MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #18

TOPICS:

I. ALGEBRAS IN MONOIDAL CATEGORIES (§4.1.1)

II. DOCTRINAL ADJUNCTION & COINDUCED ALGEBRAS (§4.3.1)

III. SUBALGEBRAS AND IDEALS (§4.2.1)

IV. QUOTIENT ALGEBRAS (§4.2.2)

I. ALGEBRAS IN MONOIDAL CATEGORIES

≡ RECALL ≡

MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

I. ALGEBRAS IN MONOIDAL CATEGORIES

≡ RECALL ≡

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

CONSISTS OF:

(a) CATEGORY $\mathcal{C}$

(b) BIFUNCTOR $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(c) OBJECT $\mathbb{1} \in \mathcal{C}$

I. ALGEBRAS IN MONOIDAL CATEGORIES

≡RECALL≡

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(d, e, f) NATURAL ISOMS:

$$a = \left\{ a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \right\}_{X,Y,Z \in \mathcal{C}}$$

$$\ell = \{ \ell_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

$$r = \{ r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \}_{X \in \mathcal{C}}$$

SATISFYING THE

PENTAGON AXIOM

& TRIANGLE AXIOM

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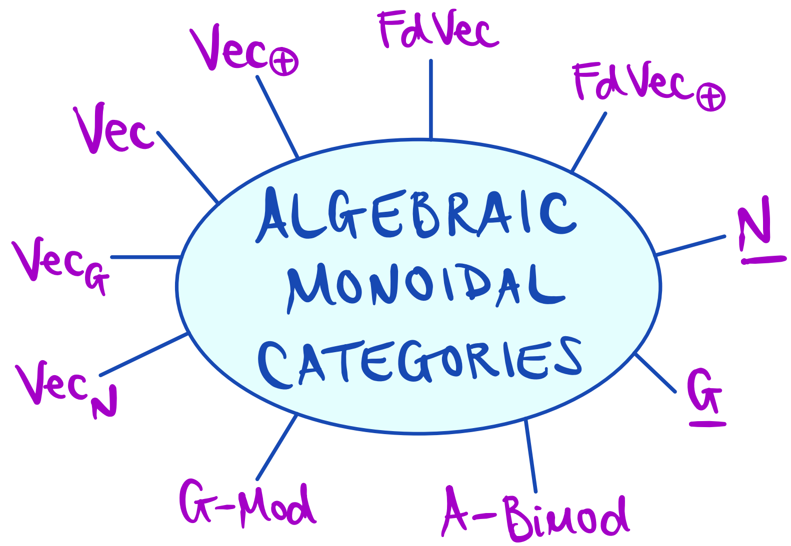
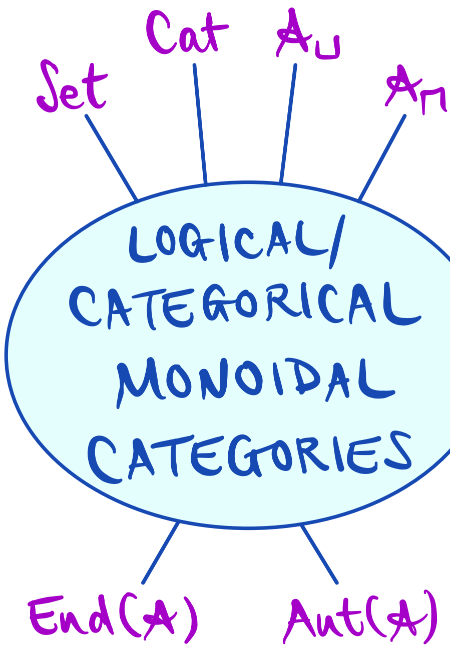
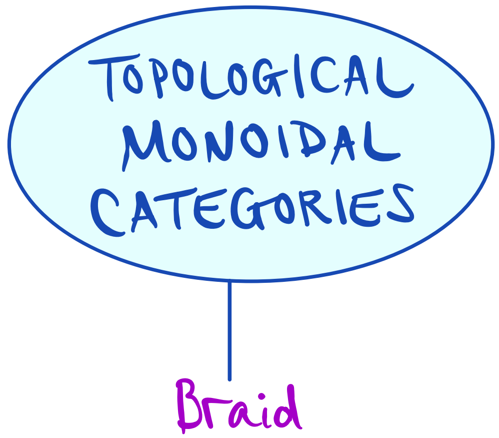
SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

LOTS OF EXAMPLES







I. ALGEBRAS IN MONOIDAL CATEGORIES

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$/\Bbbk$

$$\otimes := \otimes_{\Bbbk}$$

$$\mathbb{1} := \Bbbk$$

ALGEBRAIC MONOIDAL CATEGORIES

$\boxed{\mathsf{Vec}}$, $\mathsf{Vec}_{\oplus}$, $\boxed{\mathsf{FdVec}}$, $\mathsf{FdVec}_{\oplus}$, $\underline{\mathbb{N}}$, $\underline{G}$, A-Bimod, G-Mod, $\mathsf{Vec}_{\mathbb{N}}$, Vec_G

LOGICAL/CATEGORICAL MONOIDAL CATEGORIES

Set, Cat, $\mathcal{A}_{\sqcup}$, $\mathcal{A}_{\sqcap}$, End($\mathcal{A}$), Aut($\mathcal{A}$)

TOPOLOGICAL MONOIDAL CATEGORIES

Braid

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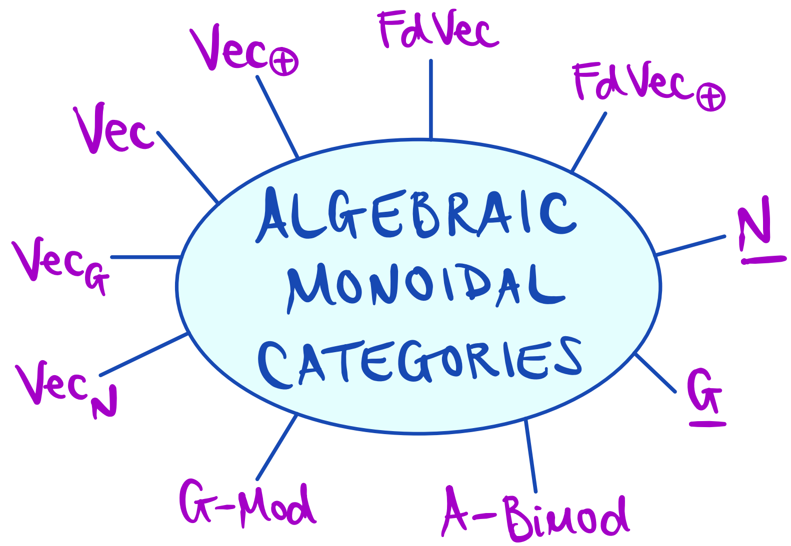
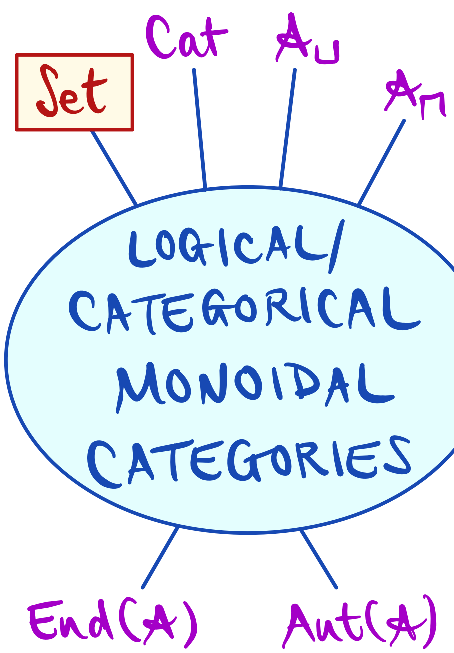
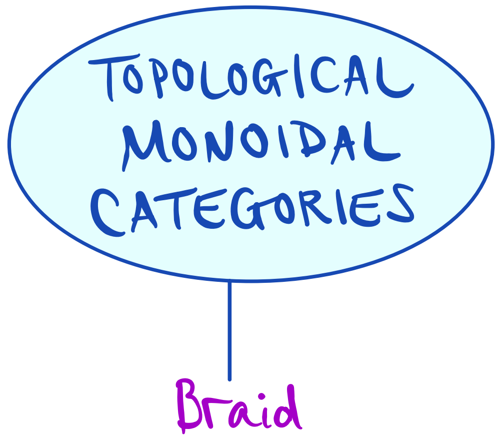
SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

$$\otimes := \times$$

$$\mathbb{1} := \{*\}$$







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SATISFYING THE

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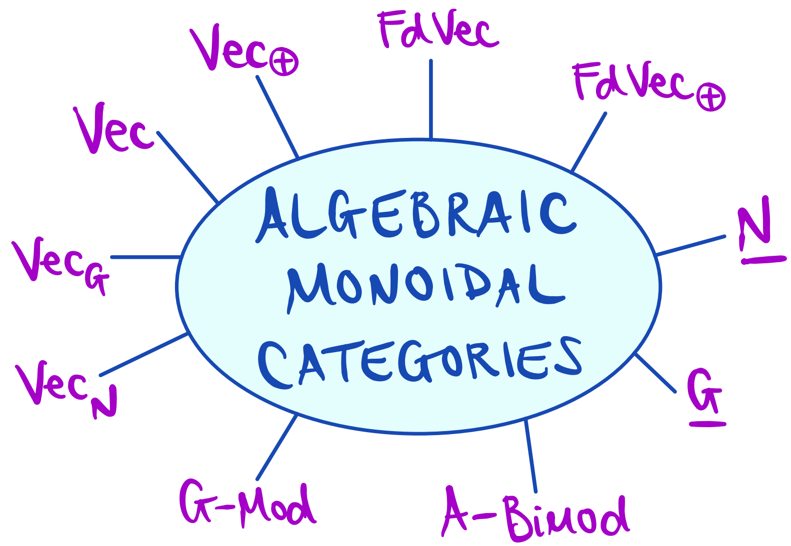
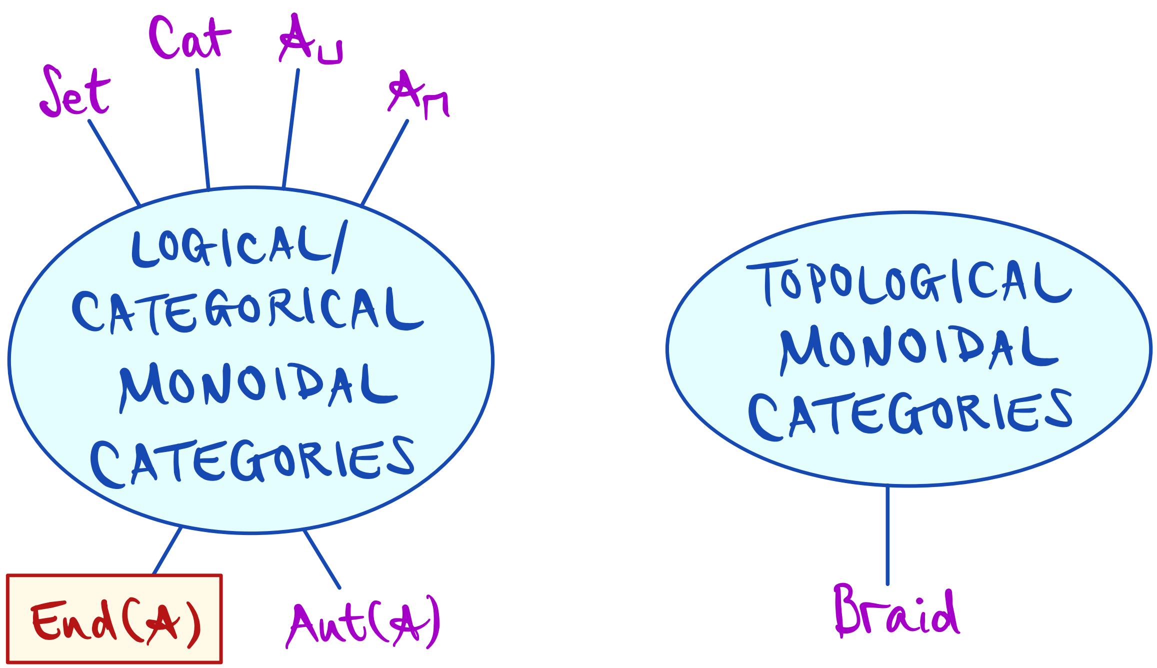
$$\otimes := \circ$$

$$\mathbb{1} := \mathrm{Id}_A$$





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SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

AN **ALGEBRA** IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ CONSISTS OF:

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> SATISFYING THE PENTAGON AXIOM & TRIANGLE AXIOM

AN **ALGEBRA** IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$ CONSISTS OF:

(a) AN OBJECT $A \in \mathcal{C}$

(b) A MORPHISM $m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$
(MULTIPLICATION MORPHISM)

(c) A MORPHISM $u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$
(UNIT MORPHISM)

SATISFYING:

I. ALGEBRAS IN MONOIDAL CATEGORIES

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(**MULTIPLICATION MORPHISM**)

(c) A MORPHISM $u := u_A : \mathbb{1} \to A \in \mathcal{C}$

(**UNIT MORPHISM**)

SATISFYING:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m \otimes id"'] & & A \otimes (A \otimes A) \arrow[d, "id \otimes m"] \\
A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(**ASSOCIATIVITY AXIOM**)

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes id"] \arrow[dr, "\ell_A"'] & A \otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$

(**LEFT UNITALITY AXIOM**)

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[r, "id \otimes u"] \arrow[dr, "r_A"'] & A \otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$

(**RIGHT UNITALITY AXIOM**)

I. Algebras in monoidal categories

Algebra in $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$:

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SATISFYING:

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A \otimes A \arrow[dr, "m"'] \arrow[rr, phantom, "\circlearrowleft"] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(**Associativity axiom**)

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes id"] \arrow[dr, "\ell_A"'] \arrow[r, phantom, shift right=3ex, "\circlearrowleft"] & A \otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$

(**Left unitality axiom**)

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[r, "id \otimes u"] \arrow[dr, "r_A"'] \arrow[r, phantom, shift right=3ex, "\circlearrowleft"] & A \otimes A \arrow[d, "m"] \\
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$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes id"'] & & A\otimes(A\otimes A) \arrow[d, "id\otimes m"] \\
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& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1}\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "\ell_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A\otimes\mathbb{1} \arrow[r, "id\otimes u"] \arrow[dr, "r_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

EXERCISE 4.1

- ALGEBRAS IN $(\mathsf{Vec}, \otimes_{\Bbbk}, \Bbbk)$ ARE $\Bbbk$-ALGEBRAS.

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SATISFYING:

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes id"'] & & A\otimes(A\otimes A) \arrow[d, "id\otimes m"] \\
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\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

EXERCISE 4.1

- ALGEBRAS IN $(\mathrm{Vec}, \otimes_{\Bbbk}, \Bbbk)$ ARE $\Bbbk$-ALGEBRAS.

INDEED:

A $\Bbbk$-ALGEBRA IS A $\Bbbk$-VSPACE MADE INTO A UNITAL RING.

FROM LECTURE #2

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH LINEAR MAPS $m: A\otimes A \to A$ (MULTIPLICATION) & $u: \Bbbk \to A$ (UNIT)
$a\otimes b \mapsto ab$ $\qquad 1_{\Bbbk} \mapsto 1_A$

SUCH THAT THE FOLLOWING DIAGRAMS COMMUTE:

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "m\otimes id"] \arrow[d, "id\otimes m"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes\Bbbk \cong A \cong \Bbbk\otimes A \arrow[r, "u\otimes id"] \arrow[d, "id\otimes u"'] \arrow[dr, "id"] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"] & A
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A\otimes A \arrow[dr, "m"'] \arrow[rr, phantom, "\circlearrowleft"] & & A\otimes A \arrow[dl, "m"] \\
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\end{tikzcd}$$

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(LEFT UNITALITY AXIOM)

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(RIGHT UNITALITY AXIOM)

EXERCISE 4.1

- ALGEBRAS IN $(\mathsf{Vec}, \otimes_{\Bbbk}, \Bbbk)$ ARE $\Bbbk$-ALGEBRAS.
- ALGEBRAS IN $(\mathsf{Set}, \times, \{*\})$ ARE ???

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\end{tikzcd}$$

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$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes id"] \arrow[dr, "\ell_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
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\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

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A \otimes \mathbb{1} \arrow[r, "id \otimes u"] \arrow[dr, "r_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
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$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes id"] \arrow[dr, "l_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
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\end{tikzcd}$$

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- ALGEBRAS IN $(\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$ ARE "**MONADS**"
(STUDIED LATER)

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EXERCISE 4.1

- ALGEBRAS IN $(\mathrm{Vec}, \otimes_{\Bbbk}, \Bbbk)$ ARE $\Bbbk$-ALGEBRAS.
- ALGEBRAS IN $(\mathrm{Set}, \times, \{*\})$ ARE MONOIDS

- ALGEBRAS IN $(\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$ ARE "MONADS"
(STUDIED LATER)

TRIVIAL EXAMPLES

- UNIT ALGEBRA: $(\mathbb{1},\ m_{\mathbb{1}} : \mathbb{1}\otimes\mathbb{1} \longrightarrow \mathbb{1},\ u_{\mathbb{1}} : \mathbb{1} \longrightarrow \mathbb{1})$

$m_{\mathbb{1}} = \ell_{\mathbb{1}} = r_{\mathbb{1}}$, $u_{\mathbb{1}} = id_{\mathbb{1}}$

I. Algebras in monoidal categories

Algebra in $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$:

(a) object $A \in \mathcal{C}$

(b) multip'n morphism
$m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) unit morphism
$u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

Satisfying:

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes id"'] \arrow[drr, phantom, "\circlearrowleft"] & & A\otimes(A\otimes A) \arrow[d, "id\otimes m"] \\
A\otimes A \arrow[dr, "m"'] & & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(Associativity axiom)

$$\begin{tikzcd}
\mathbb{1}\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "\ell_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(Left unitality axiom)

$$\begin{tikzcd}
A\otimes \mathbb{1} \arrow[r, "id\otimes u"] \arrow[dr, "r_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(Right unitality axiom)

Exercise 4.1

- Algebras in $(\mathrm{Vec}, \otimes_{\Bbbk}, \Bbbk)$ are $\Bbbk$-algebras.
- Algebras in $(\mathrm{Set}, \times, \{*\})$ are monoids

- Algebras in $(\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$ are "**monads**"
(studied later)

Trivial examples

- **Unit algebra**: $(\mathbb{1},\ m_{\mathbb{1}} : \mathbb{1}\otimes\mathbb{1} \longrightarrow \mathbb{1},\ u_{\mathbb{1}} : \mathbb{1} \longrightarrow \mathbb{1})$
$m_{\mathbb{1}} = \ell_{\mathbb{1}} = r_{\mathbb{1}}$, $\quad u_{\mathbb{1}} = id_{\mathbb{1}}$

- **Zero algebra**: $(0,\ m_0 : 0\otimes 0 \longrightarrow 0,\ u_0 : \mathbb{1} \longrightarrow 0)$
if $\exists$ zero obj. $\in \mathcal{C}$
$m_0 = \vec{0}_{0\otimes 0}$, $\quad u_0 = \vec{0}_{\mathbb{1}}$

I. ALGEBRAS IN MONOIDAL CATEGORIES

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$:

(a) OBJECT $A \in \mathcal{C}$

(b) MULTIP'N MORPHISM

$m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) UNIT MORPHISM

$u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SATISFYING:

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes \mathrm{id}"'] & & A\otimes(A\otimes A) \arrow[d, "\mathrm{id}\otimes m"] \\
A\otimes A \arrow[dr, "m"'] & \circlearrowleft & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1}\otimes A \arrow[r, "u\otimes \mathrm{id}"] \arrow[dr, "\ell_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A\otimes \mathbb{1} \arrow[r, "\mathrm{id}\otimes u"] \arrow[dr, "r_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

EXERCISE 4.1

- ALGEBRAS IN $(\mathrm{Vec}, \otimes_{\Bbbk}, \Bbbk)$ ARE $\Bbbk$-ALGEBRAS.
- ALGEBRAS IN $(\mathrm{Set}, \times, \{*\})$ ARE MONOIDS

- ALGEBRAS IN $(\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$ ARE "MONADS" (STUDIED LATER)

TRIVIAL EXAMPLES

DETAILS = EXERCISE 4.5

- UNIT ALGEBRA: $(\mathbb{1},\ m_{\mathbb{1}} : \mathbb{1}\otimes\mathbb{1} \longrightarrow \mathbb{1},\ u_{\mathbb{1}} : \mathbb{1} \longrightarrow \mathbb{1})$

  $m_{\mathbb{1}} = \ell_{\mathbb{1}} = r_{\mathbb{1}}$, $\quad u_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$

- ZERO ALGEBRA: $(0,\ m_0 : 0\otimes 0 \longrightarrow 0,\ u_0 : \mathbb{1} \longrightarrow 0)$

  IF $\exists$ ZERO OBJ. $\in \mathcal{C}$

  $m_0 = \vec{0}_{0\otimes 0}$, $\quad u_0 = \vec{0}_{\mathbb{1}}$

I. ALGEBRAS IN MONOIDAL CATEGORIES

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$: ← GETTING FROM ONE TO ANOTHER...

(a) OBJECT $A \in \mathcal{C}$

(b) MULTIP'N MORPHISM
$m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) UNIT MORPHISM
$u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SATISFYING:

$$\begin{tikzcd} (A\otimes A)\otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes id"'] & & A\otimes(A\otimes A) \arrow[d, "id\otimes m"] \\ A\otimes A \arrow[dr, "m"'] & \circlearrowleft & A\otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd} \mathbb{1}\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "\ell_A"'] & A\otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd} A\otimes \mathbb{1} \arrow[r, "id\otimes u"] \arrow[dr, "r_A"'] & A\otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

I. ALGEBRAS IN MONOIDAL CATEGORIES

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$:

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(A\otimes A)\otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes id"'] & & A\otimes(A\otimes A) \arrow[d, "id\otimes m"] \\
A\otimes A \arrow[dr, "m"'] & \arrow[d, phantom, "2"] & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1}\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "\ell_A"'] & A\otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$
$\quad 2$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A\otimes\mathbb{1} \arrow[r, "id\otimes u"] \arrow[dr, "r_A"'] & A\otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$
$\quad 2$

(RIGHT UNITALITY AXIOM)

← GETTING FROM ONE TO ANOTHER...

GIVEN ALGEBRAS (A, m, u) & (A', m', u') IN $\mathcal{C}$,

AN ALGEBRA MORPHISM $(A, m, u) \longrightarrow (A', m', u')$

IS A MORPHISM $\phi : A \longrightarrow A' \in \mathcal{C}$ SUCH THAT:

[ϕ IS COMPATIBLE WITH m, m'] [ϕ IS COMPATIBLE WITH u, u']

I. ALGEBRAS IN MONOIDAL CATEGORIES

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A \otimes A \arrow[dr, "m"'] & \circlearrowleft & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes id"] \arrow[dr, "\ell_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[r, "id \otimes u"] \arrow[dr, "r_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

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IS A MORPHISM $\phi : A \longrightarrow A' \in \mathcal{C}$ SUCH THAT:

[ϕ IS COMPATIBLE WITH m, m']

$$\begin{tikzcd}
A \otimes A \arrow[r, "m"] \arrow[d, "\phi \otimes \phi"'] & A \arrow[d, "\phi"] \\
A' \otimes A' \arrow[r, "m'"] \arrow[ur, phantom, "\circlearrowleft"] & A'
\end{tikzcd}$$

[ϕ IS COMPATIBLE WITH u, u']

$$\begin{tikzcd}
& A \arrow[d, "\phi"] \\
\mathbb{1} \arrow[ur, "u"] \arrow[dr, "u'"'] \arrow[r, phantom, "\circlearrowleft"] & A'
\end{tikzcd}$$

I. ALGEBRAS IN MONOIDAL CATEGORIES

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$:

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$m := m_A : A \otimes A \to A \in \mathcal{C}$

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SATISFYING:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m \otimes id"'] & & A \otimes (A \otimes A) \arrow[d, "id \otimes m"] \\
A \otimes A \arrow[dr, "m"'] \arrow[rr, phantom, "\circlearrowleft"] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes id"] \arrow[dr, "\ell_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[r, "id \otimes u"] \arrow[dr, "r_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
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\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

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$$\begin{tikzcd}
A \otimes A \arrow[r, "m"] \arrow[d, "\phi \otimes \phi"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "\phi"] \\
A' \otimes A' \arrow[r, "m'"] & A'
\end{tikzcd}$$

[ϕ IS COMPATIBLE WITH u, u']

$$\begin{tikzcd}
& A \arrow[d, "\phi"] \\
\mathbb{1} \arrow[ur, "u"] \arrow[dr, "u'"'] \arrow[r, phantom, "\circlearrowleft"] & A'
\end{tikzcd}$$

$\mathrm{Alg}(\mathcal{C})$: CATEGORY OF ALGEBRAS IN $\mathcal{C}$ & THEIR MORPHISMS.

I. ALGEBRAS IN MONOIDAL CATEGORIES

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$:

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(LEFT UNITALITY AXIOM)

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(RIGHT UNITALITY AXIOM)

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[ϕ IS COMPATIBLE WITH m, m']

$$\begin{tikzcd}
A \otimes A \arrow[r, "m"] \arrow[d, "\phi \otimes \phi"'] & A \arrow[d, "\phi"] \arrow[dl, phantom, "\circlearrowright"] \\
A' \otimes A' \arrow[r, "m'"'] & A'
\end{tikzcd}$$

[ϕ IS COMPATIBLE WITH u, u']

$$\begin{tikzcd}
& A \arrow[d, "\phi"] \\
\mathbb{1} \arrow[ur, "u"] \arrow[dr, "u'"'] \arrow[r, phantom, "\circlearrowright"] & \\
& A'
\end{tikzcd}$$

MONIC | EPIC | ISO ALG. MAP := ALG. MAP THAT IS MONIC | EPIC | ISO IN $\mathcal{C}$

$\mathrm{Alg}(\mathcal{C})$: CATEGORY OF ALGEBRAS IN $\mathcal{C}$ & THEIR MORPHISMS.

I. ALGEBRAS IN MONOIDAL CATEGORIES

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(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
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\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A\otimes \mathbb{1} \arrow[r, "id\otimes u"] \arrow[dr, "r_A"'] & A\otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

MONOIDAL FUNCTORS SEND ALGEBRAS TO ALGEBRAS

Alg($\mathcal{C}$): CATEGORY OF ALGEBRAS IN $\mathcal{C}$ & THEIR MORPHISMS.

I. ALGEBRAS IN MONOIDAL CATEGORIES

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$:

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A\otimes A \arrow[dr, "m"'] & \circlearrowleft & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1}\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "\ell_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A\otimes \mathbb{1} \arrow[r, "id\otimes u"] \arrow[dr, "r_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

MONOIDAL FUNCTORS SEND ALGEBRAS TO ALGEBRAS

PROP: LET $(F, F^{(2)}, F^{(0)}) : (\mathcal{C}, \otimes, \mathbb{1}) \longrightarrow (\mathcal{C}', \otimes', \mathbb{1}')$ BE A MONOIDAL FUNCTOR.

IF $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$, THEN

$$\left(\begin{array}{l} F(A), \\ m_{F(A)} : \\ u_{F(A)} : \end{array} \right) \in \mathrm{Alg}(\mathcal{C}').$$

$\mathrm{Alg}(\mathcal{C})$: CATEGORY OF ALGEBRAS IN $\mathcal{C}$ & THEIR MORPHISMS.

I. Algebras in monoidal categories

Algebra in $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$:

(a) object $A \in \mathcal{C}$

(b) multip'n morphism
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(c) unit morphism
$u := u_A : \mathbb{1} \to A \in \mathcal{C}$

Satisfying:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m \otimes id"'] & & A \otimes (A \otimes A) \arrow[d, "id \otimes m"] \\
A \otimes A \arrow[dr, "m"'] & \circlearrowleft & A \otimes A \arrow[dl, "m"] \\
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\end{tikzcd}$$

(Associativity axiom)

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes id"] \arrow[dr, "\ell_A"'] & A \otimes A \arrow[d, "m"] \\
& A
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(Left unitality axiom)

$$\begin{tikzcd}
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& A
\end{tikzcd}$$

(Right unitality axiom)

Monoidal functors send algebras to algebras

Prop: Let $(F, F^{(2)}, F^{(0)}) : (\mathcal{C}, \otimes, \mathbb{1}) \longrightarrow (\mathcal{C}', \otimes', \mathbb{1}')$ be a monoidal functor.

If $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$, then

$$\left(\begin{array}{l} F(A), \\ m_{F(A)} : F(A) \otimes' F(A) \xrightarrow{F^{(2)}_{A,A}} F(A \otimes A) \xrightarrow{F(m_A)} F(A), \\ u_{F(A)} : \end{array} \right)$$

$\in \mathsf{Alg}(\mathcal{C}')$.

$\mathsf{Alg}(\mathcal{C})$: Category of algebras in $\mathcal{C}$ & their morphisms.

I. ALGEBRAS IN MONOIDAL CATEGORIES

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$:

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$m := m_A : A \otimes A \to A \in \mathcal{C}$

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SATISFYING:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m \otimes id"'] & & A \otimes (A \otimes A) \arrow[d, "id \otimes m"] \\
A \otimes A \arrow[dr, "m"'] & \circlearrowleft & A \otimes A \arrow[dl, "m"] \\
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(ASSOCIATIVITY AXIOM)

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(RIGHT UNITALITY AXIOM)

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IF $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$, THEN

$$\left(\begin{array}{l} F(A), \\ m_{F(A)} : F(A) \otimes' F(A) \xrightarrow{F^{(2)}_{A,A}} F(A \otimes A) \xrightarrow{F(m_A)} F(A), \\ u_{F(A)} : \mathbb{1}' \xrightarrow{F^{(0)}} F(\mathbb{1}) \xrightarrow{F(u_A)} F(A) \end{array} \right) \in \mathrm{Alg}(\mathcal{C}').$$

$\mathrm{Alg}(\mathcal{C})$: CATEGORY OF ALGEBRAS IN $\mathcal{C}$ & THEIR MORPHISMS.

I. ALGEBRAS IN MONOIDAL CATEGORIES

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$:

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A \otimes A \arrow[dr, "m"'] & \circlearrowleft & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes id"] \arrow[dr, "l_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[r, "id \otimes u"] \arrow[dr, "r_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
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(RIGHT UNITALITY AXIOM)

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$\in \mathrm{Alg}(\mathcal{C}')$.

THIS INDUCES A FUNCTOR:

$$\mathrm{Alg}(\mathcal{C}) \xrightarrow{F} \mathrm{Alg}(\mathcal{C}').$$

I. ALGEBRAS IN MONOIDAL CATEGORIES

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$:

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$m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) UNIT MORPHISM
$u := u_A : \mathbb{1} \to A \in \mathcal{C}$

SATISFYING:

$$\begin{tikzcd}[column sep=small] (A\otimes A)\otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes id"'] & & A\otimes(A\otimes A) \arrow[d, "id\otimes m"] \\ A\otimes A \arrow[dr, "m"'] & \circlearrowleft & A\otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd} \mathbb{1}\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "\ell_A"'] & A\otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd} A\otimes \mathbb{1} \arrow[r, "id\otimes u"] \arrow[dr, "r_A"'] & A\otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

MONOIDAL FUNCTORS SEND ALGEBRAS TO ALGEBRAS

PROP: LET $(F, F^{(2)}, F^{(0)}) : (\mathcal{C}, \otimes, \mathbb{1}) \to (\mathcal{C}', \otimes', \mathbb{1}')$ BE A MONOIDAL FUNCTOR.

IF $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$, THEN

$$\left(\begin{array}{l} F(A), \\ m_{F(A)} : F(A) \otimes' F(A) \xrightarrow{F^{(2)}_{A,A}} F(A\otimes A) \xrightarrow{F(m_A)} F(A), \\ u_{F(A)} : \mathbb{1}' \xrightarrow{F^{(0)}} F(\mathbb{1}) \xrightarrow{F(u_A)} F(A) \end{array} \right)$$

$\in \mathsf{Alg}(\mathcal{C}')$.

THIS INDUCES A FUNCTOR:

$$\mathsf{Alg}(\mathcal{C}) \xrightarrow{F} \mathsf{Alg}(\mathcal{C}').$$

PROOF = EXERCISE 4.7

BUT LET'S GO THROUGH ASSOCIATIVITY

= ON THE BOARD =

II. DOCTRINAL ADJUNCTION & COINDUCED ALGEBRAS

MONOIDAL FUNCTORS SEND ALGEBRAS TO ALGEBRAS

MORE ON THIS

II. DOCTRINAL ADJUNCTION & COINDUCED ALGEBRAS

THEOREM (SPECIAL CASE OF DOCTRINAL ADJUNCTION, KELLY)

TAKE MONOIDAL CATEGORIES $\mathcal{C}$ AND $\mathcal{D}$ WITH

$$(F : \mathcal{C} \longrightarrow \mathcal{D}) \dashv (G : \mathcal{D} \longrightarrow \mathcal{C})$$

ADJUNCTION BETWEEN UNDERLYING CATEGORIES

UNIT $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ & COUNIT $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$.

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$G^{(2)}_{Y,Y'} : G(Y) \otimes^{\mathcal{C}} G(Y')$ $\quad$ $G(Y \otimes^{\mathcal{D}} Y')$

FOR $Y, Y' \in \mathcal{D}$

$G^{(0)} : \mathbb{1}^{\mathcal{C}}$ $\quad$ $G(\mathbb{1}^{\mathcal{D}})$

II. Doctrinal Adjunction & Coinduced Algebras

Theorem (Special case of doctrinal adjunction, Kelly)

Take monoidal categories $\mathcal{C}$ and $\mathcal{D}$ with

$$(F : \mathcal{C} \longrightarrow \mathcal{D}) \dashv (G : \mathcal{D} \longrightarrow \mathcal{C})$$

adjunction between underlying categories

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If F is strong monoidal, then G is monoidal with:

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$$G(Y) \otimes^{\mathcal{C}} G(Y') \xrightarrow{\eta_{G(Y)\otimes^{\mathcal{C}} G(Y')}} GF(G(Y) \otimes^{\mathcal{C}} G(Y')) \xrightarrow{G\left(F^{(-2)}_{G(Y),G(Y')}\right)} G(FG(Y) \otimes^{\mathcal{D}} FG(Y'))$$

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$$
\begin{array}{c}
G(Y) \otimes^{\mathcal{C}} G(Y') \\
\downarrow \eta \\
GF\big(G(Y) \otimes^{\mathcal{C}} G(Y')\big) \\
\downarrow G(F^{(-2)}) \\
G\big(FG(Y) \otimes^{\mathcal{D}} FG(Y')\big) \\
\downarrow G(\varepsilon \otimes \varepsilon) \\
G(Y \otimes^{\mathcal{D}} Y')
\end{array}
$$

$$G^{(0)} : \mathbb{1}^{\mathcal{C}} \xrightarrow{\eta} GF(\mathbb{1}^{\mathcal{C}}) \xrightarrow{G(F^{(-0)})} G(\mathbb{1}^{\mathcal{D}})$$

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EXAMPLE: TAKE $\phi : H \to G$ MORPHISM OF FINITE GROUPS

HAVE ADJUNCTION —

$$\left(\mathrm{Res}^G_H : G\text{-Mod} \to H\text{-Mod}\right) \dashv \left(\mathrm{Coind}^G_H : H\text{-Mod} \to G\text{-Mod}\right)$$

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$$\mathrm{Coind}^G_H(W, \triangleright^H_W) = \mathrm{Hom}_{H\text{-Mod}}(\Bbbk G, W)$$

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COINDUCED ALGEBRA

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COINDUCED ALGEBRA

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$$\mathbb{1} = \Bbbk \in \mathrm{Alg}(\mathrm{Vec}) \rightsquigarrow \mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, \Bbbk) \in \mathrm{Alg}(G\text{-Mod})$$

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$(\Bbbk G)^* = \mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, \Bbbk)$

WILL SHOW

DUAL GROUP ALG. IS A COINDUCED ALG.

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$$\left(\mathrm{Res}^G_{\langle e \rangle}: G\text{-Mod} \to \mathrm{Vec}\right) \dashv \left(\mathrm{Coind}^G_{\langle e \rangle}: \mathrm{Vec} \to G\text{-Mod}\right)$$

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$$G^{(0)}: \mathbb{1}^{\mathcal{C}} \xrightarrow{\eta} GF(\mathbb{1}^{\mathcal{C}}) \xrightarrow{G(F^{(-0)})} G(\mathbb{1}^{\mathcal{D}})$$

EXAMPLE: TAKE $\phi: \langle e\rangle \longrightarrow G$

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Ex. $H = \langle e\rangle$

WILL SHOW

$(\Bbbk G)^*$ DUAL GROUP ALG. IS A COINDUCED ALG.

$\|$

$$\mathbb{1} = \Bbbk \in \mathrm{Alg}(\mathrm{Vec}) \rightsquigarrow \mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, \Bbbk) \in \mathrm{Alg}(G\text{-Mod})$$

II. DOCTRINAL ADJUNCTION & COINDUCED ALGEBRAS

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FOR $Y, Y' \in \mathcal{D}$

$$G^{(0)}: 1^{\mathcal{C}} \xrightarrow{\eta} GF(1^{\mathcal{C}}) \xrightarrow{G(F^{(-0)})} G(1^{\mathcal{D}})$$

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$$\underset{id_{\Bbbk}}{\overset{\shortparallel}{Res^{(0)}}} : \underset{\Bbbk}{\overset{\shortparallel}{1_{Vec}}} \longrightarrow \underset{\Bbbk}{\overset{\shortparallel}{Res^{(0)}(1_{G\text{-Mod}})}}$$

STRONG MONOIDAL

Ex. $H = \langle e \rangle$

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$$\mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, \Bbbk) \otimes_{\Bbbk} \mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, \Bbbk) \longrightarrow \mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, \Bbbk)$$

$$f \otimes f' \longmapsto [g' \mapsto f(g')\, f'(g')]$$

$$\begin{array}{rl} \mathrm{Coind}^{(2)}_{W,W'}: & \mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, W) \otimes_{\Bbbk} \mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, W') \\ & f \otimes f' \\ \eta \downarrow & \\ & \mathrm{Hom}_{\mathrm{Vec}}\big(\Bbbk G, \mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, W) \otimes_{\Bbbk} \mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, W')\big) \\ & g' \mapsto [g' \triangleright (f \otimes f')] = [(g' \triangleright f) \otimes (g' \triangleright f')] \\ G(\varepsilon \otimes \varepsilon) \downarrow & \\ & \mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, W \otimes_{\Bbbk} W') \\ & g' \mapsto (g' \triangleright f)(e_G) \otimes (g' \triangleright f')(e_G) = f(g') \otimes_{\Bbbk} f'(g') \end{array}$$

II. DOCTRINAL ADJUNCTION & COINDUCED ALGEBRAS

DOCTRINAL ADJUNCTION:

- MONOIDAL CATS $\mathcal{C}$ AND $\mathcal{D}$.
- $(F: \mathcal{C} \to \mathcal{D}) \dashv (G: \mathcal{D} \to \mathcal{C})$

  $\eta: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ & $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

IF F IS STRONG MONOIDAL, THEN G IS MONOIDAL WITH:

$$G^{(2)}_{Y,Y'}: \quad G(Y) \otimes^{\mathcal{C}} G(Y') \xrightarrow{\eta} GF\big(G(Y) \otimes^{\mathcal{C}} G(Y')\big) \xrightarrow{G(F^{(-2)})} G\big(FG(Y) \otimes^{\mathcal{D}} FG(Y')\big) \xrightarrow{G(\varepsilon \otimes \varepsilon)} G(Y \otimes^{\mathcal{D}} Y')$$

FOR $Y, Y' \in \mathcal{D}$

$$G^{(0)}: \mathbb{1}^{\mathcal{C}} \xrightarrow{\eta} GF(\mathbb{1}^{\mathcal{C}}) \xrightarrow{G(F^{(-0)})} G(\mathbb{1}^{\mathcal{D}})$$

EXAMPLE: TAKE $\phi: \langle e \rangle \to G$.

$$\left(\mathrm{Res}^{G}_{\langle e \rangle}: G\text{-Mod} \to \mathrm{Vec}\right) \dashv \left(\mathrm{Coind}^{G}_{\langle e \rangle}: \mathrm{Vec} \to G\text{-Mod}\right)$$

$$m_{\mathrm{Coind}(\Bbbk)} := \mathrm{Coind}(m_{\Bbbk}) \circ \mathrm{Coind}^{(2)}_{\Bbbk,\Bbbk}:$$

$$\mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, \Bbbk) \otimes_{\Bbbk} \mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, \Bbbk) \longrightarrow \mathrm{Hom}_{\mathrm{Vec}}(\Bbbk G, \Bbbk)$$

$$f \otimes f' \longmapsto [g' \mapsto f(g')\, f'(g')]$$

II. DOCTRINAL ADJUNCTION & COINDUCED ALGEBRAS

DOCTRINAL ADJUNCTION:

- MONOIDAL CATS $\mathcal{C}$ AND $\mathcal{D}$.
- $(F: \mathcal{C} \to \mathcal{D}) \dashv (G: \mathcal{D} \to \mathcal{C})$

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$$G(Y) \otimes^{\mathcal{C}} G(Y') \xrightarrow{\eta} GF(G(Y) \otimes^{\mathcal{C}} G(Y')) \xrightarrow{G(F^{(-2)})} G(FG(Y) \otimes^{\mathcal{D}} FG(Y')) \xrightarrow{G(\varepsilon \otimes \varepsilon)} G(Y \otimes^{\mathcal{D}} Y')$$

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$$f \otimes f' \longmapsto [g' \mapsto f(g')\, f'(g')]$$

ON THE OTHER HAND:

$$m_{(\Bbbk G)^*} : (\Bbbk G)^* \otimes (\Bbbk G)^* \longrightarrow (\Bbbk G)^*$$

$$p_{g_1} \otimes p_{g_2} \longmapsto \delta_{g_1,g_2}\, p_{g_1}$$

$$p_g(g') := \delta_{g,g'}\, 1_{\Bbbk}$$

II. DOCTRINAL ADJUNCTION & COINDUCED ALGEBRAS

DOCTRINAL ADJUNCTION:

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FOR $Y, Y' \in \mathcal{D}$

$$G^{(0)}: \mathbb{1}^{\mathcal{C}} \xrightarrow{\eta} GF(\mathbb{1}^{\mathcal{C}}) \xrightarrow{G(F^{(-0)})} G(\mathbb{1}^{\mathcal{D}})$$

EXAMPLE: TAKE $\phi: \langle e \rangle \to G$.

$$\left(\mathrm{Res}^G_{\langle e \rangle}: G\text{-Mod} \to \mathrm{Vec}\right) \dashv \left(\mathrm{Coind}^G_{\langle e \rangle}: \mathrm{Vec} \to G\text{-Mod}\right)$$

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$$m_{(\Bbbk G)^*}: (\Bbbk G)^* \otimes (\Bbbk G)^* \longrightarrow (\Bbbk G)^*$$

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II. DOCTRINAL ADJUNCTION & COINDUCED ALGEBRAS

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- $(F : \mathcal{C} \to \mathcal{D}) \dashv (G : \mathcal{D} \to \mathcal{C})$

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$G^{(2)}_{Y,Y'} : G(Y) \otimes^{\mathcal{C}} G(Y')$ FOR $Y, Y' \in \mathcal{D}$

$$\begin{array}{c} G(Y) \otimes^{\mathcal{C}} G(Y') \\ \downarrow \eta \\ GF(G(Y) \otimes^{\mathcal{C}} G(Y')) \\ \downarrow G(F^{(-2)}) \\ G(FG(Y) \otimes^{\mathcal{D}} FG(Y')) \\ \downarrow G(\varepsilon \otimes \varepsilon) \\ G(Y \otimes^{\mathcal{D}} Y') \end{array}$$

$G^{(0)} : \mathbb{1}^{\mathcal{C}} \xrightarrow{\eta} GF(\mathbb{1}^{\mathcal{C}}) \xrightarrow{G(F^{(-0)})} G(\mathbb{1}^{\mathcal{D}})$

EXAMPLE: TAKE $\phi : \langle e \rangle \to G$.

$\therefore \mathrm{Coind}^{G}_{\langle e \rangle}(\Bbbk) = (\Bbbk G)^*$

$$\left(\mathrm{Res}^{G}_{\langle e \rangle} : G\text{-Mod} \to \mathrm{Vec}\right) \dashv \left(\mathrm{Coind}^{G}_{\langle e \rangle} : \mathrm{Vec} \to G\text{-Mod}\right)$$

$$m_{\mathrm{Coind}(\Bbbk)} := \mathrm{Coind}(m_{\Bbbk}) \circ \mathrm{Coind}^{(2)}_{\Bbbk,\Bbbk} :$$

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ON THE OTHER HAND:

$$m_{(\Bbbk G)^*} : (\Bbbk G)^* \otimes (\Bbbk G)^* \longrightarrow (\Bbbk G)^*$$

$$p_{g_1} \otimes p_{g_2} \longmapsto \delta_{g_1,g_2}\, p_{g_1}$$

$=$ ✓

$$p_g(g') := \delta_{g,g'}\, 1_{\Bbbk}$$

(NEED = ON UNITS TOO...

II. DOCTRINAL ADJUNCTION & COINDUCED ALGEBRAS

DOCTRINAL ADJUNCTION:

- MONOIDAL CATS $\mathcal{C}$ AND $\mathcal{D}$.
- $(F : \mathcal{C} \to \mathcal{D}) \dashv (G : \mathcal{D} \to \mathcal{C})$

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IF F IS STRONG MONOIDAL, THEN G IS MONOIDAL WITH:

FOR $Y, Y' \in \mathcal{D}$

$$\begin{aligned}
G^{(2)}_{Y,Y'} : G(Y) \otimes^{\mathcal{C}} G(Y') &\xrightarrow{\eta} GF(G(Y) \otimes^{\mathcal{C}} G(Y')) \\
&\xrightarrow{G(F^{(-2)})} G(FG(Y) \otimes^{\mathcal{D}} FG(Y')) \\
&\xrightarrow{G(\varepsilon \otimes \varepsilon)} G(Y \otimes^{\mathcal{D}} Y')
\end{aligned}$$

$$G^{(0)} : \mathbb{1}^{\mathcal{C}} \xrightarrow{\eta} GF(\mathbb{1}^{\mathcal{C}}) \xrightarrow{G(F^{(-0)})} G(\mathbb{1}^{\mathcal{D}})$$

DETAILS ≡ EXERCISES 4.2, 4.8 & 4.18

EXAMPLE: TAKE $\phi : \langle e \rangle \to G$.

$$\left(\mathrm{Res}^{G}_{\langle e \rangle} : G\text{-Mod} \to \mathrm{Vec}\right) \dashv \left(\mathrm{Coind}^{G}_{\langle e \rangle} : \mathrm{Vec} \to G\text{-Mod}\right)$$

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$\therefore \mathrm{Coind}^{G}_{\langle e \rangle}(\Bbbk) = (\Bbbk G)^*$

ON THE OTHER HAND:

$$m_{(\Bbbk G)^*} : (\Bbbk G)^* \otimes (\Bbbk G)^* \longrightarrow (\Bbbk G)^*$$

$$p_{g_1} \otimes p_{g_2} \longmapsto \delta_{g_1,g_2}\, p_{g_1}$$

$=$ ✓

$$p_g(g') := \delta_{g,g'}\, 1_{\Bbbk}$$

NEED = ON UNITS TOO...

ALGEBRAS IN MONOIDAL CATEGORIES

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) MULTIP'N MORPHISM

$m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) UNIT MORPHISM

$u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SATISFYING:

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes id"'] & & A\otimes(A\otimes A) \arrow[d, "id\otimes m"] \\
A\otimes A \arrow[dr, "m"'] \arrow[rr, phantom, "\circlearrowright"] & & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1}\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "l_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowright"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A\otimes \mathbb{1} \arrow[r, "id\otimes u"] \arrow[dr, "r_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowright"] \\
& A
\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

ALGEBRAS IN MONOIDAL CATEGORIES

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) MULTIP'N MORPHISM
$m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) UNIT MORPHISM
$u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SATISFYING:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m \otimes id"'] & & A \otimes (A \otimes A) \arrow[d, "id \otimes m"] \\
A \otimes A \arrow[dr, "m"'] & \circlearrowleft & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes id"] \arrow[dr, "\ell_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[r, "id \otimes u"] \arrow[dr, "r_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

NOW LET'S STUDY SUBSTRUCTURES & QUOTIENT STRUCTURES

ALGEBRAS IN MONOIDAL CATEGORIES

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) MULTIP'N MORPHISM
$m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) UNIT MORPHISM
$u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SATISFYING:

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes id"'] & & A\otimes(A\otimes A) \arrow[d, "id\otimes m"] \\
A\otimes A \arrow[dr, "m"'] & \arrow[d, phantom, "\circlearrowleft"] & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1}\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "\ell_A"'] & A\otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A\otimes \mathbb{1} \arrow[r, "id\otimes u"] \arrow[dr, "r_A"'] & A\otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

NOW LET'S STUDY SUBSTRUCTURES & QUOTIENT STRUCTURES

III. Subalgebras and ideals

Assume $\mathcal{C}$ is abelian monoidal

Algebra in $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) object $A \in \mathcal{C}$

(b) multip'n morphism
$m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) unit morphism
$u := u_A : \mathbb{1} \to A \in \mathcal{C}$

Satisfying:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m \otimes id"'] & & A \otimes (A \otimes A) \arrow[d, "id \otimes m"] \\
A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(Associativity axiom)

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes id"] \arrow[dr, "\ell_A"'] & A \otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$

(Left unitality axiom)

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[r, "id \otimes u"] \arrow[dr, "r_A"'] & A \otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$

(Right unitality axiom)

A **subalgebra** of (A, m_A, u_A) in $\mathcal{C}$ is an algebra (B, m_B, u_B) in $\mathcal{C}$ such that B_{obj} is a subobject of A_{obj} via mono $\iota := \iota_B^A : B \to A$, & $\iota \in \mathrm{Alg}(\mathcal{C})$.

III. SUBALGEBRAS AND IDEALS

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) MULTIP'N MORPHISM
$m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) UNIT MORPHISM
$u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SATISFYING:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m \otimes id"'] & & A \otimes (A \otimes A) \arrow[d, "id \otimes m"] \\
A \otimes A \arrow[dr, "m"'] & \circlearrowright & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes id"] \arrow[dr, "\ell_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowright"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[r, "id \otimes u"] \arrow[dr, "r_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowright"] \\
& A
\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

A **SUBALGEBRA** OF (A, m_A, u_A) IN $\mathcal{C}$ IS AN ALGEBRA (B, m_B, u_B) IN $\mathcal{C}$ SUCH THAT B_{obj} IS A SUBOBJECT OF A_{obj} VIA MONO $\iota := \iota_B^A : B \to A$, & $\iota \in \mathrm{Alg}(\mathcal{C})$.

$$\begin{tikzcd}
B \otimes B \arrow[r, "m_B"] \arrow[d, "\iota \otimes \iota"'] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d, "\iota"] \\
A \otimes A \arrow[r, "m_A"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
& B \arrow[d, "\iota"] \\
\mathbb{1} \arrow[ur, "u_B"] \arrow[r, "u_A"'] \arrow[r, phantom, shift left=3, "\circlearrowright"] & A
\end{tikzcd}$$

III. SUBALGEBRAS AND IDEALS

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) MULTIP'N MORPHISM
$m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) UNIT MORPHISM
$u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SATISFYING:

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m \otimes id"'] & & A \otimes (A \otimes A) \arrow[d, "id \otimes m"] \\
A \otimes A \arrow[dr, "m"'] & {} \arrow[d, phantom, "="] & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes id"] \arrow[dr, "\ell_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "="] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[r, "id \otimes u"] \arrow[dr, "r_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "="] \\
& A
\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

A SUBALGEBRA OF (A, m_A, u_A) IN $\mathcal{C}$ IS AN ALGEBRA (B, m_B, u_B) IN $\mathcal{C}$ SUCH THAT B_{OBJ} IS A SUBOBJECT OF A_{OBJ} VIA MONO $\iota := \iota_B^A : B \to A$, & $\iota \in \mathrm{Alg}(\mathcal{C})$.

IF $\nexists\, u_B$ OR IF $\iota u_B \neq u_A$, B IS NONUNITAL SUBALGEBRA OF A.

III. SUBALGEBRAS AND IDEALS

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL

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$$\begin{tikzcd}
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A \otimes A \arrow[dr, "m"'] \arrow[rr, phantom, "\circlearrowleft"] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "u \otimes id"] \arrow[dr, "\ell_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[r, "id \otimes u"] \arrow[dr, "r_A"'] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

A **SUBALGEBRA** OF (A, m_A, u_A) IN $\mathcal{C}$ IS AN ALGEBRA (B, m_B, u_B) IN $\mathcal{C}$ SUCH THAT B_{obj} IS A SUBOBJECT OF A_{obj} VIA MONO $\iota := \iota_B^A : B \to A$, & $\iota \in \mathrm{Alg}(\mathcal{C})$.

IF $\nexists\, u_B$ OR IF $\iota u_B \neq u_A$, B IS **NONUNITAL SUBALGEBRA** OF A.

A **LEFT IDEAL** OF (A, m_A, u_A) CONSISTS OF A SUBOBJECT $(I, \iota := \iota_I^A : I \longrightarrow A)$ AND A MORPHISM $\lambda_I^A : A \otimes I \longrightarrow I$ SUCH THAT

$$\begin{tikzcd}
(A \otimes A) \otimes I \arrow[rr, "a_{A,A,I}"] \arrow[d, "m_A \otimes id"'] & & A \otimes (A \otimes I) \arrow[d, "id \otimes \lambda_I^A"] \\
A \otimes I \arrow[dr, "\lambda_I^A"'] \arrow[rr, phantom, "\circlearrowleft"] & & A \otimes I \arrow[dl, "\lambda_I^A"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \otimes I \arrow[rr, "u_A \otimes id"] \arrow[dr, "\ell_I"'] & \circlearrowleft & A \otimes I \arrow[dl, "\lambda_I^A"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes I \arrow[r, "id \otimes \iota_I^A"] \arrow[d, "\lambda_I^A"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes A \arrow[d, "m_A"] \\
I \arrow[r, "\iota_I^A"'] & A
\end{tikzcd}$$

III. SUBALGEBRAS AND IDEALS

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

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$u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SATISFYING:

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes id"'] & & A\otimes(A\otimes A) \arrow[d, "id\otimes m"] \\
A\otimes A \arrow[dr, "m"'] \arrow[rr, phantom, "\circlearrowright"] & & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1}\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "\ell_A"'] \arrow[r, phantom, shift right=3ex, "\circlearrowright"] & A\otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A\otimes \mathbb{1} \arrow[r, "id\otimes u"] \arrow[dr, "r_A"'] \arrow[r, phantom, shift right=3ex, "\circlearrowright"] & A\otimes A \arrow[d, "m"] \\
& A
\end{tikzcd}$$

(RIGHT UNITALITY AXIOM)

RIGHT IDEAL OF (A, m, u) $\equiv \left(I,\ \iota_I^A : I \longrightarrow A,\ \rho_I^A : I \otimes A \longrightarrow I\right)$

DEFINED LIKEWISE

A LEFT IDEAL OF (A, m_A, u_A) CONSISTS OF

A SUBOBJECT $\left(I,\ \iota := \iota_I^A : I \longrightarrow A\right)$

AND A MORPHISM $\lambda_I^A : A \otimes I \longrightarrow I$ SUCH THAT

$$\begin{tikzcd}
(A\otimes A)\otimes I \arrow[rr, "a_{A,A,I}"] \arrow[d, "m_A\otimes id"'] & & A\otimes(A\otimes I) \arrow[d, "id\otimes\lambda_I^A"] \\
A\otimes I \arrow[dr, "\lambda_I^A"'] \arrow[rr, phantom, "\circlearrowright"] & & A\otimes I \arrow[dl, "\lambda_I^A"] \\
& I &
\end{tikzcd}
\qquad
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& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes I \arrow[r, "id\otimes\iota_I^A"] \arrow[d, "\lambda_I^A"'] \arrow[dr, phantom, "\circlearrowright"] & A\otimes A \arrow[d, "m_A"] \\
I \arrow[r, "\iota_I^A"'] & A
\end{tikzcd}$$

III. SUBALGEBRAS AND IDEALS

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) MULTIP'N MORPHISM
$m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) UNIT MORPHISM
$u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SATISFYING:

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes id"'] & & A\otimes(A\otimes A) \arrow[d, "id\otimes m"] \\
A\otimes A \arrow[dr, "m"'] \arrow[rr, phantom, "\circlearrowleft"] & & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1}\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "\ell_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A\otimes \mathbb{1} \arrow[r, "id\otimes u"] \arrow[dr, "r_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

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RIGHT IDEAL OF (A, m, u) $\equiv$ $\left(I,\ \iota_I^A : I \longrightarrow A,\ \rho_I^A : I \otimes A \longrightarrow I\right)$
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IDEAL OF (A, m, u) $\equiv$ $\left(I,\ \iota_I^A : I \to A,\ \lambda_I^A : A \otimes I \to I,\ \rho_I^A : I \otimes A \to I\right)$:

$(I, \iota, \lambda) \equiv$ LEFT IDEAL &
$(I, \iota, \rho) \equiv$ RIGHT IDEAL

???

A **LEFT IDEAL** OF (A, m_A, u_A) CONSISTS OF

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A\otimes I \arrow[dr, "\lambda_I^A"'] \arrow[rr, phantom, "\circlearrowleft"] & & A\otimes I \arrow[dl, "\lambda_I^A"] \\
& I &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1}\otimes I \arrow[rr, "u_A\otimes id"] \arrow[dr, "\ell_I"'] & \circlearrowleft & A\otimes I \arrow[dl, "\lambda_I^A"] \\
& I &
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes I \arrow[r, "id\otimes \iota_I^A"] \arrow[d, "\lambda_I^A"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \arrow[d, "m_A"] \\
I \arrow[r, "\iota_I^A"'] & A
\end{tikzcd}$$

III. SUBALGEBRAS AND IDEALS

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SATISFYING:

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A\otimes A \arrow[dr, "m"'] & & A\otimes A \arrow[dl, "m"] \\
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\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

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RIGHT IDEAL OF (A, m, u) $\equiv$ $(I,\ \iota_I^A : I \longrightarrow A,\ \rho_I^A : I\otimes A \longrightarrow I)$ DEFINED LIKEWISE

IDEAL OF (A, m, u) $\equiv$ $(I,\ \iota_I^A : I \to A,\ \lambda_I^A : A\otimes I \to I,\ \rho_I^A : I\otimes A \to I)$:

$(I, \iota, \lambda) \equiv$ LEFT IDEAL

$(I, \iota, \rho) \equiv$ RIGHT IDEAL

&

$$\begin{tikzcd}
(A\otimes I)\otimes A \arrow[rr, "a_{A,I,A}"] \arrow[d, "\lambda\otimes id"'] & & A\otimes(I\otimes A) \arrow[d, "id\otimes\rho"] \\
I\otimes A \arrow[dr, "\rho"'] & & A\otimes I \arrow[dl, "\lambda"] \\
& I &
\end{tikzcd}$$

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A\otimes I \arrow[dr, "\lambda_I^A"'] & & A\otimes I \arrow[dl, "\lambda_I^A"] \\
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\qquad
\begin{tikzcd}
\mathbb{1}\otimes I \arrow[rr, "u_A\otimes id"] \arrow[dr, "\ell_I"'] & & A\otimes I \arrow[dl, "\lambda_I^A"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes I \arrow[r, "id\otimes\iota_I^A"] \arrow[d, "\lambda_I^A"'] & A\otimes A \arrow[d, "m_A"] \\
I \arrow[r, "\iota_I^A"] & A
\end{tikzcd}$$

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A\otimes A \arrow[dr, "m"'] & \circlearrowleft & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

(ASSOCIATIVITY AXIOM)

$$\begin{tikzcd}
\mathbb{1}\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "\ell_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

(LEFT UNITALITY AXIOM)

$$\begin{tikzcd}
A\otimes\mathbb{1} \arrow[r, "id\otimes u"] \arrow[dr, "r_A"'] & A\otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\
& A
\end{tikzcd}$$

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RIGHT IDEAL OF (A, m, u) $\equiv \left(I,\ \iota_I^A : I \longrightarrow A,\ \rho_I^A : I\otimes A \longrightarrow I\right)$

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&

$$\begin{tikzcd}
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I\otimes A \arrow[dr, "\rho"'] & \circlearrowleft & A\otimes I \arrow[dl, "\lambda"] \\
& I &
\end{tikzcd}$$

EXAMPLE: IDEALS OF $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$:

- A_{obj} WITH $\iota = id_A$, $\lambda = m_A$, $\rho = m_A$

III. SUBALGEBRAS AND IDEALS

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A\otimes A \arrow[dr, "m"'] & \circlearrowright & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

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\end{tikzcd}$$

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$$\begin{tikzcd}
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I\otimes A \arrow[dr, "\rho"'] & \circlearrowright & A\otimes I \arrow[dl, "\lambda"] \\
& I &
\end{tikzcd}$$

EXAMPLE: IDEALS OF $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$:

- A_{OBJ} WITH $\iota = id_A$, $\lambda = m_A$, $\rho = m_A$
- 0 (ZERO OBJ) WITH $\iota = \vec{0}_A$, $\lambda = {}_{A\otimes 0}\vec{0}$, $\rho = {}_{0\otimes A}\vec{0}$

III. SUBALGEBRAS AND IDEALS

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I\otimes A \arrow[dr, "\rho"'] & \arrow[d, phantom, "\circlearrowleft"] & A\otimes I \arrow[dl, "\lambda"] \\
& I &
\end{tikzcd}$$

EXAMPLE: IDEALS OF $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$:

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ALL OTHER IDEALS OF A $\equiv$ **PROPER IDEALS** OF A

III. SUBALGEBRAS AND IDEALS

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SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ SUBOBJ. OF } A \\ \lambda : A \otimes I \longrightarrow I \\ \rho : I \otimes A \longrightarrow I \end{cases}$$

$\ni.$

$$\begin{bmatrix} \lambda(m \otimes \mathrm{id}) = \lambda(\mathrm{id} \otimes \lambda)\, a \\ \lambda(u \otimes \mathrm{id}) = \ell \\ \lambda\iota = m(\mathrm{id} \otimes \iota) \end{bmatrix}$$

$$\begin{bmatrix} \rho(\mathrm{id} \otimes m) = \rho(\rho \otimes \mathrm{id})\, a^{-1} \\ \rho(\mathrm{id} \otimes u) = r \\ \rho\iota = m(\iota \otimes \mathrm{id}) \end{bmatrix}$$

$$[\rho\lambda = \lambda(\mathrm{id} \otimes \rho)\, a$$

COOL FACT: IDEALS ARE NONUNITAL SUBALGEBRAS.

III. SUBALGEBRAS AND IDEALS

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$\ni .$

$$\begin{bmatrix} \lambda(m \otimes id) = \lambda(id \otimes \lambda)\, a \\ \lambda(u \otimes id) = \ell \\ \lambda\iota = m(id \otimes \iota) \end{bmatrix}$$

$$\begin{bmatrix} \rho\,(id \otimes m) = \rho(\rho \otimes id)\, a^{-1} \\ \rho\,(id \otimes u) = r \\ \rho\,\iota = m(\iota \otimes id) \end{bmatrix}$$

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EXERCISE 4.13 "IF I CONTAINS 1_A, THEN $I = A$."

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∍.

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SUPPOSE I IS AN IDEAL OF AN ALGEBRA A IN $\mathcal{C}$.

IF $\exists$ MORPHISM $u_I : \mathbb{1} \to I$ IN $\mathcal{C}$ SUCH THAT

$$\begin{tikzcd} & I \arrow[d, "\iota_I^A"] \\ \mathbb{1} \arrow[ur, "u_I"] \arrow[r, "u_A"'] \arrow[ur, phantom, shift right=4, "\circlearrowleft"] & A \end{tikzcd},$$

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SAT. ASSOC. & UNIT. AXIOMS

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$$\begin{tikzcd} & I \arrow[d, "\iota_I^A"] \\ \mathbb{1} \arrow[ur, "u_I"] \arrow[r, phantom, "2"] \arrow[dr, "u_A"'] & A \\ & \end{tikzcd},$$

THEN, I IS A UNITAL SUBALGEBRA OF A WITH UNIT u_I

& $I \cong A$ AS ALGEBRAS IN $\mathcal{C}$

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$$\begin{cases} \rho(id \otimes m) = \rho(\rho \otimes id)a^{-1} \\ \rho(id \otimes u) = r \\ \rho\iota = m(\iota \otimes id) \end{cases}$$

$$[\rho\lambda = \lambda(id \otimes \rho)a$$

PROP: IF $\phi : (A, m, u) \to (A', m', u') \in \mathrm{Alg}(\mathcal{C})$, THEN THE KERNEL OF $\phi_{OBJ} : A_{OBJ} \to A'_{OBJ}$ FORMS AN IDEAL OF (A, m, u).

EXERCISE 4.13 "IF I CONTAINS 1_A, THEN $I = A$."

SUPPOSE I IS AN IDEAL OF AN ALGEBRA A IN $\mathcal{C}$.

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PROOF SKETCH:

LET $I := \ker(\phi_{\text{OBJ}})$ w/
$$\begin{tikzcd} I \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & A \arrow[r, "\phi"] & A' \end{tikzcd}.$$

III. SUBALGEBRAS AND IDEALS

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \to A \in \mathcal{C}$

SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ SUBOBJ. OF } A \\ \lambda : A \otimes I \to I \\ \rho : I \otimes A \to I \end{cases}$$

$\ni.$

$$\begin{cases} \lambda(m \otimes id) = \lambda(id \otimes \lambda) a \\ \lambda(u \otimes id) = \ell \\ \lambda \iota = m(id \otimes \iota) \end{cases}$$

$$\begin{cases} \rho(id \otimes m) = \rho(\rho \otimes id) a^{-1} \\ \rho(id \otimes u) = r \\ \rho \iota = m(\iota \otimes id) \end{cases}$$

$[\rho \lambda = \lambda(id \otimes \rho) a$

PROP: IF $\phi : (A, m, u) \to (A', m', u') \in \mathsf{Alg}(\mathcal{C})$, THEN THE KERNEL OF $\phi_{OBJ} : A_{OBJ} \to A'_{OBJ}$ FORMS AN IDEAL OF (A, m, u).

PROOF SKETCH:

LET $I := \ker(\phi_{OBJ})$ w/
$$\begin{tikzcd} I \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & A \arrow[r, "\phi"] & A \end{tikzcd}.$$

TAKE $\iota := \alpha'$.

GET λ AS FOLLOWS:

$A \otimes I$

$$\begin{tikzcd} I \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & A \arrow[r, "\phi"] & A' \end{tikzcd}$$

III. SUBALGEBRAS AND IDEALS

$\boxed{\text{ASSUME } \mathcal{C} \text{ IS ABELIAN MONOIDAL}}$

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \to A \in \mathcal{C}$

SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) & \text{SUBOBJ. OF } A \\ \lambda : A \otimes I \to I & \\ \rho : I \otimes A \to I & \end{cases}$$

$\ni$

$$\begin{cases} \lambda(m \otimes id) = \lambda(id \otimes \lambda)\, a \\ \lambda(u \otimes id) = \ell \\ \lambda \iota = m(id \otimes \iota) \end{cases}$$

$$\begin{cases} \rho\,(id \otimes m) = \rho(\rho \otimes id)\, a^{-1} \\ \rho\,(id \otimes u) = r \\ \rho\, \iota = m(\iota \otimes id) \end{cases}$$

$[\rho \lambda = \lambda(id \otimes \rho)\, a$

PROP: IF $\phi : (A, m, u) \to (A', m', u') \in \mathrm{Alg}(\mathcal{C})$, THEN THE KERNEL OF $\phi_{\mathrm{OBJ}} : A_{\mathrm{OBJ}} \to A'_{\mathrm{OBJ}}$ FORMS AN IDEAL OF (A, m, u).

PROOF SKETCH:

LET $I := \ker(\phi_{\mathrm{OBJ}})$ w/

$$\begin{tikzcd} I \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & A \arrow[r, "\phi"] & A \end{tikzcd}.$$

TAKE $\iota := \alpha'$.

GET λ AS FOLLOWS:

$[\phi \in \mathrm{Alg}(\mathcal{C})]$

$A \otimes I$

$$\begin{tikzcd} & A \otimes A \arrow[r, "\phi \otimes \phi"] \arrow[d, "m"] & A' \otimes A' \arrow[d, "m'"] \\ I \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & A \arrow[r, "\phi"] & A' \end{tikzcd}$$

III. SUBALGEBRAS AND IDEALS

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \to A \in \mathcal{C}$

SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) & \text{SUBOBJ. OF } A \\ \lambda : A \otimes I \to I \\ \rho : I \otimes A \to I \end{cases}$$

$\ni$

$$\begin{cases} \lambda(m \otimes id) = \lambda(id \otimes \lambda)\, a \\ \lambda(u \otimes id) = \ell \\ \lambda \iota = m(id \otimes \iota) \end{cases}$$

$$\begin{cases} \rho\,(id \otimes m) = \rho(\rho \otimes id)\, a^{-1} \\ \rho\,(id \otimes u) = r \\ \rho\, \iota = m(\iota \otimes id) \end{cases}$$

$[\rho \lambda = \lambda(id \otimes \rho)\, a$

PROP: IF $\phi : (A, m, u) \to (A', m', u') \in \mathrm{Alg}(\mathcal{C})$, THEN THE KERNEL OF $\phi_{\mathrm{OBJ}} : A_{\mathrm{OBJ}} \to A'_{\mathrm{OBJ}}$ FORMS AN IDEAL OF (A, m, u).

PROOF SKETCH:

LET $I := \ker(\phi_{\mathrm{OBJ}})$ w/

$$\begin{tikzcd} I \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & A \arrow[r, "\phi"] & A' \end{tikzcd} .$$

TAKE $\iota := \alpha'$.

GET λ AS FOLLOWS:

$$\begin{tikzcd}
A \otimes I \arrow[dr, "id \otimes \alpha'"'] \arrow[drr, bend left, "\phi \otimes \vec{0}"] & & & \\
& A \otimes A \arrow[r, "\phi \otimes \phi"] \arrow[d, "m"] & A' \otimes A' \arrow[dr, "m'"] & \\
I \arrow[r, "\alpha'"] \arrow[rrr, bend right, "\vec{0}"'] & A \arrow[rr, "\phi"] & & A'
\end{tikzcd}$$

$[\phi \in \mathrm{Alg}(\mathcal{C})]$

III. Subalgebras and Ideals

Assume $\mathcal{C}$ is abelian monoidal

Algebra in $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) object $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \to A \in \mathcal{C}$

sat. assoc. & unit. axioms

Ideal of (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ subobj. of } A \\ \lambda : A \otimes I \to I \\ \rho : I \otimes A \to I \end{cases}$$

$\ni$.

$$\begin{cases} \lambda(m \otimes id) = \lambda(id \otimes \lambda) a \\ \lambda(u \otimes id) = \ell \\ \lambda \iota = m(id \otimes \iota) \end{cases}$$

$$\begin{cases} \rho(id \otimes m) = \rho(\rho \otimes id) a^{-1} \\ \rho(id \otimes u) = r \\ \rho \iota = m(\iota \otimes id) \end{cases}$$

$[\rho \lambda = \lambda(id \otimes \rho) a$

Prop: If $\phi : (A, m, u) \to (A', m', u') \in \mathrm{Alg}(\mathcal{C})$, then the kernel of $\phi_{obj} : A_{obj} \to A'_{obj}$ forms an ideal of (A, m, u).

Proof sketch:

Let $I := \ker(\phi_{obj})$ w/

$$\begin{tikzcd} I \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & A \arrow[r, "\phi"] \arrow[d, phantom, "\circlearrowleft"] & A \\ & {} & \end{tikzcd}.$$

Take $\iota := \alpha'$.

Get λ as follows:

$$\begin{tikzcd} A \otimes I \arrow[dr, "id \otimes \alpha'"] \arrow[drr, bend left, "\phi \otimes \vec{0}"] \arrow[ddrr, bend left=60, "\vec{0}"] \arrow[dd, dashed, "\exists ! \lambda"'] & & \\ & A \otimes A \arrow[r, "\phi \otimes \phi"] \arrow[d, "m"] & A' \otimes A' \arrow[d, "m'"] \\ I \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & A \arrow[r, "\phi"] & A' \end{tikzcd}$$

$[\phi \in \mathrm{Alg}(\mathcal{C})]$

[By univ. prop. of kernels]

III. SUBALGEBRAS AND IDEALS

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \to A \in \mathcal{C}$

SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ SUBOBJ. OF } A \\ \lambda : A \otimes I \to I \\ \rho : I \otimes A \to I \end{cases}$$

$\ni$

$$\left[\begin{array}{l} \lambda(m \otimes id) = \lambda(id \otimes \lambda)\, a \\ \lambda(u \otimes id) = l \\ \lambda\iota = m(id \otimes \iota) \end{array}\right.$$

$$\left[\begin{array}{l} \rho(id \otimes m) = \rho(\rho \otimes id)\, a^{-1} \\ \rho(id \otimes u) = r \\ \rho\iota = m(\iota \otimes id) \end{array}\right.$$

$[\rho\lambda = \lambda(id \otimes \rho)\, a$

PROP: IF $\phi : (A, m, u) \to (A', m', u') \in \mathrm{Alg}(\mathcal{C})$, THEN THE KERNEL OF $\phi_{obj} : A_{obj} \to A'_{obj}$ FORMS AN IDEAL OF (A, m, u).

PROOF SKETCH:

LET $I := \ker(\phi_{obj})$ w/

$$\begin{tikzcd} I \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & A \arrow[r, "\phi"] & A' \end{tikzcd}.$$

TAKE $\iota := \alpha'$.

GET λ AS FOLLOWS:

$$\begin{tikzcd} A \otimes I \arrow[dr, "id \otimes \alpha'"] \arrow[drr, bend left, "\phi \otimes \vec{0}"] \arrow[ddrrr, bend left=50, "\vec{0}"] \arrow[dd, dashed, "\exists! \lambda"'] & & & \\ & A \otimes A \arrow[r, "\phi \otimes \phi"] \arrow[d, "m"] & A' \otimes A' \arrow[drr, "m'"] & \\ I \arrow[r, "\alpha'"] \arrow[rrrr, bend right, "\vec{0}"'] & A \arrow[rrr, "\phi"] & & & A' \end{tikzcd}$$

LIKEWISE, GET ρ

[BY UNIV. PROP. OF KERNELS]

THEN CHECK $(I, \iota, \lambda, \rho)$ IS INDEED AN IDEAL OF A... ///

III. SUBALGEBRAS AND IDEALS

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$
(a) OBJECT $A \in \mathcal{C}$
(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$
(c) $u := u_A : \mathbb{1} \to A \in \mathcal{C}$
SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ SUBOBJ. OF } A \\ \lambda : A \otimes I \to I \\ \rho : I \otimes A \to I \end{cases}$$

$\ni$.

$$\begin{bmatrix} \lambda(m \otimes \mathrm{id}) = \lambda(\mathrm{id} \otimes \lambda)\, a \\ \lambda(u \otimes \mathrm{id}) = \ell \\ \lambda\iota = m(\mathrm{id} \otimes \iota) \end{bmatrix}$$

$$\begin{bmatrix} \rho(\mathrm{id} \otimes m) = \rho(\rho \otimes \mathrm{id})\, a^{-1} \\ \rho(\mathrm{id} \otimes u) = r \\ \rho\iota = m(\iota \otimes \mathrm{id}) \end{bmatrix}$$

$[\rho\lambda = \lambda(\mathrm{id} \otimes \rho)\, a$

PROP: IF $\phi : (A, m, u) \to (A', m', u') \in \mathrm{Alg}(\mathcal{C})$, THEN THE KERNEL OF $\phi_{\mathrm{OBJ}} : A_{\mathrm{OBJ}} \to A'_{\mathrm{OBJ}}$ FORMS AN IDEAL OF (A, m, u).

PROOF SKETCH:

LET $I := \ker(\phi_{\mathrm{OBJ}})$ w/

$$\begin{tikzcd} I \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & A \arrow[r, "\phi"] \arrow[d, phantom, "\circlearrowright"] & A' \\ & {} & \end{tikzcd}.$$

TAKE $\iota := \alpha'$.

GET λ AS FOLLOWS:

$$\begin{tikzcd} A\otimes I \arrow[dr, "\mathrm{id}\otimes\alpha'"] \arrow[drr, bend left, "\phi\otimes\vec{0}"] \arrow[ddrrr, bend left=40, "\vec{0}"] \arrow[dd, dashed, "\exists!\lambda"'] & & & \\ & A\otimes A \arrow[r, "\phi\otimes\phi"] \arrow[d, "\mu"] & A'\otimes A' \arrow[dr, "\mu'"] & \\ I \arrow[r, "\alpha'"] \arrow[rrr, bend right, "\vec{0}"'] & A \arrow[rr, "\phi"] & & A' \end{tikzcd}$$

LIKEWISE, GET ρ

[BY UNIV. PROP. OF KERNELS]

THEN CHECK $(I, \iota, \lambda, \rho)$ IS INDEED AN IDEAL OF A... ///

III. SUBALGEBRAS AND IDEALS

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \to A \in \mathcal{C}$

SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \left\{ \begin{array}{l} (I, \iota) \text{ SUBOBJ. OF } A \\ \lambda : A \otimes I \to I \\ \rho : I \otimes A \to I \end{array} \right.$$

$\ni.$

$$\left[\begin{array}{l} \lambda(m \otimes id) = \lambda(id \otimes \lambda)\, a \\ \lambda(u \otimes id) = \ell \\ \lambda\iota = m(id \otimes \iota) \end{array} \right.$$

$$\left[\begin{array}{l} \rho(id \otimes m) = \rho(\rho \otimes id)\, a^{-1} \\ \rho(id \otimes u) = r \\ \rho\iota = m(\iota \otimes id) \end{array} \right.$$

$$[\rho\lambda = \lambda(id \otimes \rho)\, a$$

PROP: IF $\phi : (A, m, u) \to (A', m', u') \in \mathrm{Alg}(\mathcal{C})$, THEN THE KERNEL OF $\phi_{OBJ} : A_{OBJ} \to A'_{OBJ}$ FORMS AN IDEAL OF (A, m, u).

DETAILS = EXERCISE 4.14

PROOF SKETCH:

LET $I := \ker(\phi_{OBJ})$ w/

$$\begin{tikzcd} I \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & A \arrow[d, phantom, "\circlearrowleft"] \arrow[r, "\phi"] & A' \\ & {} & \end{tikzcd}.$$

TAKE $\iota := \alpha'$.

GET λ AS FOLLOWS:

$$\begin{tikzcd}[column sep=large] A \otimes I \arrow[dd, dashed, "\exists ! \lambda"'] \arrow[dr, "id \otimes \alpha'"] \arrow[drr, bend left, "\phi \otimes \vec{0}"] \arrow[ddrrr, bend left=40, "\vec{0}"] & & & \\ & A \otimes A \arrow[r, "\phi \otimes \phi"] \arrow[dr, "m"] & A' \otimes A' \arrow[dr, "m'"] & \\ I \arrow[rr, "\alpha'"] \arrow[rrr, bend right, "\vec{0}"'] & & A \arrow[r, "\phi"] & A' \end{tikzcd}$$

LIKEWISE, GET ρ

[BY UNIV. PROP. OF KERNELS]

THEN CHECK $(I, \iota, \lambda, \rho)$ IS INDEED AN IDEAL OF A... ///

IV. QUOTIENT ALGEBRAS

BUILT WITH COKERNELS

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ SUBOBJ. OF } A \\ \lambda : A \otimes I \longrightarrow I \\ \rho : I \otimes A \longrightarrow I \end{cases}$$

$\ni$.

$$\begin{cases} \lambda(m \otimes id) = \lambda(id \otimes \lambda)\, a \\ \lambda(u \otimes id) = \ell \\ \lambda \iota = m(id \otimes \iota) \end{cases}$$

$$\begin{cases} \rho(id \otimes m) = \rho(\rho \otimes id)\, a^{-1} \\ \rho(id \otimes u) = r \\ \rho \iota = m(\iota \otimes id) \end{cases}$$

$$\left[\rho \lambda = \lambda(id \otimes \rho)\, a \right.$$

IV. QUOTIENT ALGEBRAS

BUILT WITH COKERNELS

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL
& $(X\otimes -), (-\otimes X)$ ARE RIGHT EXACT $\forall X$.

AS RIGHT EXACT FUNCTORS PRESERVE COKERNELS

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$
(a) OBJECT $A \in \mathcal{C}$
(b) $m := m_A : A\otimes A \longrightarrow A \in \mathcal{C}$
(c) $u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$
SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ SUBOBJ. OF } A \\ \lambda : A\otimes I \longrightarrow I \\ \rho : I\otimes A \longrightarrow I \end{cases}$$

$\ni$.

$$\begin{cases} \lambda(m\otimes id) = \lambda(id\otimes\lambda)\,a \\ \lambda(u\otimes id) = l \\ \lambda\iota = m(id\otimes\iota) \end{cases}$$

$$\begin{cases} \rho\,(id\otimes m) = \rho(\rho\otimes id)\,a^{-1} \\ \rho\,(id\otimes u) = r \\ \rho\,\iota = m(\iota\otimes id) \end{cases}$$

$$\left[\rho\lambda = \lambda(id\otimes\rho)\,a\right.$$

IV. QUOTIENT ALGEBRAS

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ SUBOBJ. OF } A \\ \lambda : A \otimes I \longrightarrow I \\ \rho : I \otimes A \longrightarrow I \end{cases}$$

$\ni.$

$$\left[\begin{array}{l} \lambda(m \otimes id) = \lambda(id \otimes \lambda)\, a \\ \lambda(u \otimes id) = \ell \\ \lambda\iota = m(id \otimes \iota) \end{array}\right.$$

$$\left[\begin{array}{l} \rho\,(id \otimes m) = \rho(\rho \otimes id)\, a^{-1} \\ \rho\,(id \otimes u) = r \\ \rho\,\iota = m(\iota \otimes id) \end{array}\right.$$

$$[\rho\lambda = \lambda(id \otimes \rho)\, a$$

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL & $(X \otimes -), (- \otimes X)$ ARE RIGHT EXACT $\forall X$.

PROP: TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$ WITH IDEAL $(I, \iota, \lambda, \rho)$.

THEN THE COKERNEL OF $\iota : I \longrightarrow A$:

$$\begin{tikzcd} I \arrow[r, "\iota"] \arrow[rr, bend left, "\vec{0}"] & A \arrow[r, "\pi"] & \mathrm{coker}(\iota) =: A/I \end{tikzcd}$$

FORMS AN ALGEBRA IN $\mathcal{C}$ SUCH THAT $\pi \in \mathrm{Alg}(\mathcal{C})$.

IV. Quotient algebras

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL & $(X\otimes -), (-\otimes X)$ ARE RIGHT EXACT $\forall X$.

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A\otimes A \longrightarrow A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ SUBOBJ. OF } A \\ \lambda : A\otimes I \longrightarrow I \\ \rho : I\otimes A \longrightarrow I \end{cases}$$

$\ni.$

$$\begin{cases} \lambda(m\otimes id) = \lambda(id\otimes\lambda)\, a \\ \lambda(u\otimes id) = \ell \\ \lambda\iota = m(id\otimes\iota) \end{cases}$$

$$\begin{cases} \rho(id\otimes m) = \rho(\rho\otimes id)\, a^{-1} \\ \rho(id\otimes u) = r \\ \rho\iota = m(\iota\otimes id) \end{cases}$$

$$[\rho\lambda = \lambda(id\otimes\rho)\, a$$

PROP: TAKE $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$ WITH IDEAL $(I, \iota, \lambda, \rho)$. THEN THE COKERNEL OF $\iota : I \longrightarrow A$:

$$\begin{tikzcd} I \arrow[r, "\iota"] \arrow[rr, bend left, "\vec{0}"] & A \arrow[r, "\pi"] & \mathrm{coker}(\iota) =: A/I \end{tikzcd}$$

FORMS AN ALGEBRA IN $\mathcal{C}$ SUCH THAT $\pi \in \mathsf{Alg}(\mathcal{C})$.

PROOF SKETCH: $\exists$ MORPHISM $\iota \sqcup \iota$ $\ni.$ $A/I \otimes A/I \cong \mathrm{coker}(\iota\sqcup\iota)$.

IV. QUOTIENT ALGEBRAS

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \to A \in \mathcal{C}$

SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ SUBOBJ. OF } A \\ \lambda : A \otimes I \to I \\ \rho : I \otimes A \to I \end{cases}$$

s.t.

$$\left[\begin{array}{l} \lambda(m \otimes id) = \lambda(id \otimes \lambda)\, a \\ \lambda(u \otimes id) = l \\ \lambda \iota = m(id \otimes \iota) \end{array}\right.$$

$$\left[\begin{array}{l} \rho(id \otimes m) = \rho(\rho \otimes id)\, a^{-1} \\ \rho(id \otimes u) = r \\ \rho \iota = m(\iota \otimes id) \end{array}\right.$$

$$[\rho \lambda = \lambda(id \otimes \rho)\, a$$

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL & $(X \otimes -), (- \otimes X)$ ARE RIGHT EXACT $\forall X$.

PROP: TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$ WITH IDEAL $(I, \iota, \lambda, \rho)$.
THEN THE COKERNEL OF $\iota : I \to A$:

$$\begin{tikzcd} I \arrow[r, "\iota"] \arrow[rr, bend left, "0"] & A \arrow[r, "\pi"] & \mathrm{coker}(\iota) =: A/I \end{tikzcd}$$

FORMS AN ALGEBRA IN $\mathcal{C}$ SUCH THAT $\pi \in \mathrm{Alg}(\mathcal{C})$.

PROOF SKETCH: $\exists$ MORPHISM $\iota \sqcup \iota$ s.t. $A/I \otimes A/I \cong \mathrm{coker}(\iota \sqcup \iota)$.

HERE

$$\begin{tikzcd} (A \otimes I) \sqcup_{I \otimes I} (I \otimes A) \arrow[r, "\iota \sqcup \iota"] \arrow[rr, bend left, "0"] & A \otimes A \arrow[r, "\alpha"] & \mathrm{coker}(\iota \sqcup \iota) \end{tikzcd}$$

IV. QUOTIENT ALGEBRAS

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \to A \in \mathcal{C}$

SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ SUBOBJ. OF } A \\ \lambda : A \otimes I \to I \\ \rho : I \otimes A \to I \end{cases}$$

∋.

$$\begin{cases} \lambda(m \otimes id) = \lambda(id \otimes \lambda)\, a \\ \lambda(u \otimes id) = \ell \\ \lambda\iota = m(id \otimes \iota) \end{cases}$$

$$\begin{cases} \rho\,(id \otimes m) = \rho(\rho \otimes id)\, a^{-1} \\ \rho\,(id \otimes u) = r \\ \rho\,\iota = m(\iota \otimes id) \end{cases}$$

$$[\rho\lambda = \lambda(id \otimes \rho)\, a$$

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL & $(X \otimes -), (- \otimes X)$ ARE RIGHT EXACT $\forall X$.

PROP: TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$ WITH IDEAL $(I, \iota, \lambda, \rho)$.

THEN THE COKERNEL OF $\iota : I \to A$:

$$\begin{tikzcd} I \arrow[r, "\iota"] \arrow[rr, bend left, "\vec{0}"] & A \arrow[r, "\pi"] \arrow[l, phantom, "\circlearrowleft"] & \mathrm{coker}(\iota) =: A/I \end{tikzcd}$$

FORMS AN ALGEBRA IN $\mathcal{C}$ SUCH THAT $\pi \in \mathrm{Alg}(\mathcal{C})$.

PROOF SKETCH: $\exists$ MORPHISM $\iota \sqcup \iota$ ∋. $A/I \otimes A/I \cong \mathrm{coker}(\iota \sqcup \iota)$.

HERE

$$\begin{tikzcd} (A \otimes I) \sqcup_{I \otimes I} (I \otimes A) \arrow[r, "\iota \sqcup \iota"] \arrow[rr, bend left, "\vec{0}"] & A \otimes A \arrow[r, "\alpha"', "\pi \otimes \pi \,=\, \alpha"] & \mathrm{coker}(\iota \sqcup \iota) \end{tikzcd}$$

IV. QUOTIENT ALGEBRAS

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ SUBOBJ. OF } A \\ \lambda : A \otimes I \longrightarrow I \\ \rho : I \otimes A \longrightarrow I \end{cases}$$

∋.

$$\begin{cases} \lambda(m \otimes id) = \lambda(id \otimes \lambda)\, a \\ \lambda(u \otimes id) = \ell \\ \lambda\iota = m(id \otimes \iota) \end{cases}$$

$$\begin{cases} \rho\,(id \otimes m) = \rho(\rho \otimes id)\, a^{-1} \\ \rho\,(id \otimes u) = r \\ \rho\,\iota = m(\iota \otimes id) \end{cases}$$

$$[\rho\lambda = \lambda(id \otimes \rho)\, a$$

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL & $(X \otimes -), (- \otimes X)$ ARE RIGHT EXACT $\forall X$.

PROP: TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$ WITH IDEAL $(I, \iota, \lambda, \rho)$. THEN THE COKERNEL OF $\iota : I \longrightarrow A$:

$$\begin{tikzcd}
I \arrow[r, "\iota"] \arrow[rr, bend left, "\vec{0}"] & A \arrow[r, "\pi"] & \mathrm{coker}(\iota) =: A/I
\end{tikzcd}$$

2

FORMS AN ALGEBRA IN $\mathcal{C}$ SUCH THAT $\pi \in \mathrm{Alg}(\mathcal{C})$.

PROOF SKETCH: $\exists$ MORPHISM $\iota \sqcup \iota$ ∋. $A/I \otimes A/I \cong \mathrm{coker}(\iota \sqcup \iota)$.

HERE

$$\begin{tikzcd}
(A \otimes I) \sqcup_{I \otimes I} (I \otimes A) \arrow[r, "\iota \sqcup \iota"] \arrow[rr, bend left, "\vec{0}"] \arrow[drrr, bend right, "\vec{0}"'] & A \otimes A \arrow[r, "\pi \otimes \pi = \alpha"] \arrow[dr, "m"] & \mathrm{coker}(\iota \sqcup \iota) \\
& 2 & A \arrow[r, "\pi"'] & A/I
\end{tikzcd}$$

IV. Quotient algebras

Algebra in $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) object $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

sat. assoc. & unit. axioms

Ideal of (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ subobj. of } A \\ \lambda : A \otimes I \longrightarrow I \\ \rho : I \otimes A \longrightarrow I \end{cases}$$

s.t.

$$\begin{cases} \lambda(m \otimes id) = \lambda(id \otimes \lambda)\, a \\ \lambda(u \otimes id) = \ell \\ \lambda\iota = m(id \otimes \iota) \end{cases}$$

$$\begin{cases} \rho\,(id \otimes m) = \rho(\rho \otimes id)\, a^{-1} \\ \rho\,(id \otimes u) = r \\ \rho\,\iota = m(\iota \otimes id) \end{cases}$$

$$[\rho\lambda = \lambda(id \otimes \rho)\, a$$

Assume $\mathcal{C}$ is abelian monoidal & $(X \otimes -), (- \otimes X)$ are right exact $\forall X$.

Prop: Take $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$ with ideal $(I, \iota, \lambda, \rho)$.

Then the cokernel of $\iota : I \longrightarrow A$:

$$\begin{tikzcd} I \arrow[r, "\iota"] \arrow[rr, bend left, "\vec{0}"] & A \arrow[u, phantom, "\circlearrowleft"] \arrow[r, "\pi"] & \mathrm{coker}(\iota) =: A/I \end{tikzcd}$$

forms an algebra in $\mathcal{C}$ such that $\pi \in \mathrm{Alg}(\mathcal{C})$.

Proof sketch: $\exists$ morphism $\iota \sqcup \iota$ s.t. $A/I \otimes A/I \cong \mathrm{coker}(\iota \sqcup \iota)$.

Here

$$\begin{tikzcd} (A \otimes I) \sqcup_{I \otimes I} (I \otimes A) \arrow[r, "\iota \sqcup \iota"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & A \otimes A \arrow[r, "\pi \otimes \pi = \alpha"] \arrow[d, "m"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{coker}(\iota \sqcup \iota) \arrow[d, dashed, "\exists!\, \overline{m}"] \\ & A \arrow[r, "\pi"'] & A/I \end{tikzcd}$$

[By univ. prop of cokernels]

IV. QUOTIENT ALGEBRAS

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \to A \in \mathcal{C}$

SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ SUBOBJ. OF } A \\ \lambda : A \otimes I \to I \\ \rho : I \otimes A \to I \end{cases}$$

s.t.

$$\left[\begin{array}{l} \lambda(m \otimes id) = \lambda(id \otimes \lambda)a \\ \lambda(u \otimes id) = \ell \\ \lambda\iota = m(id \otimes \iota) \end{array}\right.$$

$$\left[\begin{array}{l} \rho(id \otimes m) = \rho(\rho \otimes id)a^{-1} \\ \rho(id \otimes u) = r \\ \rho\iota = m(\iota \otimes id) \end{array}\right.$$

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ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL & $(X \otimes -), (- \otimes X)$ ARE RIGHT EXACT $\forall X$.

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$$\begin{tikzcd} I \arrow[r, "\iota"] \arrow[rr, bend left, "0"] & A \arrow[r, "\pi"] & \mathrm{coker}(\iota) =: A/I \end{tikzcd}$$

FORMS AN ALGEBRA IN $\mathcal{C}$ SUCH THAT $\pi \in \mathrm{Alg}(\mathcal{C})$.

PROOF SKETCH: $\exists$ MORPHISM $\iota \sqcup \iota$ s.t. $A/I \otimes A/I \cong \mathrm{coker}(\iota \sqcup \iota)$.

HERE

$$\begin{tikzcd} (A \otimes I) \sqcup_{I \otimes I} (I \otimes A) \arrow[r, "\iota \sqcup \iota"] \arrow[rr, bend left, "0"] \arrow[rrd, bend right, "0"'] & A \otimes A \arrow[r, "\pi \otimes \pi"] \arrow[d, "m"] & \mathrm{coker}(\iota \sqcup \iota) \arrow[d, dashed, "\exists! \overline{m}"] \\ & A \arrow[r, "\pi"] & A/I \end{tikzcd}$$

[BY UNIV. PROP OF COKERNELS]

MOREOVER, TAKE $\overline{u} : \mathbb{1} \xrightarrow{u_A} A \xrightarrow{\pi} A/I$.

IV. Quotient algebras

Algebra in $(\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$

(a) object $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

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Sat. assoc. & unit. axioms

Ideal of (A, m, u)

$$\equiv \begin{cases} (I, \iota) \text{ subobj. of } A \\ \lambda : A \otimes I \to I \\ \rho : I \otimes A \to I \end{cases}$$

s.t.

$$\left[\begin{array}{l} \lambda(m \otimes id) = \lambda(id \otimes \lambda)\, a \\ \lambda(u \otimes id) = \ell \\ \lambda\iota = m(id \otimes \iota) \end{array}\right.$$

$$\left[\begin{array}{l} \rho\,(id \otimes m) = \rho(\rho \otimes id)\, a^{-1} \\ \rho\,(id \otimes u) = r \\ \rho\,\iota = m(\iota \otimes id) \end{array}\right.$$

$$[\rho\lambda = \lambda(id \otimes \rho)\, a$$

Assume $\mathcal{C}$ is abelian monoidal & $(X \otimes -), (- \otimes X)$ are right exact $\forall X$.

Prop: Take $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$ with ideal $(I, \iota, \lambda, \rho)$.

Then the cokernel of $\iota : I \to A$:

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forms an algebra in $\mathcal{C}$ such that $\pi \in \mathsf{Alg}(\mathcal{C})$.

Proof sketch: $\exists$ morphism $\iota \sqcup \iota$ s.t. $A/I \otimes A/I \cong \mathrm{coker}(\iota \sqcup \iota)$.

Here

$$\begin{tikzcd}
(A \otimes I) \sqcup_{I \otimes I} (I \otimes A) \arrow[r, "\iota \sqcup \iota"] \arrow[rrr, bend left, "0"] \arrow[rrrd, bend right, "0"'] & A \otimes A \arrow[rr, "\substack{\pi \otimes \pi \\ \shortparallel \\ \alpha}"] \arrow[rd, "m"] & & \mathrm{coker}(\iota \sqcup \iota) \arrow[d, dashed, "\exists!\, \overline{m}"] \\
& & A \arrow[r, "\pi"] & A/I
\end{tikzcd}$$

[By univ. prop of cokernels]

Moreover, take $\overline{u} : \mathbb{1} \xrightarrow{u_A} A \xrightarrow{\pi} A/I$.

Check $(A/I, \overline{m}, \overline{u}) \in \mathsf{Alg}(\mathcal{C})$. ///

IV. QUOTIENT ALGEBRAS

FULL DETAILS IN §4.2.2

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL & $(X\otimes -), (-\otimes X)$ ARE RIGHT EXACT $\forall X$.

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A\otimes A \to A \in \mathcal{C}$

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SAT. ASSOC. & UNIT. AXIOMS

IDEAL OF (A, m, u)

$$\equiv \begin{cases} (I,\iota) \text{ SUBOBJ. OF } A \\ \lambda : A\otimes I \to I \\ \rho : I \otimes A \to I \end{cases}$$

s.t.

$$\begin{cases} \lambda(m\otimes id) = \lambda(id \otimes \lambda)a \\ \lambda(u\otimes id) = l \\ \lambda\iota = m(id\otimes\iota) \end{cases}$$

$$\begin{cases} \rho(id\otimes m) = \rho(\rho\otimes id)a^{-1} \\ \rho(id\otimes u) = r \\ \rho\iota = m(\iota\otimes id) \end{cases}$$

$[\rho\lambda = \lambda(id\otimes\rho)a$

PROP: TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$ WITH IDEAL $(I, \iota, \lambda, \rho)$.

THEN THE COKERNEL OF $\iota : I \to A$:

$$\begin{tikzcd} I \arrow[r, "\iota"] \arrow[rr, bend left, "0"] & A \arrow[r, "\pi"] & \mathrm{coker}(\iota) =: A/I \end{tikzcd}$$

FORMS AN ALGEBRA IN $\mathcal{C}$ SUCH THAT $\pi \in \mathrm{Alg}(\mathcal{C})$.

PROOF SKETCH: $\exists$ MORPHISM $\iota\sqcup\iota$ s.t. $A/I \otimes A/I \cong \mathrm{coker}(\iota\sqcup\iota)$.

HERE

$$\begin{tikzcd} (A\otimes I)\sqcup_{I\otimes I}(I\otimes A) \arrow[r, "\iota\sqcup\iota"] \arrow[rr, bend left, "0"] \arrow[drr, bend right, "0"'] & A\otimes A \arrow[r, "\pi\otimes\pi"] \arrow[d, "m"] & \mathrm{coker}(\iota\sqcup\iota) \arrow[d, dashed, "\exists!\,\overline{m}"] \\ & A \arrow[r, "\pi"] & A/I \end{tikzcd}$$

[BY UNIV. PROP OF COKERNELS]

MOREOVER, TAKE $\overline{u} : \mathbb{1} \xrightarrow{u_A} A \xrightarrow{\pi} A/I$.

CHECK $(A/I, \overline{m}, \overline{u}) \in \mathrm{Alg}(\mathcal{C})$. ///

IV. QUOTIENT ALGEBRAS

FULL DETAILS IN §4.2.2

ASSUME $\mathcal{C}$ IS ABELIAN MONOIDAL & $(X\otimes -), (-\otimes X)$ ARE RIGHT EXACT $\forall X$.

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∍.

$$\left[\begin{array}{l} \lambda(m\otimes id) = \lambda(id\otimes\lambda)a \\ \lambda(u\otimes id) = l \\ \lambda\iota = m(id\otimes\iota) \end{array} \right.$$

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FORMS AN ALGEBRA IN $\mathcal{C}$ SUCH THAT $\pi \in \mathrm{Alg}(\mathcal{C})$.

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 & A \arrow[r, "\pi"'] & A/I
\end{tikzcd}$$

[By UNIV. PROP OF COKERNELS]

MOREOVER, TAKE $\overline{u} : \mathbb{1} \xrightarrow{u_A} A \xrightarrow{\pi} A/I$.

COULD DO THIS ON THE BOARD IF YOU'D LIKE?

CHECK $(A/I, \overline{m}, \overline{u}) \in \mathrm{Alg}(\mathcal{C})$. ///

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #18

NEXT TIME

- (BI)MODULES OVER $A \in \mathrm{Alg}(\mathcal{C})$
- MONADS & THEIR MODULES

TOPICS:

I. ALGEBRAS IN MONOIDAL CATEGORIES (§4.1.1)

II. DOCTRINAL ADJUNCTION & COINDUCED ALGEBRAS (§4.3.1)

III. SUBALGEBRAS AND IDEALS (§4.2.1)

IV. QUOTIENT ALGEBRAS (§4.2.2)

Lecture #18 keywords: algebra in a monoidal category, coinduced algebra, Doctrinal Adjunction, ideal in a monoidal category, monad, quotient algebra in a monoidal category, subalgebra in a monoidal category