MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #17

LAST TIME

- FUSION CATEGORIES
- FUSION RULES & RANK
- FROBENIUS-PERRON DIMENSION

TOPICS:

I. TENSOR CATEGORIES (§3.10)

II. ENRICHED CATEGORIES (§§3.11.1, 3.11.2)

III. CLOSED MONOIDAL CATEGORIES (§3.11.3)

IV. INTERNAL HOMS (§3.11.4)

V. SUMMARY OF CHAPTER 3

I. Tensor categories

I. TENSOR CATEGORIES

SUPER NICE MONOIDAL CATEGORIES

≡ RECALL ≡

A MONOIDAL CATEGORY $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ IS FUSION IF:

(a) $\mathcal{C}$ IS ABELIAN

(b) $\mathcal{C}$ IS $(\Bbbk\text{-})$LINEAR

(c) $\mathcal{C}$ IS LOC. FINITE

(d) $\mathbb{1}$ IS ABS. SIMPLE

(e) $\mathcal{C}$ IS RIGID

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(g) $\mathcal{C}$ IS FINITE

I. TENSOR CATEGORIES

≡ RECALL ≡

SUPER NICE MONOIDAL CATEGORIES

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EXAMPLES

FdVec

~~Vec~~

~~Ab~~

~~Set~~

G-FdMod

FdVec_G

G FINITE GROUP

~~G-FdMod~~

G ARBITRARY GROUP

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EXAMPLES

FdVec

$\cancel{\text{Vec}}$ $\cancel{(e)}$

$\cancel{\text{Ab}}$ $\cancel{(f)}$

$\cancel{\text{Set}}$ $\cancel{(e)}$

G-FdMod

G FINITE GROUP

FdVec_G

SOME REASONS

$\cancel{\text{G-FdMod}}$ $\cancel{(f)}$ $\cancel{(g)}$

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SUPER NICE MONOIDAL CATEGORIES

EXAMPLES

FdVec

~~Vec~~ ~~(e)~~

~~Ab~~ ~~(f)~~

~~Set~~ ~~(e)~~

G-FdMod*

FdVec_G^*

G FINITE GROUP

~~G-FdMod~~ ~~(f)~~ ~~(g)~~ — G ARBITRARY GROUP

SOME REASONS

*: GET (f) ✓
SINCE $\mathrm{ch}(\Bbbk) = 0$

I. TENSOR CATEGORIES

WEAKENING TO GET MORE EXAMPLES

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EXAMPLES

FdVec

~~Vec~~ (e)

~~Ab~~ (f)

~~Set~~ (e)

G-FdMod*

FdVec$_G^*$

G FINITE GROUP

SOME REASONS

~~G-FdMod~~ (f) (g)

G ARBITRARY GROUP

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EXAMPLES

- FdVec
- ~~$\mathrm{Vec}_{(e)}$~~
- ~~$\mathrm{Ab}_{(f)}$~~
- ~~$\mathrm{Set}_{(e)}$~~
- $G\text{-FdMod}^{*}$
- FdVec_G^{*}

G FINITE GROUP

- ~~$G\text{-FdMod}_{(f)(g)}$~~

G ARBITRARY GROUP

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~~$\mathsf{Ab}_{(f)}$~~

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G-FdMod*

FdVec^*_G

G FINITE GROUP

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EXAMPLES

FdVec

$\cancel{\mathrm{Vec}}_{\cancel{(e)}}$

$\cancel{\mathrm{Ab}}_{\cancel{(f)}}$

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$G\text{-}\mathrm{FdMod}^*$

FdVec^*_G

G FINITE GROUP

$\cancel{G\text{-}\mathrm{FdMod}}_{\cancel{(f)}\,\cancel{(g)}}$

G ARBITRARY GROUP

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SOME REASONS

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G-FdMod*

FdVec_G^*

G FINITE GROUP

~~$G\text{-}\mathrm{FdMod}_{(f)(g)}$~~

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THINK ABOUT WHY

EXAMPLES

FdVec

~~Vec~~

~~Ab~~

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G-FdMod*

G FINITE GROUP

FdVec$_G^*$

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G ARBITRARY GROUP

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EXAMPLES

FdVec

~~Vec~~

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G-FdMod* OVER $\Bbbk$

FdVec$_G^*$ OVER $\Bbbk$

G FINITE GROUP

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EXAMPLES

FdVec

~~Vec~~

~~Ab~~

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$G\text{-FdMod}^*$ OVER $\Bbbk$

FdVec_G^* OVER $\Bbbk$

G FINITE GROUP

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EXAMPLES

FdVec

~~Vec~~

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~~Set~~

G-FdMod* OVER $\mathbb{F}$

FdVec_G^* OVER $\mathbb{F}$

G FINITE GROUP

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G-FdMod* OVER $\mathbb{F}$

FdVec$_G^*$ OVER $\mathbb{F}$

G FINITE GROUP

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G ARBITRARY GROUP

*: $\mathbb{F}$ FIELD OF ARBITRARY CHAR. (STILL ALG. CLOSED)

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EXAMPLES

FdVec

~~Vec~~

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G-FdMod* OVER $\mathbb{F}$

G FINITE GROUP

FdVec_G^* OVER $\mathbb{F}$

~~G-FdMod~~ ~~(f)~~ ~~(g)~~

G ARBITRARY GROUP

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G FINITE GROUP

G-FdMod

G ARBITRARY GROUP

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WHEN ARE TWO TENSOR CATEGORIES CONSIDERED THE SAME?

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IF ∃ LINEAR FUNCTOR BETWEEN THE TWO THAT'S AN ISOM./EQUIV. OF MONOIDAL CATEGORIES

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CLASSIFICATION ??

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CLASSIFICATION ⁇

RECALL FOR FUSION CATEGORIES, THIS WAS APPROACHED VIA FUSION RULES / RANK

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NEED

FUSION RULES /

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NEED FUSION RULES /

NEED RANK

OFTEN (f) OR (g) IS IMPOSED TO GET RESULTS (TOWARDS CLASSIFICATION OR OTHERWISE)

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NEED

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OFTEN (f) OR (g) IS IMPOSED TO GET RESULTS (TOWARDS CLASSIFICATION OR OTHERWISE)

ESP. ↑

I. TENSOR CATEGORIES

RATHER NICE MONOIDAL CATEGORIES

A MONOIDAL CATEGORY

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

IS FINITE TENSOR IF:

(a) $\mathcal{C}$ IS ABELIAN

(b) $\mathcal{C}$ IS ($\Bbbk$-)LINEAR

(c) $\mathcal{C}$ IS LOC. FINITE

(d) $\mathbb{1}$ IS ABS. SIMPLE

(e) $\mathcal{C}$ IS RIGID

~~(f) $\mathcal{C}$ IS SEMISIMPLE~~

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LOCALLY FINITE

$\exists$ ENOUGH PROJECTIVES IN $\mathcal{C}$: $\forall z \in \mathcal{C}$ $\exists$ PROJ. OBJ $P(z) \in \mathcal{C}$ WITH EPI $P(z) \twoheadrightarrow z$ IN $\mathcal{C}$.

$\exists$ ONLY FINITELY MANY ISOCLASSES OF SIMPLE OBJECTS IN $\mathcal{C}$

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HAVE FROBENIUS-PERRON DIMENSION IN THIS SETTING:

I. Tensor categories

Quite nice monoidal categories

A monoidal category $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ is **finite tensor** if:

(a) $\mathcal{C}$ is abelian

(b) $\mathcal{C}$ is ($\Bbbk$-)linear

(c) $\mathcal{C}$ is loc. finite

(d) $\mathbb{1}$ is abs. simple

(e) $\mathcal{C}$ is rigid

~~(f) $\mathcal{C}$ is semisimple~~

(g) $\mathcal{C}$ is finite

Have Frobenius-Perron dimension in this setting:

For $\mathrm{Irr}(\mathcal{C}) = \{[X_i]\}_i$

$\mathrm{FPdim}_{\mathcal{C}}(X_i) \leftarrow$ defined as fusion case

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HAVE FROBENIUS-PERRON DIMENSION IN THIS SETTING:

FOR $\mathrm{Irr}(\mathcal{C}) = \{[X_i]\}_i$

$\mathrm{FPdim}_{\mathcal{C}}(X_i)$ ← DEFINED AS FUSION CASE

$\mathrm{FPdim}_{\mathcal{C}}(X)$ ← DEFINED LIKEWISE

↖ ARBITRARY OBJECT OF $\mathcal{C}$

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X = ARBITRARY OBJECT OF $\mathcal{C}$

$$\mathrm{FPdim}(\mathcal{C}) = \sum_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}_{\mathcal{C}}(X_i) \cdot \mathrm{FPdim}_{\mathcal{C}}(\underbrace{P(X_i)}_{\text{PROJECTIVE COVER OF } X_i})$$

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↗ PROJECTIVE COVER OF X_i

INDEED $P(X_i) = X_i$ WHEN (f) IS IMPOSED (FUSION CASE)

I. TENSOR CATEGORIES

WANT MORE?

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IMPORTANT REFERENCES FOR THE THEORY OF (FINITE) TENSOR CATEGORIES & FUSION CATEGORIES:

"TENSOR CATEGORIES"

BOOK BY ETINGOF–GELAKI–NIKSHYCH–OSTRIK

"FUSION CATEGORIES"

ARTICLE BY ETINGOF–NIKSHYCH–OSTRIK

II. ENRICHED CATEGORIES

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≡ RECALL ≡

$\forall X, Y \in \mathcal{C}$

$\mathcal{C}$ LOCALLY SMALL $\rightsquigarrow$ $\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathbf{Set}$

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... ENRICHMENT $\rightsquigarrow$ FRAMEWORK FOR THE CATEGORIES ABOVE

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A **$\mathcal{V}$-CATEGORY $\mathcal{A}$** CONSISTS OF:

(a) A COLLECTION OF **OBJECTS $\mathrm{Ob}(\mathcal{A})$**.
(WRITE $X \in \mathcal{A}$)

(b) A **HOM OBJECT $\mathcal{A}(X,Y)$** $\in \mathcal{V}$.
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(c) A COMPOSITION MORPHISM IN $\mathcal{V}$
$$\gamma_{X,Y,Z} : \mathcal{A}(Y,Z) \otimes^{\mathcal{V}} \mathcal{A}(X,Y) \longrightarrow \mathcal{A}(X,Z)$$
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$$\upsilon_X : \mathbb{1}^{\mathcal{V}} \longrightarrow \mathcal{A}(X,X)$$
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SUCH THAT:

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SUCH THAT: $\forall W, X, Y, Z \in \mathcal{A}$

$$\begin{tikzcd}
& {[\mathcal{A}(Y,Z) \otimes^{\mathcal{V}} \mathcal{A}(X,Y)] \otimes^{\mathcal{V}} \mathcal{A}(W,X)} \arrow[dl, "\gamma \otimes^{\mathcal{V}} \mathrm{id}"'] \arrow[dr, "a^{\mathcal{V}}"] & \\
\mathcal{A}(X,Z) \otimes^{\mathcal{V}} \mathcal{A}(W,X) \arrow[d, "\gamma"'] & \circlearrowleft & \mathcal{A}(Y,Z) \otimes^{\mathcal{V}} [\mathcal{A}(X,Y) \otimes^{\mathcal{V}} \mathcal{A}(W,X)] \arrow[d, "\mathrm{id} \otimes^{\mathcal{V}} \gamma"] \\
\mathcal{A}(W,Z) & & \mathcal{A}(Y,Z) \otimes^{\mathcal{V}} \mathcal{A}(W,Y) \arrow[ll, "\gamma"']
\end{tikzcd}$$

(ASSOCIATIVITY)

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(ASSOCIATIVITY)

$$\begin{tikzcd}
\mathbb{1}^{\mathcal{V}} \otimes^{\mathcal{V}} \mathcal{A}(X,Y) \arrow[rr, "u_Y \otimes^{\mathcal{V}} \mathrm{id}"] \arrow[dr, "l^{\mathcal{V}}"'] & \circlearrowleft & \mathcal{A}(Y,Y) \otimes^{\mathcal{V}} \mathcal{A}(X,Y) \arrow[dl, "\gamma"] \\
& \mathcal{A}(X,Y) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A}(X,Y) \otimes^{\mathcal{V}} \mathbb{1}^{\mathcal{V}} \arrow[rr, "\mathrm{id} \otimes^{\mathcal{V}} u_X"] \arrow[dr, "r^{\mathcal{V}}"'] & \circlearrowleft & \mathcal{A}(X,Y) \otimes^{\mathcal{V}} \mathcal{A}(X,X) \arrow[dl, "\gamma"] \\
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(UNITALITY)

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+ ASSOCIATIVITY
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$\mathcal{C}$ LOCALLY SMALL: ENRICHED OVER Set

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WHAT CAN BE SAID ABOUT CATEGORIES ENRICHED OVER THEMSELVES?

- $\mathcal{C}$ LOCALLY SMALL: ENRICHED OVER **Set**
- $\mathcal{C}$ PREADDITIVE: ENRICHED OVER **Ab**
- $\mathcal{C}$ LINEAR: ENRICHED OVER **Vec**
- $\mathcal{C}$ LOCALLY FINITE: ENRICHED OVER **FdVec**

III. CLOSED MONOIDAL CATEGORIES

WILL STUDY

Fix $\mathcal{V} = (\mathcal{V}, \otimes^{\mathcal{V}}, \mathbb{1}^{\mathcal{V}}, a^{\mathcal{V}}, \ell^{\mathcal{V}}, r^{\mathcal{V}})$.

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III. CLOSED MONOIDAL CATEGORIES

WHAT CAN BE SAID ABOUT CATEGORIES ENRICHED OVER THEMSELVES?

A MONOIDAL CATEGORY $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

IS LEFT CLOSED MONOIDAL

IF $X \otimes - : \mathcal{C} \longrightarrow \mathcal{C}$

HAS A RIGHT ADJOINT

$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, -) : \mathcal{C} \longrightarrow \mathcal{C} \quad \forall X \in \mathcal{C}.$

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THAT IS

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Z, Y) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)) \quad \forall Y, Z \in \mathcal{C}$$

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$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \equiv$ LEFT INTERNAL HOM OF X AND Y

III. CLOSED MONOIDAL CATEGORIES

WHAT CAN BE SAID ABOUT CATEGORIES ENRICHED OVER THEMSELVES?

A MONOIDAL CATEGORY $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

IS LEFT CLOSED MONOIDAL

IF $X \otimes - : \mathcal{C} \longrightarrow \mathcal{C}$

HAS A RIGHT ADJOINT

$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, -) : \mathcal{C} \longrightarrow \mathcal{C} \quad \forall X \in \mathcal{C}$.

THAT IS

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Z, Y) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)) \quad \forall Y, Z \in \mathcal{C}$$

$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \equiv$ LEFT INTERNAL HOM OF X AND Y

IS RIGHT CLOSED MONOIDAL

IF $- \otimes Y : \mathcal{C} \longrightarrow \mathcal{C}$

HAS A RIGHT ADJOINT

$\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, -) : \mathcal{C} \longrightarrow \mathcal{C} \quad \forall Y \in \mathcal{C}$.

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$\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \equiv$ RIGHT INTERNAL HOM OF Y AND Z

III. CLOSED MONOIDAL CATEGORIES

EXER 3.41

$\boxed{\text{THESE ARE SELF-ENRICHED } \mathcal{C}(X,Y) := \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)}$

A MONOIDAL CATEGORY $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

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$\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \equiv$ RIGHT INTERNAL HOM OF Y AND Z

III. CLOSED MONOIDAL CATEGORIES

EXER 3.40: LEFT RIGID $\Rightarrow$ RIGHT CLOSED

$\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) := Z \otimes Y^*$

A MONOIDAL CATEGORY $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

IS LEFT CLOSED MONOIDAL

IF $X \otimes - : \mathcal{C} \to \mathcal{C}$

HAS A RIGHT ADJOINT

$\underline{\mathrm{Hom}}_{\mathcal{C}}(X,-) : \mathcal{C} \to \mathcal{C} \quad \forall X \in \mathcal{C}$.

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$\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \equiv$ RIGHT INTERNAL HOM OF Y AND Z

IV. INTERNAL HOMS

IMPORTANT FOR UNDERSTANDING NICE $\otimes$ CATEGORIES & THEIR MODULE CATS.

A MONOIDAL CATEGORY $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

IS LEFT CLOSED MONOIDAL

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IMPORTANT FOR UNDERSTANDING NICE $\otimes$ CATEGORIES & THEIR MODULE CATS.

A MONOIDAL CATEGORY $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

IV. INTERNAL HOMS

IMPORTANT FOR UNDERSTANDING NICE $\otimes$ CATEGORIES & THEIR MODULE CATS.

A MONOIDAL CATEGORY $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ IS CLOSED

IF $\exists$ $\underline{\mathrm{Hom}}_{\mathcal{M}}(M, -) : \mathcal{M} \longrightarrow \mathcal{C}$ $\ni$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "- \triangleright M"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, bend left, "\underline{\mathrm{Hom}}_{\mathcal{M}}(M,-)"] \end{tikzcd}$$

$\forall M \in \mathcal{M}$

IV. INTERNAL HOMS

IMPORTANT FOR UNDERSTANDING NICE $\otimes$ CATEGORIES & THEIR MODULE CATS.

A MONOIDAL CATEGORY $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ IS CLOSED	$(\mathcal{M}, \triangleleft) \in \mathcal{C}\text{-Mod}$ IS CLOSED
IF $\exists$ $\underline{\mathrm{Hom}}_{\mathcal{M}}(M,-) : \mathcal{M} \to \mathcal{C}$ s.t.	IF $\exists$ $\underline{\mathrm{Hom}}_{\mathcal{M}}(M,-) : \mathcal{M} \to \mathcal{C}$ s.t.
$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "- \triangleright M"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, bend left, "\underline{\mathrm{Hom}}_{\mathcal{M}}(M,-)"] \end{tikzcd}$	$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "M \triangleleft -"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, bend left, "\underline{\mathrm{Hom}}_{\mathcal{M}}(M,-)"] \end{tikzcd}$
$\forall M \in \mathcal{M}$	$\forall M \in \mathcal{M}$

IV. INTERNAL HOMS

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$\forall M \in \mathcal{M}$

COOL FACT: IF $\mathcal{C}$ IS FINITE TENSOR, THEN ALL EVERY $\mathcal{C}$-MOD. CAT. IS CLOSED

IV. INTERNAL HOMS

IMPORTANT FOR UNDERSTANDING NICE $\otimes$ CATEGORIES & THEIR MODULE CATS.

A MONOIDAL CATEGORY $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ IS CLOSED	$(\mathcal{M}, \triangleleft) \in \mathcal{C}\text{-Mod}$ IS CLOSED
IF $\exists$ $\underline{\mathrm{Hom}}_{\mathcal{M}}(M,-) : \mathcal{M} \to \mathcal{C}$ s.t.	IF $\exists$ $\underline{\mathrm{Hom}}_{\mathcal{M}}(M,-) : \mathcal{M} \to \mathcal{C}$ s.t.
$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "- \triangleright M"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, bend left, "\underline{\mathrm{Hom}}_{\mathcal{M}}(M,-)"] \end{tikzcd}$	$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "M \triangleleft -"] \arrow[r, phantom, "\perp"] & \mathcal{M} \arrow[l, bend left, "\underline{\mathrm{Hom}}_{\mathcal{M}}(M,-)"] \end{tikzcd}$
$\forall M \in \mathcal{M}$	$\forall M \in \mathcal{M}$

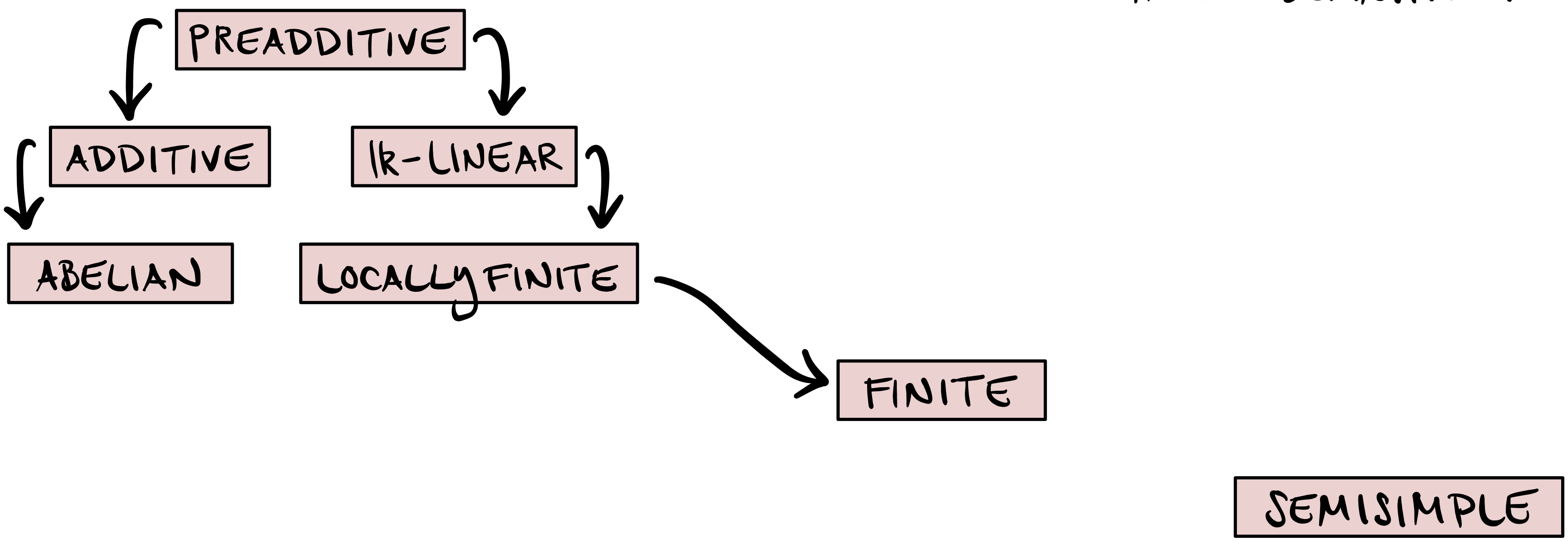
COOL FACT: IF $\mathcal{C}$ IS FINITE TENSOR, THEN ~~ALL~~ EVERY $\mathcal{C}$-MOD. CAT. IS CLOSED

INTERNAL HOMS WILL BE USED LATER TO DO ALGEBRA "IN" MONOIDAL CATEGORIES

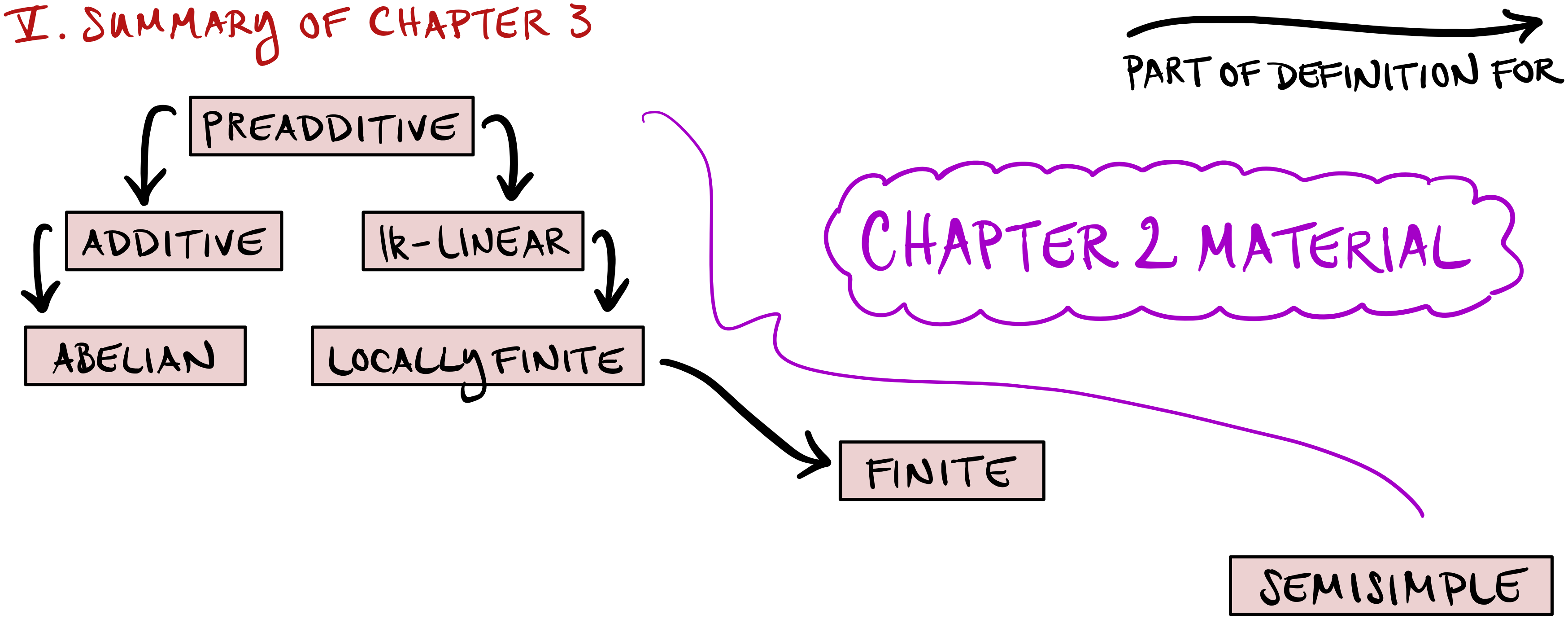
V. Summary of Chapter 3

Part of definition for

V. SUMMARY OF CHAPTER 3



V. Summary of Chapter 3



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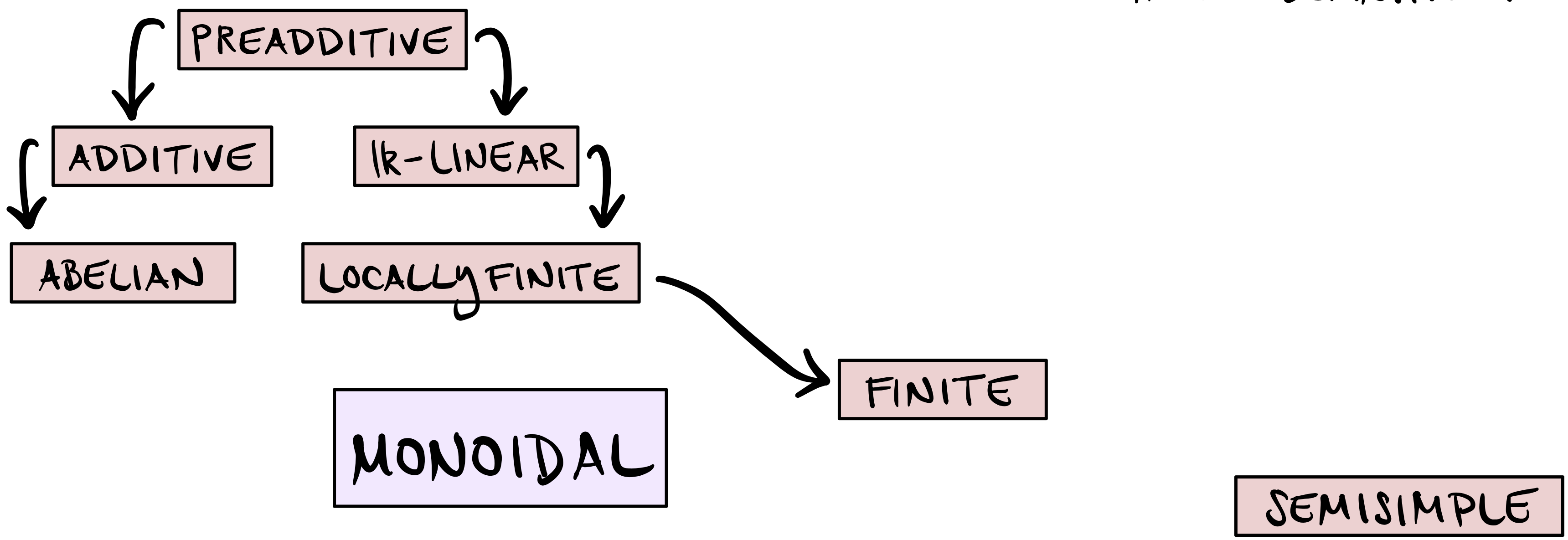


V. SUMMARY OF CHAPTER 3

PART OF DEFINITION FOR →

PREADDITIVE

ADDITIVE

ABELIAN

$\Bbbk$-LINEAR

LOCALLY FINITE

FINITE

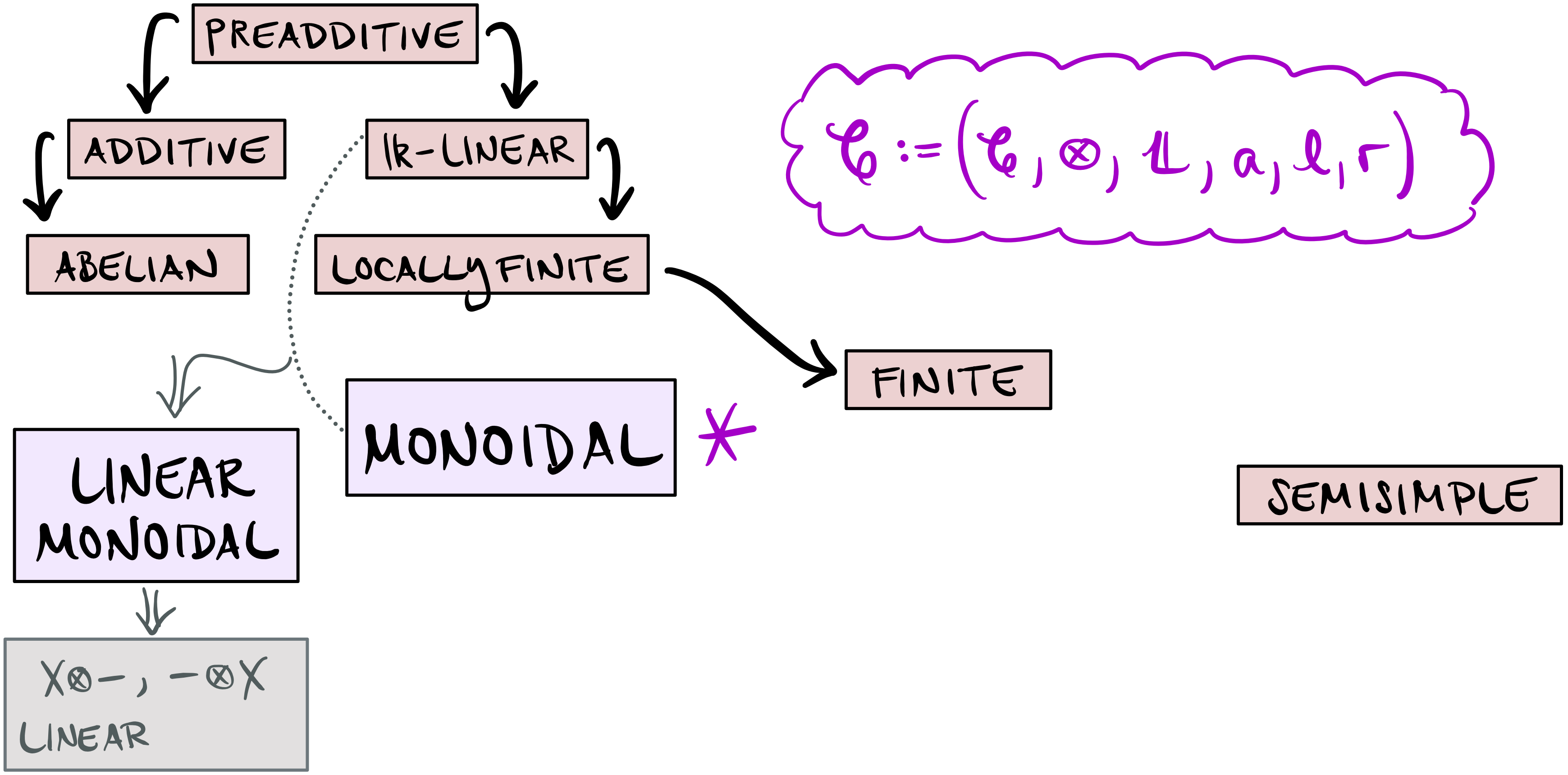
$$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$$

MONOIDAL *

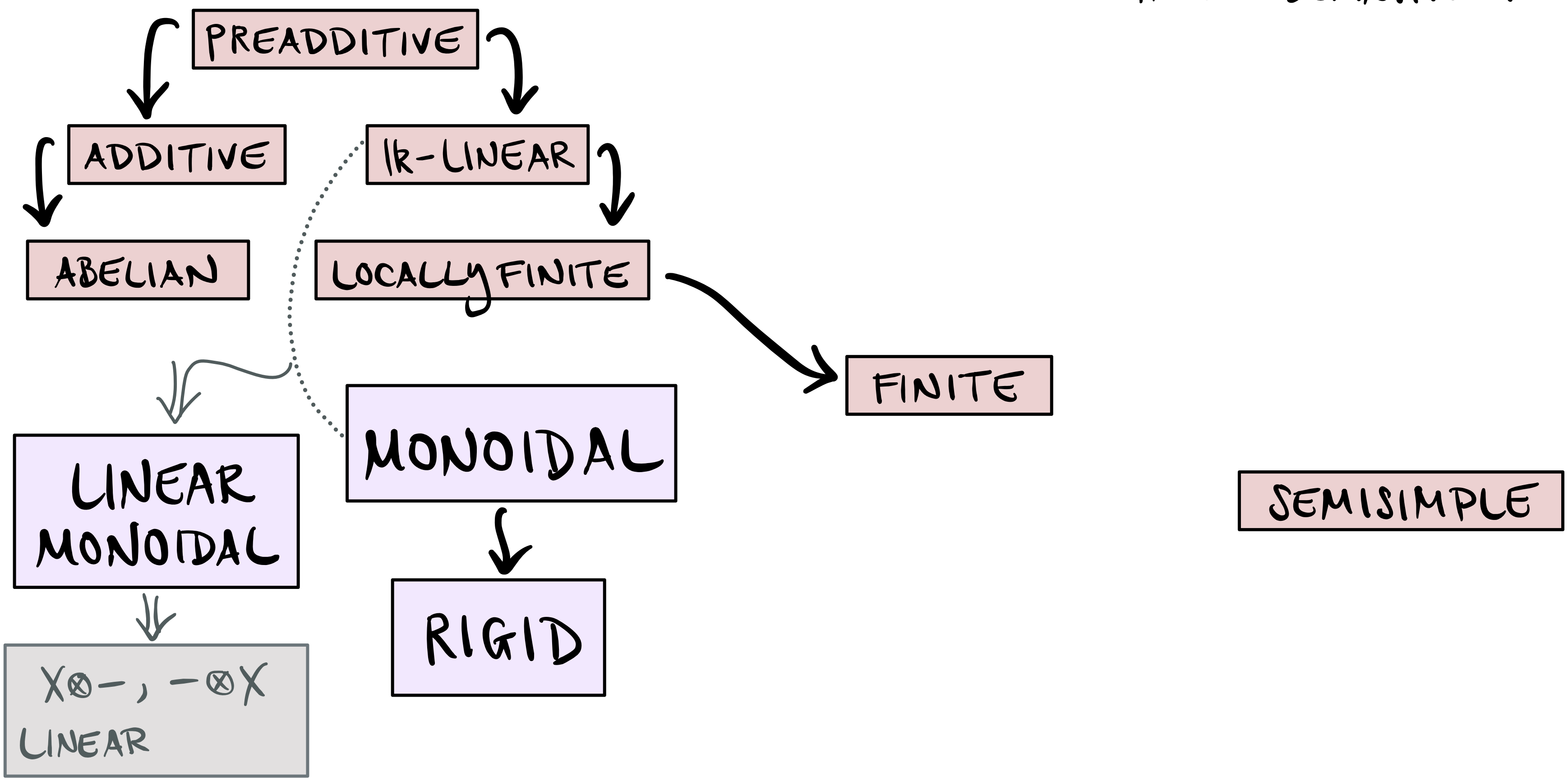
SEMISIMPLE

V. SUMMARY OF CHAPTER 3

PART OF DEFINITION FOR



I. SUMMARY OF CHAPTER 3



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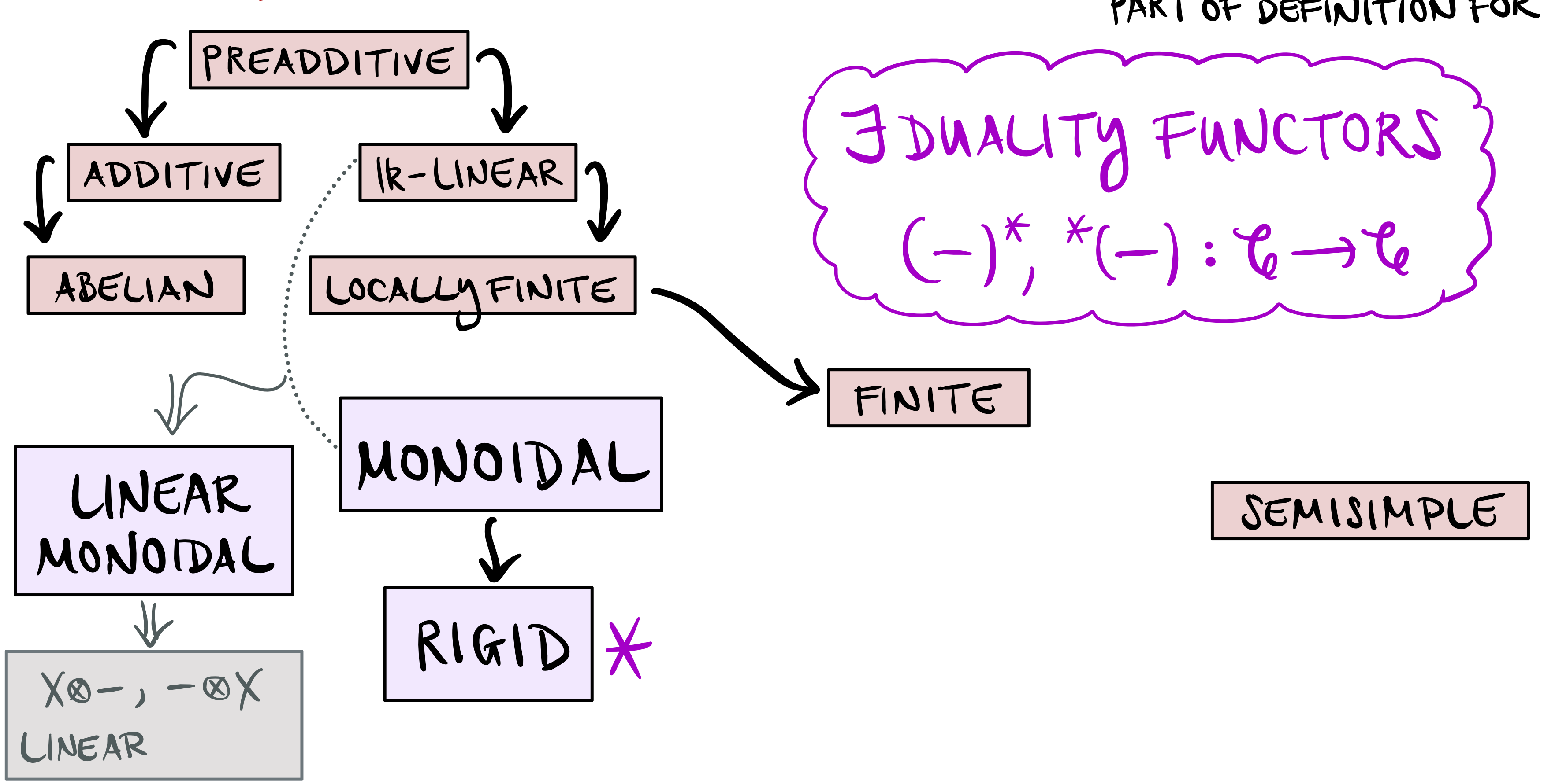

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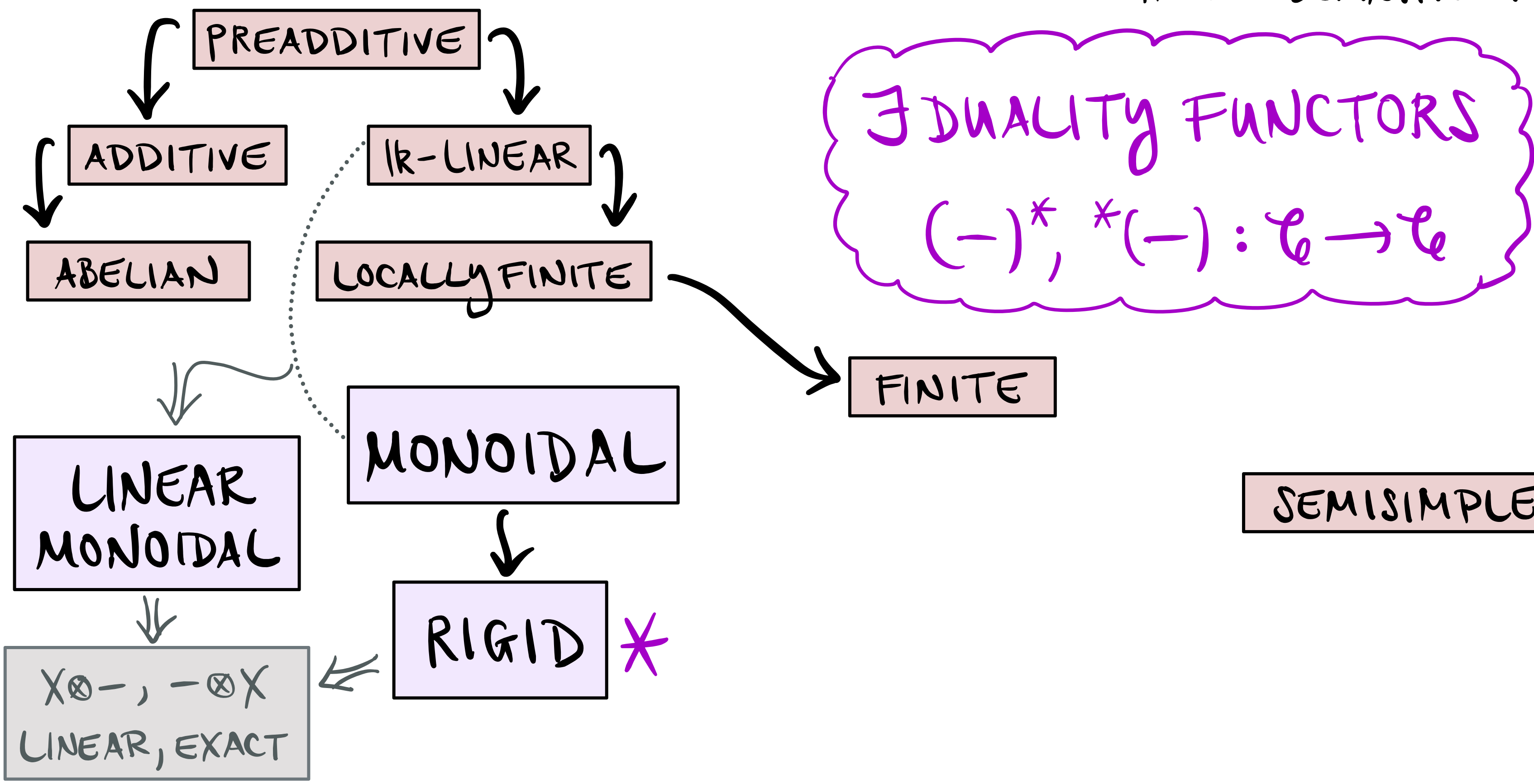

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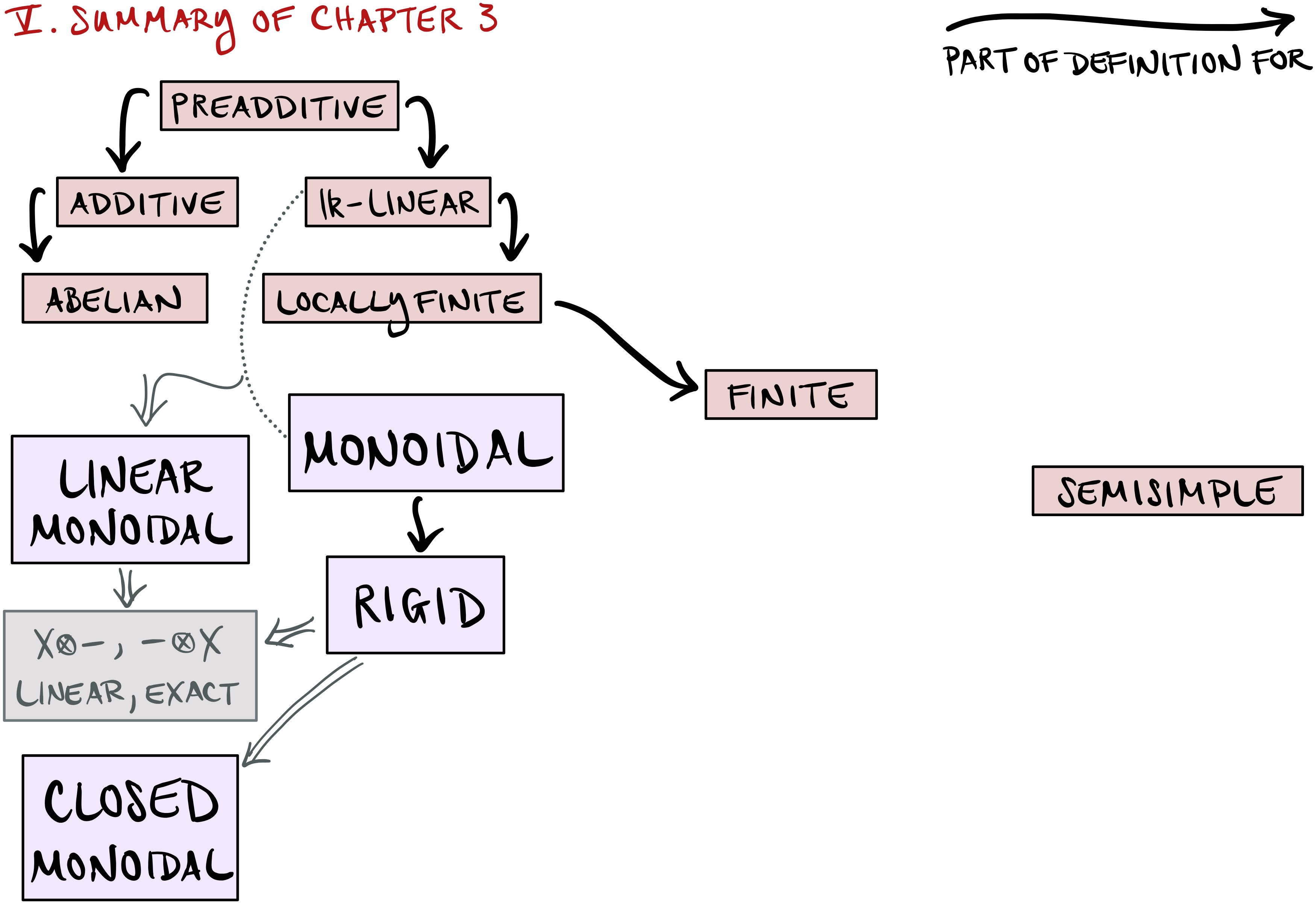

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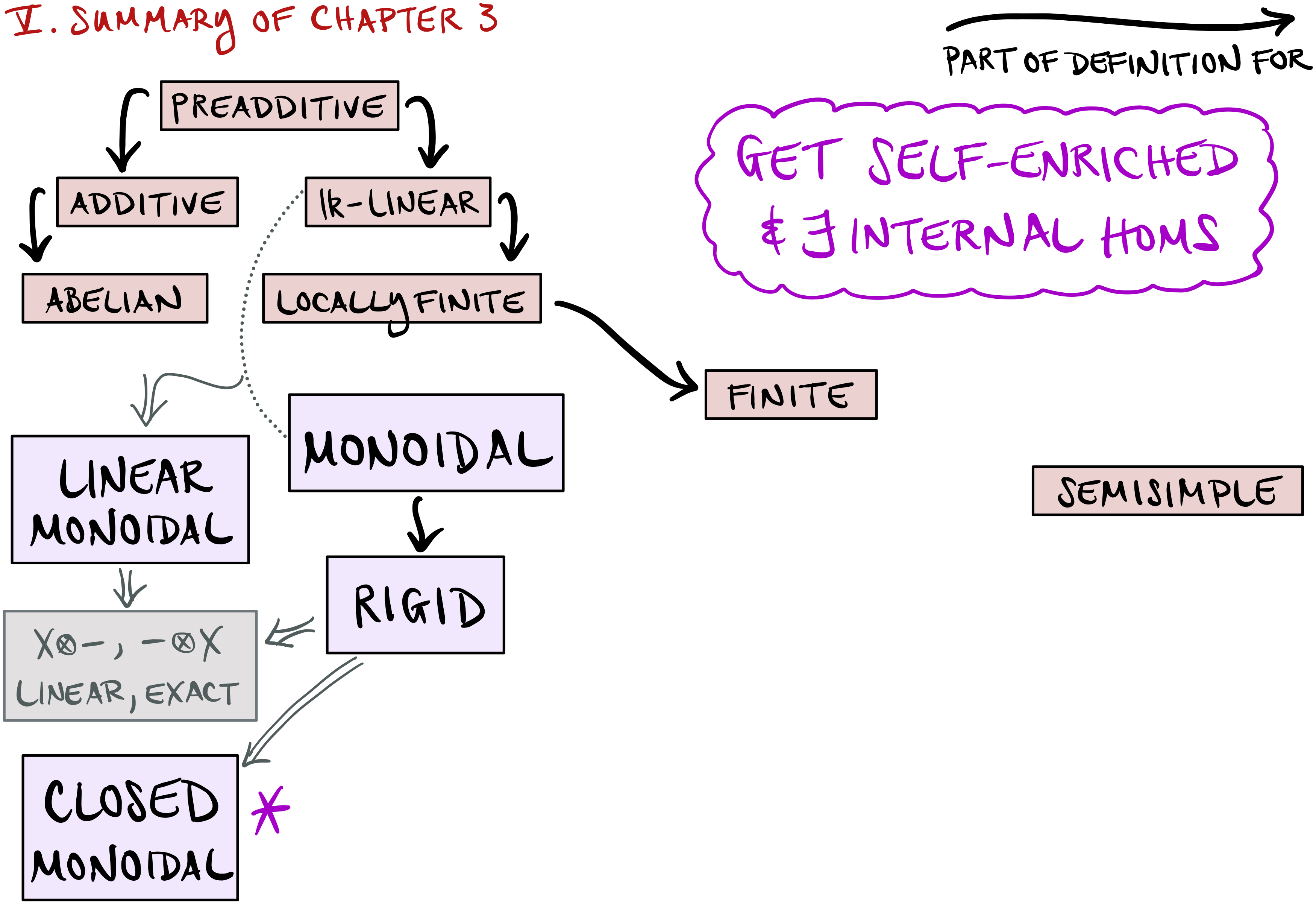

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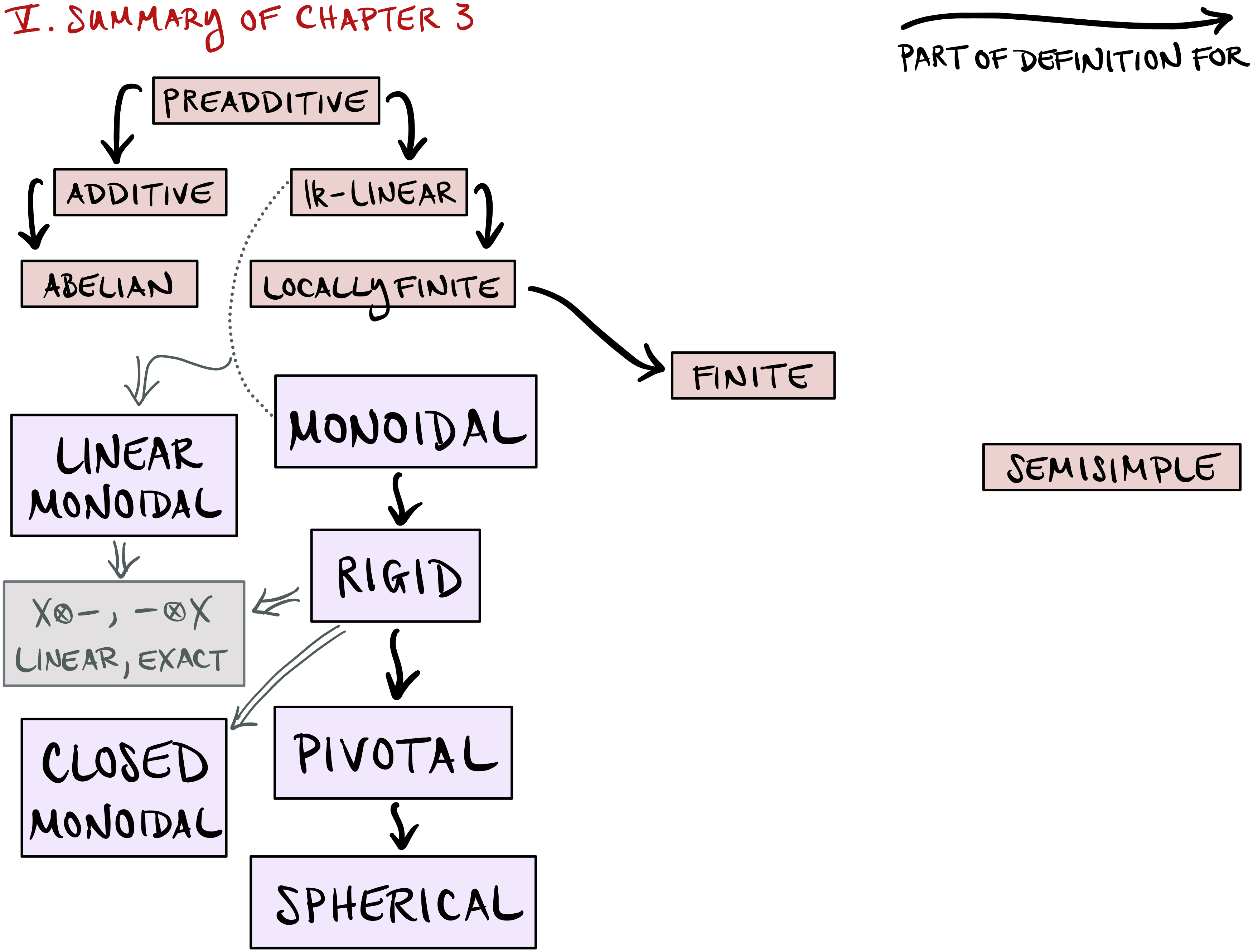

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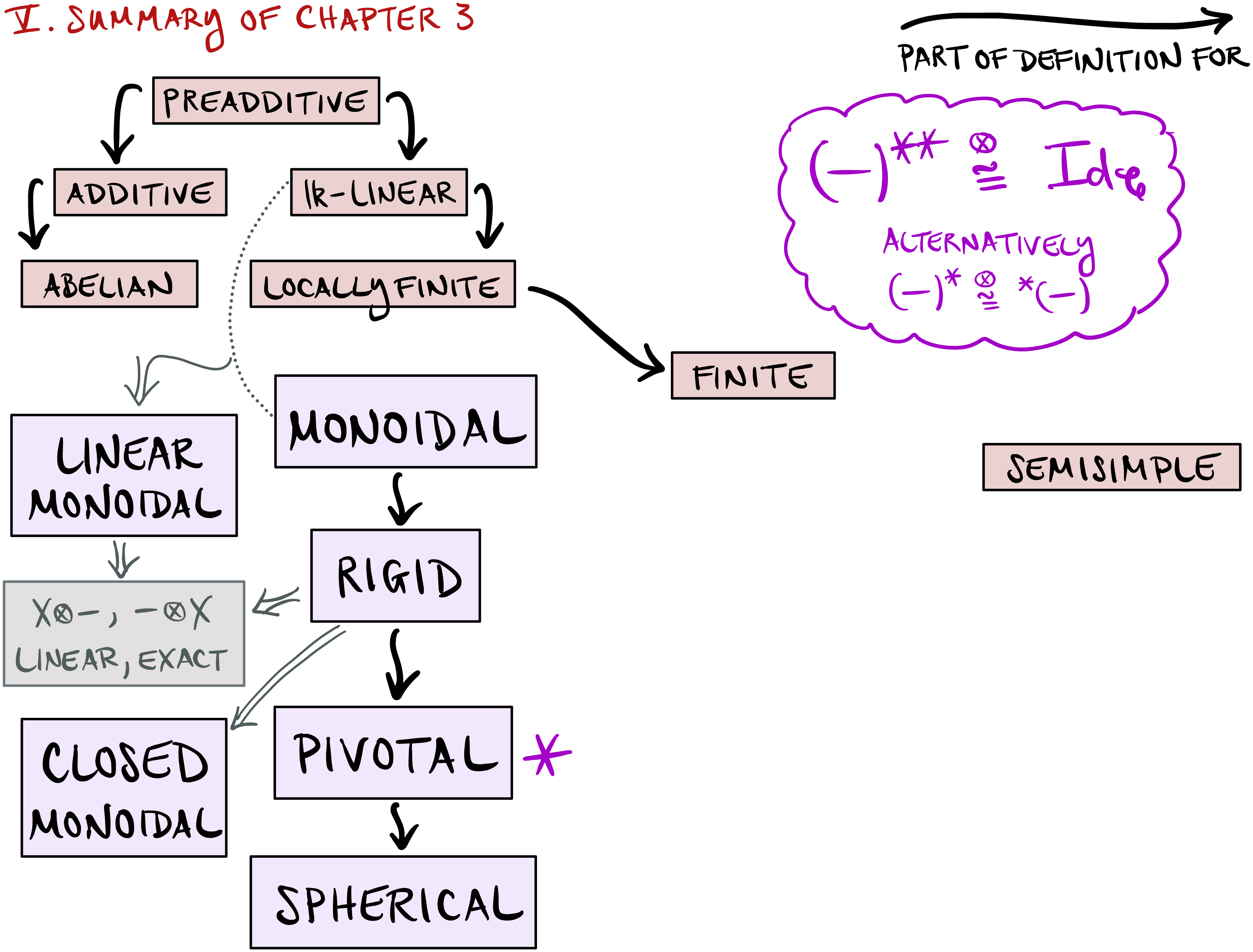

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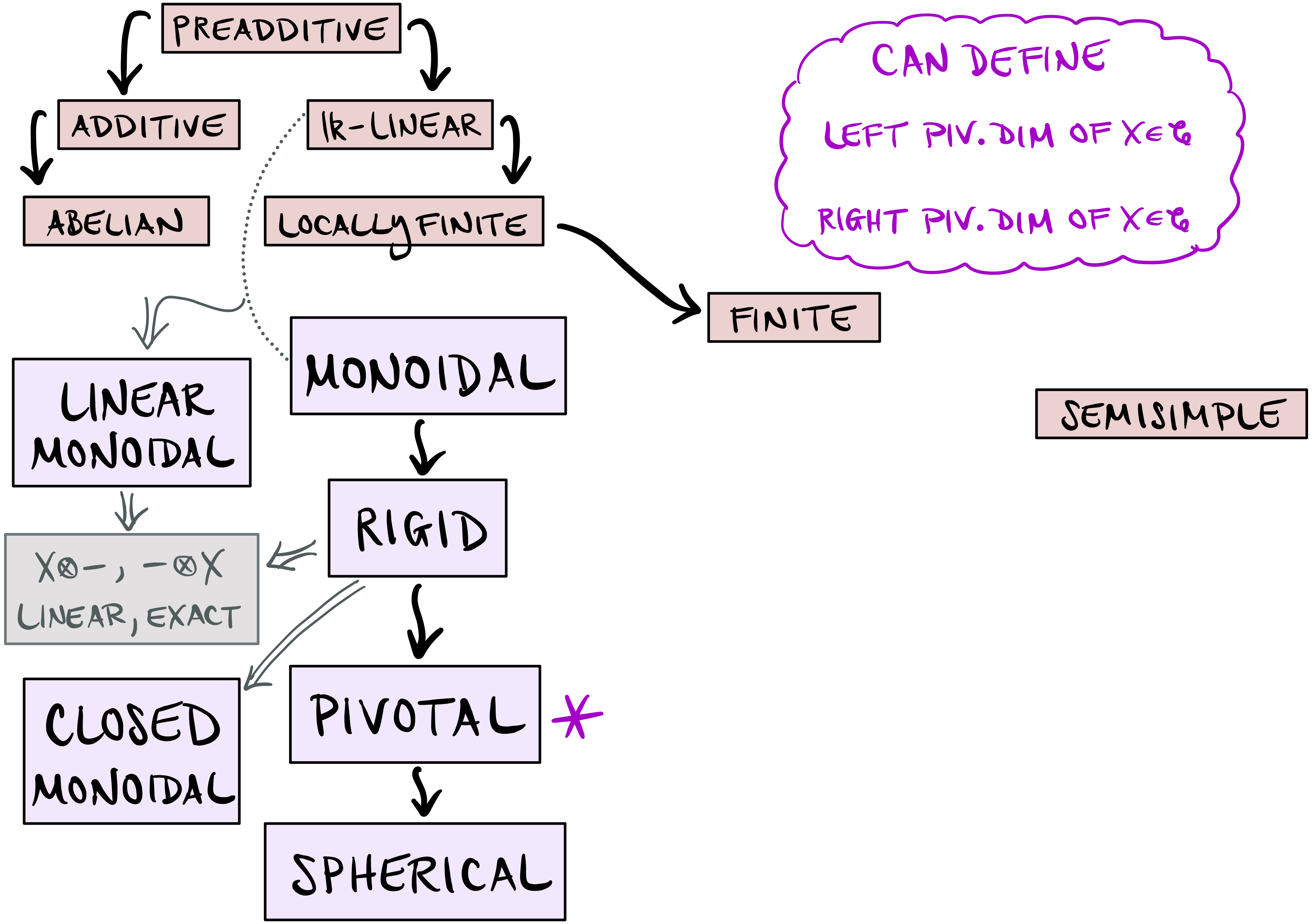

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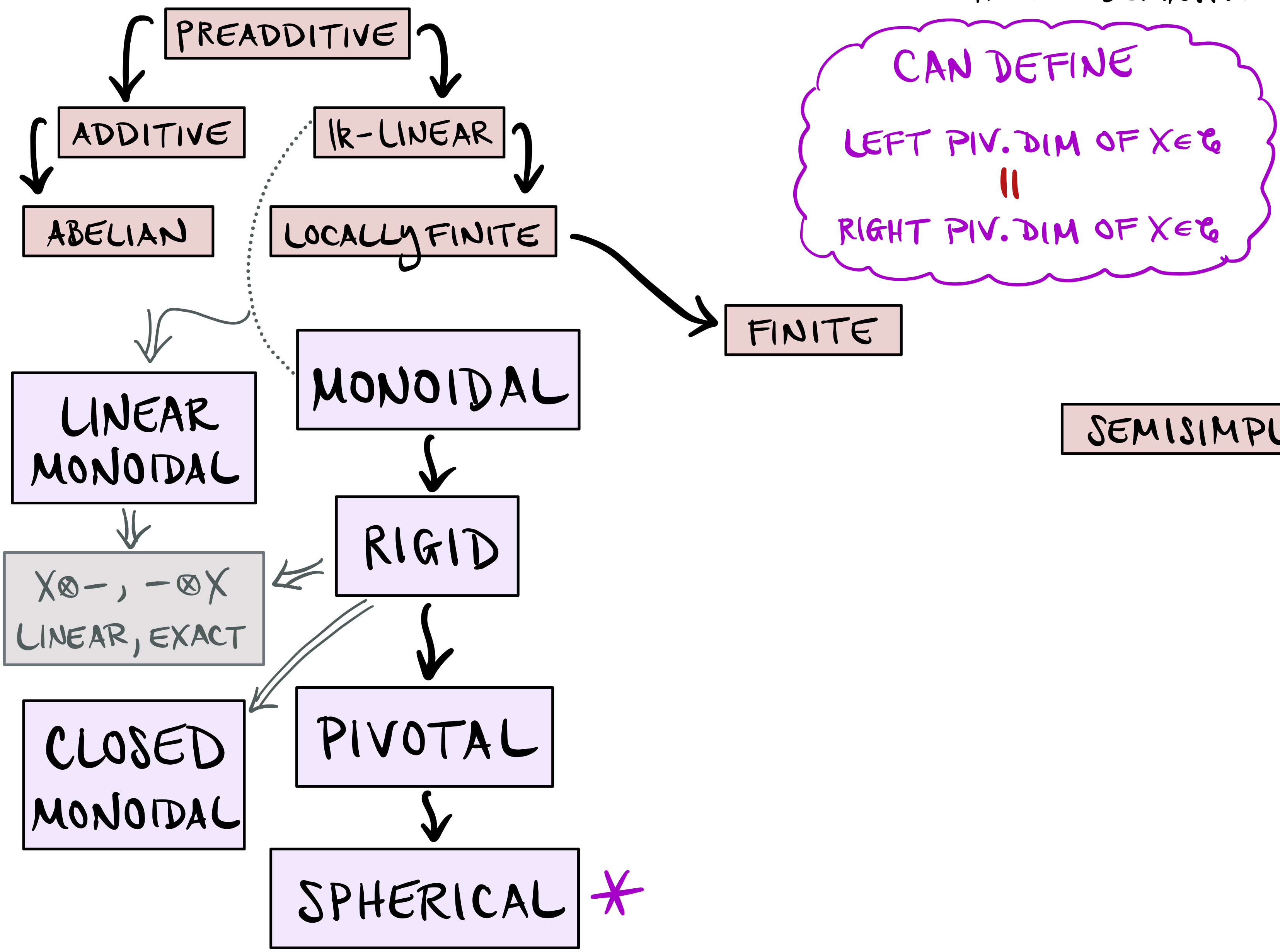

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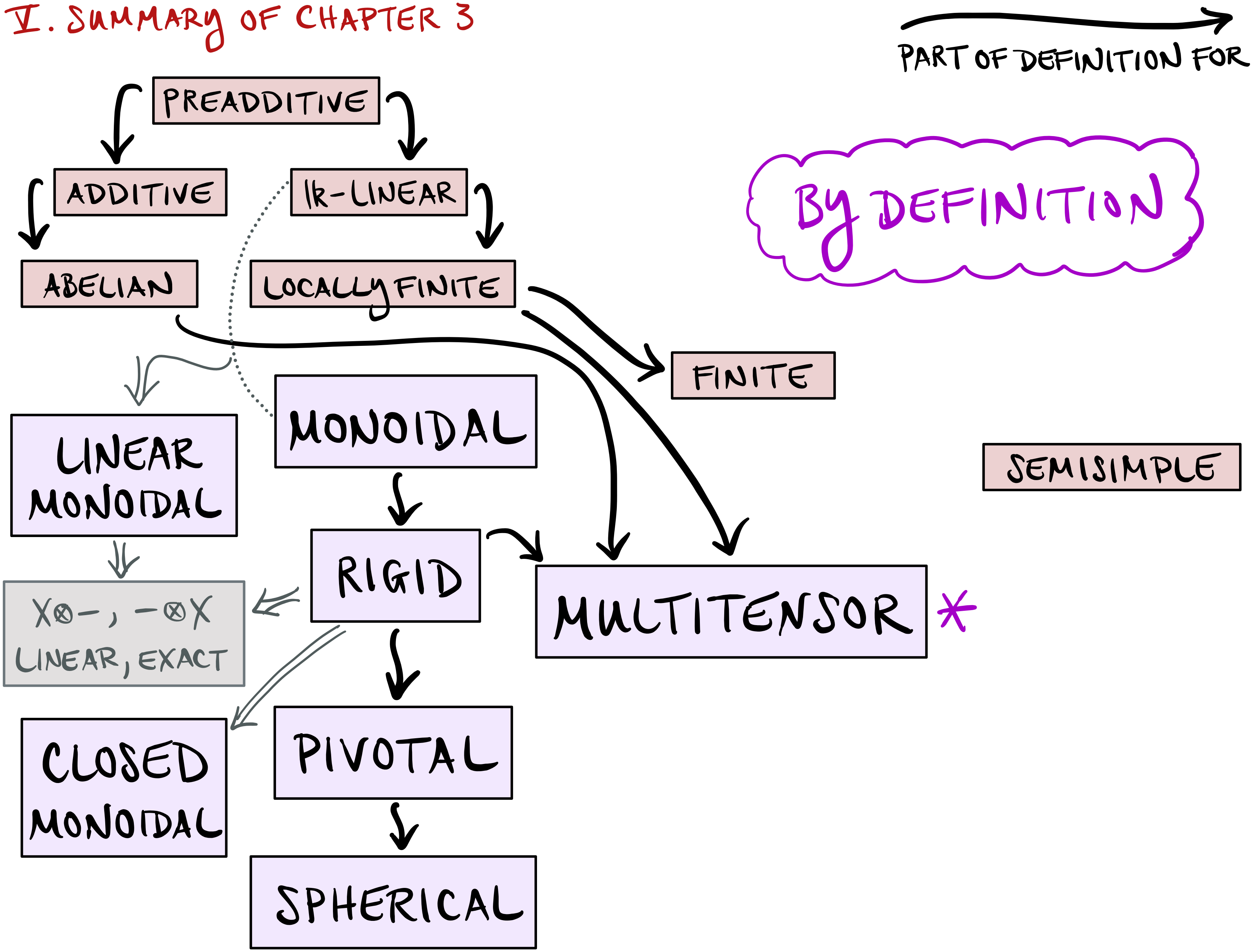

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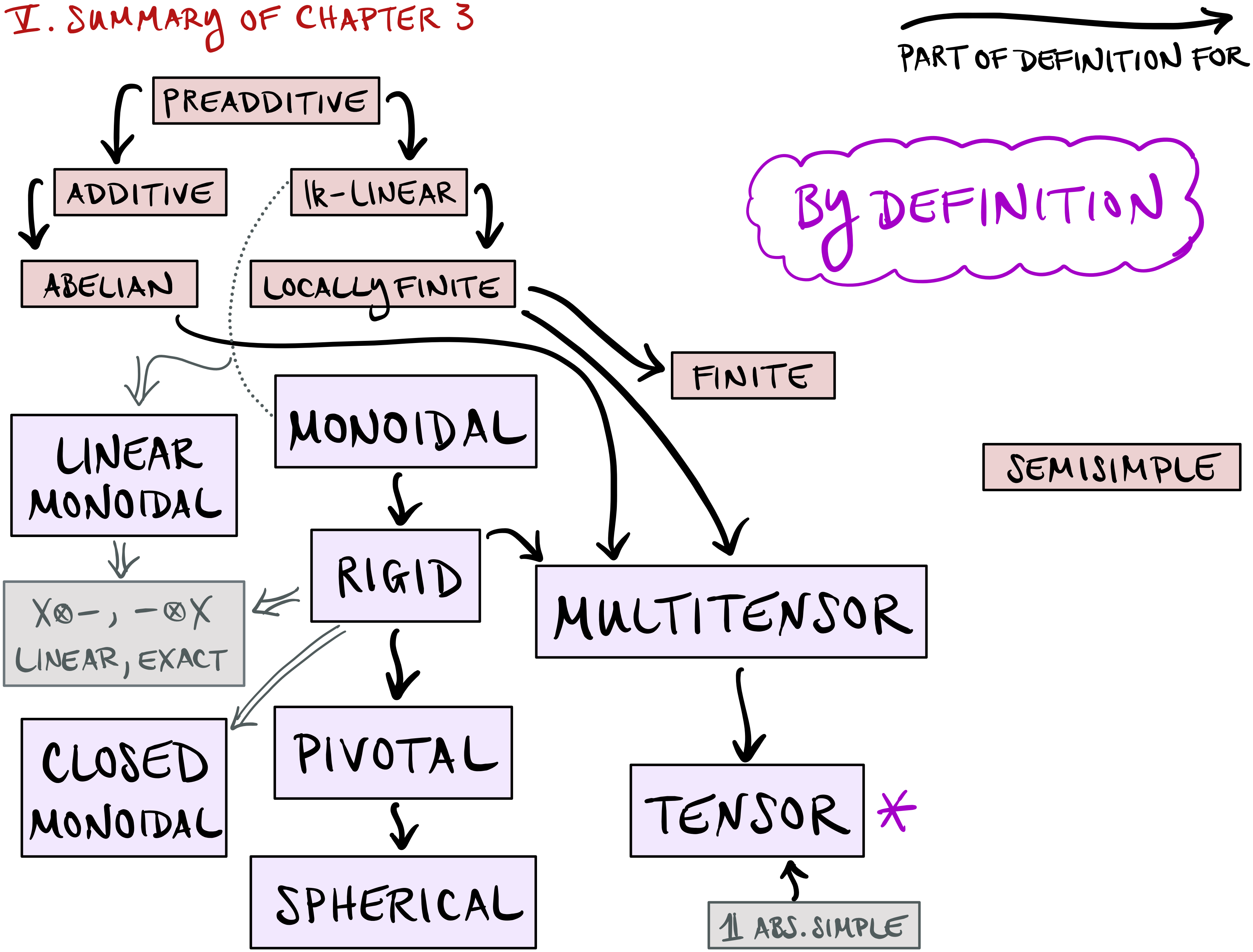

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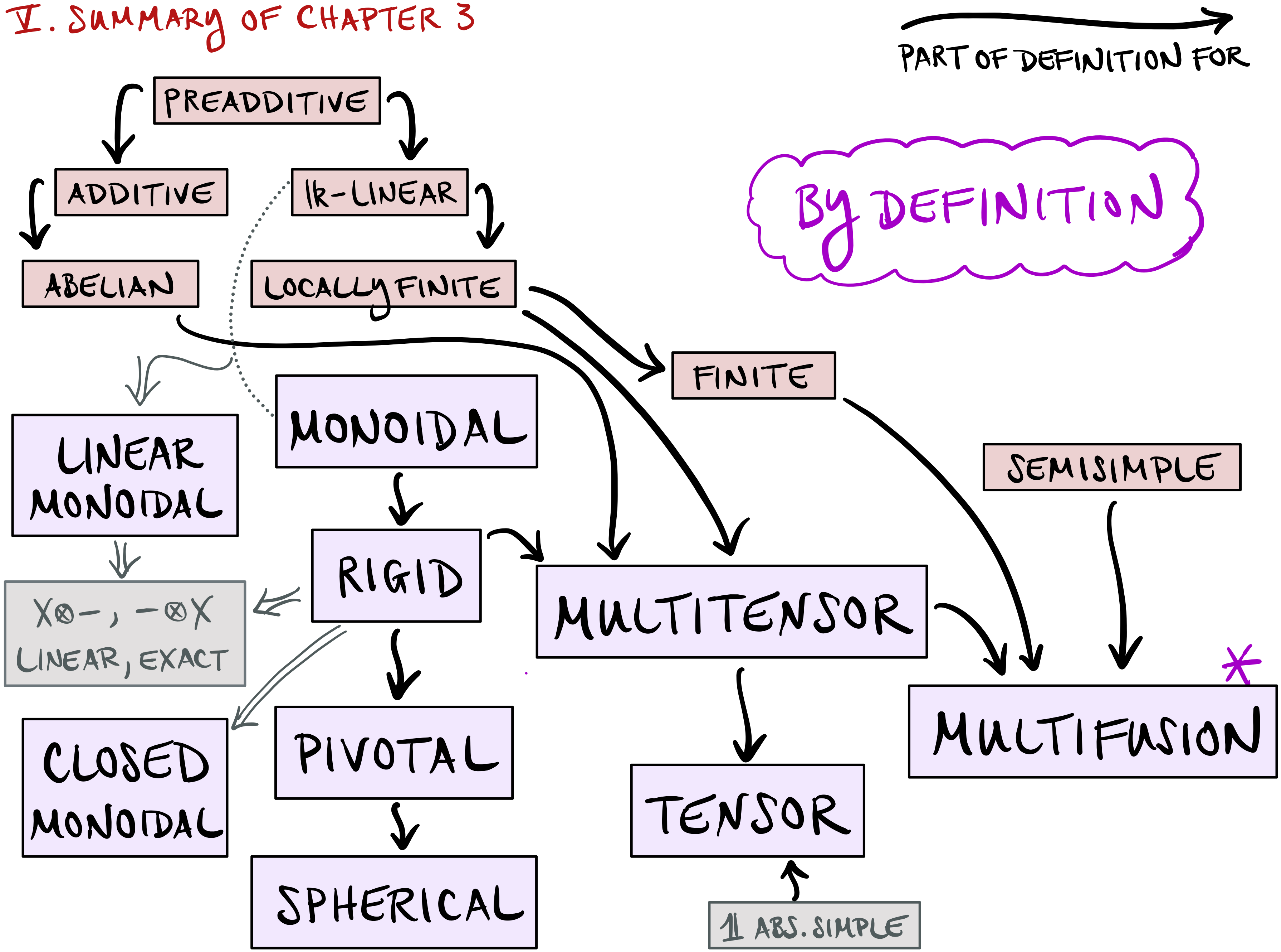

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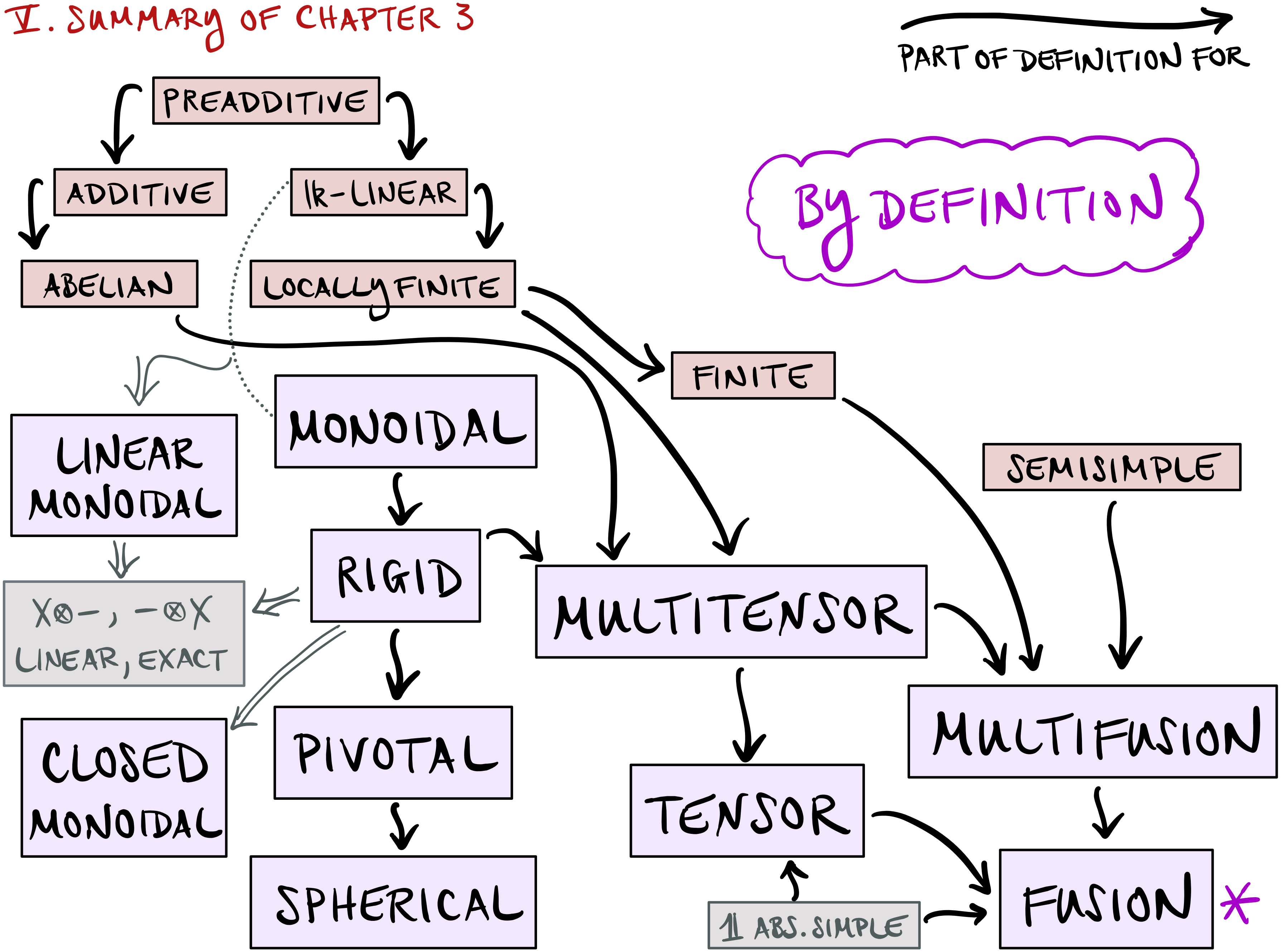

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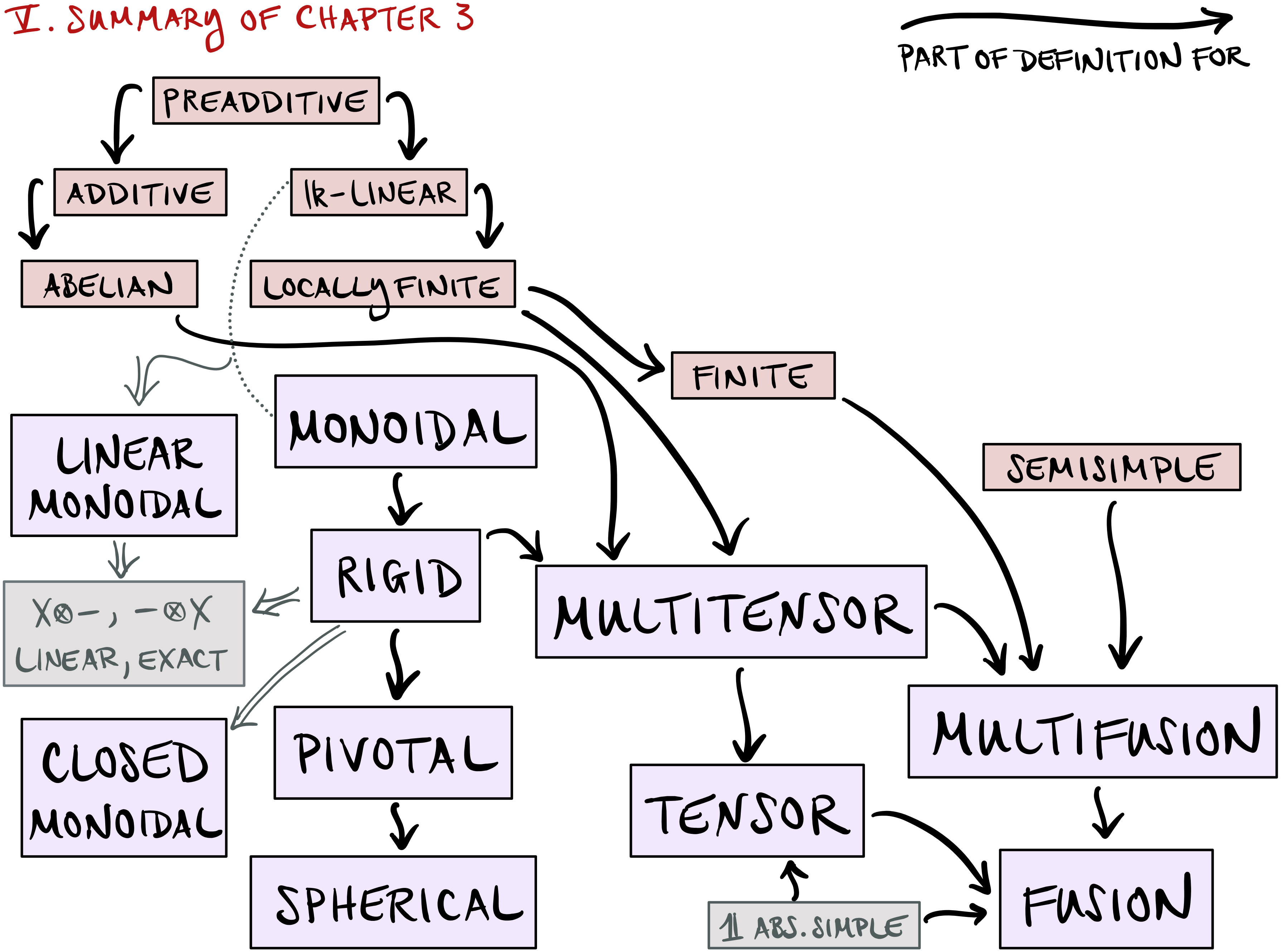

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #17

NEXT TIME:
ALGEBRAS IN $\otimes$ CATS

TOPICS:

I. TENSOR CATEGORIES (§3.10)
II. ENRICHED CATEGORIES (§§3.11.1, 3.11.2)
III. CLOSED MONOIDAL CATEGORIES (§3.11.3)
IV. INTERNAL HOMS (§3.11.4)
V. SUMMARY OF CHAPTER 3

Lecture #17 keywords: closed module category, closed monoidal category, enriched category, Frobenius-Perron dimension, internal Hom, tensor category