { schur's Lemma }\right]
$$

Ends (14) FInIte-dimensional DIVISION ALGEBRA/ $\mathbb{R}$ $\forall$

$$
\left[\begin{array}{c}
\text { LINEAR ALGEBRA } \\
\text { ARGUMENT }
\end{array}\right]
$$

$$
\operatorname{End}_{e}(\mathbb{1}) \cong \mathbb{k}
$$

III
I. Fusion categories

A MONOIDAL CATEGORY $\mathfrak{b}:=(\boldsymbol{b}, \otimes, \mathbb{L}, a, l, r)$ is FUSION IF:
(a) G is abelian
(b) G is (in-) UINEAR
(c) I is loo. FINITE
(d) $\mathbb{L}$ is Abs. simple
(e) $\mathscr{C}$ is RIGID
(f) G is semisimple
(g) Y is FINITE

When are monodic Categories

TWO FUSION CATEGORIES considered the same?
I. FUSION CATEGORIES

A MONOIDAL CATEGORY
$b:=(\boldsymbol{b}, \otimes, \mathbb{L}, a, l, r)$
is FUSION IF:
(a) I is abelian
(b) G is (in-) linear
(c) li is LOC. FINITE
(d) $\mathbb{L}$ is Abs. simple
(e) C is RIGID
(f) G is semisimple
(g) Y is finite

When are
SUPER NICE MONOIDAL CATEGORIES TWO FUSION CATEGORIES CONSIDERED THE SAME?
0
0
0
0

Which items are structural??
and which are properties??
I. Fusion categories


When are
TWO FUSION CATEGORIES
considered the same?
I. Fusion categories

when are
TWO FUSION CATEGORIES
considered the same?
I. Fusion categories

when are
TWO FUSION CATEGORIES
CONSIDERED THE SAME?

I. Fusion categories

A MONOIDAL CATEGORY $\zeta_{l}:=(\boldsymbol{l}, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
(a) Ce is abelian
(b) Cis (in-) LINEAR
(c) C is loo. FINITE
(d) $\mathbb{L}$ is Abs. SIMPLE
(e) C IS RIGID
(f) G is semisimple
(g) $\zeta$ is finite
I. Fusion categories

A MONOIDAL CATEGORY $\mathfrak{l}:=(\boldsymbol{l}, \otimes, \mathbb{U}, a, l, r)$ IS FUSION IF:
(a) G is abelian
(b) Gis (IR-) LINEAR
(c) I is loc. FINITE
(d) $\mathbb{L}$ is Abs. simplé
(e) $C$ is RIGID
(f) $\zeta$ ls semisimple
(g) C is finite



I. Fusion categories

A MONOIDAL CATEGORY $\mathfrak{l}:=(\boldsymbol{l}, \otimes, \mathbb{U}, a, l, r)$ IS FUSION IF:
(a) G is abelian
(b) Gis (IR-) LINEAR
(c) I is LOc. FINITE
(d) $\mathbb{L}$ is Abs. simplé
(e) $C$ is RIGID
(f) $\zeta$ ls semisimple
(g) C is finite


EXAMPES



I. Fusion categories

A MONOIDAL CATEGORY $\mathfrak{l}:=(\boldsymbol{l}, \otimes, \mathbb{U}, a, l, r)$ IS FUSION IF:
(a) G is abelian
(b) Gis (IR-) LINEAR
(c) I is LOc. FINITE
(d) $\mathbb{L}$ is Abs. simplé
(e) $C$ is RIGID
(f) $\zeta$ ls semisimple
(g) C is finite

I. Fusion categories

A MONOIDAL CATEGORY $\mathfrak{l}:=(\boldsymbol{l}, \otimes, \mathbb{U}, a, l, r)$ IS FUSION IF:
(a) G is abelian
(b) Gis (ik-) LINEAR
(c) I is LOc. FINITE
(d) $\mathbb{L}$ is Abs. dimplé
(e) $C$ is RIGID
(f) 6 is semisimple
(g) $\zeta$ Is finite
(Just one reason)

I. Fusion categories

A MONOIDAL CATEGORY $\mathfrak{l}:=(\boldsymbol{l}, \otimes, \mathbb{U}, a, l, r)$ IS FUSION IF:
(a) G is abelian
(b) Ge is (IR-) LINEAR
(c) I is LOc. FINITE
(d) $\mathbb{L}$ is Abs. simplé
(e) $C$ is RIGID
(f) $\zeta$ ls semisimple
(g) C is finite

I. Fusion categories

A MONOIDAL CATEGORY $\mathfrak{l}:=(\boldsymbol{l}, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
(a) G is abelian
(b) Gis (ik-) LINEAR
(c) I is LOc. FINITE
(d) $\mathbb{L}$ is Abs. dimplé
(e) Ce is RIGID
(f) द is semisimple $X$
(g) C is finite
(Just one reason)

I. Fusion categories

A MONOIDAL CATEGORY $\mathfrak{l}:=(\boldsymbol{l}, \otimes, \mathbb{U}, a, l, r)$ IS FUSION IF:
(a) G is abelian
(b) Gis (ik-) LINEAR
(c) I is LOc. FINITE
(d) $\mathbb{L}$ is Abs. dimplé
(e) $C$ is RIGID
(f) $\zeta$ ls semisimple
(g) C is finite

I. Fusion categories

A MONOIDAL CATEGORY $\mathfrak{l}:=(\boldsymbol{l}, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
(a) G is abelian
(b) G is (in-) LINEAR
(c) lis loo. FINITE
(d) $\mathbb{L}$ is abs. simple
(e) $C$ is rigid $X$
(f) G is semisimple
(g) $\zeta$ Is finite
(Just one reason)

I. FUSION CATEGORIES

A MONOIDAL CATEGORY $\zeta:=(b, \otimes, \mathbb{L}, a, l, r)$

IS FUSION IF:
(a) G is abelian
(b) Ge is (ik-) LINEAR
(c) l is loc. FINITE
(d) $\mathbb{L}$ is Abs. simplé
(e) $C$ is RIGID
(f) G is semisimple
$(g)$ e is finite

I. FUSION CATEGORIES

A MONOIDAL CATEGORY $\zeta:=(b, \otimes, \mathbb{L}, a, l, r)$

IS FUSION IF:
(a) G is abelian
(b) Ge is (ik-)linear
(c) l is loo. FINITE
(d) $\mathbb{L}$ is Abs. SIMPLE
(e) $C$ is RIGID
(f) $\zeta$ is semisimple $X$
$(g)$ IS FINITE
(Just one reason)


G arbitrary group

I. FUSION CATEGORIES

A MONOIDAL CATEGORY $\zeta:=(b, \otimes, \mathbb{L}, a, l, r)$

IS FUSION IF:
(a) G is abelian
(b) Ge is (ik-) LINEAR
(c) l is loc. FINITE
(d) $\mathbb{L}$ is Abs. Simplé
(e) $C$ is RIGID
(f) G is semisimple
$(g)$ e is finite



II. FUSION RULES \& RANK

EXAMPLES


SIMPLE OBJECTS G FINITE GROUP
II. FUSIONRULES \& RANK

A MONOIDAL CATEGORY Irr $\left(\zeta_{l}\right):=$ SET OF ISOCLASSES [X] $\mathfrak{C}:=(\boldsymbol{b}, \otimes, \mathbb{Q}, a, l, r)$
is FUSION IF:
(a) I is abelian
(b) Gis (lk-) UINEAR
(c) l is loc. FINITE
(d) $\mathbb{L}$ is Abs. SIMple
(e) $C$ is RIGID
(f) 6 is semisimple
(g) Y is finite
$\uparrow$
GOVERNED BY simple objects
II. FUSION RULES \& RANK

A MONOIDAL CATEGORY

$$
\mathfrak{b}:=(\boldsymbol{b}, \otimes, \mathbb{\mathbb { L }}, a, l, r)
$$

is FUSION IF:
(a) G is abelian
(b) G is (k-) linear
(c) I is LDc. FINITE
(d) $\mathbb{L}$ is Abs. Simple
(e) Ce is RIGID
(f) C is semisimple
(g) Y is finite
$\uparrow$
GOVERNED BY
simple objects
$\operatorname{Irr}\left(\zeta_{e}\right):=$ SET OF ISOCLASSES [X] OF SIMPLE OBJECTS OF $\zeta^{6}$
$\zeta$ FINITE $\Rightarrow \underbrace{|\operatorname{Irr}(\zeta)|}_{\text {RANK OF } \zeta}<\infty$
FdVec
$\uparrow$
RANK 1
$\operatorname{Irr}(r)=\{[\mathbb{R}]\}$
II. FUSION RULES \& RANK


GOVERNED BY
simple objects
II. FUSION RULES \& RANK

II. FUSION RULES \& RANK

II. FUSION RULES \& RANK

II. FUSION RULES \& RANK

II. FUSION RULES \& RANK

II. FUSION RULES \& RANK

$$
\begin{aligned}
& \text { A MONOIDAL CATEGORY } \\
& \boldsymbol{b}:=(\boldsymbol{b}, \otimes, \mathbb{L}, a, l, r)
\end{aligned}
$$

is FUSION IF:
(a) C is abelian
(b) G is (k-) linear
(c) I is LDc. FINITE
(d) $\mathbb{L}$ is Abs. SIMple
(e) G is RIGID
(f) G is semisimple
(g) Y is FINITE
$\uparrow$
GOVERNED BY
simple objects
$\operatorname{Irr}\left(\zeta_{e}\right):=$ SET OF ISOCLASSES [X] OF SIMPLE OBJECTS OF $\zeta^{6}$
$\zeta$ FINITE $\Rightarrow \underbrace{|\operatorname{Irr}(\zeta)|}_{\text {RANK OF } \zeta}<\infty$

Research problem
CLASSIFY ALL FUSION CATEGORIES OF A GIVEN RANK, UP TO EQUIV.
II. FUSION RULES \& RANK

$$
\begin{aligned}
& \text { A MONOIDAL CATEGORY } \\
& \boldsymbol{b}:=(\boldsymbol{l}, \otimes, \mathbb{l}, a, l, r)
\end{aligned}
$$

is FUSION IF:
(a) G is abelian
(b) G is (k-) linear
(c) I is LDc. FINITE
(d) $\mathbb{L}$ is Abs. SIMple
(e) G is RIGID
(f) G is semisimple
(g) Y IS FINITE
$\uparrow$
GOVERNED BY
SIMPLE OBJECTS
$\operatorname{Irr}\left(\zeta_{e}\right):=$ SET OF ISOCLASSES [X] OF SIMPLE OBJECTS OF $\zeta$
$\zeta$ FINITE $\Rightarrow \underbrace{|\operatorname{Irr}(\zeta)|}_{\text {RANK OF } \zeta}<\infty$

Research problem
CLASSIFY ALL FUSION CATEGORIES OF A GIVEN RANK, UP TO EQUIV.
$\checkmark$ RANK 1: FdVec
II. FUSION RULES \& RANK


$$
\begin{aligned}
& \text { A MONOIDAL CATEGORY } \\
& b:=(\boldsymbol{l}, \otimes, l, a, l, r)
\end{aligned}
$$

is FUSION IF:
(a) C is abelian
(b) G is (ak-) linear
(c) le is hoc. FINITE
(d) $\mathbb{L}$ is Abs. dimple
(e) $C$ is RIGID
(f) C is semisimple
(g) C IS FINITE
$\uparrow$
GOVERNED BY
simple objects
$\operatorname{Irr}\left(\zeta_{e}\right):=$ SET OF ISOCLASSES [X] OF SIMPLE OBJECTS OF C
$\zeta$ FINITE $\Rightarrow \underbrace{|\operatorname{Irr}(\zeta)|}_{\text {RANK OF } \zeta}<\infty$
research problem
CLASSIFY ALL FUSION CATEGORIES OF A GIVEN RANK, UP TO EQUIV.
$\checkmark$ RANK 1: FdVec
$\checkmark$ RANK 2 [OSTRIK]
子 4 Equiv. CLASSES
II. FUSION RULES \& RANK

II. FUSION RULES \& RANK

$$
\begin{aligned}
& \text { A MONOIDAL CATEGORY } \\
& \boldsymbol{b}:=(\boldsymbol{b}, \otimes, \mathbb{U}, a, l, r)
\end{aligned}
$$

is FUSION IF:
(a) I is abelian
(b) G is (k-) linear
(c) I is LDc. FINITE
(d) $\mathbb{L}$ is Abs. SIMple
(e) Ce is RIGID
(f) G is semisimple
(g) Y is FINITE
$\uparrow$
GOVERNED BY
simple objects
$\operatorname{Irr}\left(\zeta_{e}\right):=$ SET OF ISOCLASSES [X] OF SIMPLE OBJECTS OF $\zeta$
$\zeta$ FINITE $\Rightarrow \underbrace{|\operatorname{Irr}(\zeta)|}_{\text {RANK OF } \zeta}<\infty$

RESEARCH PROBLEM ミRATHER IMPOSSIbLE $\equiv$ CLASSIFY ALL FUSION CATEGORIES OF A GIVEN RANK, UP TO EQUIV.
II. FUSION RULES \& RANK

$$
\begin{aligned}
& \text { A MONOIDAL CATEGORY } \\
& \boldsymbol{b}:=(\boldsymbol{b}, \otimes, \mathbb{U}, a, l, r)
\end{aligned}
$$

is FUSION IF:
(a) C is abelian
(b) G is (mk-) linear
(c) lis soc. FINITE
(d) $\mathbb{L}$ is Abs. Simple
(e) G is RIGID
(f) G is semisimple
(g) Y is finite
$\uparrow$
GOVERNED BY
SIMPLE OBJECTS
$\operatorname{Irr}\left(\zeta_{e}\right):=$ SET OF ISOCLASSES [X] OF SIMPLE OBJECTS OF $\zeta^{6}$
$\zeta$ FINITE $\Rightarrow \underbrace{|\operatorname{Irr}(\zeta)|}_{\text {RANK OF } \zeta}<\infty$
三 WEAKEN $\equiv$
Research problem show that JOLLY FINITELY MANY FUSION CATEGORIES OF A GIVEN RANK, UP TO EQUIV.
II. FUSION RULES \& RANK

$$
\begin{aligned}
& \text { A MONOIDAL CATEGORY } \\
& \boldsymbol{b}:=(\boldsymbol{b}, \otimes, \mathbb{U}, a, l, r)
\end{aligned}
$$

is FUSION IF:
(a) C is abelian
(b) G is (k-) linear
(c) I is LDc. FINITE
(d) $\mathbb{L}$ is Abs. SIMple
(e) Ce is RIGID
(f) G is semisimple
(g) Y is FINITE
$\uparrow$
GOVERNED BY
simple objects
$\operatorname{Irr}\left(\zeta_{e}\right):=$ SET OF ISOCLASSES [X] OF SIMPLE OBJECTS OF $\zeta^{6}$
$\zeta$ FINITE $\Rightarrow \underbrace{|\operatorname{Irr}(\zeta)|}_{\text {RANK OF } \zeta}<\infty$

RANK -FINITENESS CONJECTURE
JOLLY FINITELY MANY FUSION CATEGORIES OF A GIVEN RANK, UP TO EQUIV.
II. FUSION RULES \& RANK

II. FUSION RULES \& RANK

II. FUSION RULES \& RANK

A MONOIDAL CATEGORY $b:=(b, \otimes, \mathbb{L}, a, l, r)$ is FUSION IF:
(a) C is abelian
(b) G is (kkk-) linear
(c) lis hoc. FINite
(d) $\mathbb{L}$ is Abs. Simple
(e) Ce is RIGID
(f) G is semisimple
(g) Y is finite
$\uparrow$
GOVERNED BY
SImple objects
$\operatorname{Irr}\left(\zeta_{e}\right):=$ SET OF ISOCLASSES [X] OF SIMPLE OBJECTS OF C
$\zeta$ FINITE $\Rightarrow \underbrace{|\operatorname{Irr}(\zeta)|}_{\text {RANK OF } \zeta}<\infty$
TAKE $\left\{X_{i}\right\}_{i \in I_{r(e)}}$ isoclass representatives OF SIMPLE OBJECTS OF $\zeta$
II. FUSION RULES \& RANK

II. FUSION RULES \& RANK

A MONOIDAL CATEGORY $\zeta:=(\zeta, \otimes, \mathbb{L}, a, l, r)$

IS FUSION IF:
(a) G is abelian
(b) Ge is (ik-)LINEAR
(c) lis soc. FINITE
(d) $\mathbb{L}$ is Abs. SIMPLE
(e) $C$ is RIGID
(f) G is semisimple
$(g)$ I is finite $\uparrow$
GOVERNED BY SIMPLE OBJECTS

OCNEANU RIGIDITY [ETINGOF-NIKSHYCH-OSTRIK] F only finitely many fusion categories with a given fusion rule.

TAKE $\left\{X_{i}\right\}_{i \in \operatorname{Irr}(e)}$ ISOClasS REPRESENTATIVES OF SIMPLE OBJECTS OF $\zeta$
GET $\quad X_{i} \otimes X_{j} \cong \Perp X_{k}{ }^{4 N_{k}^{\iota_{k}}}$
$k \in \operatorname{Irr}(6) \quad$ FOR SOME $N_{k}^{i, j} \in \mathbb{Z} \geqslant 0$
$\left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Irr}(e)}:\right.$ FUSION RULES OF $\zeta_{l}$
$N_{k}^{i, j}:=\left[X_{i} \otimes X_{j}: X_{k}\right]$ muLTIPLICITy of $X_{k}$ IN $X_{i} \otimes X_{j}$
II. FUSION RULES \& RANK

A MONOIDAL CATEGORY $\mathfrak{l}:=(\boldsymbol{l}, \otimes, \mathbb{L}, a, l, r)$
is FUSION IF:
(a) I is abelian
(b) G is (in-) UINEAR
(c) I is LOC. FINITE
(d) $\mathbb{L}$ is Abs. סIMPLE
(e) $\zeta$ is RIGID
(f) G is semisimple
(g) $\zeta$ is finite
$\uparrow$
GOVERNED BY SIMPLE OBJECTS $\quad\left\{N_{k}^{i, j} j_{i, j, k \in I r r(e)}\right.$ : FUSION RULES of $\epsilon_{l}$

OCNEANU RIGIDITY [ETNGOF-NIKSHYCH-OSTRIK] WITH A GIVEN FUSION RULE.

LET'S RETURN To
THE RANK 2 CASE

$$
\begin{aligned}
& \zeta \text { FINITE } \Rightarrow|\operatorname{Irr}(\zeta)|<\infty \text { RANK OF } \zeta \\
& X_{i} \otimes X_{j} \cong \Perp_{k \in \operatorname{Ir}(e)} X_{k} \text { UN } N_{k}^{i_{j} j} \text { For SOME } N_{k}^{i, j} \in \mathbb{Z}_{\geqslant 0} \\
& \left\{N_{k}^{i, j} S_{i, j, k \in \operatorname{Ir}(t)} \text { : FUSION RULES OF } \zeta\right.
\end{aligned}
$$ only finitely many fusion categories

II. FUSION RULES \& RANK
[OSTRIK] J4 EQUIV. CLASSES OF RANK 2 FUSION CATS
A MONOIDAL CATEGORY LET'S LOOK AT TWO OF THEM...

$$
\zeta:=(b, \otimes, \mathbb{U}, a, l, r)
$$

IS FUSION IF:
(a) G is abelian
(b) Ge is (ik-)LINEAR
(c) I is soc. FINITE
(d) $\mathbb{L}$ is Abs. simple
(e) C is RIGID
(f) G is semisimple
$(g)$ I IS FINITE
$\uparrow$
GOVERNED BY SIMPLE OBJECTS $\left\{N_{k}^{i, j} j_{i, j, k \in J r(e)}\right.$ : FUSION RULES of $\xi$
II. FUSION RULES \& RANK
[OSTRIK] J 4 EQUIV. CLASSES OF RANK 2 FUSION CATS
A MONOIDAL CATEGORY LET'S LOOK AT TWO OF THEM...

$$
\mathfrak{l}:=(\boldsymbol{l}, \otimes, \mathbb{L}, a, l, r)
$$

IS FUSION IF:
(a) C is abelian
(b) Ce is (IR-) LINEAR
(c) I is soc. FINITE
(d) $\mathbb{L}$ is Abs. simple
(e) $C$ is RIGID
(f) G is semisimple
$(g)$ I is finite
$\uparrow$
GOVERNED BY SIMPLE OBJECTS $\left\{N_{k}^{i, j} j_{i, j, k \in I_{r}(e)}\right.$ : FUSION RUCES of $\xi$
II. FUSION RULES \& RANK
[OSTRIK] J 4 EQUIV. CLASSES OF RANK 2 FUSION CATS
A MONOIDAL CATEGORY LET'S LOOK AT TWO OF THEM...

$$
\mathfrak{b}:=(b, \otimes, \mathbb{L}, a, l, r)
$$

IS FUSION IF:
(a) G is abelian
(b) Ge is (ik-)LINEAR
(c) l is loo. FINITE
(d) $\mathbb{L}$ is Abs. SIMPLE
(e) C is RIGID
(f) G is semisimple
(g) C is finite

$$
C_{2}=\langle g\rangle
$$

$$
\operatorname{Irr}(h)=\left\{\left[V_{0}\right],\left[V_{1}\right]\right\}
$$

TRIVIAL REP SIGN REP
$\operatorname{lkv} \operatorname{s\cdot g} \cdot v=v \quad$ Ike $\operatorname{sig} g \cdot v=-v$

$$
\begin{aligned}
& V_{0} \otimes V_{0} \cong V_{0} \\
& V_{0} \otimes V_{1} \cong V_{1} \\
& V_{1} \otimes V_{0} \cong V_{1} \\
& V_{1} \otimes V_{1} \cong V_{0}
\end{aligned}
$$

$\uparrow$
GOVERNED BY SIMPLE OBJECTS $\left\{N_{k}^{i, j} j_{i, j, k \in I_{r}(r)}\right.$ : FUSION RULES of $\xi_{e}$
II. FUSION RULES \& RANK
[OSTRIK] J 4 EQUIV. CLASSES OF RANK 2 FUSION CATS

A MONOIDAL CATEGORY

$$
l:=(l, \otimes, \mathbb{l}, a, l, r)
$$

IS FUSION IF:
(a) G is abelian
(b) Ge is (ik-)LINEAR
(c) l is soc. FINITE
(d) $\mathbb{L}$ is Abs. SIMPLE
(e) C is RIGID
(f) G is semisimple
$(g)$ I is finite


$$
\operatorname{Irr}(l)=\left\{\left[V_{0}\right],\left[V_{1}\right]\right\}
$$

TRIVIAL REP SIGN REP


$$
V_{0} \otimes V_{0} \cong V_{0}
$$

$$
V_{0} \otimes V_{1} \cong V_{1}
$$

$$
V_{1} \otimes V_{0} \cong V_{1}
$$

$$
V_{1} \otimes V_{1} \cong V_{0}
$$

$$
\zeta \text { FINITE } \Rightarrow|\operatorname{Irr}(\zeta)|<\infty \quad \text { RANK OF } \zeta
$$

$$
X_{i} \otimes X_{j} \cong \mathbb{1}_{k \in \operatorname{Ir}(G)} X_{k}{ }^{U N_{k}^{i, j}} \text { For some } N_{k}^{i, j} \in \mathbb{Z} \geqslant 0
$$

SIMPLE OBJECTS $\left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Ir}(\varepsilon)}\right.$ : FUSION RULES OF $\xi_{l}$
II. FUSION RULES \& RANK
[OSTRIK] J 4 EQUIV. CLASSES OF RANK 2 FUSION CATS

A MONOIDAL CATEGORY

$$
\mathfrak{l}:=(\boldsymbol{l}, \otimes, \mathbb{u}, a, l, r)
$$

IS FUSION IF:
(a) G is abelian
(b) Ge is (ilk-) LINEAR
(c) lis soc. FINITE
(d) $\mathbb{L}$ is Abs. simple
(e) $C$ is RIGID
(f) G is semisimple
(g) $\zeta$ is finite


$$
\operatorname{Irr}(b)=\left\{\left[V_{0}\right],\left[V_{1}\right]\right\}
$$

TRivial rep sign rep


$$
V_{0} \otimes V_{0} \cong V_{0}
$$

$$
V_{0} \otimes V_{1} \cong V_{1}
$$

$$
V_{1} \otimes V_{0} \cong V_{1}
$$

$$
V_{1} \otimes V_{1} \cong V_{0}
$$

$$
\zeta \text { FINITE } \Rightarrow|\operatorname{Irr}(\zeta)|<\infty \quad \text { RANK OF } \zeta
$$

SIMPLE OBJECTS $\left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Ir}(\varepsilon)}\right.$ : FUSION RULES OF $\xi_{l}$

Fib

$$
\operatorname{Irr}(6)=\{\mathbb{1}, x\}
$$

DEFINED WITHFWSION RULES

$$
\begin{array}{r}
\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1} \\
\mathbb{1} \otimes X \cong X \\
X \otimes \mathbb{1} \cong X \\
X \otimes X \cong \mathbb{1} \cup X
\end{array}
$$

$$
X_{i} \otimes X_{j} \cong \mathbb{1}_{k \in \operatorname{Ir}(G)} X_{k}{ }^{U N_{k}^{L_{j} j}} \text { For some } N_{k}^{i, j} \in \mathbb{Z} \geqslant 0
$$

II. FUSION RULES \& RANK
[OSTRIK] J 4 EQUIV. CLASSES OF RANK 2 FUSION CATS

II. FUSION RULES \& RANK related to the "dIe" of XeFib [OSTRIK]
A MONOIDAL CATEGORY J 4 EQUIV. CLASSES OF RANK 2 FUSION CATS $\mathfrak{b}:=(\boldsymbol{b}, \otimes, \mathbb{L}, a, l, r)$
is FUSION IF:
(a) I is abelian
(b) G is (IR-) UINEAR
(c) I is LOP. FINITE
(d) $\mathbb{L}$ is Abs. SIMple
(e) $C$ is RIGID
(f) G is semisimple
(g) Y IS FINITE

$\operatorname{Ir}(t)=\left\{\left[V_{0}\right],\left[\nu_{1}\right]\right\}$
trivial rep dian rep $\mathbb{k v} . g \cdot g \cdot v=v$ livia. $g \cdot v=-v$

$$
V_{0} \otimes V_{0} \cong V_{0}
$$

$$
V_{0} \otimes V_{1} \cong V_{1}
$$

$$
V_{1} \otimes V_{0} \cong V_{1}
$$

$$
V_{1} \otimes V_{1} \cong V_{0}
$$

governed by
SIMPLE OBJECTS

$$
\zeta \text { FINITE } \Rightarrow|\operatorname{Irr}(\zeta)|<\infty \text { RANK OF } \zeta
$$

$$
X_{i} \otimes X_{j} \cong \mathbb{U}_{k \in \operatorname{Ir}(t)} X_{k}{ }^{\| N_{k}^{i j j}} \text { For some } N_{k}^{i_{i j} j} \in \mathbb{Z} \geqslant 0
$$

$\left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Ir}(e)}\right.$ : FUSION RULES OF $\xi_{e}$
III. FROBENIUS-PERRON DIMENSION

A MONOIDAL CATEGORY

$$
\mathfrak{l}:=(\boldsymbol{l}, \otimes, \mathbb{L}, a, l, r)
$$

IS FUSION IF:
(a) G is abelian
(b) Ge is (ik-)LINEAR
(c) lis soc. FINITE
(d) $\mathbb{L}$ is Abs. simple
(e) C is RIGID
(f) G is semisimple
(g) $\zeta$ is finite
$\uparrow$
GOVERNED BY SIMPLE OBJECTS $\left\{N_{k}^{i, j} j_{i, j, k \in J r(e)}\right.$ : FUSION RULES of $\xi$
III. FROBENIUS-PERRON DIMENSION

A MONOIDAL CATEGORY $\mathfrak{b}:=(\boldsymbol{l}, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
(a) G is abelian
(b) G is (Ik-) LINEAR
(c) l is loc. FINITE
(d) $\mathbb{L}$ is Abs. simplé
(e) $C$ is RIGID
(f) G is semisimple
$(g)$ IS FINITE

$$
\uparrow
$$

GOVERNED BY SIMPLE OBJECTS $\sum_{k}^{i,} j_{i, j, k \in J r(t)}$ : FUSION RUCES of $\xi$

FROBEN US-PERRON THEOREM TAKE $M \in \mu_{\text {ath }}(\mathbb{R} \geqslant 0)$.
THEN J Non-Negative real eigenvalue
FP(M) That is $\geqslant$ absocute value of all OTHER EIGENVALLES OF $M$.
III. FROBENIUS-PERRON DIMENSION

A MONOIDAL CATEGORY $\mathfrak{b}:=(\boldsymbol{l}, \otimes, \mathbb{L}, a, l, r)$

IS FUSION IF:
(a) C is abelian
(b) Ge is (ik-)LINEAR
(c) l is loc. FINITE
(d) $\mathbb{L}$ is Abs. simple
(e) $C$ is RIGID
(f) G is semisimple
(g) Y is finite

$$
\uparrow
$$

GOVERNED BY SIMPLE OBJECTS

FROBEN US-PERRON THEOREM TAKE $M \in \mu_{\text {ath }}(\mathbb{R} \geqslant 0)$.
THEN J NON-NEGATIVE REAL EIGENVALUE
FP(M) That is $\geqslant$ absocute value of all OTHER EIGENVALMES OF M.

FP-DIMENSION OF $X_{i} \in \operatorname{Ir}(\mathcal{C}):$

$$
F P \operatorname{dim}_{l}\left(X_{i}\right):=F P\left(N_{k}^{i, j}\right)_{j, k \in \operatorname{Ir}(e)} .
$$

III. FROBENIUS-PERRON DIMENSION

A MONOIDAL CATEGORY $\mathfrak{b}:=(\boldsymbol{l}, \otimes, \mathbb{L}, a, l, r)$ IS FUSION IF:
(a) C is abelian
(b) Ge is (ik-)LINEAR
(c) l is loo. FINITE
(d) $\mathbb{L}$ is Abs. simple
(e) $C$ is RIGID
(f) G is semisimple
$(g)$ I IS FINITE
$\uparrow$
GOVERNED BY
THEN J NON-NEGATIVE REAL EIGENValue
FP (M) That $1 s \geqslant$ absolute value of all

FP-DIMENSION OF $X_{i} \in \operatorname{Ir}(\zeta):$

FP-DIMENSION OF G ITSELF:

$$
\zeta \text { FINITE } \Rightarrow|\operatorname{Irr}(\zeta)|<\infty \quad \text { RANK OF } \zeta
$$

$$
X_{i} \otimes X_{j} \cong \mathbb{1}_{k \in \operatorname{Ir}(e)} X_{k}{ }^{\| N_{k}^{i, j}} \text { For SOME } N_{k}^{i, j} \in \mathbb{Z} \geqslant 0
$$ SIMPLE OBJECTS $\left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Ir}(r)}\right.$ : FUSION RULES OF $\xi_{l}$

$$
\operatorname{FPdim}(e):=\sum_{i \in \operatorname{Ir}(e)} F_{P d_{i}}\left(X_{i}\right)^{2}
$$

FROBENIUS-PERRON THEOREM TAKE $M \in \operatorname{Mat}_{n}(\mathbb{R} \geqslant 0)$. OTHER EIGENVALUES OF M.

$$
F P \operatorname{dim}_{\epsilon}\left(X_{i}\right):=F P\left(N_{k}^{i, j}\right)_{j, k \in \operatorname{Ir}(e)} .
$$

III. FROBENIUS-PERRON DIMENSION

$$
\begin{gathered}
\text { FP THEOREM } \\
\text { TAKE } M \in M \text { Mat }(\mathbb{R} \geqslant 0) . \\
\Rightarrow \exists \in V A L U E \\
F P(M) \in \mathbb{R} \geqslant 0 \\
\text { THAT IS } \\
\geqslant \begin{array}{c}
\text { ABS VALUE OF AU L } \\
E . V A L U E S \text { OF } M .
\end{array}
\end{gathered}
$$

$$
\begin{gathered}
F P \operatorname{dim}_{e}\left(X_{i}\right) \\
\ddot{i} \\
F P\left(N_{k}^{i, j}\right)_{j, k \in \operatorname{Ir}(e)}
\end{gathered}
$$

$$
\begin{aligned}
& \zeta \text { FINITE } \Rightarrow|\operatorname{Irr}(\zeta)|<\infty \quad \text { RANK OF } \zeta \\
& X_{i} \otimes X_{j} \cong \Perp_{k \in \operatorname{Irr}(\zeta)} X_{k}{ }^{4 N_{k}^{i, j}} \text { FOR SOME } N_{k}^{i, j} \in \mathbb{Z}_{\geqslant 0} \\
& \left\{N_{k}^{i, j} ~_{i, j, k \in \operatorname{Irr}(\zeta)}: \text { FUSION RULES OF } \zeta\right.
\end{aligned}
$$

III. FROBENIUS-PERRON DIMENSION


$$
\begin{gathered}
C_{2}-F d \mu_{0 d} \\
\operatorname{Irr}(6)=\left\{\left[V_{0}\right],\left[V_{1}\right]\right\} \\
V_{0} \otimes V_{0} \cong V_{0} \\
V_{0} \otimes V_{1} \cong V_{1} \\
V_{1} \otimes V_{0} \cong V_{1} \\
V_{1} \otimes V_{1} \cong V_{0}
\end{gathered}
$$

$$
\begin{aligned}
& \zeta \text { FINITE } \Rightarrow|\operatorname{Irr}(\zeta)|<\infty \quad \text { RANK OF } \zeta \\
& X_{i} \otimes X_{j} \cong \mathbb{1}_{k \in \operatorname{Irr}(\zeta)} X_{k}{ }^{4 N_{k}^{i, j}} \text { FOR SOME } N_{k}^{i, j} \in \mathbb{Z} \geqslant 0 \\
& \left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Irr}(\zeta)}: \text { FUSION RULES OF } \zeta\right.
\end{aligned}
$$

III. FROBENIUS-PERRON DIMENSION

FPdim (e) ii $\sum \operatorname{FPdine}\left(X_{i}\right)^{2}$ $i \in \operatorname{Ir}(e)$


$$
\therefore F \operatorname{FPdim}(u)=
$$

$$
\begin{aligned}
& \text { FP THEOREM } \\
& \text { TAKE } M \in \operatorname{Mat}_{n}(\mathbb{R} \geqslant 0) \text {. } \\
& \Rightarrow \exists \text { VALUE } \\
& \Rightarrow \underset{\text { FP(M) } \in \mathbb{R} \geqslant 0}{\substack{\operatorname{THAT} \\
\operatorname{EVALUE}}} \quad \operatorname{Irr}\left(\zeta_{e}\right)=\left\{\left[V_{0}\right],\left[V_{1}\right]\right\} \\
& \text { That is } \\
& \geqslant \text { Abs ache of all } \\
& \text { e.values of } M \text {. } \\
& \begin{array}{c}
F P \operatorname{dim}_{\mathrm{l}}\left(X_{i}\right) \\
\ddot{i l} \\
\operatorname{FP}\left(N_{k}^{i, j}\right)_{j, k \in \operatorname{Ir}(e)}
\end{array} \\
& \mathrm{C}_{2}-\mathrm{Fd}_{\mathrm{Mod}} \\
& \Rightarrow \underset{\text { FP(M) } \in \mathbb{R} \geqslant 0}{\substack{\operatorname{THAT} \\
\operatorname{EVALUE}}} \quad \operatorname{Irr}\left(\zeta_{e}\right)=\left\{\left[V_{0}\right],\left[V_{1}\right]\right\} \\
& \begin{array}{cl}
\begin{array}{c}
\text { That ache of all } \\
\text { E.VALMES of M. }
\end{array} & V_{0} \otimes V_{0} \cong V_{0} \\
\begin{array}{cl}
\text { FPdin }\left(X_{0}\right) & V_{0} \otimes V_{1} \cong V_{1}
\end{array}
\end{array} \\
& \begin{array}{cl}
\begin{array}{c}
\text { abs Value of } a l \\
\text { E.values of } M .
\end{array} & V_{0} \otimes V_{0} \cong V_{0} \\
& V_{0} \otimes V_{1} \cong V_{1}
\end{array} \\
& V_{1} \otimes V_{0} \cong V_{1} \\
& V_{1} \otimes V_{1} \cong V_{0}
\end{aligned}
$$

$$
\zeta \text { FINITE } \Rightarrow|\operatorname{Irr}(\zeta)|<\infty \quad \text { RANK OF } \zeta
$$

$$
X_{i} \otimes X_{j} \cong \mathbb{U}_{k \in \operatorname{Irr}(e)} X_{k}^{U N_{k}^{i_{i j}}} \text { For some } N_{k}^{i, j} \in \mathbb{Z}_{\mathbb{C}}
$$

$$
\left\{N_{k}^{i, j} g_{i, j, k \in \operatorname{Irr}(e)}: \text { FUSION RULES OF } \zeta_{e}\right.
$$

III. FROBENIUS-PERRON DIMENSION

III. FROBENIUS-PERRON DIMENSION


$$
\therefore \operatorname{FPdim}(l)=
$$

$$
\begin{aligned}
& \text { TAKE } \underset{\text { FP } M \in O R E M}{ } \operatorname{Mat}_{n}(\mathbb{R} \geqslant 0) \text {. } \\
& \mathrm{C}_{2}-\mathrm{Fd}_{\mathrm{Mod}} \\
& \Rightarrow \underset{F P(M) \in \mathbb{R} \geqslant 0}{\exists \in V A L U E} \quad \operatorname{Irr}\left(\zeta_{e}\right)=\left\{\left[V_{0}\right],\left[V_{1}\right]\right\} \\
& \text { That is } \\
& \geqslant \text { AbS VALUE OF All } \\
& \text { e.values of } M \text {. } \\
& \begin{array}{c}
\text { FPdime }\left(X_{i}\right) \\
\ddot{\ddot{l}} \\
\operatorname{FP}\left(N_{k}^{i, j}\right)_{j, k \in \operatorname{Ir}(e) .}
\end{array} \\
& V_{0} \otimes V_{0} \cong V_{0} \\
& V_{0} \otimes V_{1} \cong V_{1} \\
& V_{1} \otimes V_{0} \cong V_{1} \\
& V_{1} \otimes V_{1} \cong V_{0}
\end{aligned}
$$

$$
\zeta \text { FINITE } \Rightarrow|\operatorname{Irr}(\zeta)|<\infty \quad \text { RANK OF } \zeta
$$

$$
X_{i} \otimes X_{j} \cong \mathbb{U}_{k \in \operatorname{Irr}(e)} X_{k}^{U N_{k}^{i, j}} \text { For some } N_{k}^{i, j} \in \mathbb{Z}_{\mathbb{C}}
$$

$$
\left\{N_{k}^{i, j} g_{i, j, k \in \operatorname{Irr}(e)}: \text { FUSION RULES OF } \zeta_{e}\right.
$$

III. FROBENIUS-PERRON DIMENSION

III. FROBENIUS-PERRON DIMENSION


$$
\therefore \operatorname{FPdim}(\zeta)=
$$

$$
\begin{aligned}
& \text { FP THEOREM } \\
& \text { TAKE } M \in \operatorname{Mat}_{n}(\mathbb{R} \geqslant 0) \text {. } \\
& \Rightarrow \exists \text { VALUE } \\
& F P(M) \in \mathbb{R} \geqslant 0 \\
& \text { That is } \\
& \geqslant \text { Abs value of all } \\
& \text { e.values of } M \text {. } \\
& \begin{array}{c}
\operatorname{FPdim}_{\epsilon}\left(X_{i}\right) \\
\ddot{i} \\
\operatorname{FP}\left(N_{k}^{i, j}\right)_{j, k \in \operatorname{Ir}(e)} .
\end{array} \\
& C_{2}-F d M o d \\
& \operatorname{Irr}\left(\zeta_{6}\right)=\left\{\left[V_{0}\right],\left[V_{1}\right]\right\} \\
& V_{0} \otimes V_{0} \cong V_{0} \\
& V_{0} \otimes V_{1} \cong V_{1} \\
& V_{1} \otimes V_{0} \cong V_{1} \\
& V_{1} \otimes V_{1} \cong V_{0}
\end{aligned}
$$

$$
\zeta \text { FINITE } \Rightarrow|\operatorname{Irr}(\zeta)|<\infty \quad \text { RANK OF } \zeta
$$

$$
X_{i} \otimes X_{j} \cong \|_{k \in \operatorname{Irr}(6)} X_{k}^{U N_{k}^{i, j}} \text { For some } N_{k}^{i, j} \in \mathbb{Z}_{0}
$$

$$
\left\{N_{k}^{i, j} g_{i, j, k \in \operatorname{Irr}(e)}: \text { FUSION RULES OF } \zeta_{e}\right.
$$

III. FROBENIUS-PERRON DIMENSION


FP THEOREM
TAKE $M \in \operatorname{Mat}_{n}(\mathbb{R} \geqslant 0)$.

$$
\operatorname{Irr}\left(C_{e}\right)=\left\{\left[V_{0}\right],\left[V_{1}\right]\right\}
$$

$$
V_{0} \otimes V_{0} \cong V_{0}
$$

$$
V_{0} \otimes V_{1} \cong V_{1}
$$

$$
V_{1} \otimes V_{0} \cong V_{1}
$$

$$
V_{1} \otimes V_{1} \cong V_{0}
$$



$$
\begin{aligned}
& \begin{array}{l}
i=0
\end{array}\left(\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right) \\
& \text { HAS EVACUES }=\{1\} \\
& \\
& \Rightarrow \text { FPdime }\left(V_{0}\right)=1
\end{aligned}
$$

$$
i=1:\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)
$$

HAS EVACUES $=$ ?

$$
\Rightarrow \text { FPdime }\left(V_{1}\right)=
$$

$\zeta$ FINITE $\Rightarrow|\operatorname{Irr}(\zeta)|<\infty \quad$ RANK OF $\zeta$
$X_{i} \otimes X_{j} \cong \|_{k \in \operatorname{Irr}(6)} X_{k} \mu_{k}^{i, j}$ For some $N_{k}^{i, j} \in \mathbb{Z} \geqslant 0$
$\left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Irr}(e)}:\right.$ FUSION RULES OF $\zeta_{l}$
III. FROBENIUS-PERRON DIMENSION


FP THEOREM
TAKE $M \in \operatorname{Mat}_{n}(\mathbb{R} \geqslant 0)$.

$$
\operatorname{Irr}(6)=\left\{\left[V_{0}\right],\left[V_{1}\right]\right\}
$$

$$
V_{0} \otimes V_{0} \cong V_{0}
$$

$$
V_{0} \otimes V_{1} \cong V_{1}
$$

$$
V_{1} \otimes V_{0} \cong V_{1}
$$

$$
V_{1} \otimes V_{1} \cong V_{0}
$$



$$
\begin{aligned}
& \begin{array}{l}
i=0
\end{array}\left(\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right) \\
& \text { HAS EVACUES }=\{1\} \\
& \Rightarrow \text { FPdime }\left(V_{0}\right)=1
\end{aligned}
$$

$$
i=1:\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)
$$

$$
\text { HAS EVACUES }=\{1,-1\}
$$

$$
\Rightarrow \operatorname{FPdime}_{e}\left(V_{1}\right)=1
$$

$\zeta$ FINITE $\Rightarrow\left|\operatorname{Irr}\left(\epsilon_{l}\right)\right|<\infty \quad$ RANK OF $\zeta$
$X_{i} \otimes X_{j} \cong \Perp_{k \in \operatorname{Irr}(6)} X_{k} 4 N_{k}^{i_{i j} j}$ For sOME $N_{k}^{i, j} \in \mathbb{Z} \geqslant 0$
$\left\{N_{k}^{i, j} g_{i, j, k \in \operatorname{Irr}(e)}: F U S I O N\right.$ RULES OF $\epsilon_{l}$
III. FROBENIUS-PERRON DIMENSION


$$
F P \operatorname{dim}(e)
$$

$$
\ddot{i}
$$

$$
\sum F P \operatorname{din}_{e}\left(X_{i}\right)^{2}
$$

$$
i \in \operatorname{Irr}(e)
$$

$$
\begin{aligned}
& \text { FP THEOREM } \\
& \text { TAKE } M \in M_{a t_{n}}(\mathbb{R} \geqslant 0) \text {. } \\
& \Rightarrow \exists \text { value } \\
& \Rightarrow \underset{F P(M) \in \mathbb{R} \geqslant 0}{\exists \operatorname{EVALLUE}} \quad \operatorname{Irr}\left(\zeta_{e}\right)=\left\{\left[V_{0}\right],\left[V_{1}\right]\right\} \\
& \text { That is } \\
& \geqslant \text { ABSVACUE OF ALL } \\
& \text { e.values of } M \text {. } \\
& \begin{array}{c}
F P \operatorname{dim}_{e}\left(X_{i}\right) \\
\ddot{i} \\
\operatorname{FP}\left(N_{k}^{i, j}\right)_{j, k \in \operatorname{Ir}(e)} .
\end{array} \\
& \mathrm{C}_{2}-\mathrm{Fd}_{\mathrm{Mod}} \\
& \Rightarrow \underset{F P(M) \in \mathbb{R} \geqslant 0}{\exists \operatorname{EVALLUE}} \quad \operatorname{Irr}\left(\zeta_{e}\right)=\left\{\left[V_{0}\right],\left[V_{1}\right]\right\} \\
& V_{0} \otimes V_{0} \cong V_{0} \\
& V_{0} \otimes V_{1} \cong V_{1} \\
& V_{1} \otimes V_{0} \cong V_{1} \\
& V_{1} \otimes V_{1} \cong V_{0}
\end{aligned}
$$

$$
\begin{aligned}
& \begin{array}{l}
i=0
\end{array}\left(\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right) \\
& \text { HAS EVACUES }=\{1\} \\
& \Rightarrow \text { FPdime }\left(V_{0}\right)=1
\end{aligned}
$$

$$
\underline{i=1}:\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)
$$

$$
\text { HAS EVACUES }=\{1,-1\}
$$

$$
\Rightarrow \operatorname{FPdime}_{e}\left(V_{1}\right)=1
$$

$\zeta$ FINITE $\Rightarrow\left|\operatorname{Irr}\left(\epsilon_{i}\right)\right|<\infty \quad$ RANK OF $\zeta$ $X_{i} \otimes X_{j} \cong \Perp_{k \in \operatorname{Irr}(e)} X_{k}{ }^{U N_{k}^{i, j}}$ For some $N_{k}^{i, j} \in \mathbb{Z}_{\mathbb{Z}} 0$ $\left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Irr}(e)}: F U S I O N\right.$ RULES OF $\epsilon_{e}$
III. FROBENIUS-PERRON DIMENSION


Fibonacci fusion categ.
Fib
$\operatorname{Irr}(t)=\{\mathbb{1}, X\}$
defined with fusion rules
$\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$
$1 \otimes X \cong X$
$X \otimes \mathbb{1} \cong X$
$X \otimes X \cong \mathbb{1} \cup X$

$$
\begin{aligned}
& \zeta \text { FINITE } \Rightarrow|\operatorname{Irr}(\zeta)|<\infty \quad \text { RANK OF } \zeta \\
& X_{i} \otimes X_{j} \cong \mathbb{1}_{k \in \operatorname{Ir}(\zeta)} X_{k}{ }^{4 N_{k}^{i, j}} \text { FOR SOME } N_{k}^{i, j} \in \mathbb{Z} \geqslant 0 \\
& \left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Irr}(\zeta)}: \text { FUSION RULES OF } \zeta\right.
\end{aligned}
$$

III. FROBENIUS-PERRON DIMENSION


$\Rightarrow$ FPdime $_{\text {e }}(X)=$

Fibonacci fusion categ.

$$
\begin{aligned}
& \text { Fib } \\
& \operatorname{Irr}(e)=\left\{\mathbb{1}_{1}^{-x_{0}} X^{=x_{1}}\right\}^{\prime}
\end{aligned}
$$

deFined witt fusion rules

$$
\begin{aligned}
& \mathbb{1} \otimes \mathbb{1} \cong \mathbb{1} \\
& \mathbb{1} \otimes X \cong X \\
& X \otimes \mathbb{1} \cong X \\
& X \otimes X \cong \mathbb{1} \cup X
\end{aligned}
$$

$$
\text { HAS Evacues }=\{\quad g
$$

$$
\therefore \operatorname{FPdim}(u)=
$$

$\zeta$ FINITE $\Rightarrow|\operatorname{Irr}(\zeta)|<\infty \quad$ RANK OF $\zeta$ $X_{i} \otimes X_{j} \cong \mathbb{1}_{k \in \operatorname{Ir}(\epsilon)} X_{k}{ }^{\text {LN }}{ }_{k}^{\iota_{j}^{j}}$ For SOME $N_{k}^{i, j} \in \mathbb{Z} \geqslant 0$ $\left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Ir}(a)}\right.$ : FUSION RULES OF $\epsilon_{e}$
III. FROBENIUS-PERRON DIMENSION



Fibonacci fusion categ.

$$
\begin{aligned}
& \text { Fib } \\
& \operatorname{Irr}(b)=\left\{\mathbb{1}_{1}^{-x_{0}} X^{=x_{1}}\right\}_{1}
\end{aligned}
$$

defined with fusion rules

$$
\begin{aligned}
& \mathbb{1} \otimes \mathbb{1} \cong \mathbb{1} \\
& \mathbb{1} \otimes X \cong X \\
& X \otimes \mathbb{1} \cong X \\
& X \otimes X \cong \mathbb{1} \cup X
\end{aligned}
$$

$$
\therefore \operatorname{FPdim}(u)=
$$

$\zeta$ FINITE $\Rightarrow|\operatorname{Irr}(\boldsymbol{e})|<\infty \quad$ RANK OF $\zeta$ $X_{i} \otimes X_{j} \cong \mathbb{1}_{k \in \operatorname{Ir}(\varphi)} X_{k}{ }^{\| N_{k}^{i, j}}$ For SOME $N_{k}^{i, j} \in \mathbb{Z} \geqslant 0$ $\left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Ir}(a)}\right.$ : FUSION RULES OF $\epsilon_{e}$
III. FROBENIUS-PERRON DIMENSION


$$
\underline{i=1}:\left(\begin{array}{ll}
\mathcal{N}_{0}^{1,0} & \mathcal{N}_{1}^{1,0} \\
\mathcal{N}_{0}^{1,1} & \mathcal{N}_{1}^{1,1}
\end{array}\right)
$$

$$
\Rightarrow \text { FPdime }(X)=
$$

Fibonacci fusion categ.

$$
\begin{aligned}
& \text { Fib } \\
& \operatorname{Irr}(t)=\left\{\mathbb{1}^{\prime \prime}, x_{0}^{\prime=}\right\}_{1}
\end{aligned}
$$

defined with fusion rules

$$
\Rightarrow F P d_{i m e}(\mathbb{L})=1
$$

$$
\text { HAS EVACKES }=\{\quad j
$$

$$
\therefore F \operatorname{FPdim}(u)=
$$

$\zeta$ FINITE $\Rightarrow|\operatorname{Irr}(\boldsymbol{e})|<\infty \quad$ RANK OF $\zeta$ $X_{i} \otimes X_{j} \cong \mathbb{1}_{k \in \operatorname{Irr}(\epsilon)} X_{k}{ }^{\text {UN }}{ }_{k}^{i_{j}^{j}}$ For SOME $N_{k}^{i, j} \in \mathbb{Z} \geqslant 0$ $\left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Ir}(a)}\right.$ : FUSION RULES OF $\zeta_{e}$
III. FROBENIUS-PERRON DIMENSION

> FP THEOREM
> TAKE $M \in M_{a} T_{n}(\mathbb{R} \geqslant 0)$.
> $\Rightarrow \exists$ E.VALUE FP (M) $\in \mathbb{R} \geqslant 0$
> THAT IS
> $\geqslant$ ABS VALUE OF AU L E.VALUES OF $M$.

$$
\begin{gathered}
F P d_{i m}\left(X_{i}\right) \\
\ddot{i} \\
F P\left(N_{k}^{i, j}\right)_{j, k \in \operatorname{Ir}(e)}
\end{gathered}
$$

$\operatorname{FPdim}(e)$
ii
$\sum \operatorname{FPdin}_{e}\left(X_{i}\right)^{2}$ $i \in \operatorname{Irr}(E)$

$$
\begin{aligned}
& i=0:\left(\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right) \\
& \text { HAS EVACUES }=\left\{\begin{array}{l}
1
\end{array}\right] \\
& \quad \Rightarrow \text { FPdime }(14)=1
\end{aligned}
$$

$$
i=1:\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right)
$$

$$
\text { HAS EVACUES }=\{
$$

$\Rightarrow \operatorname{FPdime}_{\mathrm{C}}(X)=$
$\zeta$ FINITE $\Rightarrow|\operatorname{Irr}(\zeta)|<\infty \quad$ RANK OF $\zeta$
$X_{i} \otimes X_{j} \cong \|_{k \in \operatorname{Irr}(6)} X_{k}^{\mu N_{k}^{i, j}}$ For some $N_{k}^{i, j} \in \mathbb{Z}_{\geqslant 0}$
$\left\{N_{K}^{i, j}\right\}_{i, j, k \in \operatorname{Irr}(e)}:$ FUSION RULES OF $\zeta_{l}$
III. FROBENIUS-PERRON DIMENSION

> FP THEOREM
> TAKE $M \in M a T_{n}(\mathbb{R} \geqslant 0)$.
> $\Rightarrow \exists \in \cdot V A L U E$ $F P(M) \in \mathbb{R} \geqslant 0$
> THAT IS
> $\geqslant$ ABS VALUE OF AU E.VALUES OF $M$.

$$
\begin{gathered}
F P d_{i M}\left(X_{i}\right) \\
\ddot{i} \\
F P\left(N_{k}^{i, j}\right)_{j, k \in \operatorname{Ir}(l)}
\end{gathered}
$$



Fibonacci fusion categ.

$$
\begin{aligned}
& \left.\begin{array}{l}
i=0: \\
0
\end{array} 1 \begin{array}{cc}
1 & 0 \\
\text { HAS EVACUES }=\{1
\end{array}\right) \\
& \quad \Rightarrow \text { FPdime }_{6}(14)=1
\end{aligned}
$$

$$
i=1:\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right)
$$

$$
\text { HAS EVACUES }=\left\{\frac{1}{2}(1 \pm \sqrt{5})\right\}
$$

$$
\Rightarrow \text { FPdime }(X)=\frac{1}{2}(1+\sqrt{5})
$$

$\therefore \operatorname{FPdim}(u)=$
$\zeta$ FINITE $\Rightarrow|\operatorname{Irr}(\zeta)|<\infty \quad$ RANK OF $\zeta$ $X_{i} \otimes X_{j} \cong \|_{k \in \operatorname{Irr}(6)} X_{k}^{山 N_{k}^{i, j}}$ For some $N_{k}^{i, j} \in \mathbb{Z}_{\geqslant 0}$ $\left\{N_{K}^{i, j}\right\}_{i, j, k \in \operatorname{Irr}(e)}:$ FUSION RULES OF $\zeta_{l}$
III. FROBENIUS-PERRON DIMENSION

fibonacci fusion categ.


DEFINED WITH FUSION RULES

$$
\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}
$$

$1 \otimes x \cong X$
$i=1:\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right)$
HAS EVACUES $=\left\{\frac{1}{2}(1 \pm \sqrt{5})\right\}$
$\frac{\Rightarrow \text { FPdime }(X)=\frac{1}{2}(1+\sqrt{5}): \operatorname{FPdim}(u)=}{\zeta \text { FINITE } \Rightarrow|\operatorname{Irr}(\zeta)|<\infty \text { RANK OF } \zeta}$
$X_{i} \otimes X_{j} \cong \mathbb{1}_{k \in \operatorname{Irr}(\varphi)} X_{k}{ }^{\text {LN }}{ }_{k}^{i_{j} j}$ For SOME $N_{k}^{i, j} \in \mathbb{Z} \geqslant 0$
$\left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Ir}(a)}\right.$ : FUSION RULES OF $\epsilon_{e}$
III. FROBENIUS-PERRON DIMENSION

> FP THEOREM
> TAKE $M \in M_{a} T_{n}(\mathbb{R} \geqslant 0)$.
> $\Rightarrow \exists$ E.VALUE FP (M) $\in \mathbb{R} \geqslant 0$
> THAT IS
> $\geqslant$ ABS VALUE OF AU E.VALUES OF $M$.

$$
\begin{gathered}
F \operatorname{dim}_{\epsilon}\left(X_{i}\right) \\
\ddot{i} \\
F P\left(N_{k}^{i, j}\right)_{j, k \in \operatorname{Ir}(e) .}
\end{gathered}
$$



Fibonacci fusion categ.

$$
\begin{aligned}
& \begin{array}{l}
i=0:\left(\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right) \\
\text { HAS EVACUES }=\left\{\begin{array}{l}
1
\end{array}\right\} \\
\quad \Rightarrow \text { FPdimer }(14)=1
\end{array} ~
\end{aligned}
$$

$$
i=1:\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right)
$$

$$
\text { HAS EVACUES }=\left\{\frac{1}{2}(1 \pm \sqrt{5})\right\}
$$

$$
\Rightarrow \text { FPdime }(X)=\frac{1}{2}(1+\sqrt{5})
$$

$$
\therefore F \operatorname{Fdim}(c)=\frac{5+\sqrt{5}}{2}
$$

$\zeta$ FINITE $\Rightarrow\left|\operatorname{Irr}\left(\zeta_{l}\right)\right|<\infty \quad$ RANK OF $\zeta^{6}$
$X_{i} \otimes X_{j} \cong \|_{k \in \operatorname{Irr}(\epsilon)} X_{k} \mu_{k}^{i, j}$ For some $N_{k}^{i, j} \in \mathbb{Z}_{\mathbb{}} 0$
$\left\{N_{K}^{i, j}\right\}_{i, j, k \in \operatorname{Irr}(e)}:$ FUSION RULES OF $\zeta_{l}$
III. FROBENIUS-PERRON DIMENSION
FP THEOREM
TAKE $M \in M_{\text {a }}^{n}(\mathbb{R}(\mathbb{R} \geqslant 0)$.
$\Rightarrow \exists \in V A L U E$

$F P(M) \in \mathbb{R} \geqslant 0$
THAT IS
$\geqslant$
ABSVACUE OF AU
E. VALUES OF $M$.


Fibonacci fusion categ.
Fib

$$
\operatorname{Irr}(6)=\{\mathbb{1}, X\}
$$

defined with fusion rules

$$
\begin{aligned}
& \mathbb{1} \otimes \mathbb{1} \cong \mathbb{1} \\
& \mathbb{1} \otimes X \cong X \\
& X \otimes \mathbb{1} \cong X \\
& X \otimes X \cong \mathbb{1} \cup X
\end{aligned}
$$

$$
\therefore \operatorname{FPdim}(u)=\frac{5+\sqrt{5}}{2}
$$

$\zeta$ FINITE $\Rightarrow|\operatorname{Irr}(\varphi)|<\infty \quad$ RANK OF $\varphi$
$X_{i} \otimes X_{j} \cong \mathbb{1}_{k \in \operatorname{Irr}(6)} X_{k} 4 N_{k}^{i, j}$ For sOME $N_{k}^{i, j} \in \mathbb{Z} \geqslant 0$
$\left\{N_{k}^{i, j} j_{i, j, k \in \operatorname{Irr}(e)}: F U S I O N\right.$ RULES OF $\zeta_{l}$

READ FOR MORE ON:
LECTURE \#16

FP-DIMENSION

II. FUSION RULES \& RANK
LI. FROBENIUS-PERRON DIMENSION
( $\$ 83.9 .1,3.9 .3$ )
( $853.9 .1,3.9 .3$ )
( $\$ 3.9 .2$ )

READ FOR MORE ON:
LECTURE \#16

FP-DIMENSION,
more examples of fusion categories,
\& about their module categories


TOPICS:
A. Fusion categories
( $\$ 83.9 .1,3.9 .3$ )
II. FUSION RULES \& RANK
( $\$ 83.9 .1,3.9 .3$ )
LII. FROBENIUS-PERRON DIMENSION
( $\$ 3.9 .2$ )

READ FOR MORE ON:
LECTURE \#16

FP-DIMENSION,
more examples of fusion categories,
\& about their module categories


TopIcs:
$(\$ 83.9 .1,3.9 .3)$
II. FUSION RULES \& RANK
( $\$ \S 3.9 .1,3.9 .3$ )
III. FROBENIUS-PERRON DIMENSION
( $\$ 3.9 .2$ )

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Lecture \#16 keywords: Frobenius-Perron dimension, Frobenius-Perron Theorem, fusion category, fusion rules, multiplicity, Ocneanu rigidity, rank, Rank-Finiteness Conjecture

