MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LAST TIME

- RIGID CATEGORIES
- PIVOTAL CATEGORIES

LECTURE #16

TOPICS:

I. FUSION CATEGORIES (§§3.9.1, 3.9.3)

II. FUSION RULES & RANK (§§3.9.1, 3.9.3)

III. FROBENIUS-PERRON DIMENSION (§3.9.2)

I. FUSION CATEGORIES

SUPER NICE MONOIDAL CATEGORIES

I. FUSION CATEGORIES

SUPER NICE MONOIDAL CATEGORIES

A MONOIDAL CATEGORY

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

IS FUSION IF:

(a) $\mathcal{C}$ IS ABELIAN

(b) $\mathcal{C}$ IS ($\Bbbk$-)LINEAR

(c) $\mathcal{C}$ IS LOCALLY FINITE

(d) $\mathbb{1}$ IS ABSOLUTELY SIMPLE

(e) $\mathcal{C}$ IS RIGID

(f) $\mathcal{C}$ IS SEMISIMPLE

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$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ab}$ $\forall X, Y \in \mathcal{C}$ $\xrightarrow{\text{DEF}}$ PREADDITIVE CATEGORY

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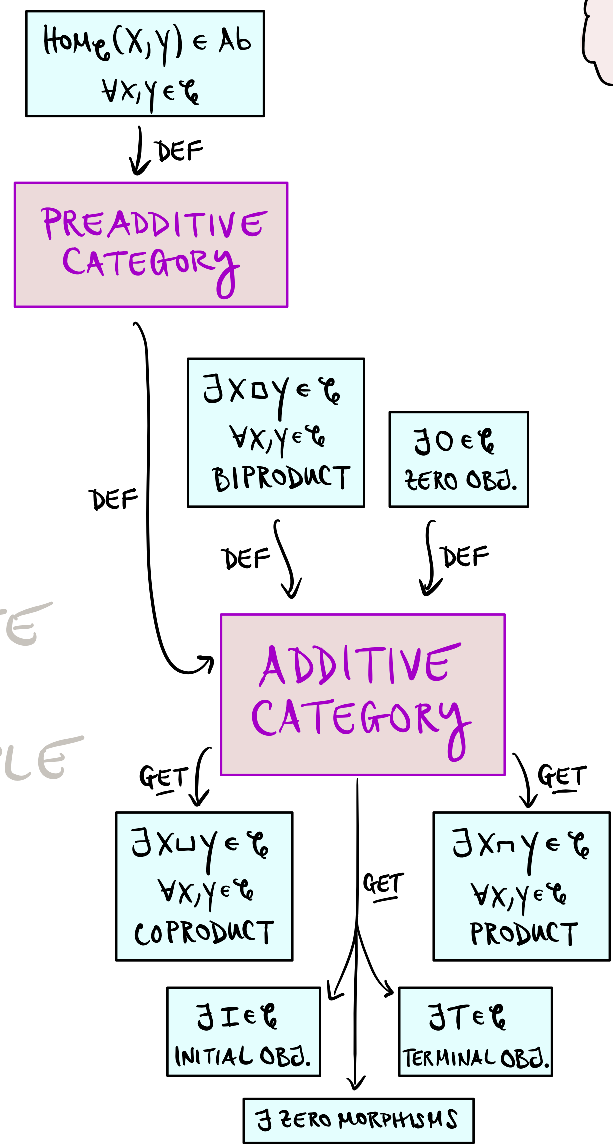

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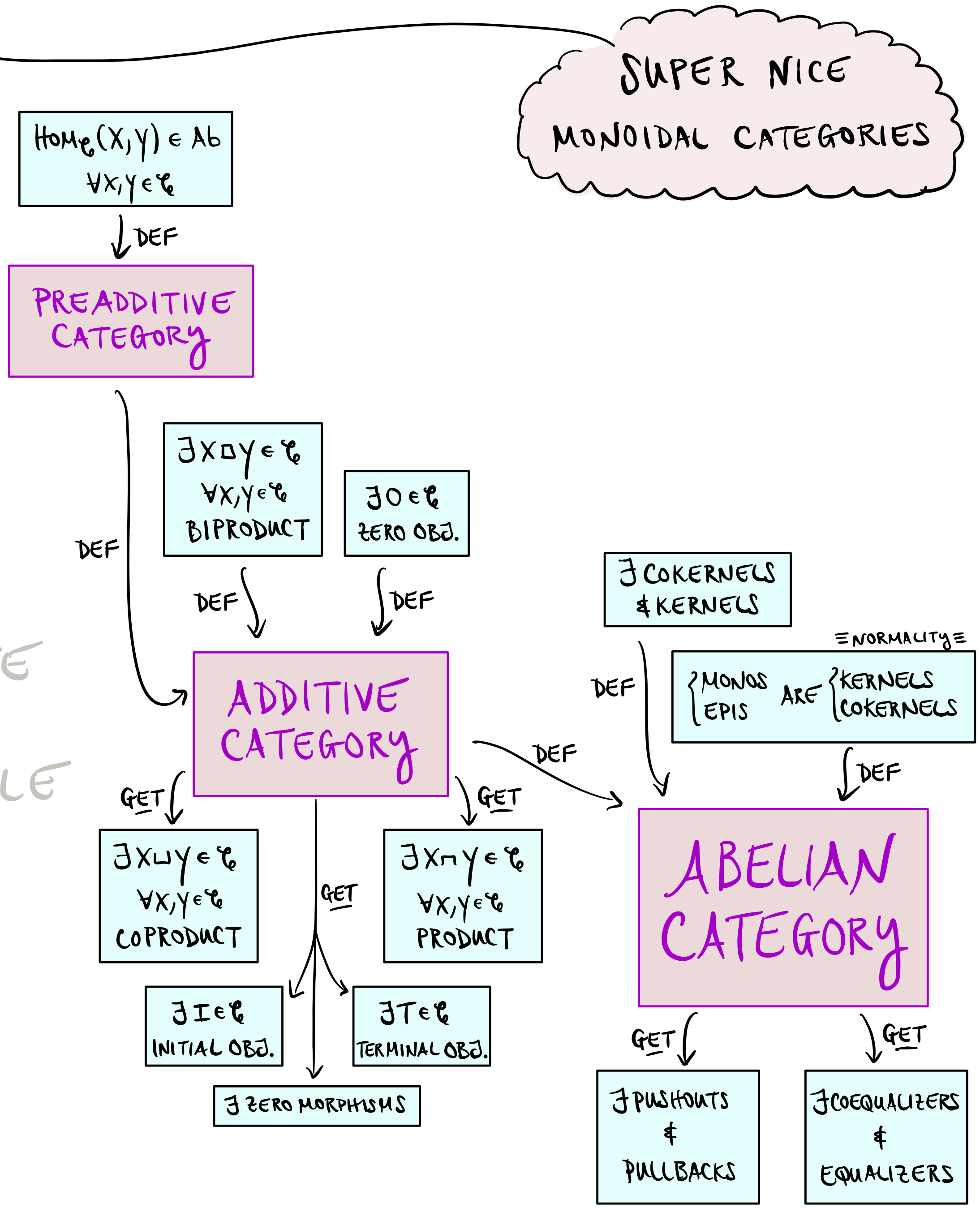

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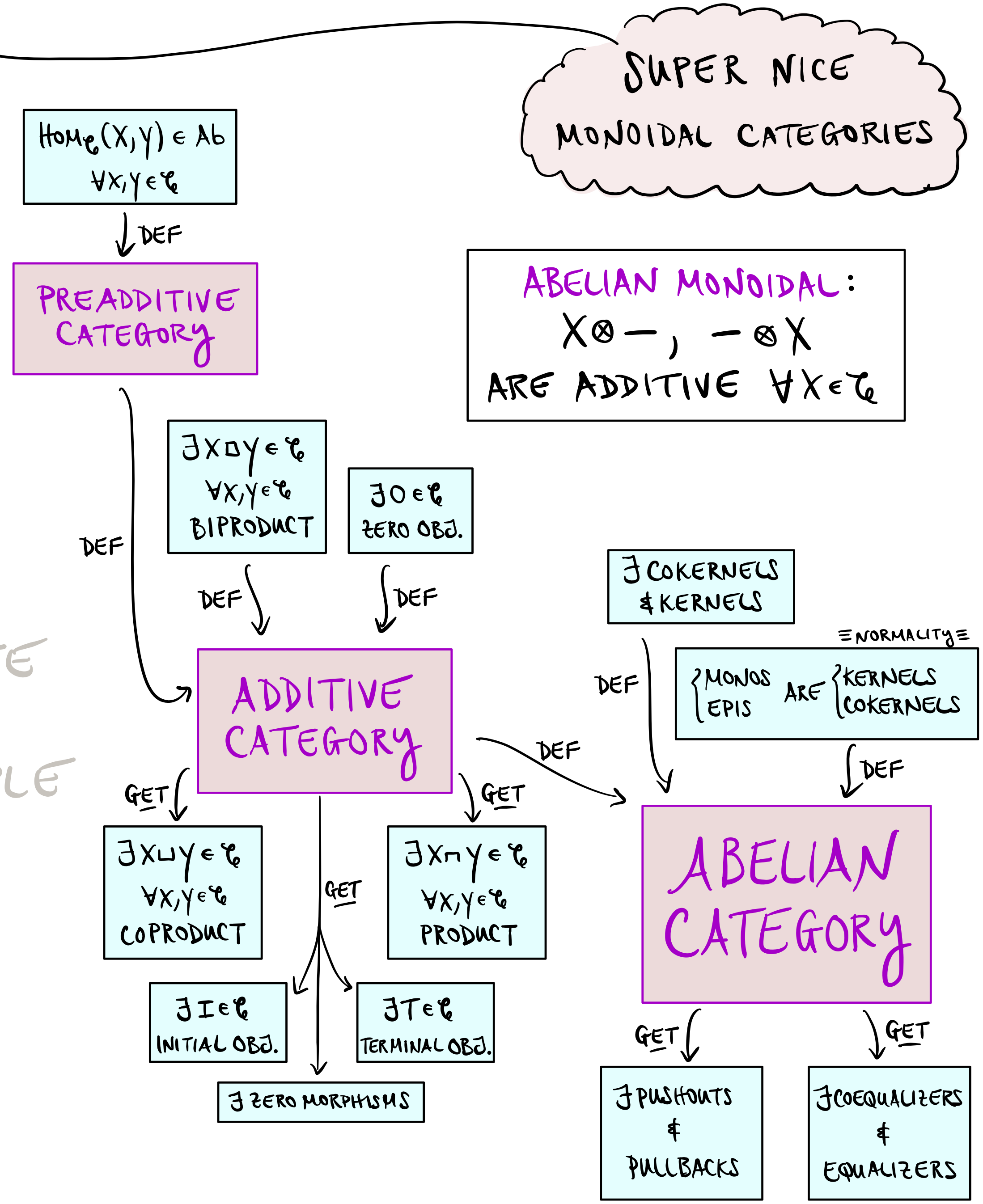

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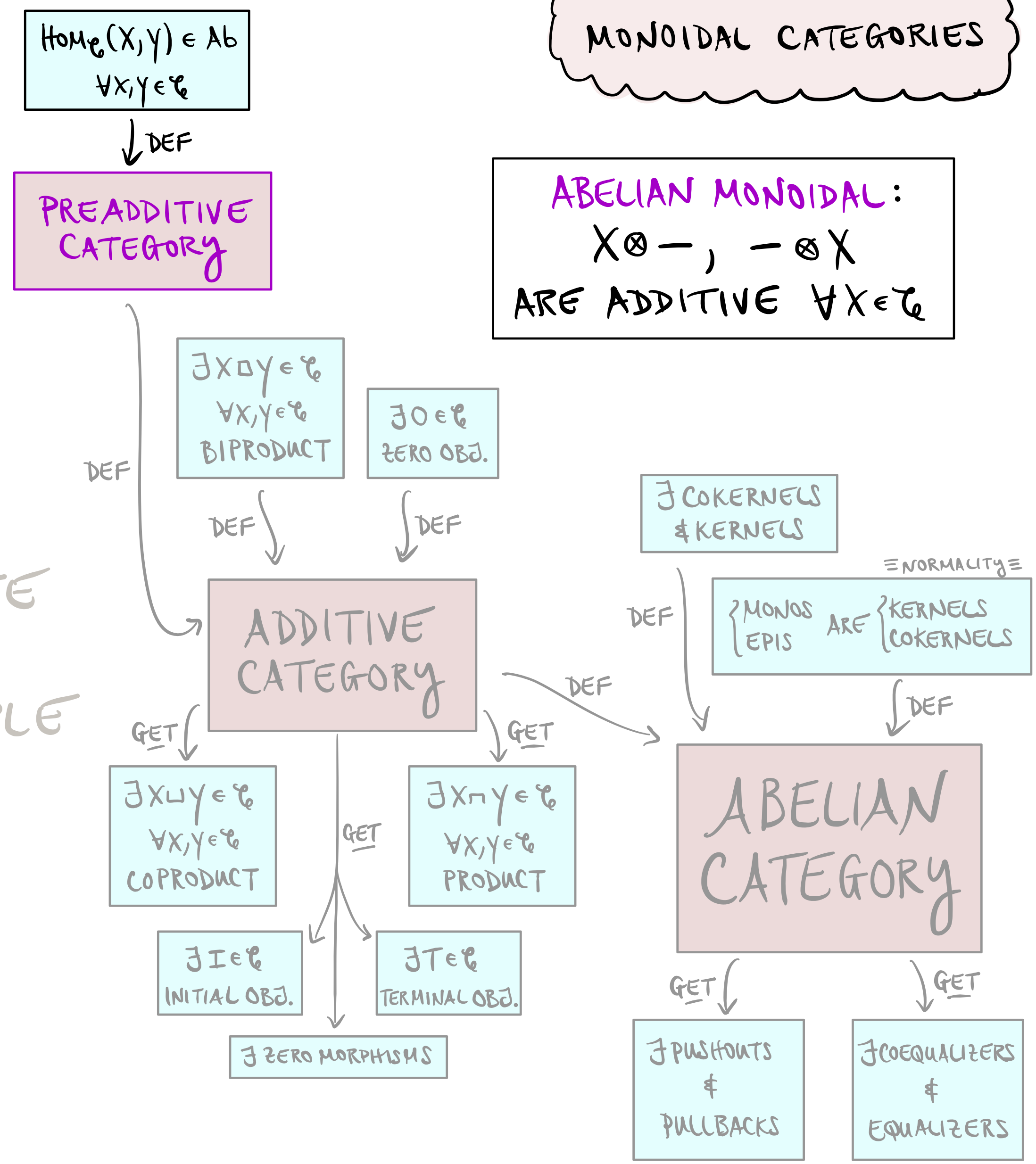

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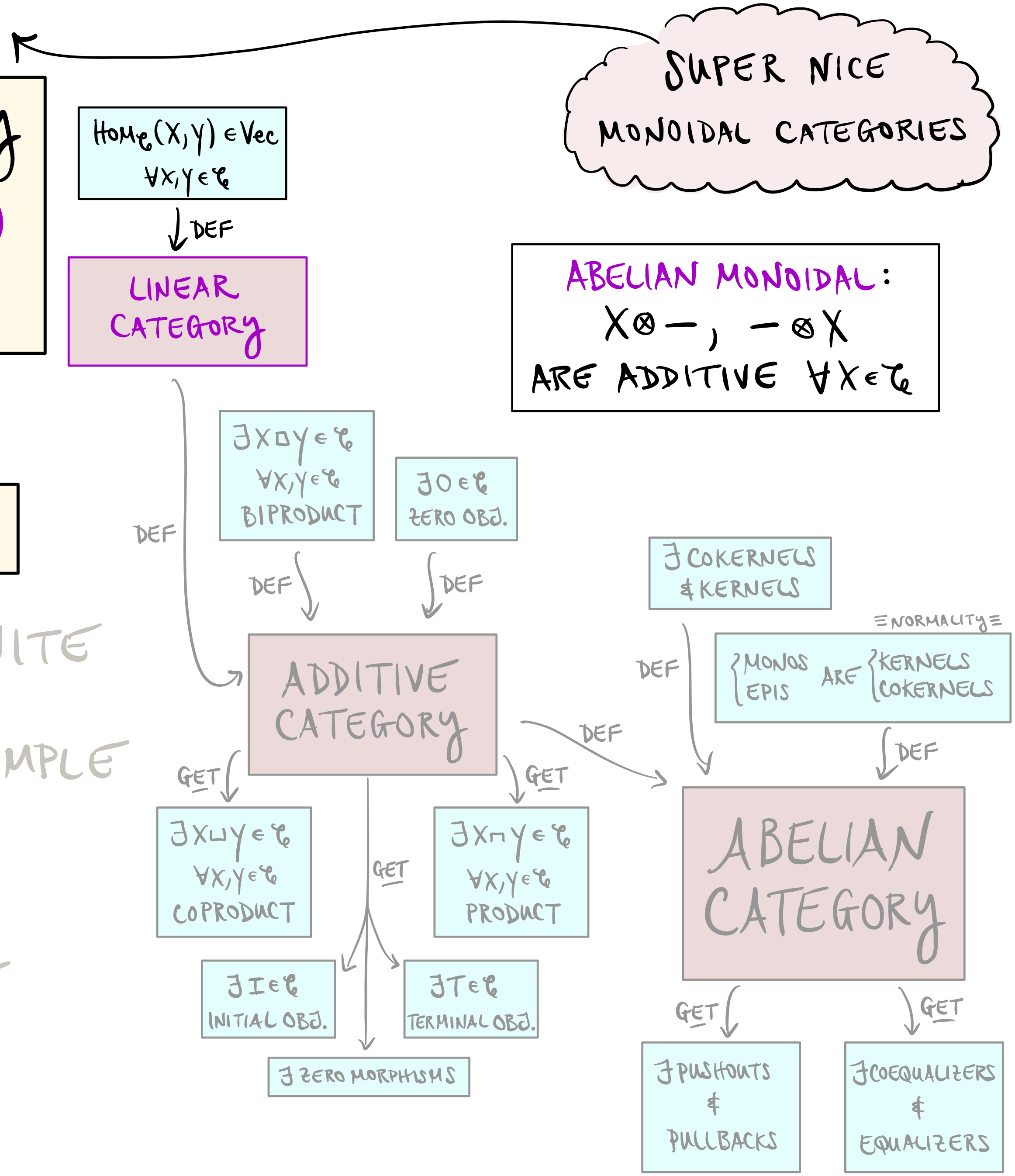

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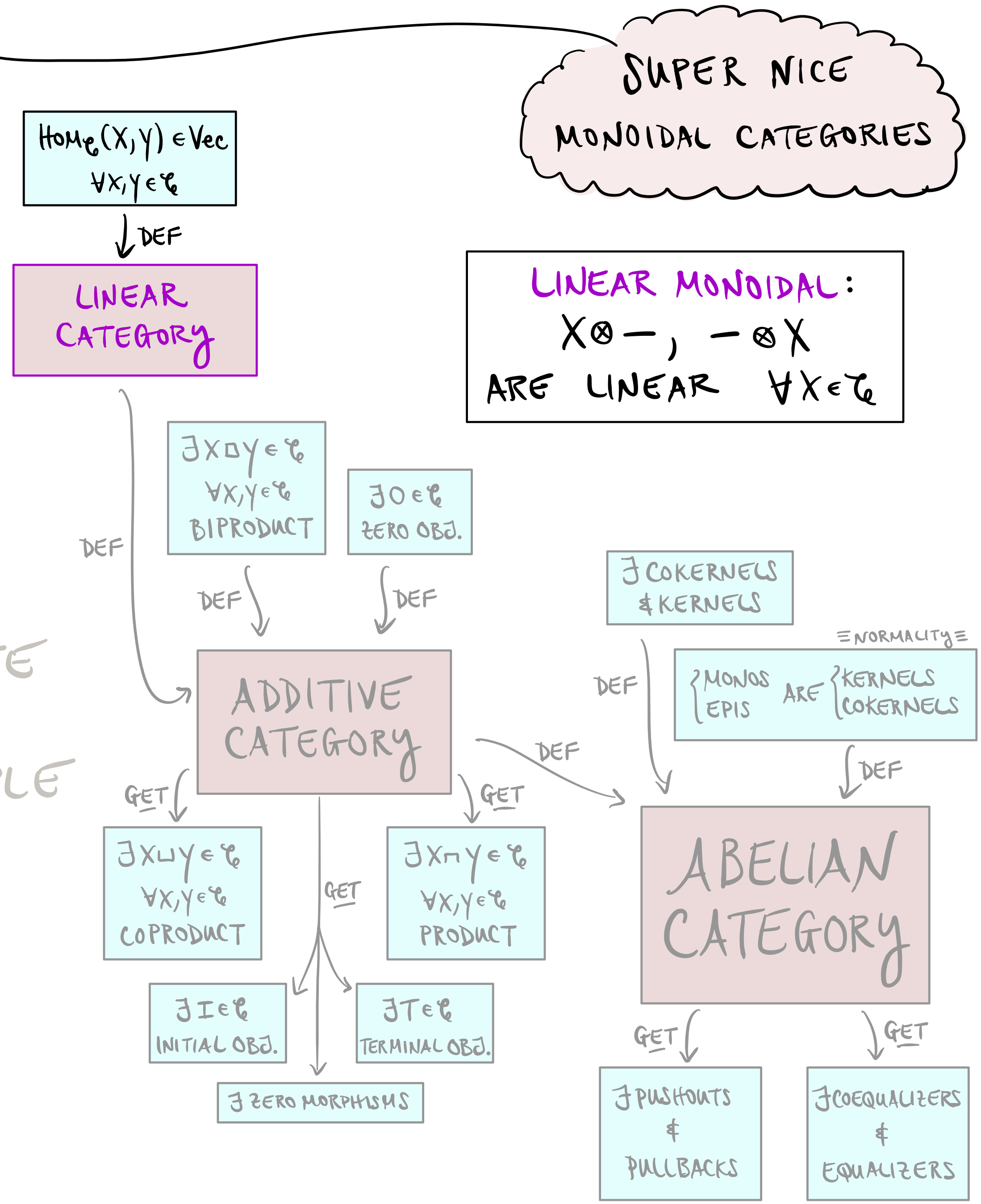

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$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Vec}$ $\forall X, Y \in \mathcal{C}$

$\downarrow$ DEF

LINEAR CATEGORY

LINEAR MONOIDAL: $X \otimes -$, $- \otimes X$ ARE LINEAR $\forall X \in \mathcal{C}$

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$\downarrow$ DEF

LINEAR CATEGORY

LINEAR MONOIDAL: $X \otimes -$, $- \otimes X$ ARE LINEAR $\forall X \in \mathcal{C}$

$\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS A FINITE DIM'L $\Bbbk$-VSPACE $\forall X,Y \in \mathcal{C}$

$\downarrow$ DEF

ALL OBJECTS IN $\mathcal{C}$ HAVE FINITE LENGTH

$\downarrow$ DEF

LOCALLY FINITE

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$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Vec}$ $\forall X,Y \in \mathcal{C}$ $\xrightarrow{\text{DEF}}$ LINEAR CATEGORY

LINEAR MONOIDAL: $X \otimes -$, $- \otimes X$ ARE LINEAR $\forall X \in \mathcal{C}$

$\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS A FINITE DIM'L $\Bbbk$-VSPACE $\forall X,Y \in \mathcal{C}$ $\xrightarrow{\text{DEF}}$ LOCALLY FINITE

ALL OBJECTS IN $\mathcal{C}$ HAVE FINITE LENGTH $\xrightarrow{\text{DEF}}$ LOCALLY FINITE

LOCALLY FINITE $\xrightarrow{\text{DEF}}$ FINITE

$\exists$ ENOUGH PROJECTIVES IN $\mathcal{C}$: $\forall Z \in \mathcal{C}$ $\exists$ PROJ. OBJ $P(Z) \in \mathcal{C}$ WITH EPI $P(Z) \twoheadrightarrow Z$ IN $\mathcal{C}$. $\xrightarrow{\text{DEF}}$ FINITE

$\exists$ ONLY FINITELY MANY ISOCLASSES OF SIMPLE OBJECTS IN $\mathcal{C}$ $\xrightarrow{\text{DEF}}$ FINITE

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$X^{\neq 0} \in \mathcal{C}$ IS SEMISIMPLE IF $X \cong \coprod_{i \in I} X_i$ FOR SIMPLE OBJECTS X_i.

$\mathcal{C}$ IS SEMISIMPLE IF ALL OBJECTS ARE SS

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$\mathcal{C}$ IS SEMISIMPLE IF ALL OBJECTS ARE SS

SIMPLE OBJECTS

$X^{\neq 0} \in \mathcal{C}$ IS SIMPLE IF THE ONLY SUBOBJS OF X ARE X & 0

INDECOMPOSABLE OBJECTS

$X^{\neq 0} \in \mathcal{C}$ IS INDECOMPOSABLE IF $X \ncong X_1 \sqcup X_2$ $\forall$ NONZERO SUBOBJ. X_1, X_2 OF X

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SIMPLE OBJECTS

$X \overset{\neq 0}{\in} \mathcal{C}$ IS SIMPLE IF THE ONLY SUBOBJS OF X ARE X & 0

IN GENERAL

INDECOMPOSABLE OBJECTS

$X \overset{\neq 0}{\in} \mathcal{C}$ IS INDECOMPOSABLE IF $X \ncong X_1 \sqcup X_2$ $\forall$ NONZERO SUBOBJ. X_1, X_2 OF X

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$X^{\neq 0} \in \mathcal{C}$ IS **SEMISIMPLE** IF $X \cong \coprod_{i \in I} X_i$ FOR SIMPLE OBJECTS X_i.

$\mathcal{C}$ IS **SEMISIMPLE** IF ALL OBJECTS ARE SS

SIMPLE OBJECTS

$X^{\neq 0} \in \mathcal{C}$ IS **SIMPLE** IF THE ONLY SUBOBJS OF X ARE X & 0

IN GENERAL $\not\Leftarrow$

INDECOMPOSABLE OBJECTS

$X^{\neq 0} \in \mathcal{C}$ IS **INDECOMPOSABLE** IF $X \not\cong X_1 \sqcup X_2$ $\forall$ NONZERO SUBOBJ. X_1, X_2 OF X

FINITE LENGTH OBJECTS

PROP: IN SEMISIMPLE CATEGORIES, INDECOMPOSABLE OBJECTS OF FINITE LENGTH ARE SIMPLE.

SEMISIMPLE CATEGORIES

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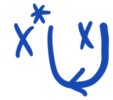
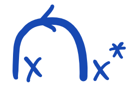
$\forall X \in \mathcal{C}$:

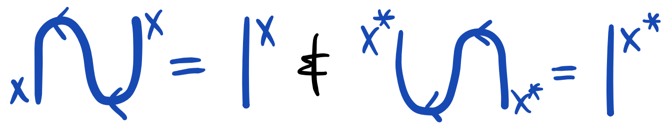
$\exists X^* \in \mathcal{C}$ WITH

$ev^L_X : X^* \otimes X \longrightarrow \mathbb{1}$



$coev^L_X : \mathbb{1} \longrightarrow X \otimes X^*$





-AND-

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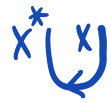
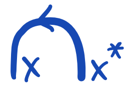
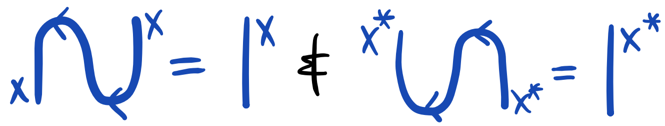
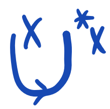
$ev_X^L : X^* \otimes X \longrightarrow \mathbb{1}$ 

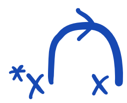
$coev_X^L : \mathbb{1} \longrightarrow X \otimes X^*$ 

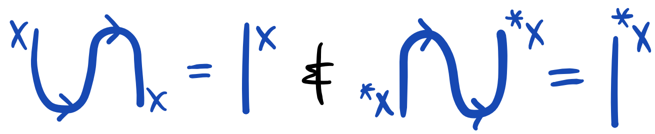


$-$AND$-$

$\exists {}^*X \in \mathcal{C}$ WITH

$ev_X^R : X \otimes {}^*X \longrightarrow \mathbb{1}$ 

$coev_X^R : \mathbb{1} \longrightarrow {}^*X \otimes X$ 



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$\boxed{\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk}$

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$\Downarrow$ ASSUME *

$\mathbb{1}$ IS A SIMPLE OBJECT OF $\mathcal{C}$

(a) $\mathcal{C}$ IS ABELIAN *

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$\boxed{\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk}$

$\Downarrow$ ASSUME *

$\boxed{\mathbb{1} \text{ IS A SIMPLE OBJECT OF } \mathcal{C}}$

PF/ TAKE A SUBOBJECT OF $\mathbb{1}$:

$(X, \overset{(\text{MONO})}{\iota} : X \to \mathbb{1})$ WLOG NONZERO, SIMPLE

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$\boxed{\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk}$

$\Downarrow$ ASSUME *

$\boxed{\mathbb{1} \text{ IS A SIMPLE OBJECT OF } \mathcal{C}}$

PF/ TAKE A SUBOBJECT OF $\mathbb{1}$:

$(X, \overset{\text{(MONO)}}{\iota} : X \to \mathbb{1})$ WLOG NONZERO, SIMPLE

GET S.E.S. $0 \to X \xrightarrow{\iota} \mathbb{1} \to \mathrm{coker}(\iota) \to 0.$

I. FUSION CATEGORIES

SUPER NICE MONOIDAL CATEGORIES

A MONOIDAL CATEGORY $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ IS FUSION IF:

(a) $\mathcal{C}$ IS ABELIAN $*$

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$\Rightarrow$ $\mathrm{id}_X \otimes \iota^*$ ISO.

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$\Rightarrow$ ι EPIC $\therefore X \cong \mathbb{1}$ ///

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$\Downarrow$ [SCHUR'S LEMMA]

$\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ DIVISION ALGEBRA$/\Bbbk$

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$\Downarrow$ [LINEAR ALGEBRA ARGUMENT]

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WHEN ARE TWO FUSION CATEGORIES CONSIDERED THE SAME?

⋮

WHICH ITEMS ARE STRUCTURAL??

AND WHICH ARE PROPERTIES??

I. FUSION CATEGORIES

SUPER NICE MONOIDAL CATEGORIES

A MONOIDAL CATEGORY — STRUCTURE

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

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. . . THE ANSWER??

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WHEN ARE TWO FUSION CATEGORIES CONSIDERED THE SAME?

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IF $\exists$ LINEAR FUNCTOR BETWEEN THE TWO THAT'S AN ISOM./EQUIV. OF MONOIDAL CATEGORIES

I. FUSION CATEGORIES

EXAMPLES

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? Set ?

I. FUSION CATEGORIES

A MONOIDAL CATEGORY

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

IS **FUSION** IF:

(a) $\mathcal{C}$ IS ABELIAN

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(d) $\mathbb{1}$ IS ABS. SIMPLE

(e) $\mathcal{C}$ IS RIGID ✗

(f) $\mathcal{C}$ IS SEMISIMPLE

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(JUST ONE REASON)

EXAMPLES

FdVec

~~Vec~~

~~Ab~~

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G-FdMod

G ARBITRARY GROUP

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FdVec

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FdVec_G

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FdVec

RANK 1

$\mathrm{Irr}(\mathcal{C}) = \{[\Bbbk]\}$

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G FINITE GRP

$\cong \Bbbk G$-FdMod

RANK DEPENDS ON ARTIN-WEDDERBURN DECOMPOSITION OF $\Bbbk G$

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RANK DEPENDS ON ARTIN-WEDDERBURN DECOMPOSITION OF $\Bbbk G$

$$\Bbbk G \cong \prod_{i=1}^{r} \mathrm{Mat}_{n_i}(\Bbbk)$$

AS $\Bbbk$-ALGS

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Ex.

G FINITE ABELIAN

$\Downarrow$

$\mathrm{rk}(G\text{-}\mathrm{FdMod}) = ??$

$G\text{-}\mathrm{FdMod}$

G FINITE GRP

$\cong \Bbbk G\text{-}\mathrm{FdMod}$

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Ex.

G FINITE ABELIAN

$\Downarrow$

$\Bbbk G$ COMMUTATIVE

$\Downarrow$

$\mathrm{rk}(G\text{-FdMod}) = |G|$

G-FdMod

G FINITE GRP

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RANK DEPENDS ON ARTIN–WEDDERBURN DECOMPOSITION OF $\Bbbk G$

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Ex.

$G \cong S_3$

$\Downarrow$

$\mathrm{rk}(G\text{-FdMod}) = ??$

$G\text{-FdMod}$ G FINITE GRP

$\cong \Bbbk G\text{-FdMod}$

RANK DEPENDS ON ARTIN–WEDDERBURN DECOMPOSITION OF $\Bbbk G$

$\Bbbk G \cong \prod_{i=1}^{r} \mathrm{Mat}_{n_i}(\Bbbk)$

AS $\Bbbk$-ALGS

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Ex.

$$G \cong S_3$$
$$\Downarrow$$
$$\Bbbk G \cong \Bbbk^{\times 2} \times \mathrm{Mat}_2(\Bbbk)$$
$$\Downarrow$$
$$\mathrm{rk}(G\text{-FdMod}) = 3$$

G-FdMod

G FINITE GRP

$\cong \Bbbk G$-FdMod

RANK DEPENDS ON ARTIN–WEDDERBURN DECOMPOSITION OF $\Bbbk G$

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RESEARCH PROBLEM

CLASSIFY ALL FUSION CATEGORIES OF A GIVEN RANK, UP TO EQUIV.

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✓ RANK 1: FdVec

✓ RANK 2 [OSTRIK]
$\exists$ 4 EQUIV. CLASSES

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RESEARCH PROBLEM ≡ RATHER IMPOSSIBLE ≡

CLASSIFY ALL FUSION CATEGORIES OF A GIVEN RANK, UP TO EQUIV.

INVOLVES KNOWING THE ARTIN-WEDDERBURN DECOMP. OF ALL FINITE GROUPS

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≡WEAKEN≡

RESEARCH PROBLEM SHOW THAT $\exists$ ONLY FINITELY MANY FUSION CATEGORIES OF A GIVEN RANK, UP TO EQUIV.

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SETTLED IN SPECIAL CASES

[BRUILLARD–NG–ROWELL–WANG IN "MODULAR FUSION" CASE (2013)]

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RANK-FINITENESS CONJECTURE

$\exists$ ONLY FINITELY MANY FUSION CATEGORIES OF A GIVEN RANK, UP TO EQUIV.

... HAVE A FINITENESS RESULT USING THIS FACT

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TAKE $\{X_i\}_{i \in \mathrm{Irr}(\mathcal{C})}$ ISOCLASS REPRESENTATIVES OF SIMPLE OBJECTS OF $\mathcal{C}$

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GET $X_i \otimes X_j \cong \coprod_{k \in \mathrm{Irr}(\mathcal{C})} X_k^{\amalg N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

$N_k^{i,j} := [X_i \otimes X_j : X_k]$ MULTIPLICITY OF X_k IN $X_i \otimes X_j$

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OCNEANU RIGIDITY [ETINGOF–NIKSHYCH–OSTRIK]

$\exists$ ONLY FINITELY MANY FUSION CATEGORIES WITH A GIVEN FUSION RULE.

TAKE $\{X_i\}_{i \in \mathrm{Irr}(\mathcal{C})}$ ISOCLASS REPRESENTATIVES OF SIMPLE OBJECTS OF $\mathcal{C}$

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(f) $\mathcal{C}$ IS SEMISIMPLE

(g) $\mathcal{C}$ IS FINITE

GOVERNED BY SIMPLE OBJECTS

OCNEANU RIGIDITY [ETINGOF–NIKSHYCH–OSTRIK]

$\exists$ ONLY FINITELY MANY FUSION CATEGORIES WITH A GIVEN FUSION RULE.

LET'S RETURN TO THE RANK 2 CASE

$\mathcal{C}$ FINITE $\Rightarrow$ $|\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in \mathrm{Irr}(\mathcal{C})} X_k^{\sqcup N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

II. FUSION RULES & RANK

[OSTRIK] $\exists$ 4 EQUIV. CLASSES OF RANK 2 FUSION CATS

LET'S LOOK AT TWO OF THEM...

A MONOIDAL CATEGORY

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

IS FUSION IF:

(a) $\mathcal{C}$ IS ABELIAN

(b) $\mathcal{C}$ IS ($\mathbb{k}$-)LINEAR

(c) $\mathcal{C}$ IS LOC. FINITE

(d) $\mathbb{1}$ IS ABS. SIMPLE

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(g) $\mathcal{C}$ IS FINITE

GOVERNED BY SIMPLE OBJECTS

$C_2 = \langle g \rangle$

C_2-FdMod

$\mathcal{C}$ FINITE $\Rightarrow$ $|\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in \mathrm{Irr}(\mathcal{C})} X_k^{\sqcup N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

II. FUSION RULES & RANK

A MONOIDAL CATEGORY

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

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[OSTRIK] $\exists$ 4 EQUIV. CLASSES OF RANK 2 FUSION CATS

LET'S LOOK AT TWO OF THEM...

$C_2 = \langle g \rangle$

C_2-FdMod

$\mathrm{Irr}(\mathcal{C}) = \{[V_0], [V_1]\}$

TRIVIAL REP $\Bbbk v$ s.t. $g \cdot v = v$

SIGN REP $\Bbbk v$ s.t. $g \cdot v = -v$

$\mathcal{C}$ FINITE $\Rightarrow$ $|\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in \mathrm{Irr}(\mathcal{C})} X_k^{\amalg N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

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FIBONACCI FUSION CATEG.

Fib

$\mathrm{Irr}(\mathcal{C}) = \{\mathbb{1}, X\}$

DEFINED WITH FUSION RULES

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DEFINED WITH FUSION RULES

$\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$

$\mathbb{1} \otimes X \cong X$

$X \otimes \mathbb{1} \cong X$

$X \otimes X \cong \mathbb{1} \sqcup X$

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DIFFERENT FUSION RULES

$\Downarrow$

INEQUIVALENT

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$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

II. FUSION RULES & RANK

RELATED TO THE "SIZE" OF $X \in \mathrm{Fib}$ [OSTRIK]

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DIFFERENT FUSION RULES

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$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

III. FROBENIUS-PERRON DIMENSION

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GOVERNED BY SIMPLE OBJECTS

FROBENIUS-PERRON THEOREM TAKE $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$.
THEN $\exists$ NON-NEGATIVE REAL EIGENVALUE $\mathrm{FP}(M)$ THAT IS $\geq$ ABSOLUTE VALUE OF ALL OTHER EIGENVALUES OF M.

$\mathcal{C}$ FINITE $\Rightarrow$ $|\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

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GOVERNED BY SIMPLE OBJECTS

<u>FROBENIUS-PERRON THEOREM</u> TAKE $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$. THEN $\exists$ NON-NEGATIVE REAL EIGENVALUE $FP(M)$ THAT IS $\geq$ ABSOLUTE VALUE OF ALL OTHER EIGENVALUES OF M.

FP-DIMENSION OF $X_i \in \mathrm{Irr}(\mathcal{C})$:

$$\boxed{\mathrm{FPdim}_{\mathcal{C}}(X_i) := FP\left(N_k^{i,j}\right)_{j,k \in \mathrm{Irr}(\mathcal{C})}.}$$

$\mathcal{C}$ FINITE $\Rightarrow$ $|\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in \mathrm{Irr}(\mathcal{C})} X_k^{\sqcup N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

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A MONOIDAL CATEGORY $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ IS **FUSION** IF:

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$$\boxed{\mathrm{FPdim}_{\mathcal{C}}(X_i) := FP\left(N_k^{i,j}\right)_{j,k \in \mathrm{Irr}(\mathcal{C})}.}$$

FP-DIMENSION OF $\mathcal{C}$ ITSELF:

$$\boxed{\mathrm{FPdim}(\mathcal{C}) := \sum_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}_{\mathcal{C}}(X_i)^2}$$

$\mathcal{C}$ FINITE $\Rightarrow$ $|\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in \mathrm{Irr}(\mathcal{C})} X_k^{\sqcup N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

III. FROBENIUS-PERRON DIMENSION

<u>FP THEOREM</u>

TAKE $M \in Mat_n(\mathbb{R}_{\geq 0})$.

$\Rightarrow \exists$ E.VALUE

$FP(M) \in \mathbb{R}_{\geq 0}$

THAT IS

$\geq$ ABS VALUE OF ALL E.VALUES OF M.

$$FPdim_{\mathcal{C}}(X_i) \overset{\cdot\cdot}{=} FP\left(N_k^{i,j}\right)_{j,k \in Irr(\mathcal{C})}.$$

$$FPdim(\mathcal{C}) \overset{\cdot\cdot}{=} \sum_{i \in Irr(\mathcal{C})} FPdim_{\mathcal{C}}(X_i)^2$$

$\mathcal{C}$ FINITE $\Rightarrow$ $|Irr(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in Irr(\mathcal{C})} X_k^{\sqcup N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in Irr(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

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FP THEOREM

TAKE $M \in Mat_n(\mathbb{R}_{\geq 0})$.

$\Rightarrow \exists$ E.VALUE $FP(M) \in \mathbb{R}_{\geq 0}$ THAT IS $\geq$ ABS VALUE OF ALL E.VALUES OF M.

$$FPdim_{\mathcal{C}}(X_i) \;\; \text{ii} \;\; FP(N_k^{i,j})_{j,k \in Irr(\mathcal{C})}.$$

$$FPdim(\mathcal{C}) \;\; \text{ii} \;\; \sum_{i \in Irr(\mathcal{C})} FPdim_{\mathcal{C}}(X_i)^2$$

C_2-FdMod

$Irr(\mathcal{C}) = \{[V_0], [V_1]\}$

$V_0 \otimes V_0 \cong V_0$

$V_0 \otimes V_1 \cong V_1$

$V_1 \otimes V_0 \cong V_1$

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WHAT IS

$FPdim_{\mathcal{C}}(V_0)$?

$FPdim_{\mathcal{C}}(V_1)$?

$FPdim(\mathcal{C})$?

$\mathcal{C}$ FINITE $\Rightarrow$ $|Irr(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in Irr(\mathcal{C})} X_k^{\sqcup N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

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$\Rightarrow \exists$ E.VALUE $FP(M) \in \mathbb{R}_{\geq 0}$ THAT IS $\geq$ ABS VALUE OF ALL E.VALUES OF M.

$FPdim_{\mathcal{C}}(X_i) =: FP(N_k^{i,j})_{j,k \in Irr(\mathcal{C})}$.

$FPdim(\mathcal{C}) =: \sum_{i \in Irr(\mathcal{C})} FPdim_{\mathcal{C}}(X_i)^2$

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$V_1 \otimes V_1 \cong V_0$

$\therefore FPdim(\mathcal{C}) =$

$\underline{i=0}$: $\begin{pmatrix} N_0^{0,0} & N_1^{0,0} \\ N_0^{0,1} & N_1^{0,1} \end{pmatrix}$

HAS EVALUES = $\{ \quad \}$

$\Rightarrow FPdim_{\mathcal{C}}(V_0) =$

$\underline{i=1}$: $\begin{pmatrix} N_0^{1,0} & N_1^{1,0} \\ N_0^{1,1} & N_1^{1,1} \end{pmatrix}$

HAS EVALUES = $\{ \quad \}$

$\Rightarrow FPdim_{\mathcal{C}}(V_1) =$

$\mathcal{C}$ FINITE $\Rightarrow$ $|Irr(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in Irr(\mathcal{C})} X_k^{\amalg N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

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FP THEOREM

TAKE $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$.

$\Rightarrow \exists$ E.VALUE $FP(M) \in \mathbb{R}_{\geq 0}$

THAT IS

$\geq$ ABS VALUE OF ALL E.VALUES OF M.

$$\mathrm{FPdim}_{\mathcal{C}}(X_i) \stackrel{\text{def}}{=} FP\left(N_k^{i,j}\right)_{j,k \in \mathrm{Irr}(\mathcal{C})}.$$

$$\mathrm{FPdim}(\mathcal{C}) \stackrel{\text{def}}{=} \sum_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}_{\mathcal{C}}(X_i)^2$$

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$\mathrm{Irr}(\mathcal{C}) = \{[V_0], [V_1]\}$

$V_0 \otimes V_0 \cong V_0$

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$V_1 \otimes V_0 \cong V_1$

$V_1 \otimes V_1 \cong V_0$

$\therefore \mathrm{FPdim}(\mathcal{C}) =$

$\underline{i=0}$: $\begin{pmatrix} 1 & 0 \\ N_0^{0,1} & N_1^{0,1} \end{pmatrix}$

HAS EVALUES = $\{ \quad \}$

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(V_0) =$

$\underline{i=1}$: $\begin{pmatrix} N_0^{1,0} & N_1^{1,0} \\ N_0^{1,1} & N_1^{1,1} \end{pmatrix}$

HAS EVALUES = $\{ \quad \}$

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(V_1) =$

$\mathcal{C}$ FINITE $\Rightarrow$ $|\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in \mathrm{Irr}(\mathcal{C})} X_k^{\sqcup N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

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FP THEOREM

TAKE $M \in Mat_n(\mathbb{R}_{\geq 0})$.

$\Rightarrow \exists$ E.VALUE

$FP(M) \in \mathbb{R}_{\geq 0}$

THAT IS

$\geq$ ABS VALUE OF ALL E.VALUES OF M.

$FPdim_{\mathcal{C}}(X_i)$

$\overset{..}{||}$

$FP(N_k^{i,j})_{j,k \in Irr(\mathcal{C})}$.

$FPdim(\mathcal{C})$

$\overset{..}{||}$

$\sum_{i \in Irr(\mathcal{C})} FPdim_{\mathcal{C}}(X_i)^2$

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$V_1 \otimes V_1 \cong V_0$

$\therefore FPdim(\mathcal{C}) =$

$\underline{i=0}$: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

HAS EVALUES $= \{ \quad \}$

$\Rightarrow FPdim_{\mathcal{C}}(V_0) =$

$\underline{i=1}$: $\begin{pmatrix} N_0^{1,0} & N_1^{1,0} \\ N_0^{1,1} & N_1^{1,1} \end{pmatrix}$

HAS EVALUES $= \{ \quad \}$

$\Rightarrow FPdim_{\mathcal{C}}(V_1) =$

$\mathcal{C}$ FINITE $\Rightarrow$ $|Irr(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in Irr(\mathcal{C})} X_k^{\amalg N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in Irr(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

III. FROBENIUS-PERRON DIMENSION

FP THEOREM

TAKE $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$.

$\Rightarrow$ $\exists$ E.VALUE

$FP(M) \in \mathbb{R}_{\geq 0}$

THAT IS

$\geq$ ABS VALUE OF ALL E.VALUES OF M.

$$\mathrm{FPdim}_{\mathcal{C}}(X_i) \overset{\cdot\cdot}{=} FP\left(N_k^{i,j}\right)_{j,k \in \mathrm{Irr}(\mathcal{C})}.$$

$$\mathrm{FPdim}(\mathcal{C}) \overset{\cdot\cdot}{=} \sum_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}_{\mathcal{C}}(X_i)^2$$

C_2-FdMod

$\mathrm{Irr}(\mathcal{C}) = \{[V_0], [V_1]\}$

$V_0 \otimes V_0 \cong V_0$

$V_0 \otimes V_1 \cong V_1$

$V_1 \otimes V_0 \cong V_1$

$V_1 \otimes V_1 \cong V_0$

$\therefore$ $\mathrm{FPdim}(\mathcal{C}) =$

$i=0$: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

HAS EVALUES $= \{1\}$

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(V_0) = 1$

$i=1$: $\begin{pmatrix} N_0^{1,0} & N_1^{1,0} \\ N_0^{1,1} & N_1^{1,1} \end{pmatrix}$

HAS EVALUES $= \{ \quad \}$

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$V_1 \otimes V_1 \cong V_0$

$\therefore \mathrm{FPdim}(\mathcal{C}) =$

$\underline{i=0}$: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

HAS EVALUES $= \{1\}$

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(V_0) = 1$

$\underline{i=1}$: $\begin{pmatrix} 0 & 1 \\ N_0^{1,1} & N_1^{1,1} \end{pmatrix}$

HAS EVALUES $= \{ \quad \}$

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(V_1) =$

$\mathcal{C}$ FINITE $\Rightarrow |\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

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$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

III. FROBENIUS-PERRON DIMENSION

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$\mathrm{FPdim}_{\mathcal{C}}(X_i)$

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$V_1 \otimes V_0 \cong V_1$

$V_1 \otimes V_1 \cong V_0$

$\therefore\ \mathrm{FPdim}(\mathcal{C}) =$

$\underline{i=0}$: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

HAS EVALUES $= \{1\}$

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(V_0) = 1$

$\underline{i=1}$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

HAS EVALUES $= \{ \quad \}$

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$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

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$FP(M) \in \mathbb{R}_{\geq 0}$

THAT IS

$\geq$ ABS VALUE OF ALL E.VALUES OF M.

$\mathrm{FPdim}_{\mathcal{C}}(X_i)$

$\overset{\cdot\cdot}{\shortparallel}$

$FP(N_k^{i,j})_{j,k \in \mathrm{Irr}(\mathcal{C})}$.

$\mathrm{FPdim}(\mathcal{C})$

$\overset{\cdot\cdot}{\shortparallel}$

$\sum_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}_{\mathcal{C}}(X_i)^2$

C_2-FdMod

$\mathrm{Irr}(\mathcal{C}) = \{[V_0], [V_1]\}$

$V_0 \otimes V_0 \cong V_0$

$V_0 \otimes V_1 \cong V_1$

$V_1 \otimes V_0 \cong V_1$

$V_1 \otimes V_1 \cong V_0$

$\therefore\ \mathrm{FPdim}(\mathcal{C}) =$

$\underline{i=0}$: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

HAS EVALUES $= \{1\}$

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(V_0) = 1$

$\underline{i=1}$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

HAS EVALUES $= \{1, -1\}$

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(V_1) = 1$

$\mathcal{C}$ FINITE $\Rightarrow$ $|\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in \mathrm{Irr}(\mathcal{C})} X_k^{\amalg N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

III. FROBENIUS-PERRON DIMENSION

FP THEOREM

TAKE $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$.
$\Rightarrow \exists$ E.VALUE
$FP(M) \in \mathbb{R}_{\geq 0}$
THAT IS
$\geq$ ABS VALUE OF ALL E.VALUES OF M.

$$\mathrm{FPdim}_{\mathcal{C}}(X_i) \;\overset{\cdot\cdot}{=}\; FP\left(N_k^{i,j}\right)_{j,k \in \mathrm{Irr}(\mathcal{C})}.$$

$$\mathrm{FPdim}(\mathcal{C}) \;\overset{\cdot\cdot}{=}\; \sum_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}_{\mathcal{C}}(X_i)^2$$

C_2-FdMod

$\mathrm{Irr}(\mathcal{C}) = \{[V_0], [V_1]\}$

$V_0 \otimes V_0 \cong V_0$

$V_0 \otimes V_1 \cong V_1$

$V_1 \otimes V_0 \cong V_1$

$V_1 \otimes V_1 \cong V_0$

$\therefore \mathrm{FPdim}(\mathcal{C}) = 2$

$i=0$: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

HAS EVALUES $= \{1\}$

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(V_0) = 1$

$i=1$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

HAS EVALUES $= \{1, -1\}$

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(V_1) = 1$

$\mathcal{C}$ FINITE $\Rightarrow$ $|\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in \mathrm{Irr}(\mathcal{C})} X_k^{\sqcup N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

III. FROBENIUS-PERRON DIMENSION

FP THEOREM

TAKE $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$.

$\Rightarrow \exists$ E.VALUE $FP(M) \in \mathbb{R}_{\geq 0}$

THAT IS

$\geq$ ABS VALUE OF ALL E.VALUES OF M.

$$\mathrm{FPdim}_{\mathcal{C}}(X_i) = FP\left(N_k^{i,j}\right)_{j,k \in \mathrm{Irr}(\mathcal{C})}.$$

$$\mathrm{FPdim}(\mathcal{C}) = \sum_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}_{\mathcal{C}}(X_i)^2$$

WHAT IS

$\mathrm{FPdim}_{\mathcal{C}}(\mathbb{1})$?

$\mathrm{FPdim}_{\mathcal{C}}(X)$?

$\mathrm{FPdim}(\mathcal{C})$?

FIBONACCI FUSION CATEG.

Fib

$\mathrm{Irr}(\mathcal{C}) = \{\mathbb{1}, X\}$

DEFINED WITH FUSION RULES

$$\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$$

$$\mathbb{1} \otimes X \cong X$$

$$X \otimes \mathbb{1} \cong X$$

$$X \otimes X \cong \mathbb{1} \sqcup X$$

$\mathcal{C}$ FINITE $\Rightarrow$ $|\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in \mathrm{Irr}(\mathcal{C})} X_k^{\sqcup N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

III. FROBENIUS-PERRON DIMENSION

FP THEOREM

TAKE $M \in Mat_n(\mathbb{R}_{\geq 0})$.

$\Rightarrow \exists$ E.VALUE $FP(M) \in \mathbb{R}_{\geq 0}$

THAT IS $\geq$ ABS VALUE OF ALL E.VALUES OF M.

$i=0$: $\begin{pmatrix} N_0^{0,0} & N_1^{0,0} \\ N_0^{0,1} & N_1^{0,1} \end{pmatrix}$

HAS EVALUES = $\{ \quad \}$

$\Rightarrow FPdim_{\mathcal{C}}(\mathbb{1}) =$

$i=1$: $\begin{pmatrix} N_0^{1,0} & N_1^{1,0} \\ N_0^{1,1} & N_1^{1,1} \end{pmatrix}$

HAS EVALUES = $\{ \quad \}$

$\Rightarrow FPdim_{\mathcal{C}}(X) =$

FIBONACCI FUSION CATEG.

Fib

$Irr(\mathcal{C}) = \{ \underset{}{\overset{=X_0}{\mathbb{1}}}, \overset{=X_1}{X} \}$

DEFINED WITH FUSION RULES

$\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$

$\mathbb{1} \otimes X \cong X$

$X \otimes \mathbb{1} \cong X$

$X \otimes X \cong \mathbb{1} \sqcup X$

$\therefore FPdim(\mathcal{C}) =$

$FPdim_{\mathcal{C}}(X_i)$

$\vdots\vdots$

$FP(N_k^{i,j})_{j,k \in Irr(\mathcal{C})}$.

$FPdim(\mathcal{C})$

$\vdots\vdots$

$\sum_{i \in Irr(\mathcal{C})} FPdim_{\mathcal{C}}(X_i)^2$

$\mathcal{C}$ FINITE $\Rightarrow$ $|Irr(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in Irr(\mathcal{C})} X_k^{\sqcup N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in Irr(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

III. FROBENIUS-PERRON DIMENSION

FP THEOREM

TAKE $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$.
$\Rightarrow \exists$ E.VALUE $FP(M) \in \mathbb{R}_{\geq 0}$
THAT IS
$\geq$ ABS VALUE OF ALL E.VALUES OF M.

$$\mathrm{FPdim}_{\mathcal{C}}(X_i) \;\; \text{is} \;\; FP\left(N_k^{i,j}\right)_{j,k \in \mathrm{Irr}(\mathcal{C})}.$$

$$\mathrm{FPdim}(\mathcal{C}) \;\; \text{is} \;\; \sum_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}_{\mathcal{C}}(X_i)^2$$

$\underline{i=0}$: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

HAS EVALUES = { }

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(\mathbb{1}) =$

$\underline{i=1}$: $\begin{pmatrix} N_0^{1,0} & N_1^{1,0} \\ N_0^{1,1} & N_1^{1,1} \end{pmatrix}$

HAS EVALUES = { }

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(X) =$

FIBONACCI FUSION CATEG.

Fib

$\mathrm{Irr}(\mathcal{C}) = \{ \underset{}{\overset{=X_0}{\mathbb{1}}}, \overset{=X_1}{X} \}$

DEFINED WITH FUSION RULES

$\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$

$\mathbb{1} \otimes X \cong X$

$X \otimes \mathbb{1} \cong X$

$X \otimes X \cong \mathbb{1} \sqcup X$

$\therefore \mathrm{FPdim}(\mathcal{C}) =$

$\mathcal{C}$ FINITE $\Rightarrow$ $|\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in \mathrm{Irr}(\mathcal{C})} X_k^{\sqcup N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

III. FROBENIUS-PERRON DIMENSION

FP THEOREM

TAKE $M \in Mat_n(\mathbb{R}_{\geq 0})$.

$\Rightarrow \exists$ E.VALUE

$FP(M) \in \mathbb{R}_{\geq 0}$

THAT IS

$\geq$ ABS VALUE OF ALL E.VALUES OF M.

$\underline{i=0}$: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

HAS EVALUES = { 1 }

$\Rightarrow FPdim_{\mathcal{C}}(\mathbb{1}) = 1$

$FPdim_{\mathcal{C}}(X_i)$

$\shortparallel$

$FP(N_k^{i,j})_{j,k \in Irr(\mathcal{C})}$.

$\underline{i=1}$: $\begin{pmatrix} N_0^{1,0} & N_1^{1,0} \\ N_0^{1,1} & N_1^{1,1} \end{pmatrix}$

HAS EVALUES = { }

$\Rightarrow FPdim_{\mathcal{C}}(X) =$

$FPdim(\mathcal{C})$

$\shortparallel$

$\sum_{i \in Irr(\mathcal{C})} FPdim_{\mathcal{C}}(X_i)^2$

FIBONACCI FUSION CATEG.

Fib

$Irr(\mathcal{C}) = \{ \underbrace{\mathbb{1}}_{=X_0}, \underbrace{X}_{=X_1} \}$

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$\therefore FPdim(\mathcal{C}) =$

$\mathcal{C}$ FINITE $\Rightarrow$ $|Irr(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

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$$FPdim_{\mathcal{C}}(X_i) \overset{\cdot\cdot}{=} FP(N_k^{i,j})_{j,k \in Irr(\mathcal{C})}.$$

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$\underline{i=0}$: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

HAS EVALUES $= \{ 1 \}$

$\Rightarrow FPdim_{\mathcal{C}}(\mathbb{1}) = 1$

$\underline{i=1}$: $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

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III. FROBENIUS-PERRON DIMENSION

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$$FPdim_{\mathcal{C}}(X_i) := FP\left(N_k^{i,j}\right)_{j,k \in Irr(\mathcal{C})}.$$

$$FPdim(\mathcal{C}) := \sum_{i \in Irr(\mathcal{C})} FPdim_{\mathcal{C}}(X_i)^2$$

$\underline{i=0}$: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ HAS EVALUES $= \{1\}$

$\Rightarrow FPdim_{\mathcal{C}}(\mathbb{1}) = 1$

$\underline{i=1}$: $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ HAS EVALUES $= \{\frac{1}{2}(1 \pm \sqrt{5})\}$

$\Rightarrow FPdim_{\mathcal{C}}(X) = \frac{1}{2}(1+\sqrt{5})$

FIBONACCI FUSION CATEG.

Fib

$Irr(\mathcal{C}) = \{\underbrace{\mathbb{1}}_{=X_0}, \underbrace{X}_{=X_1}\}$

DEFINED WITH FUSION RULES

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$FPdim_{\mathcal{C}}(X_i)$ $\stackrel{..}{=}$ $FP(N_K^{i,j})_{j,k \in Irr(\mathcal{C})}$.

$FPdim(\mathcal{C})$ $\stackrel{..}{=}$ $\sum_{i \in Irr(\mathcal{C})} FPdim_{\mathcal{C}}(X_i)^2$

$\underline{i=0}$: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

HAS EVALUES $= \{1\}$

$\Rightarrow FPdim_{\mathcal{C}}(\mathbb{1}) = 1$

$\underline{i=1}$: $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

HAS EVALUES $= \{\frac{1}{2}(1 \pm \sqrt{5})\}$

$\Rightarrow FPdim_{\mathcal{C}}(X) = \frac{1}{2}(1+\sqrt{5})$

GOLDEN RATIO

FIBONACCI FUSION CATEG.

Fib

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$\mathcal{C}$ FINITE $\Rightarrow |Irr(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

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$\{N_K^{i,j}\}_{i,j,k \in Irr(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

III. FROBENIUS-PERRON DIMENSION

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HAS EVALUES $= \{1\}$

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(\mathbb{1}) = 1$

$i=1$: $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

HAS EVALUES $= \{\frac{1}{2}(1 \pm \sqrt{5})\}$

$\Rightarrow \mathrm{FPdim}_{\mathcal{C}}(X) = \frac{1}{2}(1+\sqrt{5})$

FIBONACCI FUSION CATEG.

Fib

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DEFINED WITH FUSION RULES

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$\mathbb{1} \otimes X \cong X$

$X \otimes \mathbb{1} \cong X$

$X \otimes X \cong \mathbb{1} \sqcup X$

$\therefore \mathrm{FPdim}(\mathcal{C}) = \dfrac{5+\sqrt{5}}{2}$

$\mathcal{C}$ FINITE $\Rightarrow |\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{k \in \mathrm{Irr}(\mathcal{C})} X_k^{\sqcup N_k^{i,j}}$ FOR SOME $N_k^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_k^{i,j}\}_{i,j,k \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

III. FROBENIUS-PERRON DIMENSION

FP THEOREM

TAKE $M \in \mathrm{Mat}_n(\mathbb{R}_{\geq 0})$.

$\Rightarrow \exists$ E.VALUE $FP(M) \in \mathbb{R}_{\geq 0}$

THAT IS

$\geq$ ABS VALUE OF ALL E.VALUES OF M.

$$\mathrm{FPdim}_{\mathcal{C}}(X_i) \overset{..}{=} FP\left(N_K^{i,j}\right)_{j,K \in \mathrm{Irr}(\mathcal{C})}.$$

$$\mathrm{FPdim}(\mathcal{C}) \overset{..}{=} \sum_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}_{\mathcal{C}}(X_i)^2$$

C_2–FdMod

$$\mathrm{Irr}(\mathcal{C}) = \{[V_0], [V_1]\}$$

$$V_0 \otimes V_0 \cong V_0$$

$$V_0 \otimes V_1 \cong V_1$$

$$V_1 \otimes V_0 \cong V_1$$

$$V_1 \otimes V_1 \cong V_0$$

$$\boxed{\therefore\ \mathrm{FPdim}(\mathcal{C}) = 2}$$

FIBONACCI FUSION CATEG.

Fib

$$\mathrm{Irr}(\mathcal{C}) = \{\mathbb{1}, X\}$$

DEFINED WITH FUSION RULES

$$\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$$

$$\mathbb{1} \otimes X \cong X$$

$$X \otimes \mathbb{1} \cong X$$

$$X \otimes X \cong \mathbb{1} \sqcup X$$

$$\boxed{\therefore\ \mathrm{FPdim}(\mathcal{C}) = \frac{5+\sqrt{5}}{2}}$$

$\mathcal{C}$ FINITE $\Rightarrow$ $|\mathrm{Irr}(\mathcal{C})| < \infty$ RANK OF $\mathcal{C}$

$X_i \otimes X_j \cong \coprod_{K \in \mathrm{Irr}(\mathcal{C})} X_K^{\sqcup N_K^{i,j}}$ FOR SOME $N_K^{i,j} \in \mathbb{Z}_{\geq 0}$

$\{N_K^{i,j}\}_{i,j,K \in \mathrm{Irr}(\mathcal{C})}$: FUSION RULES OF $\mathcal{C}$

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READ FOR MORE ON:

FP-DIMENSION

LECTURE #16

TOPICS:

I. FUSION CATEGORIES (§§3.9.1, 3.9.3)

II. FUSION RULES & RANK (§§3.9.1, 3.9.3)

III. FROBENIUS-PERRON DIMENSION (§3.9.2)

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MORE EXAMPLES OF FUSION CATEGORIES,
& ABOUT THEIR MODULE CATEGORIES

TOPICS:

✓ I. FUSION CATEGORIES (§§3.9.1, 3.9.3)

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HAS A RICH THEORY

TOPICS:

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Lecture #16 keywords: Frobenius-Perron dimension, Frobenius-Perron Theorem, fusion category, fusion rules, multiplicity, Ocneanu rigidity, rank, Rank-Finiteness Conjecture