MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #19

LAST TIME

ALGEBRAS IN MONOIDAL CATEGORIES

TOPICS:

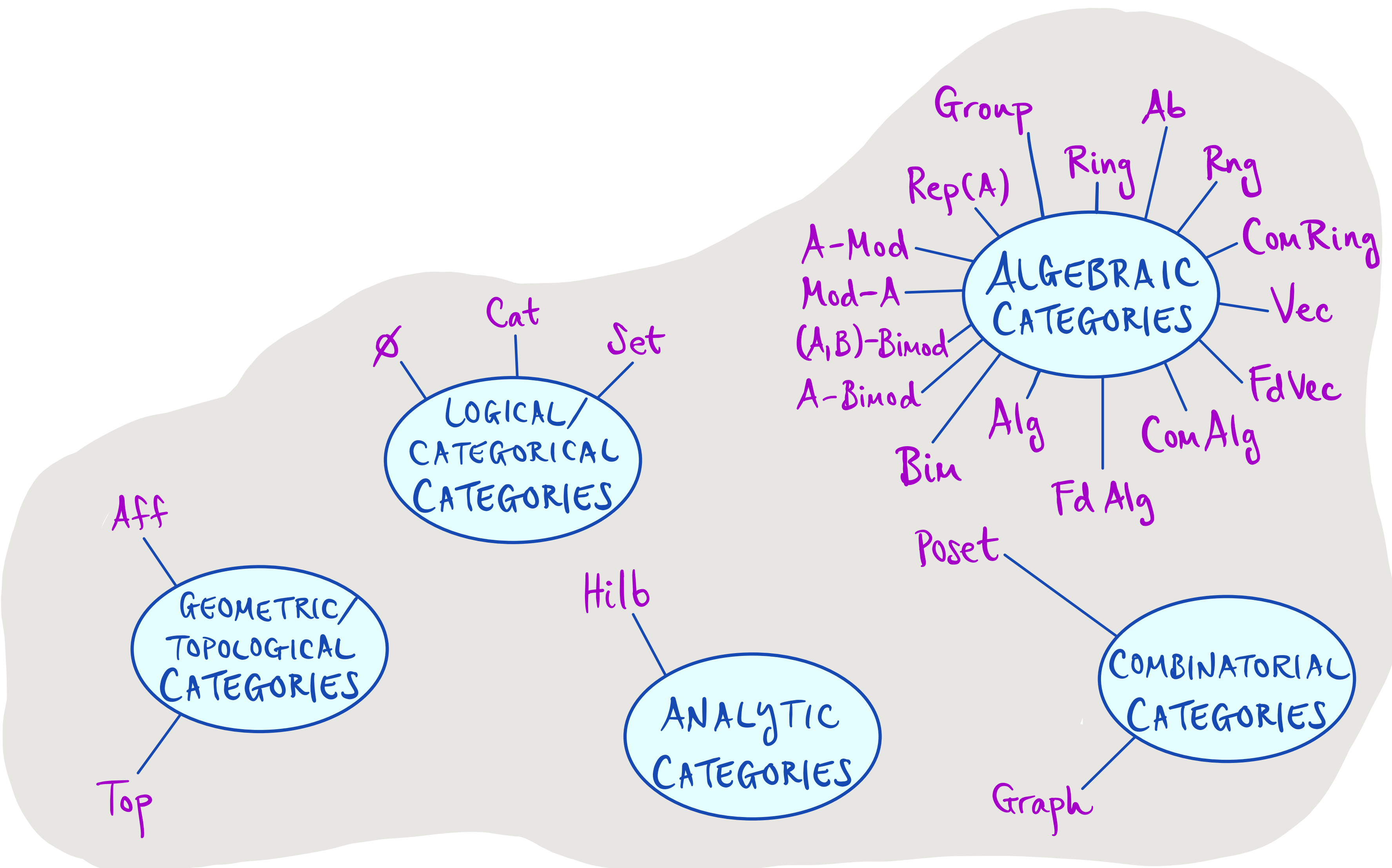
I. (BI)MODULES IN MONOIDAL CATEGORIES (§§4.4.1, 4.4.2)

II. MONADS (§4.3.2)

III. EILENBERG-MOORE CATEGORIES (§4.4.3)

≡ RECALL ≡

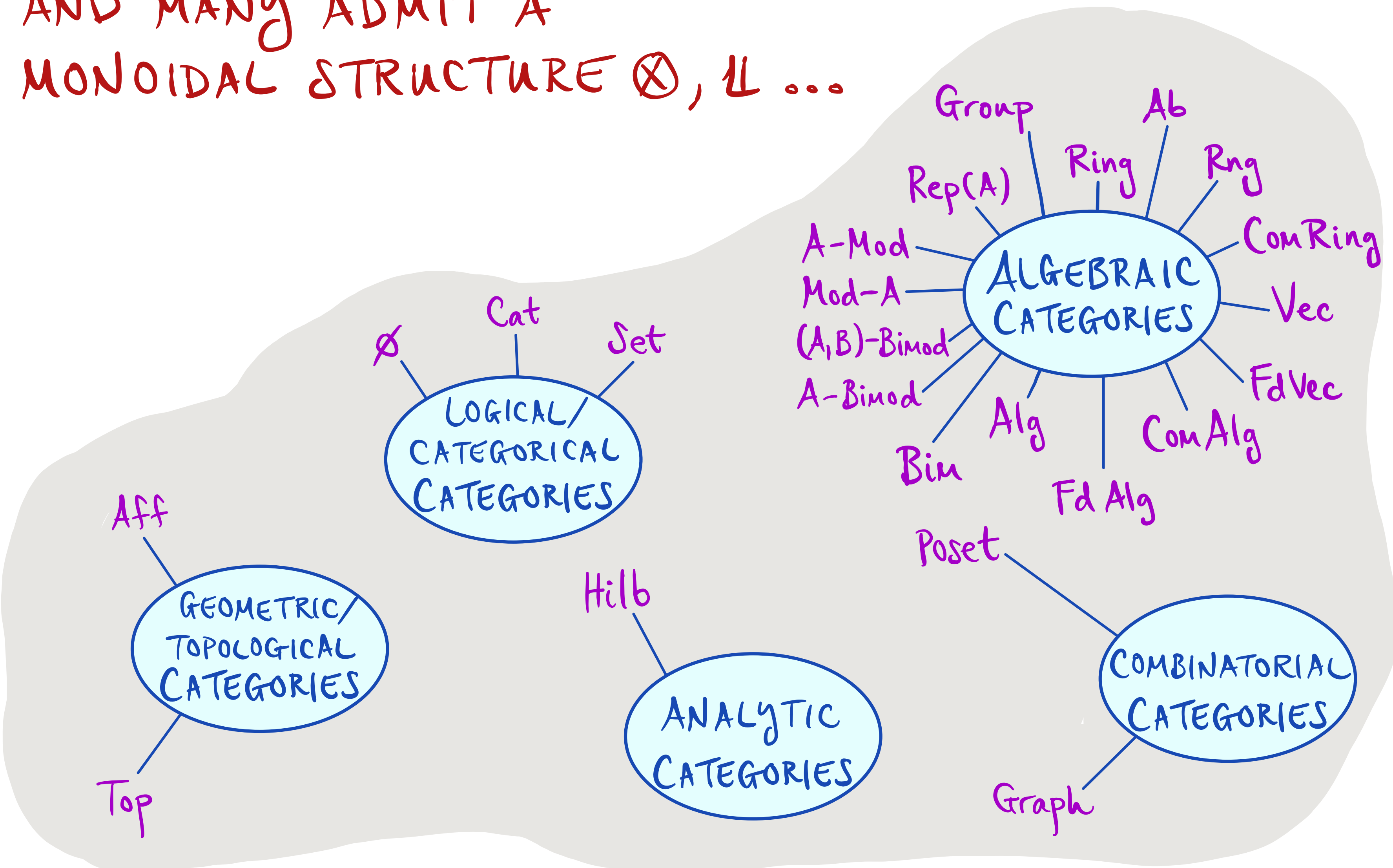
THERE ARE MANY CATEGORIES OUT THERE ...



≡ RECALL ≡

THERE ARE MANY CATEGORIES OUT THERE ...

AND MANY ADMIT A MONOIDAL STRUCTURE $\otimes$, $\amalg$ ...

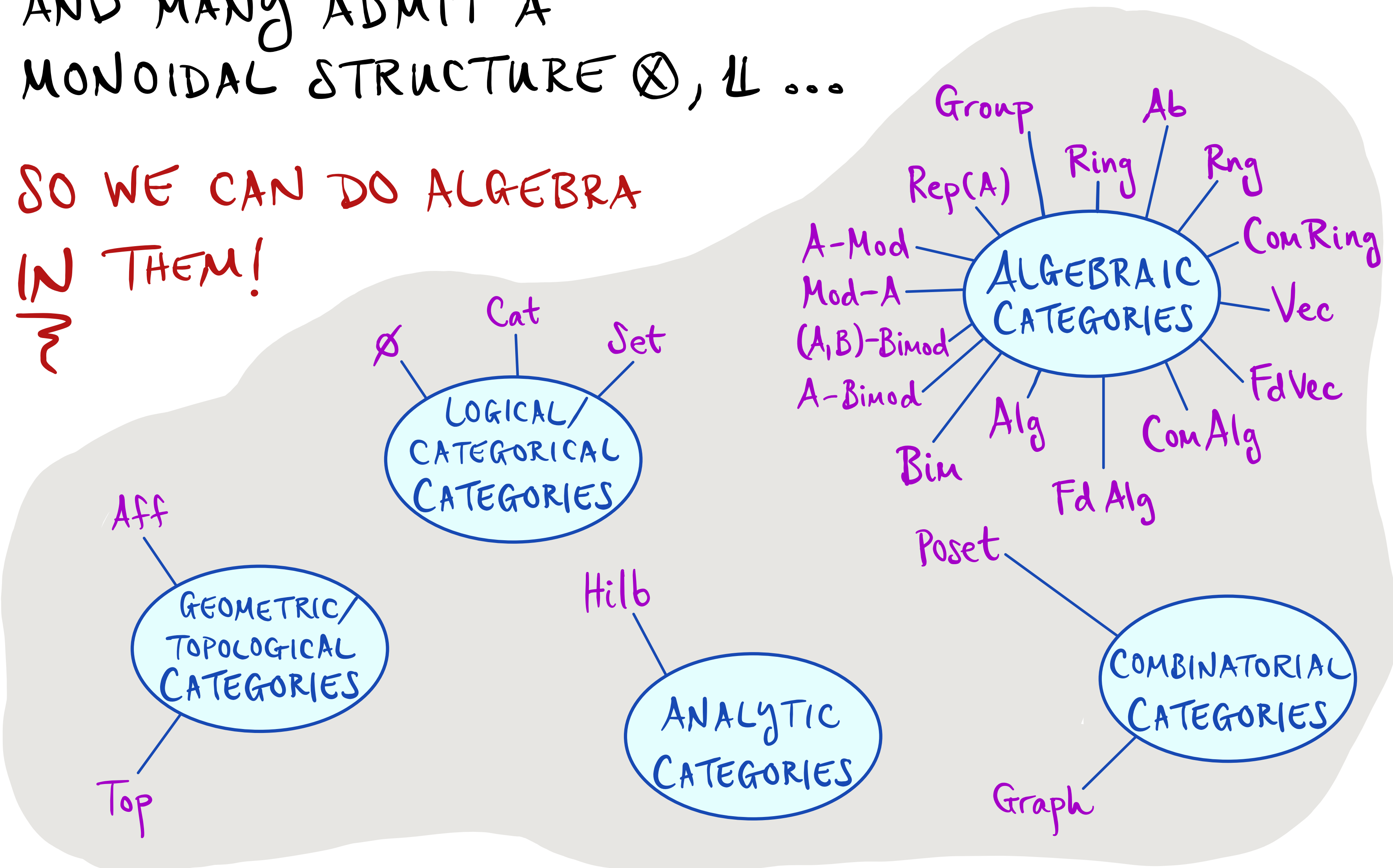


≡ RECALL ≡

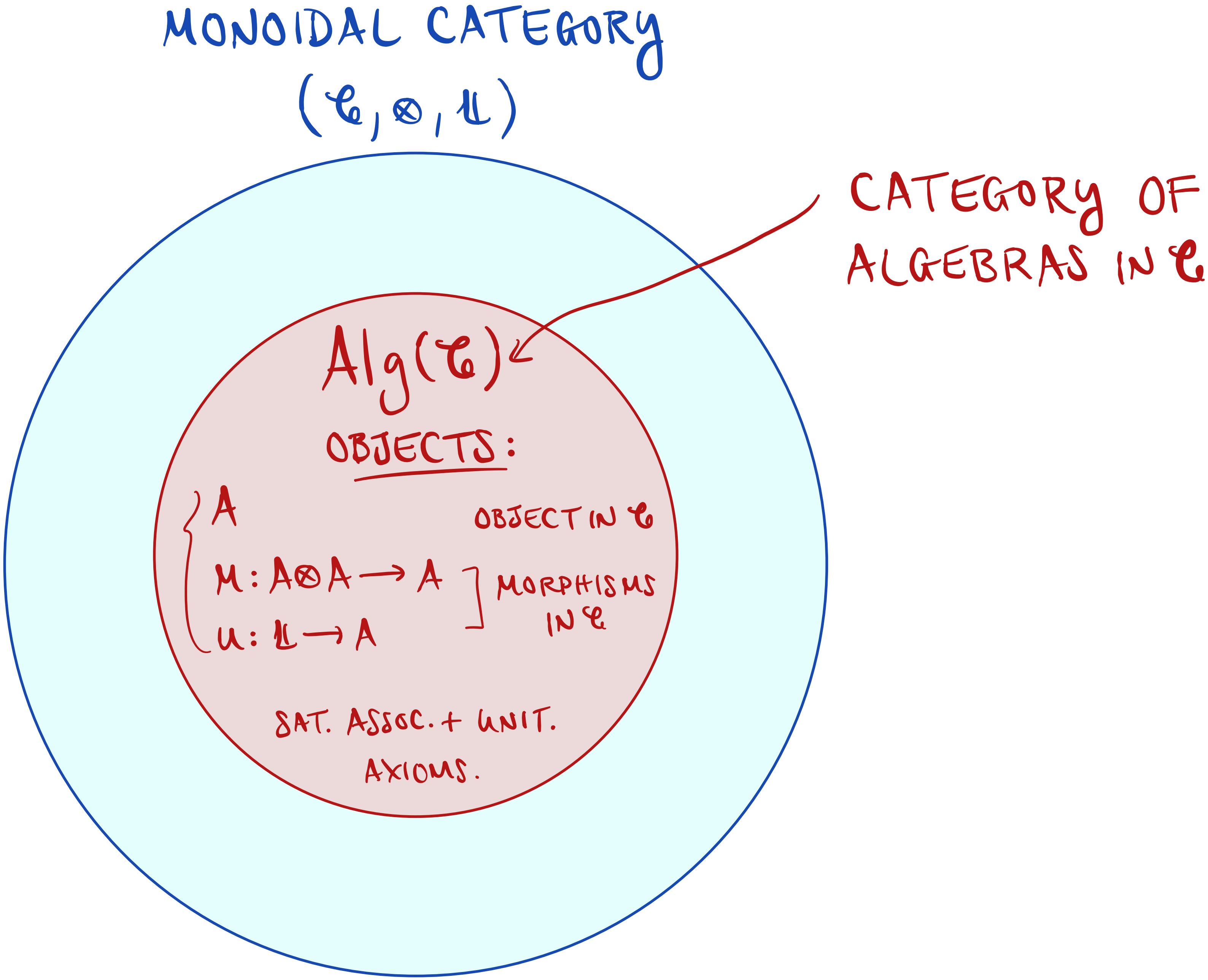
THERE ARE MANY CATEGORIES OUT THERE ...

AND MANY ADMIT A MONOIDAL STRUCTURE $\otimes, \mathbb{1}$ ...

SO WE CAN DO ALGEBRA IN THEM!



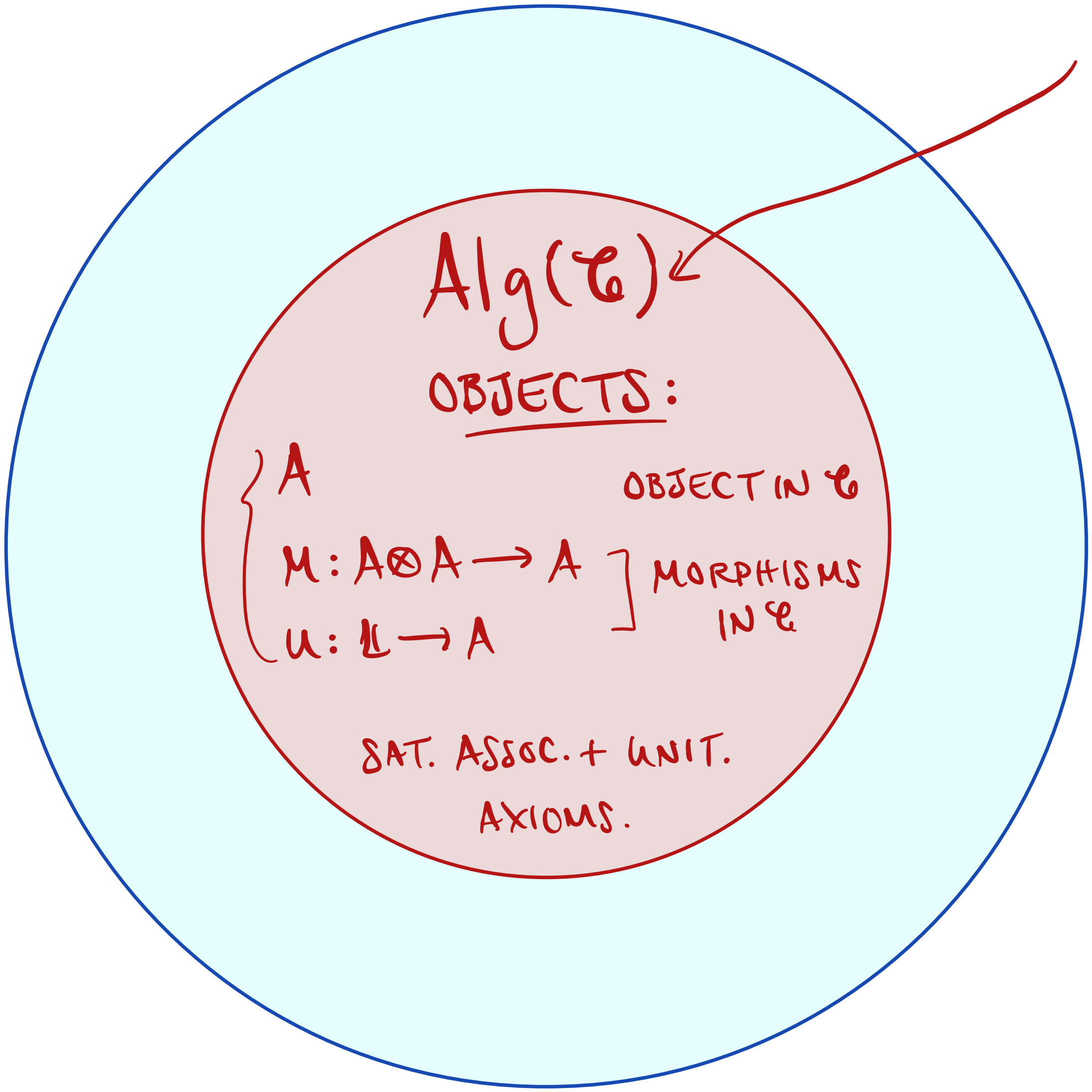
≡ RECALL ≡


MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1})$
CATEGORY OF ALGEBRAS IN $\mathcal{C}$
$\mathrm{Alg}(\mathcal{C})$
OBJECTS:
A OBJECT IN $\mathcal{C}$
$m : A \otimes A \to A$
$u : \mathbb{1} \to A$
MORPHISMS IN $\mathcal{C}$
SAT. ASSOC. + UNIT. AXIOMS.

≡ RECALL ≡

MONOIDAL CATEGORY

$(\mathcal{C}, \otimes, \mathbb{1})$

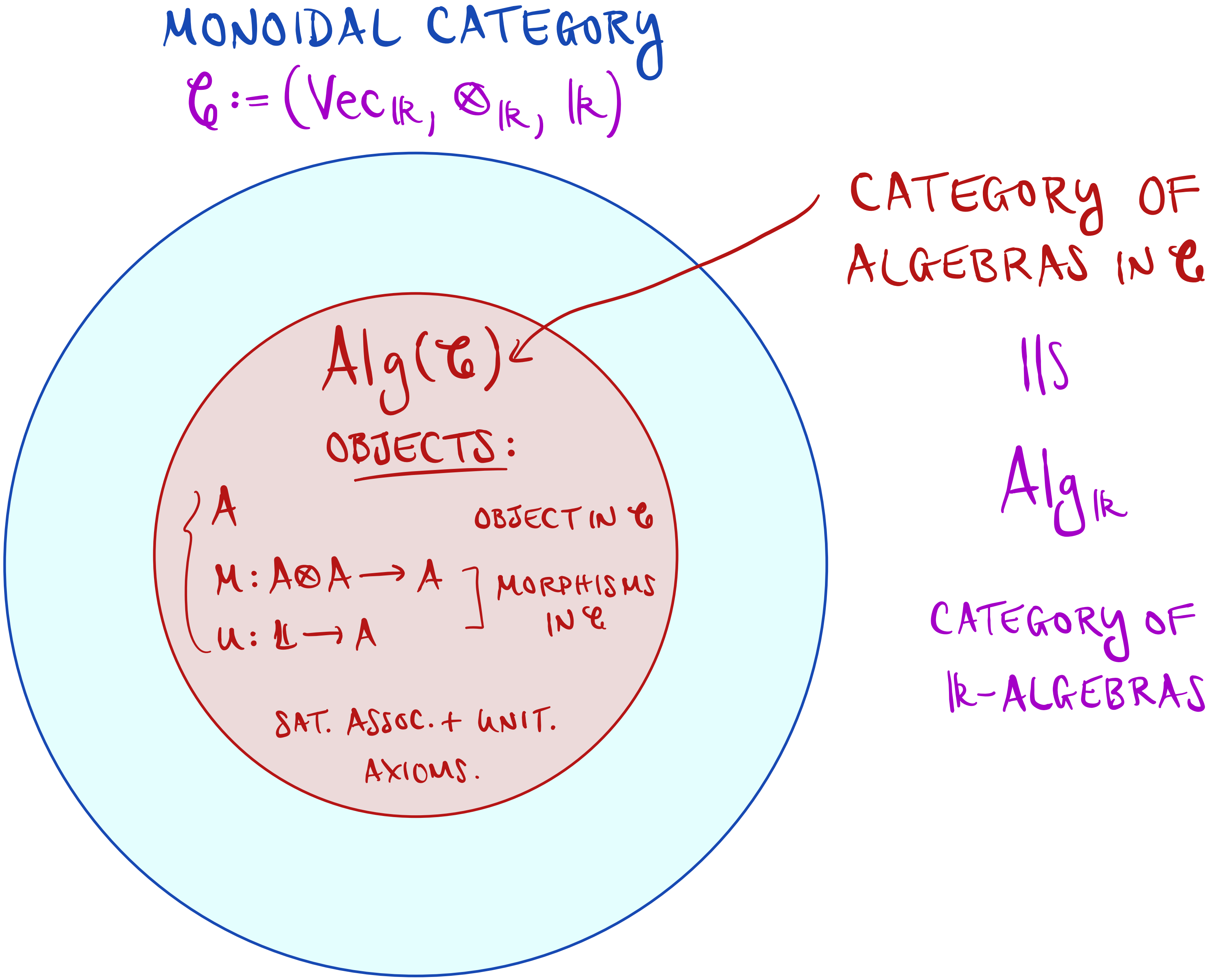


CATEGORY OF ALGEBRAS IN $\mathcal{C}$

DISCUSSED —

- MORPHISMS
- SUBSTRUCTURES
- QUOTIENT STRUCTURES

≡ RECALL ≡



MONOIDAL CATEGORY
$\mathcal{C} := (\mathrm{Vec}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk)$
$\mathrm{Alg}(\mathcal{C})$
OBJECTS:
A
OBJECT IN $\mathcal{C}$
$m : A \otimes A \to A$
$u : \mathbb{1} \to A$
MORPHISMS IN $\mathcal{C}$
SAT. ASSOC. + UNIT. AXIOMS.
CATEGORY OF ALGEBRAS IN $\mathcal{C}$
$\cong$
$\mathrm{Alg}_{\Bbbk}$
CATEGORY OF $\Bbbk$-ALGEBRAS

≡ RECALL ≡

MONOIDAL CATEGORY

$\mathcal{C} := (\mathsf{Set}, \times, \{*\})$

$\mathsf{Alg}(\mathcal{C})$

OBJECTS:

$$\left\{\begin{array}{ll} A & \text{OBJECT IN } \mathcal{C} \\ m : A \otimes A \to A & \multirow{2}{*}{MORPHISMS IN $\mathcal{C}$} \\ u : \mathbb{1} \to A & \end{array}\right.$$

SAT. ASSOC. + UNIT. AXIOMS.

CATEGORY OF ALGEBRAS IN $\mathcal{C}$

$\cong$

Monoid

CATEGORY OF MONOIDS

≡ RECALL ≡

$\mathcal{A}$: (NOT NEC. MONOIDAL) CATEGORY

MONOIDAL CATEGORY

$\mathcal{C} := (\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$

CATEGORY OF ALGEBRAS IN $\mathcal{C}$

$\mathrm{Alg}(\mathcal{C})$

OBJECTS:

A — OBJECT IN $\mathcal{C}$

$m : A \otimes A \longrightarrow A$

$u : \mathbb{1} \longrightarrow A$

MORPHISMS IN $\mathcal{C}$

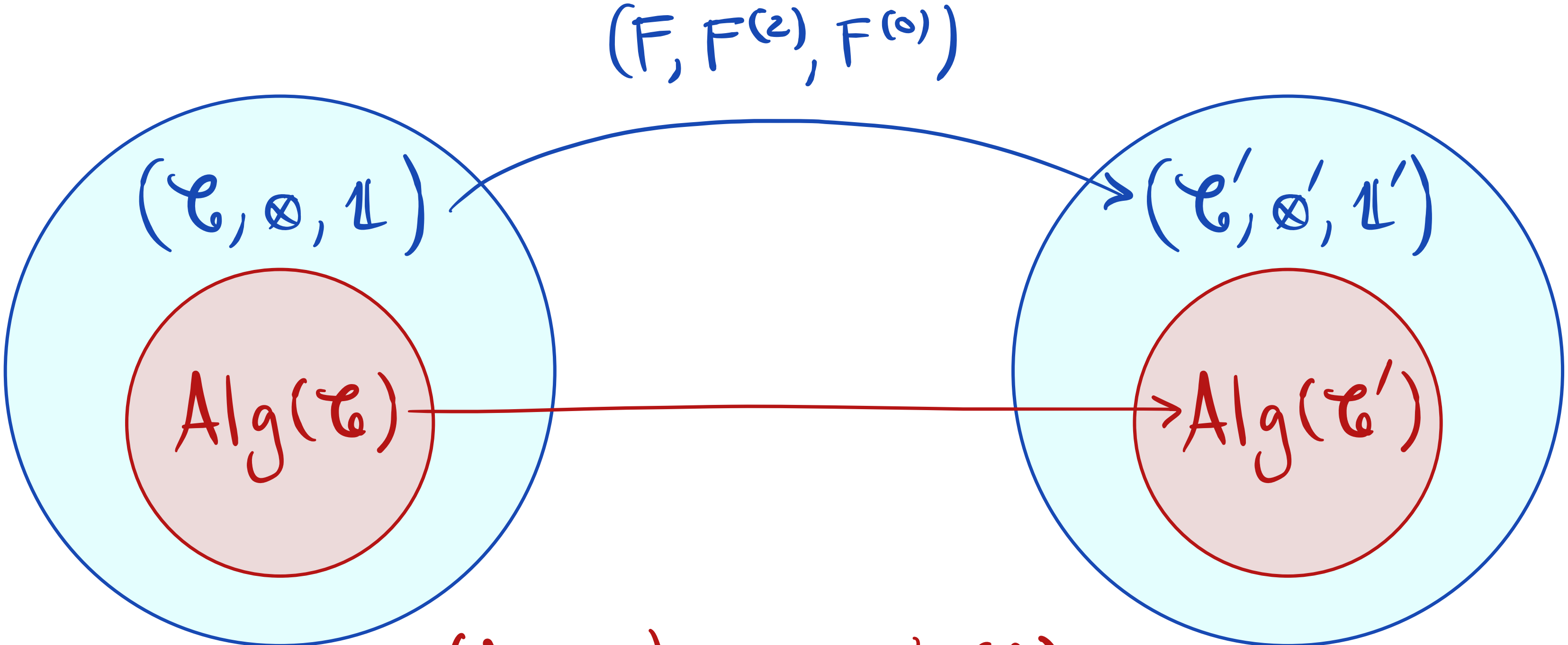
SAT. ASSOC. + UNIT. AXIOMS.

$\mathrm{Alg}(\mathcal{C}) = \mathrm{Monad}(\mathcal{A})$

CATEGORY OF "MONADS" ON $\mathcal{A}$ (STUDIED LATER)

≡ RECALL ≡

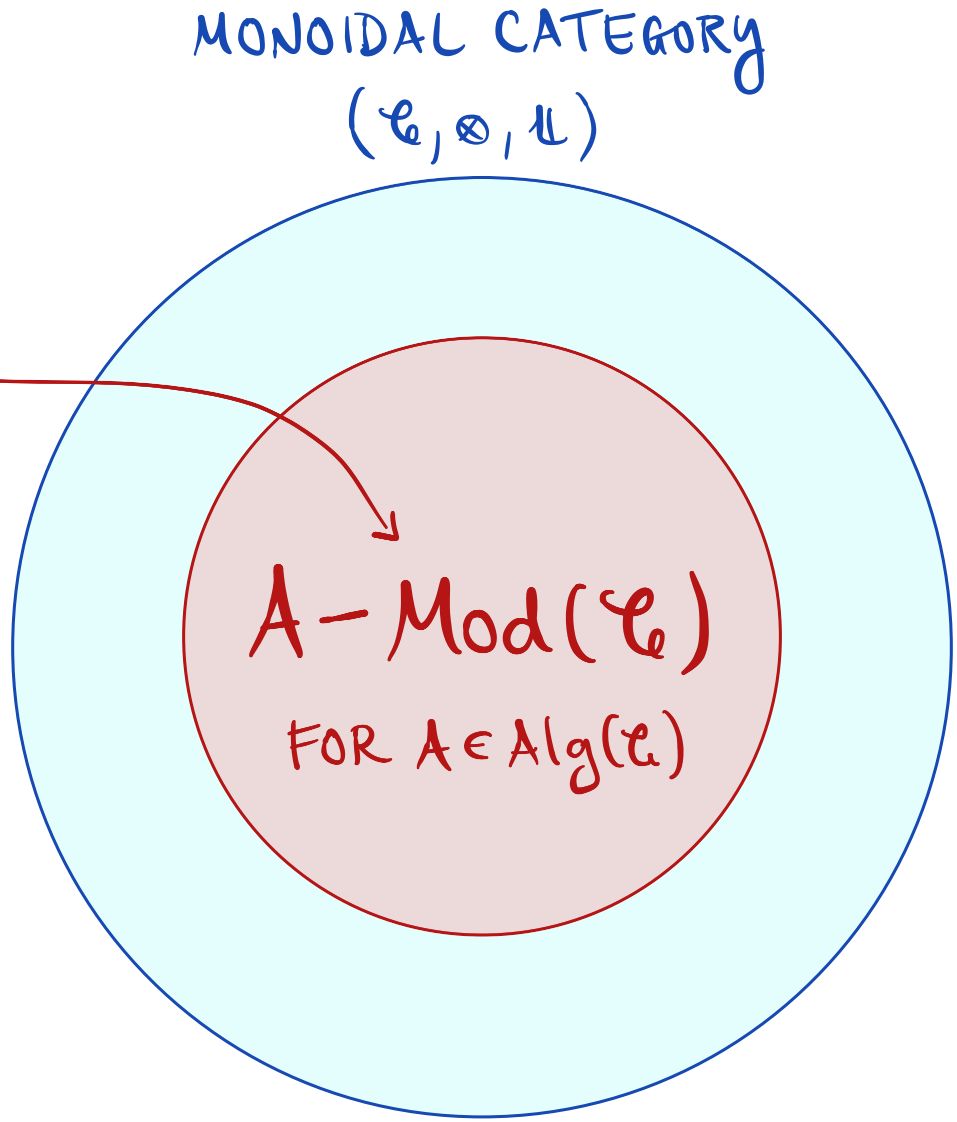
ALSO SAW THAT MONOIDAL FUNCTORS TRANSPORT ALGEBRAS...



$$(A, m, u) \longmapsto \begin{cases} F(A) \\ m' := F(m)\, F^{(2)}_{A,A} \\ u' := F(u)\, F^{(0)} \end{cases}$$

I. (BI)MODULES IN MONOIDAL CATEGORIES

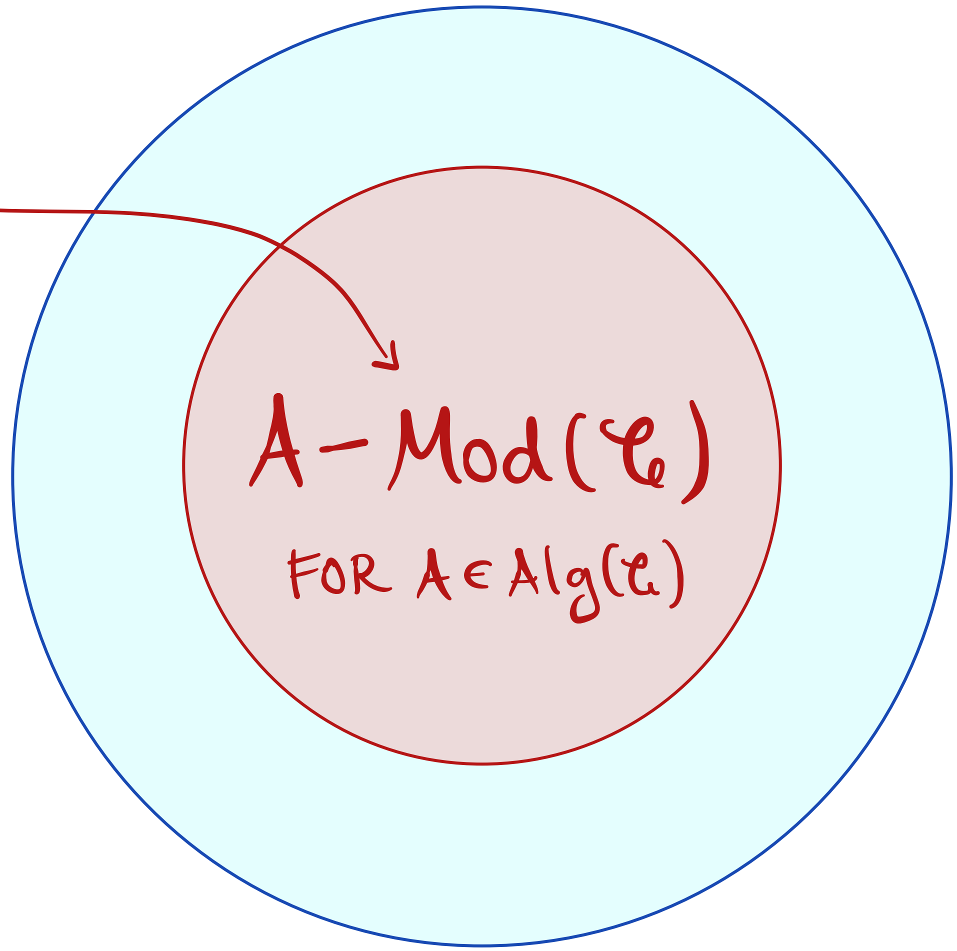
NOW LET'S STUDY THE CATEGORY OF MODULES OVER ALGEBRAS IN $\mathcal{C}$



I. (BI)MODULES IN MONOIDAL CATEGORIES

NOW LET'S STUDY THE CATEGORY OF MODULES OVER ALGEBRAS IN $\mathcal{C}$

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1})$



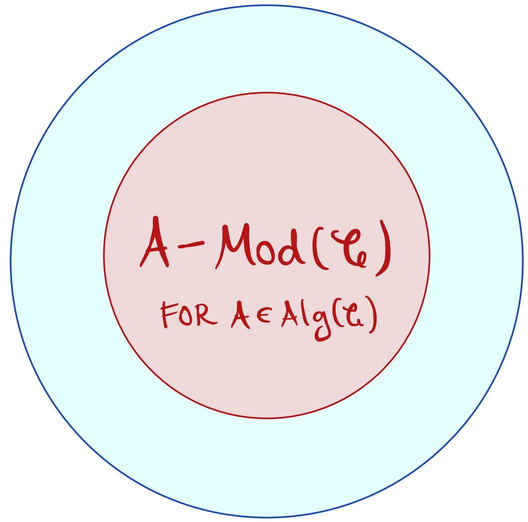
MODULE/ REPRESENTATION THEORY IN $\mathcal{C}$

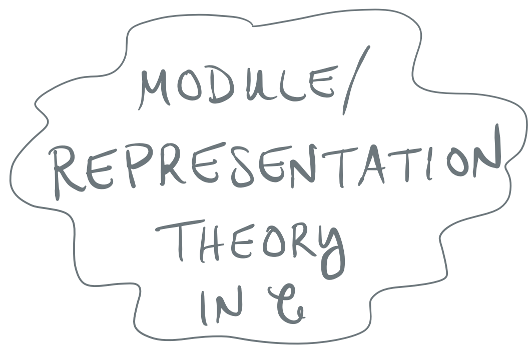
I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

A LEFT A-MODULE IN $\mathcal{C}$ CONSISTS OF

MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1})$





I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

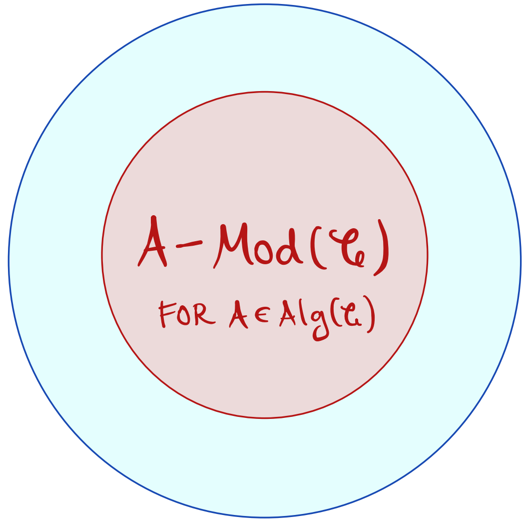
A **LEFT A-MODULE** IN $\mathcal{C}$ CONSISTS OF

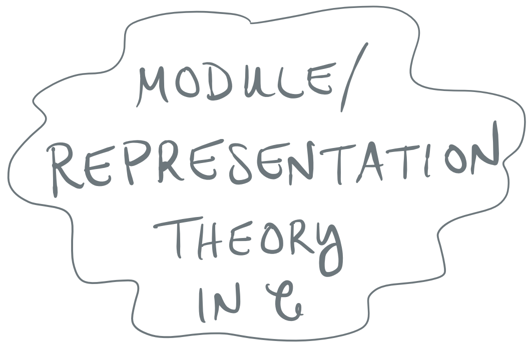
(a) AN OBJECT M IN $\mathcal{C}$

(b) A MORPHISM $\triangleright : A \otimes M \to M$ IN $\mathcal{C}$
(LEFT ACTION MORPHISM)

SATISFYING:

MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1})$





I. (BI)MODULES IN MONOIDAL CATEGORIES

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(LEFT ACTION MORPHISM)

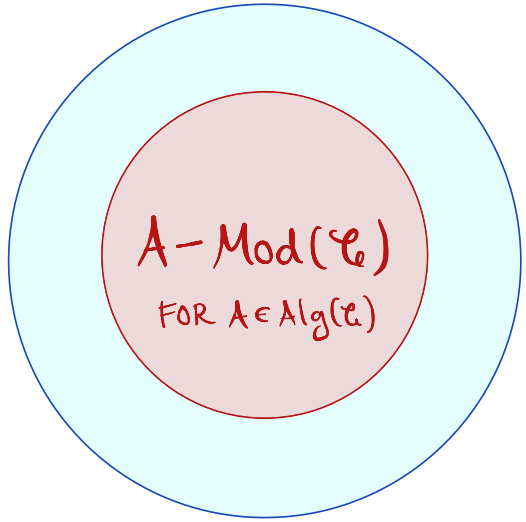
SATISFYING:

$$\begin{tikzcd}
(A\otimes A)\otimes M \arrow[rr, "a_{A,A,M}"] \arrow[d, "m\otimes id"'] & & A\otimes(A\otimes M) \arrow[d, "id\otimes\triangleright"] \\
A\otimes M \arrow[dr, "\triangleright"'] & 2 & A\otimes M \arrow[dl, "\triangleright"] \\
& M &
\end{tikzcd}$$

&

$$\begin{tikzcd}
\mathbb{1}\otimes M \arrow[r, "u\otimes id"] \arrow[dr, "l_M"'] & A\otimes M \arrow[d, "\triangleright"] \arrow[dl, phantom, "2"] \\
& M
\end{tikzcd}$$

MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1})$



MODULE/
REPRESENTATION
THEORY
IN $\mathcal{C}$

I. (BI)MODULES IN MONOIDAL CATEGORIES

MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1})$

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

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SATISFYING:

$$\begin{tikzcd}
(A\otimes A)\otimes M \arrow[rr, "a_{A,A,M}"] \arrow[d, "m\otimes id"'] & & A\otimes(A\otimes M) \arrow[d, "id\otimes \triangleright"] \\
A\otimes M \arrow[dr, "\triangleright"'] \arrow[rr, phantom, "2"] & & A\otimes M \arrow[dl, "\triangleright"] \\
& M &
\end{tikzcd}$$

&

$$\begin{tikzcd}
\mathbb{1}\otimes M \arrow[r, "u\otimes id"] \arrow[dr, "l_M"'] & A\otimes M \arrow[d, "\triangleright"] \arrow[dl, phantom, "2"] \\
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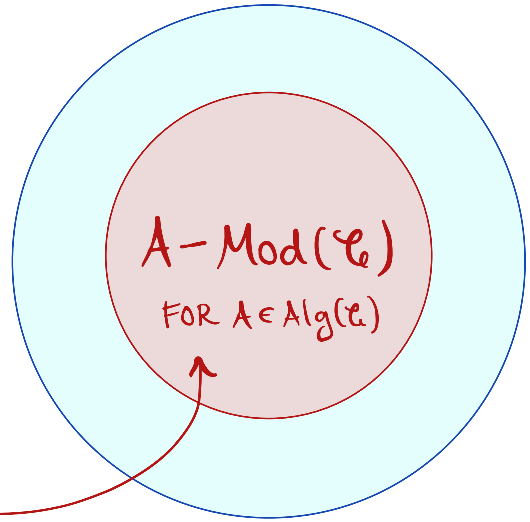
\end{tikzcd}$$



FORMS A CATEGORY WITH

$\phi \in \mathrm{Hom}_{A\text{-Mod}(\mathcal{C})}(M, M')$

I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

A LEFT A-MODULE IN $\mathcal{C}$ CONSISTS OF

(a) AN OBJECT M IN $\mathcal{C}$

(b) A MORPHISM $\triangleright : A \otimes M \to M$ IN $\mathcal{C}$
(LEFT ACTION MORPHISM)

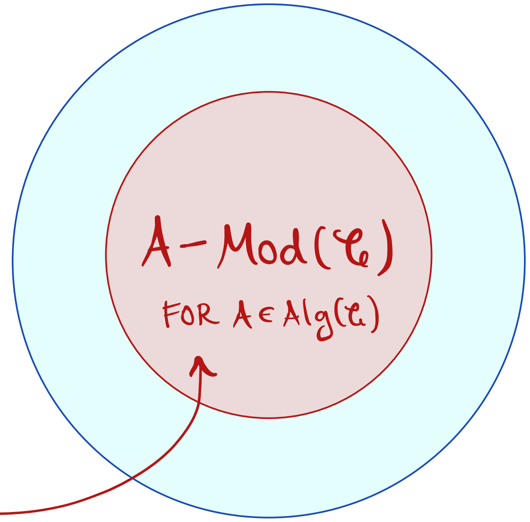
SATISFYING :

$$\begin{tikzcd}
(A\otimes A)\otimes M \arrow[rr, "a_{A,A,M}"] \arrow[d, "m\otimes id"'] & & A\otimes(A\otimes M) \arrow[d, "id\otimes\triangleright"] \\
A\otimes M \arrow[dr, "\triangleright"'] & {} \arrow[phantom, u, "2"] & A\otimes M \arrow[dl, "\triangleright"] \\
& M &
\end{tikzcd}$$

&

$$\begin{tikzcd}
\mathbb{1}\otimes M \arrow[r, "u\otimes id"] \arrow[dr, "l_M"'] & A\otimes M \arrow[d, "\triangleright"] \arrow[dl, phantom, "2"] \\
& M
\end{tikzcd}$$

MONOIDAL CATEGORY
$(\mathcal{C}, \otimes, \mathbb{1})$



FORMS A CATEGORY WITH

$\phi \in \mathrm{Hom}_{A\text{-}\mathrm{Mod}(\mathcal{C})}(M, M')$

$\cong$

$\phi : M_{\mathrm{OBJ}} \longrightarrow M'_{\mathrm{OBJ}}$ IN $\mathcal{C}$

SUCH THAT:

$$\begin{tikzcd}
A\otimes M \arrow[r, "\triangleright"] \arrow[d, "id\otimes\phi"'] & M \arrow[d, "\phi"] \\
A\otimes M' \arrow[r, "\triangleright'"] \arrow[ur, phantom, "2"] & M'
\end{tikzcd}$$

I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

LEFT A-MODULE IN $\mathcal{C}$ IS:

(a) OBJECT M IN $\mathcal{C}$

(b) MAP $\triangleright : A \otimes M \to M$ IN $\mathcal{C}$

SATISFYING :

(ASSOC.) $\triangleright(m \otimes \mathrm{id}) = \triangleright(\mathrm{id} \otimes \triangleright)\, a$

(UNIT.) $\triangleright(u \otimes \mathrm{id}) = \ell$

FORMS CATEGORY

$A\text{-}\mathrm{Mod}(\mathcal{C})$

WITH

$\phi \in \mathrm{Hom}_{A\text{-}\mathrm{Mod}(\mathcal{C})}(M, M')$

$\parallel$

$\phi \in \mathrm{Hom}_{\mathcal{C}}(M, M')$ s.t.

$\triangleright \phi = (\mathrm{id} \otimes \phi) \triangleright'$

I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

LEFT A-MODULE IN $\mathcal{C}$ IS:

(a) OBJECT M IN $\mathcal{C}$

(b) MAP $\triangleright : A \otimes M \to M$ IN $\mathcal{C}$

SATISFYING:

(ASSOC.) $\triangleright(m \otimes id) = \triangleright(id \otimes \triangleright)\,a$

(UNIT.) $\triangleright(u \otimes id) = l$

FORMS CATEGORY

$A\text{-}\mathrm{Mod}(\mathcal{C})$

WITH

$\phi \in \mathrm{Hom}_{A\text{-}\mathrm{Mod}(\mathcal{C})}(M, M')$

$\Updownarrow$

$\phi \in \mathrm{Hom}_{\mathcal{C}}(M, M')$ s.t.

$\triangleright \phi = (id \otimes \phi)\triangleright'$

RIGHT A-MODULES IN $\mathcal{C}$ DEFINED LIKEWISE

OBJECTS : $(M \in \mathcal{C},\ \triangleleft : M \otimes A \to M \in \mathcal{C})$

SATISFYING ASSOC., UNIT.

FORMS CATEGORY $\mathrm{Mod}\text{-}A(\mathcal{C})$

I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

LEFT A-MODULE IN $\mathcal{C}$ IS:

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(b) MAP $\triangleright : A \otimes M \to M$ IN $\mathcal{C}$

SATISFYING:

(ASSOC.) $\triangleright(m \otimes id) = \triangleright(id \otimes \triangleright)\, a$

(UNIT.) $\triangleright(u \otimes id) = l$

FORMS CATEGORY
$A\text{-}\mathrm{Mod}(\mathcal{C})$

WITH

$\phi \in \mathrm{Hom}_{A\text{-}\mathrm{Mod}(\mathcal{C})}(M, M')$

$\parallel\!|$

$\phi \in \mathrm{Hom}_{\mathcal{C}}(M, M')$ $\ni$.

$\triangleright \phi = (id \otimes \phi)\, \triangleright'$

RIGHT A-MODULES IN $\mathcal{C}$ — DEFINED LIKEWISE

OBJECTS : $(M \in \mathcal{C},\ \triangleleft : M \otimes A \to M \in \mathcal{C})$
SATISFYING ASSOC., UNIT.

FORMS CATEGORY $\mathrm{Mod}\text{-}A(\mathcal{C})$

A-BIMODULES IN $\mathcal{C}$:

OBJECTS : $(M,\ \triangleright : A \otimes M \to M,\ \triangleleft : M \otimes A \to M)$

I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

LEFT A-MODULE IN $\mathcal{C}$ IS:

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SATISFYING:

(ASSOC.) $\triangleright(m \otimes id) = \triangleright(id \otimes \triangleright) a$

(UNIT.) $\triangleright(u \otimes id) = \ell$

FORMS CATEGORY

$A\text{-}\mathrm{Mod}(\mathcal{C})$

WITH

$\phi \in \mathrm{Hom}_{A\text{-}\mathrm{Mod}(\mathcal{C})}(M, M')$

$\equiv$

$\phi \in \mathrm{Hom}_{\mathcal{C}}(M, M')$ s.t.

$\triangleright \phi = (id \otimes \phi) \triangleright'$

RIGHT A-MODULES IN $\mathcal{C}$ — DEFINED LIKEWISE

OBJECTS : $(M \in \mathcal{C},\ \triangleleft : M \otimes A \to M \in \mathcal{C})$

SATISFYING ASSOC., UNIT.

FORMS CATEGORY $\mathrm{Mod}\text{-}A(\mathcal{C})$

A-BIMODULES IN $\mathcal{C}$:

OBJECTS : $(M,\ \triangleright : A \otimes M \to M,\ \triangleleft : M \otimes A \to M)$

WITH : $(M, \triangleright) \in A\text{-}\mathrm{Mod}(\mathcal{C})$ &

$(M, \triangleleft) \in \mathrm{Mod}\text{-}A(\mathcal{C})$

I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

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SATISFYING:

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FORMS CATEGORY

$A\text{-}\mathrm{Mod}(\mathcal{C})$

WITH

$\phi \in \mathrm{Hom}_{A\text{-}\mathrm{Mod}(\mathcal{C})}(M, M')$

$\equiv$

$\phi \in \mathrm{Hom}_{\mathcal{C}}(M, M')$ s.t.

$\triangleright \phi = (id \otimes \phi) \triangleright'$

RIGHT A-MODULES IN $\mathcal{C}$

DEFINED LIKEWISE

OBJECTS : $(M \in \mathcal{C},\ \triangleleft : M \otimes A \to M \in \mathcal{C})$

SATISFYING ASSOC., UNIT.

FORMS CATEGORY $\mathrm{Mod}\text{-}A(\mathcal{C})$

A-BIMODULES IN $\mathcal{C}$:

OBJECTS: $(M,\ \triangleright : A \otimes M \to M,\ \triangleleft : M \otimes A \to M)$

WITH: $(M, \triangleright) \in A\text{-}\mathrm{Mod}(\mathcal{C})$

$(M, \triangleleft) \in \mathrm{Mod}\text{-}A(\mathcal{C})$

&

$$\begin{tikzcd}
(A \otimes M) \otimes A \arrow[rr, "a", "\sim"'] \arrow[d, "\triangleright \otimes id"'] & & A \otimes (M \otimes A) \arrow[d, "id \otimes \triangleleft"] \\
M \otimes A \arrow[dr, "\triangleleft"'] & \circlearrowleft & A \otimes M \arrow[dl, "\triangleright"] \\
& M &
\end{tikzcd}$$

I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

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FORMS CATEGORY

$A\text{-}\mathrm{Mod}(\mathcal{C})$

WITH

$\phi \in \mathrm{Hom}_{A\text{-}\mathrm{Mod}(\mathcal{C})}(M, M')$

$\equiv$

$\phi \in \mathrm{Hom}_{\mathcal{C}}(M, M')$ ∍.

$\triangleright \phi = (id \otimes \phi) \triangleright'$

RIGHT A-MODULES IN $\mathcal{C}$

DEFINED LIKEWISE

OBJECTS : $(M \in \mathcal{C},\ \triangleleft : M \otimes A \to M \in \mathcal{C})$

SATISFYING ASSOC., UNIT.

FORMS CATEGORY $\mathrm{Mod}\text{-}A(\mathcal{C})$

A-BIMODULES IN $\mathcal{C}$:

OBJECTS : $(M,\ \triangleright : A \otimes M \to M,\ \triangleleft : M \otimes A \to M)$

WITH : $(M, \triangleright) \in A\text{-}\mathrm{Mod}(\mathcal{C})$

$(M, \triangleleft) \in \mathrm{Mod}\text{-}A(\mathcal{C})$

&

$$\begin{tikzcd}
(A \otimes M) \otimes A \arrow[rr, "a", "\sim"'] \arrow[d, "\triangleright \otimes id"'] & & A \otimes (M \otimes A) \arrow[d, "id \otimes \triangleleft"] \\
M \otimes A \arrow[dr, "\triangleleft"'] & & A \otimes M \arrow[dl, "\triangleright"] \\
& M &
\end{tikzcd}$$

FORMS CATEGORY $A\text{-}\mathrm{Bimod}(\mathcal{C})$

$$\mathrm{Hom}_{A\text{-}\mathrm{Bimod}(\mathcal{C})}(M, M') = \mathrm{Hom}_{A\text{-}\mathrm{Mod}(\mathcal{C})}(M, M') \cap \mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(M, M')$$

I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

LEFT A-MODULE IN $\mathcal{C}$ IS:

(a) OBJECT M IN $\mathcal{C}$

(b) MAP $\triangleright : A \otimes M \to M$ IN $\mathcal{C}$

SATISFYING:

(ASSOC.) $\triangleright(m \otimes \mathrm{id}) = \triangleright(\mathrm{id} \otimes \triangleright)\, a$

(UNIT.) $\triangleright(u \otimes \mathrm{id}) = \ell$

FORMS CATEGORY

$A\text{-}\mathrm{Mod}(\mathcal{C})$

WITH

$$\phi \in \mathrm{Hom}_{A\text{-}\mathrm{Mod}(\mathcal{C})}(M, M')$$
$$\text{III}$$
$$\phi \in \mathrm{Hom}_{\mathcal{C}}(M, M') \ni.$$
$$\triangleright\, \phi = (\mathrm{id} \otimes \phi)\, \triangleright'$$

OBTAIN SUBSTRUCTURES & QUOTIENT STRUCTURES WHEN $\mathcal{C}$ IS ABELIAN MONOIDAL

RIGHT A-MODULES IN $\mathcal{C}$

DEFINED LIKEWISE

OBJECTS : $(M \in \mathcal{C},\ \triangleleft : M \otimes A \to M \in \mathcal{C})$

SATISFYING ASSOC., UNIT.

FORMS CATEGORY $\mathrm{Mod}\text{-}A(\mathcal{C})$

A-BIMODULES IN $\mathcal{C}$:

OBJECTS : $(M,\ \triangleright : A \otimes M \to M,\ \triangleleft : M \otimes A \to M)$

WITH : $(M, \triangleright) \in A\text{-}\mathrm{Mod}(\mathcal{C})$

$(M, \triangleleft) \in \mathrm{Mod}\text{-}A(\mathcal{C})$

&

$$\begin{tikzcd}
(A \otimes M) \otimes A \arrow[rr, "a", "\sim"'] \arrow[d, "\triangleright \otimes \mathrm{id}"'] & & A \otimes (M \otimes A) \arrow[d, "\mathrm{id} \otimes \triangleleft"] \\
M \otimes A \arrow[dr, "\triangleleft"'] & 2 & A \otimes M \arrow[dl, "\triangleright"] \\
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\end{tikzcd}$$

FORMS CATEGORY $A\text{-}\mathrm{Bimod}(\mathcal{C})$

$$\mathrm{Hom}_{A\text{-}\mathrm{Bimod}(\mathcal{C})}(M, M') = \begin{array}{c} \mathrm{Hom}_{A\text{-}\mathrm{Mod}(\mathcal{C})}(M, M') \\ \cap \\ \mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(M, M') \end{array}$$

I. (BI)MODULES IN MONOIDAL CATEGORIES

EXAMPLES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

LEFT A-MODULE IN $\mathcal{C}$ IS:
(a) OBJECT M IN $\mathcal{C}$
(b) MAP $\triangleright : A \otimes M \to M$ IN $\mathcal{C}$
SAT. ASSOC & UNIT. AXIOMS
FORMS CATEG. $A\text{-Mod}(\mathcal{C})$

RIGHT A-MODULE IN $\mathcal{C}$ IS:
(a) OBJECT M IN $\mathcal{C}$
(b) MAP $\triangleleft : M \otimes A \to M$ IN $\mathcal{C}$
SAT. ASSOC & UNIT. AXIOMS
FORMS CATEG. $\text{Mod-}A(\mathcal{C})$

A-BIMODULE IN $\mathcal{C}$ IS:
(a) OBJECT M IN $\mathcal{C}$
(b) MAPS $\triangleright, \triangleleft$ IN $\mathcal{C}$
SAT. $(A, \triangleright) \in A\text{-Mod}(\mathcal{C})$
$(A, \triangleleft) \in \text{Mod-}A(\mathcal{C})$
+ COMPATIBILITY
FORMS CATEG. $A\text{-Bimod}(\mathcal{C})$

I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$.
LEFT A-MODULE IN $\mathcal{C}$ IS: (a) OBJECT M IN $\mathcal{C}$ (b) MAP $\triangleright : A \otimes M \to M$ IN $\mathcal{C}$ SAT. ASSOC & UNIT. AXIOMS   FORMS CATEG. $A\text{-}\mathsf{Mod}(\mathcal{C})$
RIGHT A-MODULE IN $\mathcal{C}$ IS: (a) OBJECT M IN $\mathcal{C}$ (b) MAP $\triangleleft : M \otimes A \to M$ IN $\mathcal{C}$ SAT. ASSOC & UNIT. AXIOMS   FORMS CATEG. $\mathsf{Mod}\text{-}A(\mathcal{C})$
A-BIMODULE IN $\mathcal{C}$ IS: (a) OBJECT M IN $\mathcal{C}$ (b) MAPS $\triangleright, \triangleleft$ IN $\mathcal{C}$ SAT. $(A, \triangleright) \in A\text{-}\mathsf{Mod}(\mathcal{C})$ $(A, \triangleleft) \in \mathsf{Mod}\text{-}A(\mathcal{C})$ + COMPATIBILITY   FORMS CATEG. $A\text{-}\mathsf{Bimod}(\mathcal{C})$

EXAMPLES

REGULAR LEFT A-MODULE ${}_A(A_{reg}) = (A, \triangleright = m_A)$
REGULAR RIGHT A-MODULE $(A_{reg})_A = (A, \triangleleft = m_A)$
REGULAR A-BIMODULE ${}_A(A_{reg})_A = (A, \triangleright = \triangleleft = m_A)$

I. (BI)MODULES IN MONOIDAL CATEGORIES

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LEFT A-MODULE IN $\mathcal{C}$ IS:
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$(A, \triangleleft) \in \mathrm{Mod}\text{-}A(\mathcal{C})$
+ COMPATIBILITY
FORMS CATEG. $A\text{-}\mathrm{Bimod}(\mathcal{C})$

EXAMPLES

REGULAR LEFT A-MODULE

$${}_A(A_{reg}) = (A, \ \triangleright = m_A)$$

LEFT IDEALS OF A:

$$(\underbrace{I}_{M}, \iota, \underbrace{\lambda}_{\triangleright} : A \otimes I \to I)$$

REGULAR RIGHT A-MODULE

$$(A_{reg})_A = (A, \ \triangleleft = m_A)$$

RIGHT IDEALS OF A:

$$(\underbrace{I}_{M}, \iota, \underbrace{\rho}_{\triangleleft} : I \otimes A \to I)$$

REGULAR A-BIMODULE

$${}_A(A_{reg})_A = (A, \ \triangleright = \triangleleft = m_A)$$

IDEALS OF A:

$$(\underbrace{I}_{M}, \iota, \underbrace{\lambda}_{\triangleright}, \underbrace{\rho}_{\triangleleft})$$

I. (BI)MODULES IN MONOIDAL CATEGORIES

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SAT. ASSOC & UNIT. AXIOMS

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$(A, \triangleleft) \in \mathrm{Mod}\text{-}A(\mathcal{C})$
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FORMS CATEG. $A\text{-}\mathrm{Bimod}(\mathcal{C})$

TAKE $\mathbb{1} := (\mathbb{1}, l_{\mathbb{1}} = r_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}}) \in \mathrm{Alg}(\mathcal{C})$. THEN:

$$\mathbb{1}\text{-}\mathrm{Mod}(\mathcal{C}) \cong \mathrm{Mod}\text{-}\mathbb{1}(\mathcal{C}) \cong \mathbb{1}\text{-}\mathrm{Bimod}(\mathcal{C})$$

$\cong \mathcal{C}$ AS CATEGORIES

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TAKE $\mathbb{1} := (\mathbb{1}, \ell_{\mathbb{1}} = r_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}}) \in \mathrm{Alg}(\mathcal{C})$. THEN:

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$\cong \mathcal{C}$ AS CATEGORIES

TAKE $(M, \triangleright_M : \mathbb{1} \otimes M \to M) \in \mathbb{1}\text{-}\mathrm{Mod}$.

$$\begin{tikzcd}
\mathbb{1} \otimes M \arrow[r, "u \otimes \mathrm{id}"] \arrow[dr, "\ell_M"'] & A \otimes M \arrow[d, "\triangleright_M"] \\
& M
\end{tikzcd}$$

(UNITALITY)

$A = \mathbb{1}$
$u = \mathrm{id}_{\mathbb{1}}$

$$\begin{tikzcd}
\mathbb{1} \otimes M \arrow[r, "\mathrm{id} \otimes \mathrm{id}"] \arrow[dr, "\ell_M"'] & \mathbb{1} \otimes M \arrow[d, "\triangleright_M"] \\
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\end{tikzcd}$$

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SAT. ASSOC & UNIT. AXIOMS

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SAT. $(A, \triangleright) \in A\text{-}\mathsf{Mod}(\mathcal{C})$

$(A, \triangleleft) \in \mathsf{Mod}\text{-}A(\mathcal{C})$

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TAKE $\mathbb{1} := (\mathbb{1}, \ell_{\mathbb{1}} = r_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}}) \in \mathsf{Alg}(\mathcal{C})$. THEN:

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$\cong \mathcal{C}$ AS CATEGORIES

TAKE $(M, \triangleright_M : \mathbb{1} \otimes M \to M) \in \mathbb{1}\text{-}\mathsf{Mod}$.

THEN: $\triangleright_M = \ell_M$ ($\leftarrow$ PART OF STRUCTURE OF $\mathcal{C}$)

$\therefore \; \mathbb{1}\text{-}\mathsf{Mod}(\mathcal{C}) \cong \mathcal{C}$

$$\begin{tikzcd} \mathbb{1} \otimes M \arrow[r, "u \otimes id"] \arrow[dr, "\ell_M"'] & A \otimes M \arrow[d, "\triangleright_M"] \arrow[dl, phantom, "2"] \\ & M \end{tikzcd}$$

(UNITALITY)

$A = \mathbb{1}$

$u = \mathrm{id}_{\mathbb{1}}$

$$\begin{tikzcd} \mathbb{1} \otimes M \arrow[r, "id \otimes id"] \arrow[dr, "\ell_M"'] & \mathbb{1} \otimes M \arrow[d, "\triangleright_M"] \arrow[dl, phantom, "2"] \\ & M \end{tikzcd}$$

I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

LEFT A-MODULE IN $\mathcal{C}$ IS:
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SAT. ASSOC & UNIT. AXIOMS

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$(A, \triangleleft) \in \text{Mod-}A(\mathcal{C})$
+ COMPATIBILITY

FORMS CATEG. $A\text{-Bimod}(\mathcal{C})$

TAKE $\mathbb{1} := (\mathbb{1}, l_{\mathbb{1}} = r_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}}) \in \mathrm{Alg}(\mathcal{C})$. THEN:

$$\mathbb{1}\text{-Mod}(\mathcal{C}) \cong \text{Mod-}\mathbb{1}(\mathcal{C}) \cong \mathbb{1}\text{-Bimod}(\mathcal{C})$$

$\cong \mathcal{C}$ AS CATEGORIES

TAKE $(M, \triangleright_M : \mathbb{1} \otimes M \to M) \in \mathbb{1}\text{-Mod}$.

THEN: $\triangleright_M = l_M$ ($\leftarrow$ PART OF STRUCTURE OF $\mathcal{C}$)

$\therefore \ \mathbb{1}\text{-Mod}(\mathcal{C}) \cong \mathcal{C}$

LIKEWISE, $\text{Mod-}\mathbb{1} \cong \mathcal{C}$ SINCE $\triangleleft_M = r_M$.

I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

LEFT A-MODULE IN $\mathcal{C}$ IS:

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SAT. ASSOC & UNIT. AXIOMS

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ALSO,
$\mathbb{1}\text{-Bimod}(\mathcal{C}) \cong \mathcal{C}$
SINCE:

$$\begin{tikzcd}
(\mathbb{1} \otimes M) \otimes \mathbb{1} \arrow[rr, "a_{\mathbb{1}, M, \mathbb{1}}"] \arrow[d, "l_M \otimes \mathrm{id}"'] & & \mathbb{1} \otimes (M \otimes \mathbb{1}) \arrow[d, "\mathrm{id} \otimes r_M"] \\
M \otimes \mathbb{1} \arrow[dr, "r_M"'] & \circlearrowleft & \mathbb{1} \otimes M \arrow[dl, "l_M"] \\
& M &
\end{tikzcd}$$

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TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

LEFT A-MODULE IN $\mathcal{C}$ IS:
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SINCE:

$$\begin{tikzcd}
(\mathbb{1} \otimes M) \otimes \mathbb{1} \arrow[rr, "a_{\mathbb{1},M,\mathbb{1}}"] \arrow[d, "l_M \otimes \mathrm{id}"'] \arrow[drr, "r_{\mathbb{1}\otimes M}"'] & & \mathbb{1} \otimes (M \otimes \mathbb{1}) \arrow[d, "\mathrm{id} \otimes r_M"] \arrow[dl, phantom, "\text{EXER 3.1}\ \circlearrowleft"] \\
M \otimes \mathbb{1} \arrow[dr, "r_M"'] & & \mathbb{1} \otimes M \arrow[dl, "l_M"] \\
& M &
\end{tikzcd}$$

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LIKEWISE, $\mathrm{Mod}\text{-}\mathbb{1} \cong \mathcal{C}$ SINCE $\triangleleft_M = r_M$.

ALSO,

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SINCE:

$$
\begin{tikzcd}
(\mathbb{1}\otimes M)\otimes \mathbb{1} \arrow[rr, "a_{\mathbb{1},M,\mathbb{1}}"] \arrow[d, "l_M \otimes \mathrm{id}"'] \arrow[drr, "r_{\mathbb{1}\otimes M}"'] & & \mathbb{1}\otimes(M\otimes \mathbb{1}) \arrow[d, "\mathrm{id}\otimes r_M"] \arrow[dl, phantom, "\text{EXER 3.1} \circlearrowright"] \\
M \otimes \mathbb{1} \arrow[dr, "r_M"'] & \circlearrowright \ast & \mathbb{1}\otimes M \arrow[dl, "l_M"] \\
& M &
\end{tikzcd}
$$

(*) NATURALITY OF r AT l_M

I. (BI)MODULES IN MONOIDAL CATEGORIES

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EXERCISE 4.25 SHOW THAT:

(a) $\left(A \otimes X,\ \triangleright_{A\otimes X} : A \otimes (A \otimes X) \longrightarrow A \otimes X\right)$ FOR $X \in \mathcal{C}$

$$\begin{tikzcd} A \otimes (A \otimes X) \arrow[rr] \arrow[dr, "a^{-1}_{A,A,X}"'] & \text{DEF} & A \otimes X \\ & (A \otimes A) \otimes X \arrow[ur, "m \otimes id"'] & \end{tikzcd}$$

≡ FREE MODULE ≡ $\in A\text{-}\mathrm{Mod}(\mathcal{C})$

(b) $\mathrm{Free} : \mathcal{C} \longrightarrow A\text{-}\mathrm{Mod}(\mathcal{C})$
$X \longmapsto (A \otimes X, \triangleright_{A\otimes X})$
IS A FUNCTOR.

(c) $(\mathrm{Free}) \dashv \left(\mathrm{Forg} : A\text{-}\mathrm{Mod}(\mathcal{C}) \longrightarrow \mathcal{C},\ (M, \triangleright) \longmapsto M\right)$

(INSTANCE OF FREE-FORGET ADJUNCTION)

I. (BI)MODULES IN MONOIDAL CATEGORIES

LET'S EXPLORE THIS ON THE BOARD

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

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SAT. ASSOC & UNIT. AXIOMS

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SAT. ASSOC & UNIT. AXIOMS

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≡ FREE MODULE ≡

$\in A\text{-Mod}(\mathcal{C})$

(b) $\mathrm{Free} : \mathcal{C} \longrightarrow A\text{-Mod}(\mathcal{C})$

$X \longmapsto (A \otimes X,\ \triangleright_{A \otimes X})$

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SAT. ASSOC & UNIT. AXIOMS
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SAT. $(A, \triangleright) \in A\text{-}\mathrm{Mod}(\mathcal{C})$
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= FREE MODULE = $\in A\text{-}\mathrm{Mod}(\mathcal{C})$

ASSUME $\mathcal{C}$ STRICT WLOG. THEN NEED:

$$\begin{tikzcd} A \otimes A \otimes A \otimes X \arrow[r, "m_A \otimes id_A \otimes id_X"] \arrow[d, "id_A \otimes \triangleright_{A \otimes X}"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes A \otimes X \arrow[d, "\triangleright_{A \otimes X}"] \\ A \otimes A \otimes X \arrow[r, "\triangleright_{A \otimes X}"] & A \otimes X \end{tikzcd}$$

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "m_A \otimes id"] \arrow[d, "id \otimes \triangleright"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes M \arrow[d, "\triangleright"] \\ A \otimes M \arrow[r, "\triangleright"] & M \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{1} \otimes A \otimes X = A \otimes X \arrow[r, "u_A \otimes id_A \otimes id_X"] \arrow[dr, "id_{A \otimes X}"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & A \otimes A \otimes X \arrow[d, "\triangleright_{A \otimes X}"] \\ & A \otimes X \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{1} \otimes M \arrow[r, "u_A \otimes id"] \arrow[dr, "l_M"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & A \otimes M \arrow[d, "\triangleright"] \\ & M \end{tikzcd}$$

I. (BI)MODULES IN MONOIDAL CATEGORIES

LET'S EXPLORE THIS ON THE BOARD

TAKE $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

LEFT A-MODULE IN $\mathcal{C}$ IS:
(a) OBJECT M IN $\mathcal{C}$
(b) MAP $\triangleright : A \otimes M \to M$ IN $\mathcal{C}$
SAT. ASSOC & UNIT. AXIOMS
FORMS CATEG. $A\text{-Mod}(\mathcal{C})$

RIGHT A-MODULE IN $\mathcal{C}$ IS:
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$(A, \triangleleft) \in \text{Mod-}A(\mathcal{C})$
+ COMPATIBILITY
FORMS CATEG. $A\text{-Bimod}(\mathcal{C})$

EXERCISE 4.25 SHOW THAT:

(b) $\mathrm{Free} : \mathcal{C} \longrightarrow A\text{-Mod}(\mathcal{C})$

$$X \longmapsto \left(A \otimes X,\ \triangleright_{A \otimes X} := (m_A \otimes \mathrm{id}_X)\, a^{-1}_{A,A,X}\right)$$

IS A FUNCTOR

RECALL $\phi \in \mathrm{Hom}_{A\text{-Mod}(\mathcal{C})}(M, M')$

$\phi : M \to M' \in \mathcal{C}$ $\ni.$

$$\begin{tikzcd} A \otimes M \arrow[r, "\triangleright"] \arrow[d, "\mathrm{id} \otimes \phi"'] & M \arrow[d, "\phi"] \\ A \otimes M' \arrow[r, "\triangleright'"] & M' \end{tikzcd}$$

- FOR $f : X \to X' \in \mathcal{C}$, WHAT IS $\mathrm{Free}(f)$??

⋮

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RECALL $\phi \in \mathrm{Hom}_{A\text{-}\mathrm{Mod}(\mathcal{C})}(M, M')$
$\qquad \qquad \parallel\!\!\!\wr$
$\phi : M \to M' \in \mathcal{C}$ s.t.

$$\begin{tikzcd} A \otimes M \arrow[r, "\triangleright"] \arrow[d, "\mathrm{id} \otimes \phi"'] & M \arrow[d, "\phi"] \\ A \otimes M' \arrow[r, "\triangleright'"] & M' \end{tikzcd}$$

- FOR $f : X \to X' \in \mathcal{C}$, WHAT IS $\mathrm{Free}(f)$??
- DOES $\mathrm{Free}(\mathrm{id}_X) = \mathrm{id}_{\mathrm{Free}(X)}$ $\forall X \in \mathcal{C}$??

⋮

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- FOR $f : X \to X' \in \mathcal{C}$, WHAT IS $\mathrm{Free}(f)$??
- DOES $\mathrm{Free}(\mathrm{id}_X) = \mathrm{id}_{\mathrm{Free}(X)}$ $\forall X \in \mathcal{C}$??
- DOES $\mathrm{Free}(fg) = \mathrm{Free}(f)\,\mathrm{Free}(g)$??
$\forall$ COMPOSABLE $f, g \in \mathrm{Hom}(\mathcal{C})$

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EXERCISE 4.25 SHOW THAT:

(c) $(\mathrm{Free}) \dashv \left(\begin{array}{rcl} \mathrm{Forg} : A\text{-Mod}(\mathcal{C}) & \longrightarrow & \mathcal{C} \\ (M, \triangleright) & \longmapsto & M \end{array} \right)$

$$\left[\begin{array}{rcl} \mathrm{Free} : \mathcal{C} & \longrightarrow & A\text{-Mod}(\mathcal{C}) \\ X & \longmapsto & (A \otimes X,\ \triangleright_{A \otimes X} := (m_A \otimes \mathrm{id}_X)\, a^{-1}_{A,A,X}) \end{array} \right]$$

EITHER DEFINE —

$\boxed{\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \mathrm{Forg} \circ \mathrm{Free}}$ & $\boxed{\varepsilon : \mathrm{Free} \circ \mathrm{Forg} \Rightarrow \mathrm{Id}_{A\text{-Mod}(\mathcal{C})}}$

SATISFYING $\forall X \in \mathcal{C},\ Y \in A\text{-Mod}(\mathcal{C})$:

$F := \mathrm{Free}$
$G := \mathrm{Forg}$

$$\begin{tikzcd} F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \arrow[dl, phantom, "="] \\ & F(X) \end{tikzcd} \qquad \begin{tikzcd} G(Y) \arrow[r, "\eta_{G(Y)}"] \arrow[dr, "\mathrm{id}_{G(Y)}"'] & GFG(Y) \arrow[d, "G(\varepsilon_Y)"] \arrow[dl, phantom, "="] \\ & G(Y) \end{tikzcd}$$

I. (BI)MODULES IN MONOIDAL CATEGORIES

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OR SHOW $\boxed{\mathrm{Hom}_{A\text{-Mod}(\mathcal{C})}(\mathrm{Free}(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{Forg}(Y))}$

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WHICH DO YOU PREFER?

OR SHOW $\boxed{\mathrm{Hom}_{A\text{-}\mathrm{Mod}(\mathcal{C})}(\mathrm{Free}(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{Forg}(Y))}$

I. (BI)MODULES IN MONOIDAL CATEGORIES

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≡ RECALL ≡ "MODULES OVER CATEGORIES"

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A **RIGHT MODULE CATEGORY OVER $\mathcal{C}$** IS

$$(\underbrace{\mathcal{M}}_{\text{CATEGORY}},\ \underbrace{\triangleleft : \mathcal{M} \times \mathcal{C} \to \mathcal{M}}_{\text{ACTION BIFUNCTOR}},\ \underbrace{\alpha,\ \rho}_{\text{ASSOC. UNIT. CONSTRAINTS}})$$

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A RIGHT MODULE CATEGORY OVER $\mathcal{C}$ IS

$$(\underset{\substack{\uparrow\\ \text{CATEGORY}}}{\mathcal{M}},\ \underset{\substack{\uparrow\\ \text{ACTION BIFUNCTOR}}}{\triangleleft : \mathcal{M} \times \mathcal{C} \longrightarrow \mathcal{M}},\ \underset{\substack{\uparrow\\ \text{ASSOC.}}}{n},\ \underset{\substack{\uparrow\\ \text{UNIT. CONSTRAINTS}}}{q})$$

+ COMPATIBILITY

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PROP "MODULES IN $\mathcal{C}$ ARE MODULES OVER $\mathcal{C}$"

$A\text{-Mod}(\mathcal{C}) \in \text{Mod-}\mathcal{C}$ VIA

$$\triangleleft : A\text{-Mod}(\mathcal{C}) \times \mathcal{C} \longrightarrow A\text{-Mod}(\mathcal{C})$$

$$((M, \triangleright), X) \longmapsto \left(M \otimes X,\ \blacktriangleright : A \otimes (M \otimes X) \xrightarrow{a^{-1}_{A,M,X}} (A \otimes M) \otimes X \xrightarrow{\triangleright \otimes \mathrm{id}_X} M \otimes X\right)$$

DEF

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DEF

$\text{Mod-}A(\mathcal{C}) \in \mathcal{C}\text{-Mod}$ VIA

$$\triangleright : \mathcal{C} \times \text{Mod-}A(\mathcal{C}) \longrightarrow \text{Mod-}A(\mathcal{C})$$

$$(X, (M, \triangleleft)) \longmapsto \left(X \otimes M,\ \blacktriangleleft : (X \otimes M) \otimes A \xrightarrow{a_{X,M,A}} X \otimes (M \otimes A) \xrightarrow{\mathrm{id}_X \otimes \triangleleft} X \otimes M\right)$$

DEF

A **LEFT MODULE CATEGORY OVER** $\mathcal{C}$ IS

$(\mathcal{M},\ \triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M},\ \mu,\ p)$ + COMPATIBILITY

CATEGORY — ACTION BIFUNCTOR — ASSOC. UNIT. CONSTRAINTS

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FORMS CATEG. $A\text{-}\mathrm{Mod}(\mathcal{C})$

RIGHT A-MODULE IN $\mathcal{C}$ IS:
(a) OBJECT M IN $\mathcal{C}$
(b) MAP $\triangleleft : M \otimes A \to M$ IN $\mathcal{C}$

SAT. ASSOC & UNIT. AXIOMS

FORMS CATEG. $\mathrm{Mod}\text{-}A(\mathcal{C})$

A-BIMODULE IN $\mathcal{C}$ IS:
(a) OBJECT M IN $\mathcal{C}$
(b) MAPS $\triangleright, \triangleleft$ IN $\mathcal{C}$

SAT. $(A, \triangleright) \in A\text{-}\mathrm{Mod}(\mathcal{C})$
$(A, \triangleleft) \in \mathrm{Mod}\text{-}A(\mathcal{C})$
+ COMPATIBILITY

FORMS CATEG. $A\text{-}\mathrm{Bimod}(\mathcal{C})$

PROP "MODULES IN $\mathcal{C}$ ARE MODULES OVER $\mathcal{C}$"

$A\text{-}\mathrm{Mod}(\mathcal{C}) \in \mathrm{Mod}\text{-}\mathcal{C}$ VIA

$$\triangleleft : A\text{-}\mathrm{Mod}(\mathcal{C}) \times \mathcal{C} \longrightarrow A\text{-}\mathrm{Mod}(\mathcal{C})$$

$$((M, \triangleright), X) \longmapsto \left(M \otimes X,\ \blacktriangleright : A \otimes (M \otimes X) \xrightarrow{a^{-1}_{A,M,X}} (A \otimes M) \otimes X \xrightarrow{\triangleright \otimes \mathrm{id}_X} M \otimes X\right)$$

DEF

$\mathrm{Mod}\text{-}A(\mathcal{C}) \in \mathcal{C}\text{-}\mathrm{Mod}$ VIA

$$\triangleright : \mathcal{C} \times \mathrm{Mod}\text{-}A(\mathcal{C}) \longrightarrow \mathrm{Mod}\text{-}A(\mathcal{C})$$

$$(X, (M, \triangleleft)) \longmapsto \left(X \otimes M,\ \blacktriangleleft : (X \otimes M) \otimes A \xrightarrow{a_{X,M,A}} X \otimes (M \otimes A) \xrightarrow{\mathrm{id}_X \otimes \triangleleft} X \otimes M\right)$$

DEF

PROOF ≡ EXERCISE 4.28

A **LEFT MODULE CATEGORY OVER** $\mathcal{C}$ IS

$$(\mathcal{M},\ \triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M},\ m,\ p)$$

CATEGORY — ACTION BIFUNCTOR — ASSOC. UNIT. CONSTRAINTS

+ COMPATIBILITY

I. (BI)MODULES IN MONOIDAL CATEGORIES

TAKE $(A, m, u) \in \mathrm{Alg}(\mathscr{C})$.

LEFT A-MODULE IN $\mathscr{C}$ IS:
(a) OBJECT M IN $\mathscr{C}$
(b) MAP $\triangleright : A \otimes M \to M$ IN $\mathscr{C}$
SAT. ASSOC & UNIT. AXIOMS
FORMS CATEG. $A\text{-}\mathrm{Mod}(\mathscr{C})$

RIGHT A-MODULE IN $\mathscr{C}$ IS:
(a) OBJECT M IN $\mathscr{C}$
(b) MAP $\triangleleft : M \otimes A \to M$ IN $\mathscr{C}$
SAT. ASSOC & UNIT. AXIOMS
FORMS CATEG. $\mathrm{Mod}\text{-}A(\mathscr{C})$

A-BIMODULE IN $\mathscr{C}$ IS:
(a) OBJECT M IN $\mathscr{C}$
(b) MAPS $\triangleright, \triangleleft$ IN $\mathscr{C}$
SAT. $(A, \triangleright) \in A\text{-}\mathrm{Mod}(\mathscr{C})$
$(A, \triangleleft) \in \mathrm{Mod}\text{-}A(\mathscr{C})$
+ COMPATIBILITY
FORMS CATEG. $A\text{-}\mathrm{Bimod}(\mathscr{C})$

READ ABOUT WHEN THESE CATEGORIES SATISFY:

$$\begin{tikzcd}
& \boxed{\text{PREADDITIVE}} \arrow[dl] \arrow[dr] & \\
\boxed{\text{ADDITIVE}} \arrow[d] & & \boxed{\Bbbk\text{-LINEAR}} \\
\boxed{\text{ABELIAN}} & &
\end{tikzcd}$$

II. MONADS

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1})$ STRICT.

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SATISFYING:

(ASSOC) $m(m \otimes id_A) = m(id_A \otimes m)$

(UNIT.) $m(u \otimes id_A) = id_A$

$m(id_A \otimes u) = id_A$

FORMS CATEGORY
$\mathrm{Alg}(\mathcal{C})$

WITH:

$\phi \in \mathrm{Hom}_{\mathrm{Alg}(\mathcal{C})}(A, A')$

$\parallel\!\!\!|$

$\phi : A \longrightarrow A' \in \mathcal{C}$

$\ni.$

$\phi m = m'(\phi \otimes \phi)$

$\phi u = u'$

II. MONADS

TAKE A (NOT NEC. MONOIDAL) CATEGORY $\mathcal{A}$.

A MONAD ON $\mathcal{A}$ IS BY DEFINITION

AN ALGEBRA IN $(\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$ STRICT

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1})$ STRICT.

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \longrightarrow A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \longrightarrow A \in \mathcal{C}$

SATISFYING:

(ASSOC) $m(m \otimes id_A) = m(id_A \otimes m)$

(UNIT.) $m(u \otimes id_A) = id_A$

$m(id_A \otimes u) = id_A$

FORMS CATEGORY $\mathrm{Alg}(\mathcal{C})$

WITH:

$$\phi \in \mathrm{Hom}_{\mathrm{Alg}(\mathcal{C})}(A, A')$$
$$\parallel\!\!\mid$$
$$\phi : A \longrightarrow A' \in \mathcal{C}$$
$$\ni.$$
$$\phi m = m'(\phi \otimes \phi)$$
$$\phi u = u'$$

II. MONADS

TAKE A (NOT NEC. MONOIDAL) CATEGORY $\mathcal{A}$.

A MONAD ON $\mathcal{A}$ IS BY DEFINITION

AN ALGEBRA IN $(\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$ ← STRICT

THAT IS:

AN ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$ EQUIPPED W/

NAT'L TRANSF'NS $\mu: T \circ T \Rightarrow T$

$\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

SATISFYING

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1})$ ← STRICT.

(a) OBJECT $A \in \mathcal{C}$

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FORMS CATEGORY

$\mathrm{Alg}(\mathcal{C})$

WITH:

$\phi \in \mathrm{Hom}_{\mathrm{Alg}(\mathcal{C})}(A, A')$

$\rotatebox{90}{$\cong$}$

$\phi: A \to A' \in \mathcal{C}$

s.t.

$\phi m = m'(\phi \otimes \phi)$

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II. MONADS

TAKE A (NOT NEC. MONOIDAL) CATEGORY $\mathcal{A}$.

A **MONAD ON $\mathcal{A}$** IS BY DEFINITION

AN ALGEBRA IN $(\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$ ← STRICT

THAT IS:

AN ENDOFUNCTOR $T : \mathcal{A} \to \mathcal{A}$ EQUIPPED W/

NAT'L TRANSF'NS $\mu : T \circ T \Rightarrow T$

$\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

SATISFYING $\forall X \in \mathcal{A}$:

$T^n := \overbrace{T \circ \cdots \circ T}^{n}$

$$\begin{tikzcd}
T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "2"] & T^2(X) \arrow[d, "\mu_X"] \\
T^2(X) \arrow[r, "\mu_X"] & T(X)
\end{tikzcd}$$

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1})$ ← STRICT.

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \to A \in \mathcal{C}$

SATISFYING:

(ASSOC) $\boxed{m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m)}$

(UNIT.) $m(u \otimes \mathrm{id}_A) = \mathrm{id}_A$

$m(\mathrm{id}_A \otimes u) = \mathrm{id}_A$

FORMS CATEGORY $\mathrm{Alg}(\mathcal{C})$

WITH:

$\phi \in \mathrm{Hom}_{\mathrm{Alg}(\mathcal{C})}(A, A')$

$\equiv$

$\phi : A \to A' \in \mathcal{C}$

$\ni$

$\phi m = m'(\phi \otimes \phi)$

$\phi u = u'$

II. MONADS

TAKE A (NOT NEC. MONOIDAL) CATEGORY $\mathcal{A}$.

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T^2(X) \arrow[r, "\mu_X"'] & T(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\
 & T(X)
\end{tikzcd}$$

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1})$ ← STRICT.

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

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SATISFYING:

(ASSOC) $m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m)$

(UNIT.) $\boxed{m(u \otimes \mathrm{id}_A) = \mathrm{id}_A}$

$m(\mathrm{id}_A \otimes u) = \mathrm{id}_A$

FORMS CATEGORY $\mathrm{Alg}(\mathcal{C})$

WITH:

$\phi \in \mathrm{Hom}_{\mathrm{Alg}(\mathcal{C})}(A, A')$

$\parallel$

$\phi: A \to A' \in \mathcal{C}$

s.t.

$\phi m = m'(\phi \otimes \phi)$

$\phi u = u'$

II. MONADS

TAKE A (NOT NEC. MONOIDAL) CATEGORY $\mathcal{A}$.

A MONAD ON $\mathcal{A}$ IS BY DEFINITION

AN ALGEBRA IN $(\mathrm{End}(\mathcal{A}), \circ, \mathrm{Id}_{\mathcal{A}})$ ← STRICT

THAT IS:

AN ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$ EQUIPPED W/

NAT'L TRANSF'NS $\mu : T \circ T \Rightarrow T$

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SATISFYING $\forall X \in \mathcal{A}$:

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T^2(X) \arrow[r, "\mu_X"] & T(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] \arrow[dr, phantom, "2", near start, shift left=3] & T^2(X) \arrow[d, "\mu_X"] \\
& T(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
T(X) \arrow[r, "T(\eta_X)"] \arrow[dr, "\mathrm{id}_{T(X)}"'] \arrow[dr, phantom, "2", near start, shift left=3] & T^2(X) \arrow[d, "\mu_X"] \\
& T(X)
\end{tikzcd}$$

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1})$ ← STRICT.

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \to A \in \mathcal{C}$

SATISFYING:

(ASSOC) $m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m)$

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FORMS CATEGORY $\mathsf{Alg}(\mathcal{C})$

WITH:

$\phi \in \mathrm{Hom}_{\mathsf{Alg}(\mathcal{C})}(A, A')$

$\parallel$

$\phi : A \to A' \in \mathcal{C}$

$\ni.$

$\phi m = m'(\phi \otimes \phi)$

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II. MONADS

TAKE A (NOT NEC. MONOIDAL) CATEGORY $\mathcal{A}$.

A MONAD ON $\mathcal{A}$ IS BY DEFINITION

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T^2(X) \arrow[r, "\mu_X"] & T(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\
& T(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
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& T(X)
\end{tikzcd}$$

HAVE CATEGORY $\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1})$ STRICT.

(a) OBJECT $A \in \mathcal{C}$

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$m(\mathrm{id}_A \otimes u) = \mathrm{id}_A$

FORMS CATEGORY $\mathrm{Alg}(\mathcal{C})$

WITH:

$\phi \in \mathrm{Hom}_{\mathrm{Alg}(\mathcal{C})}(A, A')$

$\parallel\!\!\!\;|$

$\phi : A \to A' \in \mathcal{C}$

$\ni.$

$\phi m = m'(\phi \otimes \phi)$

$\phi u = u'$

II. MONADS

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

$$\left[\begin{array}{l} \text{ENDOFUNCTOR } T : \mathcal{A} \to \mathcal{A} \\ \text{NAT TRANS: } \mu : T \circ T \Rightarrow T \\ \qquad\qquad\qquad\; \eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T \end{array}\right.$$

SATISFYING $\forall X \in \mathcal{A}$:

$$\begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "2"] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\ & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "T(\eta_X)"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\ & T(X) \end{tikzcd}$$

ALGEBRA IN $(\mathcal{C}, \otimes, \mathbb{1})$ STRICT.

(a) OBJECT $A \in \mathcal{C}$

(b) $m := m_A : A \otimes A \to A \in \mathcal{C}$

(c) $u := u_A : \mathbb{1} \to A \in \mathcal{C}$

SATISFYING:

(ASSOC) $m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m)$

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FORMS CATEGORY

$\mathrm{Alg}(\mathcal{C})$

WITH:

$\phi \in \mathrm{Hom}_{\mathrm{Alg}(\mathcal{C})}(A, A')$

$\mathrel{\rotatebox{90}{$\equiv$}}$

$\phi : A \to A' \in \mathcal{C}$

s.t.

$\phi m = m'(\phi \otimes \phi)$

$\phi u = u'$

II. MONADS

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

ENDOFUNCTOR $T : \mathcal{A} \to \mathcal{A}$

NAT TRANS: $\mu : T \circ T \Rightarrow T$

$\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

SATISFYING $\forall X \in \mathcal{A}$:

$$\begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "="] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "="] \\ & T(X) \end{tikzcd}$$

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EXAMPLES

II. MONADS

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

$\begin{cases} \text{ENDOFUNCTOR } T: \mathcal{A} \to \mathcal{A} \\ \text{NAT TRANS: } \mu: T \circ T \Rightarrow T \\ \qquad\qquad\quad \eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T \end{cases}$

SATISFYING $\forall X \in \mathcal{A}$:

$$\begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "2"] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

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EXAMPLES

IDENTITY MONAD ON $\mathcal{A}$

$T := \mathrm{Id}_{\mathcal{A}} \qquad \underset{\mathrm{id}_X}{\overset{\mu_X}{\shortparallel}} : X \longrightarrow X \qquad \underset{\mathrm{id}_X}{\overset{\eta_X}{\shortparallel}} : X \longrightarrow X$

$\forall X \in \mathcal{A}$

II. MONADS

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

- ENDOFUNCTOR $T : \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu : T \circ T \Rightarrow T$
- $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

SATISFYING $\forall X \in \mathcal{A}$:

$$\begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "2"] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\ & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "T(\eta_X)"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\ & T(X) \end{tikzcd}$$

EXAMPLES

IDENTITY MONAD ON $\mathcal{A}$

$T := \mathrm{Id}_{\mathcal{A}}$ $\quad \underset{\mathrm{id}_X}{\underset{\shortparallel}{\mu_X}} : X \to X \quad \underset{\mathrm{id}_X}{\underset{\shortparallel}{\eta_X}} : X \to X$ $\quad \forall X \in \mathcal{A}$

GIVEN $A \in \mathrm{Alg}(\mathcal{C}, \otimes, \mathbb{1})$ STRICT

GET $(A \otimes -) \in \mathrm{Monad}(\mathcal{C})$ WITH

$$\mu_X : A \otimes A \otimes X \xrightarrow{??} A \otimes X$$

$$\eta_X : X \xrightarrow{??} A \otimes X$$

$\forall X \in \mathcal{C}$

II. MONADS

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu: T \circ T \Rightarrow T$
- $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

SATISFYING $\forall X \in \mathcal{A}$:

$$\begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "2"] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\ & T(X) \end{tikzcd}$$

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EXAMPLES

IDENTITY MONAD ON $\mathcal{A}$

$T := \mathrm{Id}_{\mathcal{A}}$ $\quad$ $\mu_X : X \to X$ $\quad$ $\eta_X : X \to X$

$\mu_X = \mathrm{id}_X$ $\quad$ $\eta_X = \mathrm{id}_X$ $\quad$ $\forall X \in \mathcal{A}$

GIVEN $A \in \mathrm{Alg}(\mathcal{C}, \otimes, \mathbb{1})$ STRICT

GET $(A \otimes -) \in \mathrm{Monad}(\mathcal{C})$ WITH

$$\mu_X : A \otimes A \otimes X \xrightarrow{m_A \otimes \mathrm{id}_X} A \otimes X$$

$$\eta_X : X \xrightarrow{u_A \otimes \mathrm{id}_X} A \otimes X$$

$\forall X \in \mathcal{C}$

II. MONADS

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

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$$\left[\begin{array}{l} \text{ENDOFUNCTOR } T: \mathcal{A} \to \mathcal{A} \\ \text{NAT TRANS: } \mu: T \circ T \Rightarrow T \\ \qquad\qquad\quad\; \eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T \end{array}\right.$$

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\end{tikzcd}$$

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T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\
 & T(X)
\end{tikzcd}$$

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 & T(X)
\end{tikzcd}$$

EXAMPLES

IDENTITY MONAD ON $\mathcal{A}$

$T := \mathrm{Id}_{\mathcal{A}}$ $\qquad$ $\underset{\mathrm{id}_X}{\overset{\mu_X}{\shortparallel}} : X \to X$ $\qquad$ $\underset{\mathrm{id}_X}{\overset{\eta_X}{\shortparallel}} : X \to X$

$\forall X \in \mathcal{A}$

DETAILS ≡ EXERCISE 4.20

GIVEN $A \in \mathrm{Alg}(\mathcal{C}, \otimes, \mathbb{1})$ STRICT

GET $(A \otimes -) \in \mathrm{Monad}(\mathcal{C})$ WITH

$$\mu_X : A \otimes A \otimes X \xrightarrow{m_A \otimes \mathrm{id}_X} A \otimes X$$

$$\eta_X : X \xrightarrow{u_A \otimes \mathrm{id}_X} A \otimes X$$

$\forall X \in \mathcal{C}$

II. MONADS

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu: T \circ T \Rightarrow T$
- $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

SATISFYING $\forall X \in \mathcal{A}$:

$$\begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "2"] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\ & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "T(\eta_X)"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\ & T(X) \end{tikzcd}$$

EXAMPLES

MOST IMPORTANT

TAKE ADJUNCTION

$(F: \mathcal{A} \longrightarrow \mathcal{B}) \dashv (G: \mathcal{B} \longrightarrow \mathcal{A})$

UNIT $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow GF$

COUNIT $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{B}}$

DETAILS $\equiv$ EXERCISE 4.20

GIVEN $A \in \mathrm{Alg}(\mathcal{C}, \otimes, \mathbb{1})$ STRICT

GET $(A \otimes -) \in \mathrm{Monad}(\mathcal{C})$ WITH

$$\mu_X : A \otimes A \otimes X \xrightarrow{m_A \otimes \mathrm{id}_X} A \otimes X$$

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$\forall X \in \mathcal{C}$

II. MONADS

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

- ENDOFUNCTOR $T : \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu : T \circ T \Rightarrow T$
- $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

SATISFYING $\forall X \in \mathcal{A}$:

$$\begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "\circlearrowleft"] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "id_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "T(\eta_X)"] \arrow[dr, "id_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & T(X) \end{tikzcd}$$

EXAMPLES

MOST IMPORTANT

TAKE ADJUNCTION

$(F : \mathcal{A} \to \mathcal{B}) \dashv (G : \mathcal{B} \to \mathcal{A})$

UNIT $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow GF$

COUNIT $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{B}}$

GET $GF \in \mathrm{Monad}(\mathcal{A})$ WITH

$$\mu : GFGF \overset{??}{\Longrightarrow} GF \quad \& \quad \eta : \mathrm{Id}_{\mathcal{A}} \overset{??}{\Longrightarrow} GF$$

DETAILS ≡ EXERCISE 4.20

GIVEN $A \in \mathrm{Alg}(\mathcal{C}, \otimes, \mathbb{1})$ STRICT

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$$\mu_X : A \otimes A \otimes X \xrightarrow{m_A \otimes id_X} A \otimes X$$

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$\forall X \in \mathcal{C}$

II. MONADS

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu: T \circ T \Rightarrow T$
- $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

SATISFYING $\forall X \in \mathcal{A}$:

$$\begin{tikzcd}
T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "2"] & T^2(X) \arrow[d, "\mu_X"] \\
T^2(X) \arrow[r, "\mu_X"] & T(X)
\end{tikzcd}$$

$$\begin{tikzcd}
T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\
 & T(X)
\end{tikzcd}$$

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T(X) \arrow[r, "T(\eta_X)"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\
 & T(X)
\end{tikzcd}$$

EXAMPLES

MOST IMPORTANT

TAKE ADJUNCTION

$(F: \mathcal{A} \to \mathcal{B}) \dashv (G: \mathcal{B} \to \mathcal{A})$

UNIT $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow GF$

COUNIT $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{B}}$

GET $GF \in \mathrm{Monad}(\mathcal{A})$ WITH

NO ABUSE OF NOTATION ☺

$\mu: GFGF \xRightarrow{G\varepsilon F} GF$ & $\eta: \mathrm{Id}_{\mathcal{A}} \xRightarrow{\eta} GF$

DETAILS ≡ EXERCISE 4.20

GIVEN $A \in \mathrm{Alg}(\mathcal{C}, \otimes, \mathbb{1})$ STRICT

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$\forall X \in \mathcal{C}$

II. MONADS

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu: T \circ T \Rightarrow T$, $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

SATISFYING $\forall X \in \mathcal{A}$:

$$\begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "2"] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\ & T(X) \end{tikzcd}$$

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EXAMPLES

MOST IMPORTANT

TAKE ADJUNCTION

$(F: \mathcal{A} \to \mathcal{B}) \dashv (G: \mathcal{B} \to \mathcal{A})$

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GET $GF \in \mathrm{Monad}(\mathcal{A})$ WITH

$\mu: GFGF \xRightarrow{G\varepsilon F} GF$ & $\eta: \mathrm{Id}_{\mathcal{A}} \xRightarrow{\eta} GF$

NO ABUSE OF NOTATION ☺

DETAILS ≡ EXERCISE 4.24

GIVEN $A \in \mathrm{Alg}(\mathcal{C}, \otimes, \mathbb{1})$ STRICT

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$\eta_X: X \xrightarrow{u_A \otimes \mathrm{id}_X} A \otimes X$

$\forall X \in \mathcal{C}$

II. MONADS

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

- ENDOFUNCTOR $T : \mathcal{A} \to \mathcal{A}$
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EXAMPLES

MOST IMPORTANT

TAKE ADJUNCTION

$(F : \mathcal{A} \to \mathcal{B}) \dashv (G : \mathcal{B} \to \mathcal{A})$

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NO ABUSE OF NOTATION ☺

THIS IS A SPECIAL CASE VIA EXERCISE 4.25

DETAILS ≡ EXERCISE 4.24

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II. MONADS

TAKE A CATEGORY $\mathcal{A}$.

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OBJECTS:

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
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$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \\ & T(X) \end{tikzcd} \quad 2$$

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EXAMPLES

MOST IMPORTANT

FUN EXAMPLES ≡ EXERCISES 4.21 – 4.23

TAKE ADJUNCTION

$(F: \mathcal{A} \to \mathcal{B}) \dashv (G: \mathcal{B} \to \mathcal{A})$

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$\forall X \in \mathcal{C}$

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

Monad($\mathcal{A}$) := Alg(End($\mathcal{A}$))

OBJECTS:

- ENDOFUNCTOR $T : \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu : T \circ T \Rightarrow T$
- $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

SATISFYING $\forall X \in \mathcal{A}$:

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T^2(X) \arrow[r, "\mu_X"] & T(X)
\end{tikzcd}$$

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\end{tikzcd}$$

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TAKE ADJUNCTION

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UNIT $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow GF$

COUNIT $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{B}}$

GET $GF \in$ Monad($\mathcal{A}$) WITH

$$\mu : GFGF \xRightarrow{G\varepsilon F} GF \quad \& \quad \eta : \mathrm{Id}_{\mathcal{A}} \xRightarrow{\eta} GF$$

ADJUNCTIONS $\rightsquigarrow$ MONADS

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

$\left[\begin{array}{l} \text{ENDOFUNCTOR } T : \mathcal{A} \to \mathcal{A} \\ \text{NAT TRANS: } \mu : T \circ T \Rightarrow T \\ \qquad\qquad\qquad \eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T \end{array}\right.$

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$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "\circlearrowright"] \\ & T(X) \end{tikzcd}$$

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TAKE ADJUNCTION

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$$\begin{tikzcd} \text{ADJUNCTIONS} \arrow[rr, bend left] & & \text{MONADS} \arrow[ll, bend left, "\text{WILL SEE}"] \end{tikzcd}$$

III. EILENBERG-MOORE CATEGORIES

A NICE SUBCATEGORY OF MODULES OVER MONADS

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

- ENDOFUNCTOR $T : \mathcal{A} \to \mathcal{A}$
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SATISFYING $\forall X \in \mathcal{A}$:

$$\begin{tikzcd}
T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "2"] & T^2(X) \arrow[d, "\mu_X"] \\
T^2(X) \arrow[r, "\mu_X"] & T(X)
\end{tikzcd}$$

$$\begin{tikzcd}
T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\
 & T(X)
\end{tikzcd}$$

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 & T(X)
\end{tikzcd}$$

$$\begin{tikzcd}
\text{ADJUNCTIONS} \arrow[r, bend left] & \text{MONADS} \arrow[l, bend left]
\end{tikzcd}$$

WILL SEE

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu: T \circ T \Rightarrow T$
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T^2(X) \arrow[r, "\mu_X"] & T(X)
\end{tikzcd}$$

$$\begin{tikzcd}
T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\
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\end{tikzcd}$$

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\end{tikzcd}$$

A NICE SUBCATEGORY OF MODULES OVER MONADS

$T\text{-}\mathrm{Mod}(\mathrm{End}(\mathcal{A}))$ FOR $\mathbb{T} := (T, \mu, \eta)$ IS QUITE LARGE

$$\begin{tikzcd}
\text{ADJUNCTIONS} \arrow[r, bend left] & \text{MONADS} \arrow[l, bend left, "\text{WILL SEE}"]
\end{tikzcd}$$

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

- ENDOFUNCTOR $T : \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu : T \circ T \Rightarrow T$
  $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

SATISFYING $\forall X \in \mathcal{A}$:

$$\begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "2"] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\ & T(X) \end{tikzcd}$$

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A NICE SUBCATEGORY OF MODULES OVER MONADS

$T\text{-}\mathrm{Mod}(\mathrm{End}(\mathcal{A}))$ FOR $\mathbb{T} := (T, \mu, \eta)$ IS QUITE LARGE

MUCH BETTER THEORY FOR EM CATEGORIES

ADJUNCTIONS $\rightleftarrows$ MONADS

WILL SEE

III. EILENBERG-MOORE CATEGORIES

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$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\ & T(X) \end{tikzcd}$$

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A NICE SUBCATEGORY OF MODULES OVER MONADS

$T\text{-}\mathrm{Mod}(\mathrm{End}(\mathcal{A}))$ FOR $\mathbb{T} := (T, \mu, \eta)$ IS QUITE LARGE

MUCH BETTER THEORY FOR EM CATEGORIES

ADJUNCTIONS

MONADS

WILL SEE

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

$\left[\begin{array}{l} \text{ENDOFUNCTOR } T : \mathcal{A} \to \mathcal{A} \\ \text{NAT TRANS}: \mu : T \circ T \Rightarrow T \\ \qquad\qquad\qquad \eta : Id_{\mathcal{A}} \Rightarrow T \end{array}\right.$

SATISFYING $\forall X \in \mathcal{A}$:

$$\begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "2"] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

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GIVEN A MONAD (T, μ, η) ON $\mathcal{A}$,

AN EM OBJECT IS AN OBJECT $Y \in \mathcal{A}$ EQUIPPED

WITH A MORPHISM $\xi := \xi_Y : T(Y) \longrightarrow Y \in \mathcal{A}$

SUCH THAT

III. EILENBERG–MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

$\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$

OBJECTS:

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$\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

SATISFYING $\forall X \in \mathcal{A}$:

$$\begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "2"] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\ & T(X) \end{tikzcd}$$

$$\begin{tikzcd} T(X) \arrow[r, "T(\eta_X)"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "2"] \\ & T(X) \end{tikzcd}$$

GIVEN A MONAD (T, μ, η) ON $\mathcal{A}$,

AN **EM OBJECT** IS AN OBJECT $Y \in \mathcal{A}$ EQUIPPED

WITH A MORPHISM $\xi := \xi_Y : T(Y) \to Y \in \mathcal{A}$

SUCH THAT

$$\begin{tikzcd} T^2(Y) \arrow[r, "\mu_Y"] \arrow[d, "T(\xi)"'] \arrow[dr, phantom, "2"] & T(Y) \arrow[d, "\xi"] \\ T(Y) \arrow[r, "\xi"] & Y \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[dr, "\mathrm{id}_Y"'] & T(Y) \arrow[d, "\xi"] \arrow[dl, phantom, "2"] \\ & Y \end{tikzcd}$$

III. EILENBERG-MOORE CATEGORIES

> TAKE A CATEGORY $\mathcal{A}$.
>
> $\mathrm{Monad}(\mathcal{A}) := \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$
>
> OBJECTS:
>
> ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
>
> NAT TRANS: $\mu: T \circ T \Rightarrow T$
>
> $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$
>
> SATISFYING $\forall X \in \mathcal{A}$:
>
> $$\begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] \arrow[dr, phantom, "\circlearrowleft"] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$
>
> $$\begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & T(X) \end{tikzcd}$$
>
> $$\begin{tikzcd} T(X) \arrow[r, "T(\eta_X)"] \arrow[dr, "\mathrm{id}_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & T(X) \end{tikzcd}$$

GIVEN A MONAD (T, μ, η) ON $\mathcal{A}$,

AN **EM OBJECT** IS AN OBJECT $Y \in \mathcal{A}$ EQUIPPED

WITH A MORPHISM $\xi := \xi_Y : T(Y) \to Y \in \mathcal{A}$

SUCH THAT

$$\begin{tikzcd} T^2(Y) \arrow[r, "\mu_Y"] \arrow[d, "T(\xi)"'] \arrow[dr, phantom, "\circlearrowleft"] & T(Y) \arrow[d, "\xi"] \\ T(Y) \arrow[r, "\xi"] & Y \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[dr, "\mathrm{id}_Y"'] & T(Y) \arrow[d, "\xi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & Y \end{tikzcd}$$

$\boxed{\text{THESE FORM THE } \textbf{EM CATEGORY } \mathcal{A}^T}$ WITH

$$\phi \in \mathrm{Hom}_{\mathcal{A}^T}\big((Y, \xi), (Y', \xi')\big)$$

$\parallel\!\!\mid$

$$\phi : Y \to Y' \in \mathcal{A} \;\; \ni. \quad \begin{tikzcd} T(Y) \arrow[r, "\xi"] \arrow[d, "T(\phi)"'] & Y \arrow[d, "\phi"] \\ T(Y') \arrow[r, "\xi'"] & Y' \end{tikzcd}$$

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

ENDOFUNCTOR $T : \mathcal{A} \to \mathcal{A}$

NAT TRANS: $\mu : T \circ T \Rightarrow T$

$\eta : Id_{\mathcal{A}} \Rightarrow T$

s.t. $\forall X \in \mathcal{A}$:

$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

$$\mu_X \circ \eta_{T(X)} = id_{T(X)}$$

$$\mu_X \circ T(\eta_X) = id_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi : T(Y) \to Y \in \mathcal{A})$

s.t. $\xi \circ \mu_Y = \xi \circ T(\xi)$

$\xi \circ \eta_Y = id_Y$

MORPHISMS: $\phi : Y \to Y' \in \mathcal{A}$

s.t. $\phi \circ \xi = \xi' \circ T(\phi)$

EXAMPLES

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
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$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

$$\mu_X \circ \eta_{T(X)} = \mathrm{id}_{T(X)}$$

$$\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi: T(Y) \to Y \in \mathcal{A})$

$\ni$. $\xi \circ \mu_Y = \xi \circ T(\xi)$

$\xi \circ \eta_Y = \mathrm{id}_Y$

MORPHISMS: $\phi: Y \to Y' \in \mathcal{A}$

$\ni$. $\phi \circ \xi = \xi' \circ T(\phi)$

EXAMPLES

IDENTITY MONAD ON $\mathcal{A}$

$$T := \mathrm{Id}_{\mathcal{A}} \qquad \underset{\mathrm{id}_X}{\overset{\mu_X}{\parallel}} : X \longrightarrow X \qquad \underset{\mathrm{id}_X}{\overset{\eta_X}{\parallel}} : X \longrightarrow X$$

$\forall X \in \mathcal{A}$

WHAT IS $\mathcal{A}^{\mathrm{Id}}$??

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu: T \circ T \Rightarrow T$
- $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

∋. $\forall X \in \mathcal{A}$:

$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

$$\mu_X \circ \eta_{T(X)} = \mathrm{id}_{T(X)}$$

$$\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi: T(Y) \to Y \in \mathcal{A})$

∋. $\xi \circ \mu_Y = \xi \circ T(\xi)$

$\xi \circ \eta_Y = \mathrm{id}_Y$

MORPHISMS: $\phi: Y \to Y' \in \mathcal{A}$

∋. $\phi \circ \xi = \xi' \circ T(\phi)$

EXAMPLES

$\mathcal{A}^{\mathrm{Id}} \cong \mathcal{A}$

IDENTITY MONAD ON $\mathcal{A}$

$$T := \mathrm{Id}_{\mathcal{A}} \qquad \underset{\mathrm{id}_X}{\underset{\shortparallel}{\mu_X}} : X \to X \qquad \underset{\mathrm{id}_X}{\underset{\shortparallel}{\eta_X}} : X \to X \qquad \forall X \in \mathcal{A}$$

WHAT IS $\mathcal{A}^{\mathrm{Id}}$??

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu: T \circ T \Rightarrow T$
- $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

s.t. $\forall X \in \mathcal{A}$:

$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

$$\mu_X \circ \eta_{T(X)} = \mathrm{id}_{T(X)}$$

$$\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi: T(Y) \to Y \in \mathcal{A})$

s.t. $\xi \circ \mu_Y = \xi \circ T(\xi)$

$\xi \circ \eta_Y = \mathrm{id}_Y$

MORPHISMS: $\phi: Y \to Y' \in \mathcal{A}$

s.t. $\phi \circ \xi = \xi' \circ T(\phi)$

EXAMPLES

$\mathcal{A}^{\mathrm{Id}} \cong \mathcal{A}$

IDENTITY MONAD ON $\mathcal{A}$

$T := \mathrm{Id}_{\mathcal{A}}$ $\qquad \underset{\mathrm{id}_X}{\overset{\mu_X}{\|}} : X \to X$ $\qquad \underset{\mathrm{id}_X}{\overset{\eta_X}{\|}} : X \to X$

$\forall X \in \mathcal{A}$

WHAT IS $\mathcal{C}^{(A \otimes -)}$??

GIVEN $A \in \mathrm{Alg}(\mathcal{C}, \otimes, \mathbb{1})$ STRICT

GET $(A \otimes -) \in \mathrm{Monad}(\mathcal{C})$ WITH

$$\mu_X : A \otimes A \otimes X \xrightarrow{m_A \otimes \mathrm{id}_X} A \otimes X$$

$$\eta_X : X \xrightarrow{u_A \otimes \mathrm{id}_X} A \otimes X$$

$\forall X \in \mathcal{C}$

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$

NAT TRANS: $\mu: T \circ T \Rightarrow T$

$\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

∋. $\forall X \in \mathcal{A}$:

$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

$$\mu_X \circ \eta_{T(X)} = \mathrm{id}_{T(X)}$$

$$\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi: T(Y) \to Y \in \mathcal{A})$

∋. $\xi \circ \mu_Y = \xi \circ T(\xi)$

$\xi \circ \eta_Y = \mathrm{id}_Y$

MORPHISMS: $\phi: Y \to Y' \in \mathcal{A}$

∋. $\phi \circ \xi = \xi' \circ T(\phi)$

EXAMPLES

$\mathcal{A}^{\mathrm{Id}} \cong \mathcal{A}$

IDENTITY MONAD ON $\mathcal{A}$

$$T := \mathrm{Id}_{\mathcal{A}} \qquad \underbrace{\mu_X}_{= \mathrm{id}_X} : X \to X \qquad \underbrace{\eta_X}_{= \mathrm{id}_X} : X \to X$$

$\forall X \in \mathcal{A}$

WHAT IS $\mathcal{C}^{(A \otimes -)}$??

$\mathcal{C}^{(A \otimes -)} \cong A\text{-Mod}(\mathcal{C})$

GIVEN $A \in \mathrm{Alg}(\mathcal{C}, \otimes, \mathbb{1})$ STRICT

GET $(A \otimes -) \in \mathrm{Monad}(\mathcal{C})$ WITH

$$\mu_X : A \otimes A \otimes X \xrightarrow{m_A \otimes \mathrm{id}_X} A \otimes X$$

$$\eta_X : X \xrightarrow{u_A \otimes \mathrm{id}_X} A \otimes X$$

$\forall X \in \mathcal{C}$

III. EILENBERG-MOORE CATEGORIES

ADJUNCTIONS $\rightleftarrows$ MONADS

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

$$\begin{cases} \text{ENDOFUNCTOR } T : \mathcal{A} \to \mathcal{A} \\ \text{NAT TRANS: } \mu : T \circ T \Rightarrow T \\ \qquad\qquad\quad\;\; \eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T \end{cases}$$

$\ni.\ \forall X \in \mathcal{A}$:

$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

$$\mu_X \circ \eta_{T(X)} = \mathrm{id}_{T(X)}$$

$$\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi : T(Y) \to Y \in \mathcal{A})$

$\ni.$ $\xi \circ \mu_Y = \xi \circ T(\xi)$

$\xi \circ \eta_Y = \mathrm{id}_Y$

MORPHISMS: $\phi : Y \to Y' \in \mathcal{A}$

$\ni.$ $\phi \circ \xi = \xi' \circ T(\phi)$

III. EILENBERG-MOORE CATEGORIES

ADJUNCTIONS $\rightleftarrows$ MONADS

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$

NAT TRANS: $\mu: T \circ T \Rightarrow T$

$\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

s.t. $\forall X \in \mathcal{A}$:

$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

$$\mu_X \circ \eta_{T(X)} = \mathrm{id}_{T(X)}$$

$$\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi: T(Y) \to Y \in \mathcal{A})$

s.t. $\xi \circ \mu_Y = \xi \circ T(\xi)$

$\xi \circ \eta_Y = \mathrm{id}_Y$

MORPHISMS: $\phi: Y \to Y' \in \mathcal{A}$

s.t. $\phi \circ \xi = \xi' \circ T(\phi)$

THEOREM: TAKE A MONAD (T, μ, η) ON $\mathcal{A}$.

(a) $\exists$ FUNCTORS:

(b)

(c)

III. EILENBERG-MOORE CATEGORIES

ADJUNCTIONS $\rightleftarrows$ MONADS

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

- ENDOFUNCTOR $T : \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu : T \circ T \Rightarrow T$
- $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

s.t. $\forall X \in \mathcal{A}$:

$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

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$$\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$$(Y \in \mathcal{A},\ \xi : T(Y) \to Y \in \mathcal{A})$$

s.t. $\xi \circ \mu_Y = \xi \circ T(\xi)$

$\xi \circ \eta_Y = \mathrm{id}_Y$

MORPHISMS: $\phi : Y \to Y' \in \mathcal{A}$

s.t. $\phi \circ \xi = \xi' \circ T(\phi)$

THEOREM: TAKE A MONAD (T, μ, η) ON $\mathcal{A}$.

(a) $\exists$ FUNCTORS:

$$\mathrm{Forg}^T : \mathcal{A}^T \longrightarrow \mathcal{A}$$
$$(Y, \xi) \longmapsto Y$$

(b)

(c)

III. EILENBERG-MOORE CATEGORIES

ADJUNCTIONS $\rightleftarrows$ MONADS

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

ENDOFUNCTOR $T : \mathcal{A} \to \mathcal{A}$

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$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

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$$\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi : T(Y) \to Y \in \mathcal{A})$

s.t. $\xi \circ \mu_Y = \xi \circ T(\xi)$

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MORPHISMS: $\phi : Y \to Y' \in \mathcal{A}$

s.t. $\phi \circ \xi = \xi' \circ T(\phi)$

<u>THEOREM</u> : TAKE A MONAD (T, μ, η) ON $\mathcal{A}$.

(a) $\exists$ FUNCTORS:

$$\mathrm{Free}^T : \mathcal{A} \longrightarrow \mathcal{A}^T \qquad Y \longmapsto (T(Y), \mu_Y)$$

$$\mathrm{Forg}^T : \mathcal{A}^T \longrightarrow \mathcal{A} \qquad (Y, \xi) \longmapsto Y$$

(b)

(c)

III. EILENBERG-MOORE CATEGORIES

ADJUNCTIONS $\rightleftarrows$ MONADS

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$

NAT TRANS: $\mu: T \circ T \Rightarrow T$

$\eta: Id_{\mathcal{A}} \Rightarrow T$

s.t. $\forall X \in \mathcal{A}$:

$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

$$\mu_X \circ \eta_{T(X)} = id_{T(X)}$$

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EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi: T(Y) \to Y \in \mathcal{A})$

s.t. $\xi \circ \mu_Y = \xi \circ T(\xi)$

$\xi \circ \eta_Y = id_Y$

MORPHISMS: $\phi: Y \to Y' \in \mathcal{A}$

s.t. $\phi \circ \xi = \xi' \circ T(\phi)$

THEOREM: TAKE A MONAD (T, μ, η) ON $\mathcal{A}$.

(a) $\exists$ FUNCTORS:

$$\mathrm{Free}^T: \mathcal{A} \longrightarrow \mathcal{A}^T \qquad Y \longmapsto (T(Y), \mu_Y)$$

$$\mathrm{Forg}^T: \mathcal{A}^T \longrightarrow \mathcal{A} \qquad (Y, \xi) \longmapsto Y$$

(b) $\mathrm{Free}^T \dashv \mathrm{Forg}^T$

(c)

III. EILENBERG-MOORE CATEGORIES

ADJUNCTIONS ⇄ MONADS

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ ≡

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu: T \circ T \Rightarrow T$
- $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

s.t. $\forall X \in \mathcal{A}$:

$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

$$\mu_X \circ \eta_{T(X)} = \mathrm{id}_{T(X)}$$

$$\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi: T(Y) \to Y \in \mathcal{A})$

s.t. $\xi \circ \mu_Y = \xi \circ T(\xi)$

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s.t. $\phi \circ \xi = \xi' \circ T(\phi)$

THEOREM: TAKE A MONAD (T, μ, η) ON $\mathcal{A}$.

(a) $\exists$ FUNCTORS:

$$\mathrm{Free}^T: \mathcal{A} \longrightarrow \mathcal{A}^T \qquad Y \longmapsto (T(Y), \mu_Y)$$

$$\mathrm{Forg}^T: \mathcal{A}^T \longrightarrow \mathcal{A} \qquad (Y, \xi) \longmapsto Y$$

(b) $\mathrm{Free}^T \dashv \mathrm{Forg}^T$

(c) $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$ AS MONADS VIA

TAKE ADJUNCTION

$(F: \mathcal{A} \to \mathcal{B}) \dashv (G: \mathcal{B} \to \mathcal{A})$

UNIT $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow GF$

COUNIT $\varepsilon: FG \Rightarrow \mathrm{Id}_{\mathcal{B}}$

GET $GF \in \mathrm{Monad}(\mathcal{A})$ WITH

$$\mu: GFGF \xRightarrow{G\varepsilon F} GF \quad \& \quad \eta: \mathrm{Id}_{\mathcal{B}} \xRightarrow{\eta} FG$$

III. EILENBERG-MOORE CATEGORIES

ADJUNCTIONS $\rightleftarrows$ MONADS

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

- ENDOFUNCTOR $T : \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu : T \circ T \Rightarrow T$
- $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

s.t. $\forall X \in \mathcal{A}$:

$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

$$\mu_X \circ \eta_{T(X)} = id_{T(X)}$$

$$\mu_X \circ T(\eta_X) = id_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi : T(Y) \to Y \in \mathcal{A})$

s.t. $\xi \circ \mu_Y = \xi \circ T(\xi)$

$\xi \circ \eta_Y = id_Y$

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s.t. $\phi \circ \xi = \xi' \circ T(\phi)$

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(a) $\exists$ FUNCTORS:

$$\mathrm{Free}^T : \mathcal{A} \longrightarrow \mathcal{A}^T \qquad Y \longmapsto (T(Y), \mu_Y)$$

$$\mathrm{Forg}^T : \mathcal{A}^T \longrightarrow \mathcal{A} \qquad (Y, \xi) \longmapsto Y$$

(b) $\mathrm{Free}^T \dashv \mathrm{Forg}^T$

(c) $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$ AS MONADS VIA

TAKE ADJUNCTION

$(F : \mathcal{A} \to \mathcal{B}) \dashv (G : \mathcal{B} \to \mathcal{A})$

UNIT $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow GF$

COUNIT $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{B}}$

GET $GF \in \mathrm{Monad}(\mathcal{A})$ WITH

$\mu : GFGF \xRightarrow{G\varepsilon F} GF$ & $\eta : \mathrm{Id}_{\mathcal{B}} \xRightarrow{\eta} FG$

SOME DETAILS IN THE BOOK, REST $\equiv$ EXER 4.33

III. EILENBERG–MOORE CATEGORIES

$$\begin{tikzcd} \text{ADJUNCTIONS} \arrow[r, bend left] & \text{MONADS} \arrow[l, bend left] \end{tikzcd}$$

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu: T \circ T \Rightarrow T$
- $\eta: Id_{\mathcal{A}} \Rightarrow T$

∋. $\forall X \in \mathcal{A}$:

$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

$$\mu_X \circ \eta_{T(X)} = id_{T(X)}$$

$$\mu_X \circ T(\eta_X) = id_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$$(Y \in \mathcal{A},\ \xi : T(Y) \to Y \in \mathcal{A})$$

∋. $\xi \circ \mu_Y = \xi \circ T(\xi)$

$\xi \circ \eta_Y = id_Y$

MORPHISMS: $\phi : Y \to Y' \in \mathcal{A}$

∋. $\phi \circ \xi = \xi' \circ T(\phi)$

THEOREM:

TAKE (T, μ, η) ON $\mathcal{A}$.

(a) $\exists$ FUNCTORS:

$$\mathrm{Free}^T : \mathcal{A} \longrightarrow \mathcal{A}^T$$
$$Y \longmapsto (T(Y), \mu_Y)$$

$$\mathrm{Forg}^T : \mathcal{A}^T \longrightarrow \mathcal{A}$$
$$(Y, \xi) \longmapsto Y$$

(b) $\mathrm{Free}^T \dashv \mathrm{Forg}^T$

(c) $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

EXAMPLE

FOR $T := (A \otimes -) = \mathrm{Forg} \circ (A \otimes -)$ MONAD ON $\mathcal{C}$

III. EILENBERG-MOORE CATEGORIES

$$\text{ADJUNCTIONS} \rightleftarrows \text{MONADS}$$

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

$\left[\begin{array}{l} \text{ENDOFUNCTOR } T: \mathcal{A} \to \mathcal{A} \\ \text{NAT TRANS: } \mu: T \circ T \Rightarrow T \\ \qquad\qquad\quad \eta: Id_{\mathcal{A}} \Rightarrow T \end{array}\right.$

s.t. $\forall X \in \mathcal{A}$:

$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

$$\mu_X \circ \eta_{T(X)} = id_{T(X)}$$

$$\mu_X \circ T(\eta_X) = id_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi: T(Y) \to Y \in \mathcal{A})$

s.t. $\xi \circ \mu_Y = \xi \circ T(\xi)$

$\xi \circ \eta_Y = id_Y$

MORPHISMS: $\phi: Y \to Y' \in \mathcal{A}$

s.t. $\phi \circ \xi = \xi' \circ T(\phi)$

<u>THEOREM</u>:

TAKE (T, μ, η) ON $\mathcal{A}$.

(a) $\exists$ FUNCTORS:

$$\mathrm{Free}^T: \mathcal{A} \longrightarrow \mathcal{A}^T$$
$$Y \longmapsto (T(Y), \mu_Y)$$

$$\mathrm{Forg}^T: \mathcal{A}^T \longrightarrow \mathcal{A}$$
$$(Y, \xi) \longmapsto Y$$

(b) $\mathrm{Free}^T \dashv \mathrm{Forg}^T$

(c) $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

<u>EXAMPLE</u>

FOR $T := (A \otimes -)$ MONAD ON $\mathcal{C}$

$\mathrm{Forg} \circ (A \otimes -)$

$$\mathrm{Free}^T: \mathcal{C} \longrightarrow \mathcal{C}^T$$
$$X \longmapsto (A \otimes X,\ m_A \otimes id)$$

$\mathrm{Free}^T = A \otimes -$

III. EILENBERG-MOORE CATEGORIES

ADJUNCTIONS $\rightleftarrows$ MONADS

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu: T \circ T \Rightarrow T$
- $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$

s.t. $\forall X \in \mathcal{A}$:

$$\mu_X \circ \mu_{T(X)} = \mu_X \circ T(\mu_X)$$

$$\mu_X \circ \eta_{T(X)} = \mathrm{id}_{T(X)}$$

$$\mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}$$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi: T(Y) \to Y \in \mathcal{A})$

s.t. $\xi \circ \mu_Y = \xi \circ T(\xi)$

$\xi \circ \eta_Y = \mathrm{id}_Y$

MORPHISMS: $\phi: Y \to Y' \in \mathcal{A}$

s.t. $\phi \circ \xi = \xi' \circ T(\phi)$

THEOREM:

TAKE (T, μ, η) ON $\mathcal{A}$.

(a) $\exists$ FUNCTORS:

$$\mathrm{Free}^T: \mathcal{A} \to \mathcal{A}^T, \quad Y \mapsto (T(Y), \mu_Y)$$

$$\mathrm{Forg}^T: \mathcal{A}^T \to \mathcal{A}, \quad (Y, \xi) \mapsto Y$$

(b) $\mathrm{Free}^T \dashv \mathrm{Forg}^T$

(c) $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

EXAMPLE

FOR $T := \underbrace{(A \otimes -)}_{= \mathrm{Forg} \circ (A \otimes -)}$ MONAD ON $\mathcal{C}$

$$\mathrm{Free}^T: \mathcal{C} \to \mathcal{C}^T, \quad X \mapsto (A \otimes X,\ m_A \otimes \mathrm{id})$$

$\mathrm{Free}^T = A \otimes -$

$\mathrm{Forg}^T = \mathrm{Forg}$ FROM BEFORE

III. EILENBERG-MOORE CATEGORIES

ADJUNCTIONS $\rightleftarrows$ MONADS

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

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MORPHISMS: $\phi : Y \to Y' \in \mathcal{A}$

s.t. $\phi \circ \xi = \xi' \circ T(\phi)$

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TAKE (T, μ, η) ON $\mathcal{A}$.

(a) $\exists$ FUNCTORS:

$\mathrm{Free}^T : \mathcal{A} \longrightarrow \mathcal{A}^T$

$Y \longmapsto (T(Y), \mu_Y)$

$\mathrm{Forg}^T : \mathcal{A}^T \longrightarrow \mathcal{A}$

$(Y, \xi) \longmapsto Y$

(b) $\mathrm{Free}^T \dashv \mathrm{Forg}^T$

(c) $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

EXAMPLE

$\mathrm{Forg} \circ (A \otimes -)$

FOR $T := (A \otimes -)$ MONAD ON $\mathcal{C}$

$\mathrm{Free}^T : \mathcal{C} \longrightarrow \mathcal{C}^T$

$\|$ $\quad X \longmapsto (A \otimes X,\ \mu_A \otimes \mathrm{id})$

$A \otimes -$

$\mathrm{Forg}^T = \mathrm{Forg}$ FROM BEFORE

THIS EXAMPLE IS NOT SO ILLUMINATING BECAUSE

$$\mathcal{C}^{(A\otimes -)} \cong A\text{-}\mathrm{Mod}(\mathcal{C})$$

III. EILENBERG-MOORE CATEGORIES

ADJUNCTIONS ⇄ MONADS

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$

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EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi: T(Y) \to Y \in \mathcal{A})$

s.t. $\xi \circ \mu_Y = \xi \circ T(\xi)$

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$\mathrm{Free}^T: \mathcal{A} \longrightarrow \mathcal{A}^T$

$Y \longmapsto (T(Y), \mu_Y)$

$\mathrm{Forg}^T: \mathcal{A}^T \longrightarrow \mathcal{A}$

$(Y, \xi) \longmapsto Y$

(b) $\mathrm{Free}^T \dashv \mathrm{Forg}^T$

(c) $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

EXAMPLE

FOR $T := (A \otimes -)$ MONAD ON $\mathcal{C}$

$= \mathrm{Forg} \circ (A \otimes -)$

$\mathrm{Free}^T: \mathcal{C} \longrightarrow \mathcal{C}^T$

$X \longmapsto (A \otimes X,\ m_A \otimes id)$

$\mathrm{Free}^T \,\|\, A \otimes -$

$\mathrm{Forg}^T = \mathrm{Forg}$ FROM BEFORE

THIS EXAMPLE IS NOT SO ILLUMINATING BECAUSE

$\mathcal{C}^{(A \otimes -)} \cong A\text{-}\mathrm{Mod}(\mathcal{C})$

CODOMAIN OF LEFT ADJOINT

HERE

III. EILENBERG-MOORE CATEGORIES

ADJUNCTIONS $\rightleftarrows$ MONADS

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
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EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$$(Y \in \mathcal{A},\ \xi: T(Y) \to Y \in \mathcal{A})$$

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THEOREM:

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(b) $\mathrm{Free}^T \dashv \mathrm{Forg}^T$

(c) $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

EXAMPLE

FOR $T := (A \otimes -)$ MONAD ON $\mathcal{C}$

$\mathrm{Forg} \circ (A \otimes -)$

$$\mathrm{Free}^T: \mathcal{C} \longrightarrow \mathcal{C}^T$$
$$X \longmapsto (A \otimes X,\ m_A \otimes \mathrm{id})$$

$\mathrm{Free}^T = A \otimes -$

$\mathrm{Forg}^T = \mathrm{Forg}$ FROM BEFORE

THIS EXAMPLE IS NOT SO ILLUMINATING BECAUSE

$$\mathcal{C}^{(A \otimes -)} \cong A\text{-}\mathrm{Mod}(\mathcal{C})$$

CODOMAIN OF LEFT ADJOINT HERE

CODOMAIN OF LEFT ADJOINT OF $\dashv$ WE START WITH

III. EILENBERG-MOORE CATEGORIES

ADJUNCTIONS $\rightleftarrows$ MONADS

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
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EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi: T(Y) \to Y \in \mathcal{A})$

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$\ni.$ $\phi \circ \xi = \xi' \circ T(\phi)$

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$$\mathrm{Free}^T: \mathcal{A} \longrightarrow \mathcal{A}^T$$
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$$\mathrm{Forg}^T: \mathcal{A}^T \longrightarrow \mathcal{A}$$
$$(Y, \xi) \longmapsto Y$$

(b) $\mathrm{Free}^T \dashv \mathrm{Forg}^T$

(c) $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

WHAT IF THESE CODOMAINS ARE DIFFERENT?!

EXAMPLE

FOR $T := (A \otimes -) = \mathrm{Forg} \circ (A \otimes -)$ MONAD ON $\mathcal{C}$

$$\mathrm{Free}^T: \mathcal{C} \longrightarrow \mathcal{C}^T$$
$$X \longmapsto (A \otimes X,\ m_A \otimes \mathrm{id})$$

$\mathrm{Free}^T = A \otimes -$

$\mathrm{Forg}^T = \mathrm{Forg}$ FROM BEFORE

THIS EXAMPLE IS NOT SO ILLUMINATING BECAUSE

$$\mathcal{C}^{(A \otimes -)} \simeq A\text{-}\mathrm{Mod}(\mathcal{C})$$

CODOMAIN OF LEFT ADJOINT HERE

CODOMAIN OF LEFT ADJOINT OF $\dashv$ WE START WITH

III. Eilenberg-Moore Categories

Take a category $\mathcal{A}$.

Monad on $\mathcal{A}$ $\equiv$

- endofunctor $T : \mathcal{A} \to \mathcal{A}$
- nat trans: $\mu : T \circ T \Rightarrow T$
- $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$
- $\vdots$

EM category $\mathcal{A}^T$

Objects:

$(y \in \mathcal{A},\ \xi : T(y) \to y \in \mathcal{A})$

$\vdots$

<u>Theorem</u>: For

$$\mathrm{Free}^T : \mathcal{A} \longrightarrow \mathcal{A}^T, \quad y \longmapsto (T(y), \mu_y)$$

$$\mathrm{Forg}^T : \mathcal{A}^T \longrightarrow \mathcal{A}, \quad (y, \xi) \longmapsto y$$

Get:

- $\mathrm{Free}^T \dashv \mathrm{Forg}^T$
- $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

$T := GF$

$$\begin{tikzcd} \mathcal{A} \arrow[loop left] \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathcal{B} \arrow[ll, bend left, "G"] \end{tikzcd}$$

Monad via adjunction

What if these codomains are different??

III. Eilenberg-Moore Categories

Take a category $\mathcal{A}$.

Monad on $\mathcal{A}$ $\equiv$

$$\left[\begin{array}{l} \text{endofunctor } T : \mathcal{A} \to \mathcal{A} \\ \text{nat trans: } \mu : T \circ T \Rightarrow T \\ \qquad\qquad\quad\; \eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T \\ \vdots \end{array}\right.$$

EM category $\mathcal{A}^T$

Objects:

$(Y \in \mathcal{A},\ \xi : T(Y) \to Y \in \mathcal{A})$

$\vdots$

<u>Theorem</u>: For

$\mathrm{Free}^T : \mathcal{A} \longrightarrow \mathcal{A}^T$

$\qquad Y \longmapsto (T(Y), \mu_Y)$

$\mathrm{Forg}^T : \mathcal{A}^T \longrightarrow \mathcal{A}$

$\qquad (Y, \xi) \longmapsto Y$

Get:

- $\mathrm{Free}^T \dashv \mathrm{Forg}^T$
- $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

$\mathrm{Forg}^T \circ \mathrm{Free}^T$

$\|$

$T := GF$

Monad via adjunction

$$\begin{tikzcd} & & \mathcal{A}^T \arrow[dll, bend left, "\mathrm{Forg}^T"] \\ \mathcal{A} \arrow[loop left] \arrow[urr, bend left, "\mathrm{Free}^T"] \arrow[urr, phantom, "\perp"] \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathcal{B} \arrow[ll, bend left, "G"] \end{tikzcd}$$

What if these codomains are different??

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu: T \circ T \Rightarrow T$
- $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$
- $\vdots$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(Y \in \mathcal{A},\ \xi: T(Y) \to Y \in \mathcal{A})$

$\vdots$

THEOREM: FOR

$\mathrm{Free}^T: \mathcal{A} \longrightarrow \mathcal{A}^T$

$Y \longmapsto (T(Y), \mu_Y)$

$\mathrm{Forg}^T: \mathcal{A}^T \longrightarrow \mathcal{A}$

$(Y, \xi) \longmapsto Y$

GET:

- $\mathrm{Free}^T \dashv \mathrm{Forg}^T$
- $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

$$\mathrm{Forg}^T \circ \mathrm{Free}^T \;\;\|\;\; T := GF$$

MONAD VIA ADJUNCTION

$$\begin{tikzcd}
 & & \mathcal{A}^T = \mathcal{C}^{(A\otimes -)} \arrow[ll, bend left, "\mathrm{Forg}^T"] \arrow[d, phantom, "\cong"] \\
\mathcal{A} \arrow[loop left] \arrow[urr, bend left, "\mathrm{Free}^T"] \arrow[rr, bend left, "F = \mathrm{Free}"] \arrow[rr, phantom, "\perp"] \arrow[urr, phantom, "\perp"] & & \mathcal{B} \arrow[ll, bend left, "G = \mathrm{Forg}"] \\
\mathcal{C} \arrow[u, phantom, "\|"] & & A\text{-}\mathrm{Mod}(\mathcal{C}) \arrow[u, phantom, "\|"]
\end{tikzcd}$$

FOR $A \in \mathrm{Alg}(\mathcal{C})$

... IN THE EXAMPLE

WHAT IF THESE CODOMAINS ARE DIFFERENT??

III. Eilenberg-Moore Categories

Take a category $\mathcal{A}$.

Monad on $\mathcal{A}$ $\equiv$

- Endofunctor $T : \mathcal{A} \to \mathcal{A}$
- Nat trans: $\mu : T \circ T \Rightarrow T$
- $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow T$
- $\vdots$

EM category $\mathcal{A}^T$

Objects:

$(y \in \mathcal{A},\ \xi : T(y) \to y \in \mathcal{A})$

$\vdots$

Theorem: For

$$\mathrm{Free}^T : \mathcal{A} \longrightarrow \mathcal{A}^T, \quad y \longmapsto (T(y), \mu_y)$$

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Get:

- $\mathrm{Free}^T \dashv \mathrm{Forg}^T$
- $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

$\mathrm{Forg}^T \circ \mathrm{Free}^T$

$\|$

$T := GF$

Monad via adjunction

$$\begin{tikzcd}
 & & \mathcal{A}^T \arrow[dll, bend left, "\mathrm{Forg}^T"] \\
\mathcal{A} \arrow[loop left] \arrow[urr, bend left, "\mathrm{Free}^T"] \arrow[urr, phantom, "\perp"] \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathcal{B} \arrow[ll, bend left, "G"]
\end{tikzcd}$$

What if these codomains are different??

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

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- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
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EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(y \in \mathcal{A},\ \xi: T(y) \to y \in \mathcal{A})$

$\vdots$

THEOREM: FOR

$\mathrm{Free}^T: \mathcal{A} \longrightarrow \mathcal{A}^T$

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GET:

- $\mathrm{Free}^T \dashv \mathrm{Forg}^T$
- $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

$\mathrm{Forg}^T \circ \mathrm{Free}^T$

$\|$

$T := GF$

MONAD VIA ADJUNCTION

$$\begin{tikzcd}[row sep=large]
 & & \mathcal{A}^T \arrow[dll, bend left, "\mathrm{Forg}^T"] \arrow[d, phantom, sloped, "\cong"] \\
\mathcal{A} \arrow[loop left] \arrow[urr, bend left, "\mathrm{Free}^T"] \arrow[urr, phantom, "\perp"] \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathcal{B} \arrow[ll, bend left, "G"]
\end{tikzcd}$$

BECK'S MONADICITY THEOREM GIVES PRECISE (& TECHNICAL) CONDITIONS FOR EQUIVALENCE

WHAT IF THESE CODOMAINS ARE DIFFERENT??

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu: T \circ T \Rightarrow T$
- $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$
- $\vdots$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(y \in \mathcal{A},\ \xi: T(y) \to y \in \mathcal{A})$

$\vdots$

THEOREM: FOR

$$\mathrm{Free}^T: \mathcal{A} \longrightarrow \mathcal{A}^T, \quad y \longmapsto (T(y), \mu_y)$$

$$\mathrm{Forg}^T: \mathcal{A}^T \longrightarrow \mathcal{A}, \quad (y, \xi) \longmapsto y$$

GET:

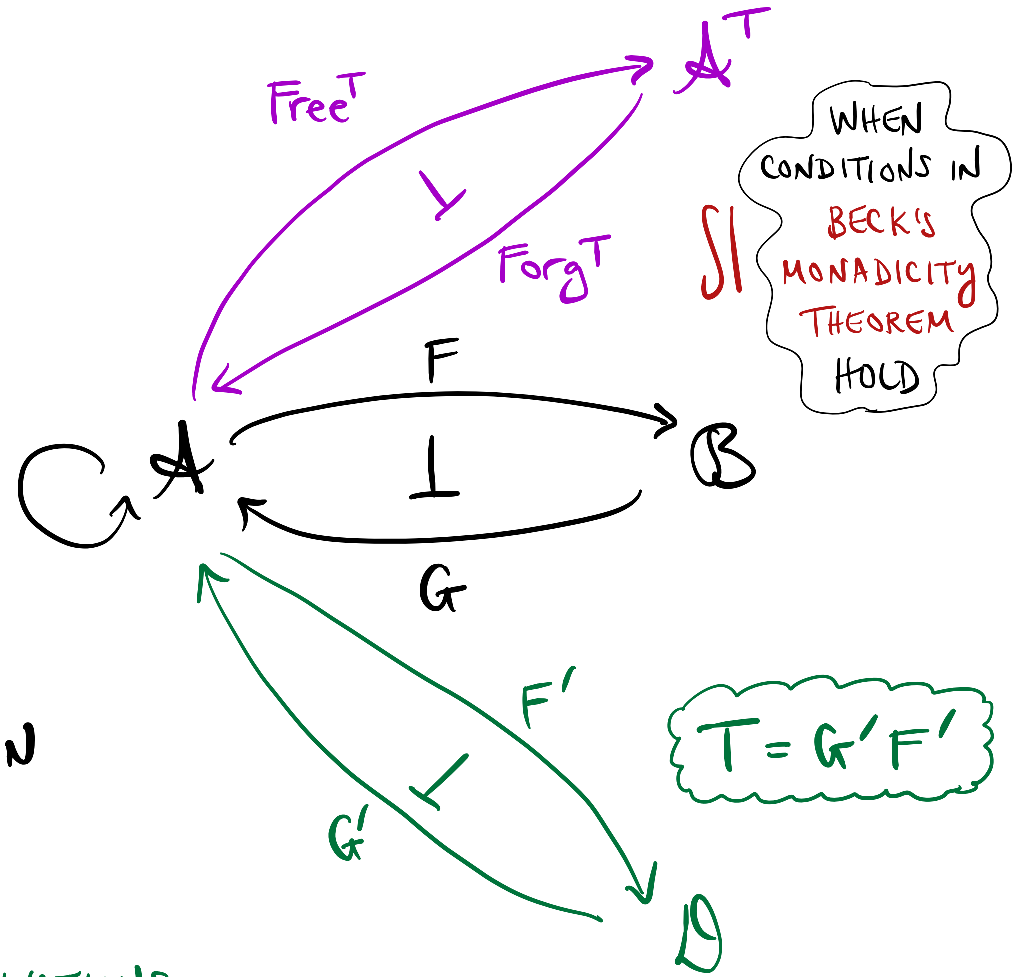
- $\mathrm{Free}^T \dashv \mathrm{Forg}^T$
- $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

$\mathrm{Forg}^T \circ \mathrm{Free}^T$

$\|$

$T := GF$

MONAD VIA ADJUNCTION



$\exists$ OTHER SOLUTIONS TO THE PROBLEM OF GETTING T VIA ADJUNCTION

III. EILENBERG-MOORE CATEGORIES

TAKE A CATEGORY $\mathcal{A}$.

MONAD ON $\mathcal{A}$ $\equiv$

- ENDOFUNCTOR $T: \mathcal{A} \to \mathcal{A}$
- NAT TRANS: $\mu: T \circ T \Rightarrow T$
- $\eta: \mathrm{Id}_{\mathcal{A}} \Rightarrow T$
- $\vdots$

EM CATEGORY $\mathcal{A}^T$

OBJECTS:

$(y \in \mathcal{A},\ \xi: T(y) \to y \in \mathcal{A})$

$\vdots$

<u>THEOREM</u>: FOR

$\mathrm{Free}^T: \mathcal{A} \longrightarrow \mathcal{A}^T$

$y \longmapsto (T(y), \mu_y)$

$\mathrm{Forg}^T: \mathcal{A}^T \longrightarrow \mathcal{A}$

$(y, \xi) \longmapsto y$

GET:

- $\mathrm{Free}^T \dashv \mathrm{Forg}^T$
- $T = \mathrm{Forg}^T \circ \mathrm{Free}^T$

$\mathrm{Forg}^T \circ \mathrm{Free}^T$

$\|$

$T := GF$

MONAD VIA ADJUNCTION

$$\begin{tikzcd}
 & \mathcal{A}^T \arrow[dl, bend left, "\mathrm{Forg}^T"] \\
\mathcal{A} \arrow[loop left] \arrow[ur, bend left, "\mathrm{Free}^T"] \arrow[ur, phantom, "\perp"] \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] \arrow[dr, bend left, "F'"] \arrow[dr, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "G"] \\
 & \mathcal{D} \arrow[ul, bend left, "G'"]
\end{tikzcd}$$

$\cong$ WHEN CONDITIONS IN BECK'S MONADICITY THEOREM HOLD

$T = G'F'$

$\exists$ OTHER SOLUTIONS TO THE PROBLEM OF GETTING T VIA ADJUNCTION

... BUT WE MUST END HERE

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #19

NEXT TIME

OPERATIONS &
SOME CAPSTONE RESULTS

TOPICS:

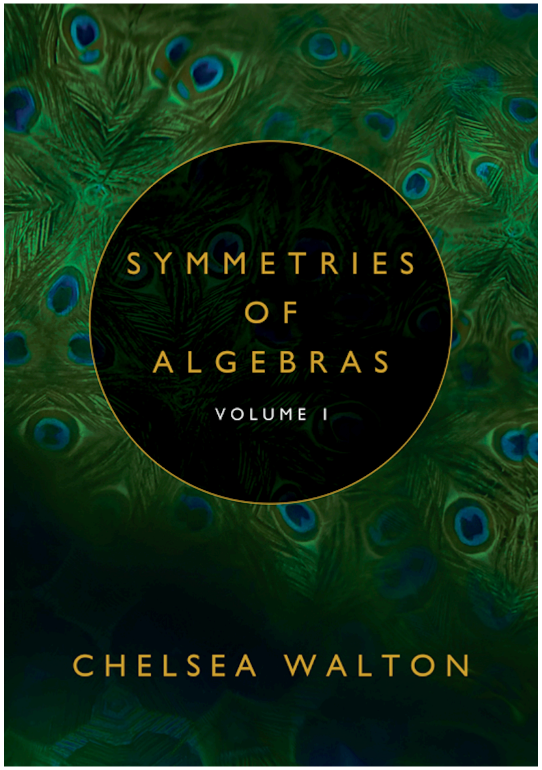
I. (BI)MODULES IN MONOIDAL CATEGORIES (§§4.4.1, 4.4.2)

II. MONADS (§4.3.2)

III. EILENBERG-MOORE CATEGORIES (§4.4.3)

Enjoy this lecture?
You'll enjoy the textbook!

C. Walton's "Symmetries of Algebras, Volume 1" (2024)



Available for purchase at :

619 Wreath (at a discount)

https://www.619wreath.com/

Also on Amazon
&
Google Play

Lecture #19 keywords: adjunction monad, Beck's Monadicity Theorem, bimodule in a monoidal category, Eilenberg-Moore category, Eilenberg-Moore object, module in a monoidal category, monad