MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #2

LAST TIME

- INTRO TO COURSE
- GROUPS, RINGS, VSPACES
- $\times$, $+$, $\oplus$, HOM, DUAL OF VSPACES

TOPICS:

I. TENSOR PRODUCT OF VECTOR SPACES (VIA QUOTIENT) (§1.1.4)

II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED (§1.1.4)

III. MORE ON TENSOR PRODUCT & HOM OF VSPACES (§1.1.4)

IV. ALGEBRAS OVER A FIELD (§1.1.5)

V. EXAMPLES/TYPES OF ALGEBRAS OVER A FIELD (§1.2)

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I. TENSOR PRODUCT OF VECTOR SPACES (VIA QUOTIENT)

VECTOR SPACE OVER A FIELD $\Bbbk$

ABELIAN GROUP WITH ANOTHER OPERATION $*$ COMPATIBLE WITH $+$

A $\Bbbk$-VS IS AN ABELIAN GROUP $(V,+,0)$ WITH AN OPERATION

$$*: \Bbbk \times V \longrightarrow V \qquad (\lambda, v) \longmapsto \lambda * v =: \lambda v$$

SUCH THAT $\forall \lambda, \lambda' \in \Bbbk$ AND $v, v' \in V$:

$$\lambda(v+v') = \lambda v + \lambda v', \quad (\lambda+\lambda')v = \lambda v + \lambda' v, \quad (\lambda\lambda')v = \lambda(\lambda' v), \quad 1_{\Bbbk} v = v$$

TENSOR PRODUCT

GIVEN VSPACES $\begin{cases} V \text{ WITH BASIS } \{b_i\}_i \\ W \text{ WITH BASIS } \{c_j\}_j \end{cases}$

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TENSOR PRODUCT GIVEN VSPACES $\begin{cases} V \text{ WITH BASIS } \{b_i\}_i \\ W \text{ WITH BASIS } \{c_j\}_j \end{cases}$

INFORMALLY —

THE TENSOR PRODUCT OF V AND W

$$V \otimes_{\Bbbk} W =: V \otimes W$$

IS THE $\Bbbk$-VECTOR SPACE WITH BASIS $\{b_i \otimes c_j\}_{i,j}$

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INFORMALLY —

THE TENSOR PRODUCT OF V AND W

$$V \otimes_{\Bbbk} W =: V \otimes W$$

"SIMPLE TENSOR"

IS THE $\Bbbk$-VECTOR SPACE WITH BASIS $\{b_i \otimes c_j\}_{i,j}$

A SYMBOL

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VECTOR SPACE OVER A FIELD $\Bbbk$

ABELIAN GROUP WITH ANOTHER OPERATION $*$ COMPATIBLE WITH $+$

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TENSOR PRODUCT GIVEN VSPACES $\begin{cases} V \text{ WITH BASIS } \{b_i\}_i \\ W \text{ WITH BASIS } \{c_j\}_j \end{cases}$

INFORMALLY —

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$$V \otimes_{\Bbbk} W =: V \otimes W$$

"SIMPLE TENSOR"

IS THE $\Bbbk$-VECTOR SPACE WITH BASIS $\{b_i \otimes c_j\}_{i,j}$

ELEMENTS OR VECTORS ARE FINITE LINEAR COMBINATIONS

OF SIMPLE TENSORS: $\sum_{i,j}^{\text{finite}} \lambda_{i,j}\, b_i \otimes c_j \qquad \lambda_{i,j} \in \Bbbk$

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ADDITION

SCALAR MULTIPLICATION

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ADDITION

SCALAR MULTIPLICATION

$$\lambda(b_i\otimes c_j) := \lambda b_i\otimes c_j$$
$$:= b_i\otimes \lambda c_j$$

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I. TENSOR PRODUCT OF VECTOR SPACES (VIA QUOTIENT)

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ELEMENTS OR **VECTORS** ARE $\sum_{i,j}^{\text{finite}} \lambda_{i,j}\, b_i \otimes c_j$, $\lambda_{i,j} \in \Bbbk$

ADDITION

SCALAR MULTIPLICATION

$$\lambda(b_i \otimes c_j) := \lambda b_i \otimes c_j \qquad \text{USE } *_V$$
$$:= b_i \otimes \lambda c_j \qquad \text{OR } *_W$$

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ADDITION

$$b_i \otimes c_j + b_i' \otimes c_j := (b_i + b_i') \otimes c_j$$

$$b_i \otimes c_j + b_i \otimes c_j' := b_i \otimes (c_j + c_j')$$

ELSE CAN'T ADD

SCALAR MULTIPLICATION

$$\lambda(b_i \otimes c_j) := \lambda b_i \otimes c_j$$

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ADDITION

$$b_i\otimes c_j + b_i'\otimes c_j := (\underbrace{b_i+b_i'}_{\text{USE } +_V})\otimes c_j$$

$$b_i\otimes c_j + b_i\otimes c_j' := b_i\otimes(\underbrace{c_j+c_j'}_{\text{USE } +_W})$$

ELSE CAN'T ADD

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(FORMALLY)

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$$V \otimes_{\Bbbk} W := V \otimes W \overset{\text{DEF}}{=} \underline{\mathrm{SPAN}_{\Bbbk}\langle (v,w) \mid v \in V, w \in W \rangle}$$

(QUOTIENT SPACE)

(FORMALLY)

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$$V \otimes_{\Bbbk} W := V \otimes W \overset{\text{DEF}}{=} \frac{\text{SPAN}_{\Bbbk}\langle (v,w) \mid v \in V, w \in W \rangle}{\text{SPAN}_{\Bbbk}\left(\lambda(v,w) - (\lambda v, w) \quad \lambda(v,w) - (v, \lambda w) \right)}$$

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(QUOTIENT SPACE)

(FORMALLY)

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(QUOTIENT SPACE)

(FORMALLY)

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(QUOTIENT SPACE)

THERE'S A CLEANER WAY OF GETTING THIS CONSTRUCTION

... HELPFUL FOR COMPARING $V \otimes W$ WITH ANOTHER VSPACE

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

GIVEN A GADGET X,

A UNIVERSAL STRUCTURE ATTACHED TO X

IS A STRUCTURE $\mathrm{Univ}(X)$

FORM I

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

GIVEN A GADGET X,

A UNIVERSAL STRUCTURE ATTACHED TO X

IS A STRUCTURE $\mathrm{Univ}(X)$

$$X \xrightarrow{\alpha} \mathrm{Univ}(X)$$

$$\mathrm{Univ}(X) \xrightarrow{\alpha'} X$$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

GIVEN A GADGET X,

A UNIVERSAL STRUCTURE ATTACHED TO X

IS A STRUCTURE $\mathrm{Univ}(X)$

SUCH THAT $\forall$ ARBITRARY STRUCTURES $\mathrm{Arb}(X)$ ATTACHED TO X

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{Univ}(X) \\ & \mathrm{Arb}(X) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Arb}(X) \arrow[dr, "\forall \beta'"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'"] & X \end{tikzcd}$$

FORM I　　　　FORM II

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GIVEN A GADGET X,

A UNIVERSAL STRUCTURE ATTACHED TO X

IS A STRUCTURE $\mathrm{Univ}(X)$

SUCH THAT $\forall$ ARBITRARY STRUCTURES $\mathrm{Arb}(X)$ ATTACHED TO X

$\exists!$ STRUCTURE-PRESERVING MAP MAKING THE DIAGRAM COMMUTE:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[dr, "\forall\beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists!\gamma'"'] \arrow[dr, "\forall\beta'"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'"] & X
\end{tikzcd}$$

FORM I $\qquad\qquad$ FORM II

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GIVEN A GADGET X,

A UNIVERSAL STRUCTURE ATTACHED TO X

IS A STRUCTURE $\mathrm{Univ}(X)$

SUCH THAT $\forall$ ARBITRARY STRUCTURES $\mathrm{Arb}(X)$ ATTACHED TO X

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$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[dr, "\forall\beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma"] \arrow[dl, phantom, "\circlearrowright"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists!\gamma'"'] \arrow[dr, "\forall\beta'"] \arrow[dr, phantom, "\circlearrowright"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'"'] & X
\end{tikzcd}$$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

GIVEN A GADGET X,

A UNIVERSAL STRUCTURE ATTACHED TO X

IS A STRUCTURE $\mathrm{Univ}(X)$

SUCH THAT $\forall$ ARBITRARY STRUCTURES $\mathrm{Arb}(X)$ ATTACHED TO X

$\exists!$ STRUCTURE-PRESERVING MAP MAKING THE DIAGRAM COMMUTE:

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathrm{Arb}(X)
\end{tikzcd}
$$

$$\{\beta : X \to \mathrm{Arb}(X)\} \overset{1-1}{\longleftrightarrow} \{\gamma : \mathrm{Univ}(X) \to \mathrm{Arb}(X)\}$$

FORM I

$$
\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[dr, phantom, "\circlearrowleft"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'"] & X
\end{tikzcd}
$$

$$\{\beta' : \mathrm{Arb}(X) \to X\} \overset{1-1}{\longleftrightarrow} \{\gamma' : \mathrm{Arb}(X) \to \mathrm{Univ}(X)\}$$

FORM II

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GIVEN A GADGET X,

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$$\begin{tikzcd}
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& \mathrm{Arb}(X)
\end{tikzcd}$$

$$\{\beta : X \to \mathrm{Arb}(X)\} \overset{1-1}{\longleftrightarrow} \{\gamma : \mathrm{Univ}(X) \to \mathrm{Arb}(X)\}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists!\gamma'"'] \arrow[dr, "\forall\beta'"] \arrow[dr, phantom, "\circlearrowright"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'"] & X
\end{tikzcd}$$

$$\{\beta' : \mathrm{Arb}(X) \to X\} \overset{1-1}{\longleftrightarrow} \{\gamma' : \mathrm{Arb}(X) \to \mathrm{Univ}(X)\}$$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

You do — EXERCISE 1.2

$\mathrm{Univ}(X)$ NEED NOT EXIST, BUT IS UNIQUE IF DOES EXIST.

CONSIDER FORM I & SUPPOSE $\exists\, \overline{\mathrm{Univ}(X)}$ UNIVERSAL STRUCTURE ATTACHED TO X.

SHOW $\mathrm{Univ}(X) \cong \overline{\mathrm{Univ}(X)}$.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

$$\{\beta : X \to \mathrm{Arb}(X)\} \overset{1-1}{\updownarrow} \{\gamma : \mathrm{Univ}(X) \to \mathrm{Arb}(X)\}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'"'] \arrow[dr, "\forall \beta'"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'"'] & X
\end{tikzcd}$$

$$\{\beta' : \mathrm{Arb}(X) \to X\} \overset{1-1}{\updownarrow} \{\gamma' : \mathrm{Arb}(X) \to \mathrm{Univ}(X)\}$$

FORM II

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

You do **EXERCISE 1.2**

$\mathrm{Univ}(X)$ NEED NOT EXIST, BUT IS UNIQUE IF DOES EXIST.

CONSIDER FORM I & SUPPOSE $\exists\, \overline{\mathrm{Univ}(X)}$ UNIVERSAL STRUCTURE ATTACHED TO X.

SHOW $\mathrm{Univ}(X) \cong \overline{\mathrm{Univ}(X)}$.

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}
$$

$$\{\beta : X \to \mathrm{Arb}(X)\} \overset{1-1}{\longleftrightarrow} \{\gamma : \mathrm{Univ}(X) \to \mathrm{Arb}(X)\}$$

FORM I

USE UNIVERSAL PROPERTY TO GET MAPS

$$\phi : \mathrm{Univ}(X) \longrightarrow \overline{\mathrm{Univ}(X)}$$

$$\psi : \overline{\mathrm{Univ}(X)} \longrightarrow \mathrm{Univ}(X)$$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

You do **EXERCISE 1.2**

$\mathrm{Univ}(X)$ NEED NOT EXIST, BUT IS UNIQUE IF DOES EXIST.

CONSIDER FORM I & SUPPOSE $\exists\, \overline{\mathrm{Univ}(X)}$ — UNIVERSAL STRUCTURE ATTACHED TO X.

SHOW $\mathrm{Univ}(X) \cong \overline{\mathrm{Univ}(X)}$.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

$$\{\beta : X \to \mathrm{Arb}(X)\}$$
$$\updownarrow \text{1-1}$$
$$\{\gamma : \mathrm{Univ}(X) \to \mathrm{Arb}(X)\}$$

FORM I

USE UNIVERSAL PROPERTY TO GET MAPS

$$\phi : \mathrm{Univ}(X) \longrightarrow \overline{\mathrm{Univ}(X)}$$
$$\psi : \overline{\mathrm{Univ}(X)} \longrightarrow \mathrm{Univ}(X)$$

USE UNIQUENESS TO GET

$$\psi\phi = \mathrm{id}_{\mathrm{Univ}(X)} \qquad \phi\psi = \mathrm{id}_{\overline{\mathrm{Univ}(X)}}$$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{Univ}(X) \arrow[d, dotted, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

USE ↑ TO CHARACTERIZE

TAKE $\mathbb{R}$-VECTOR SPACES V, W

$$V \otimes_{\mathbb{R}} W := V \otimes W \overset{\text{DEF}}{=} \frac{\mathrm{SPAN}_{\mathbb{R}} \langle (v,w) \mid v \in V, w \in W \rangle}{\mathrm{SPAN}_{\mathbb{R}} \begin{pmatrix} \lambda(v,w) - (\lambda v, w) \quad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v',w) \\ (v, w+w') - (v,w) - (v,w') \end{pmatrix}}$$

(QUOTIENT SPACE)

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

USE ↑ TO CHARACTERIZE

TAKE $\mathbb{R}$-VECTOR SPACES V, W

$$V \otimes_{\mathbb{R}} W := V \otimes W \overset{\text{DEF}}{=} \frac{\mathrm{SPAN}_{\mathbb{R}} \langle (v,w) \mid v \in V, w \in W \rangle}{\mathrm{SPAN}_{\mathbb{R}} \begin{pmatrix} \lambda(v,w) - (\lambda v, w) \quad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v', w) \\ (v, w+w') - (v,w) - (v,w') \end{pmatrix}}$$

(QUOTIENT SPACE)

HAVE MAP

$$\begin{aligned} V \times W &\xrightarrow{\ \alpha\ } V \otimes_{\mathbb{R}} W \\ (v,w) &\longmapsto v \otimes w \end{aligned}$$

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II. Universal property, tensor product of VS revisited

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

USE ↑ TO CHARACTERIZE

TAKE $\mathbb{R}$-VECTOR SPACES V, W

$$\boxed{V \otimes_{\mathbb{R}} W := V \otimes W \overset{\text{DEF}}{=} \frac{\mathrm{SPAN}_{\mathbb{R}}\langle (v,w) \mid v \in V, w \in W \rangle}{\mathrm{SPAN}_{\mathbb{R}} \begin{pmatrix} \lambda(v,w) - (\lambda v, w) \qquad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v', w) \\ (v, w+w') - (v,w) - (v,w') \end{pmatrix}}}$$

(QUOTIENT SPACE)

HAVE MAP

$$\begin{aligned} V \times W &\xrightarrow[\text{BILINEAR}]{\alpha} V \otimes_{\mathbb{R}} W \\ (v,w) &\longmapsto v \otimes w \end{aligned}$$

HERE—

$$\begin{aligned} \alpha_v : W &\longrightarrow V \otimes_{\mathbb{R}} W \\ w &\longmapsto \alpha(v,w) \end{aligned}$$

$$\begin{aligned} \alpha_w : V &\longrightarrow V \otimes_{\mathbb{R}} W \\ v &\longmapsto \alpha(v,w) \end{aligned}$$

ARE LINEAR MAPS

$\forall v \in V, w \in W$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

USE ↑ TO CHARACTERIZE

TAKE $\mathbb{k}$-VECTOR SPACES V, W

$$V \otimes_{\mathbb{k}} W := V \otimes W \overset{\text{DEF}}{=} \frac{\mathrm{SPAN}_{\mathbb{k}} \langle (v,w) \mid v \in V, w \in W \rangle}{\mathrm{SPAN}_{\mathbb{k}} \begin{pmatrix} \lambda(v,w) - (\lambda v, w) \qquad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v',w) \\ (v, w+w') - (v,w) - (v,w') \end{pmatrix}}$$

(QUOTIENT SPACE)

$$\begin{array}{ccc} V \times W & \xrightarrow[\text{BILINEAR}]{\alpha} & V \otimes_{\mathbb{k}} W \\ (v, w) & \longmapsto & v \otimes w \end{array}$$

SUCH THAT

$\forall$ BILINEAR MAPS β $\longrightarrow$ Z VS

HERE—

$$\alpha_v : W \longrightarrow V \otimes_{\mathbb{k}} W$$
$$w \longmapsto \alpha(v,w)$$

$$\alpha_w : V \longrightarrow V \otimes_{\mathbb{k}} W$$
$$v \longmapsto \alpha(v,w)$$

ARE LINEAR MAPS

$\forall v \in V, w \in W$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

USE ↑ TO CHARACTERIZE

TAKE $\Bbbk$-VECTOR SPACES V, W

$$V \otimes_{\Bbbk} W := V \otimes W \overset{\text{DEF}}{=} \frac{\mathrm{SPAN}_{\Bbbk}\langle (v,w) \mid v \in V, w \in W \rangle}{\mathrm{SPAN}_{\Bbbk}\begin{pmatrix} \lambda(v,w) - (\lambda v, w) \quad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v',w) \\ (v, w+w') - (v,w) - (v,w') \end{pmatrix}}$$

(QUOTIENT SPACE)

$$\begin{tikzcd} V \times W \arrow[r, "\alpha \ \text{BILINEAR}"] \arrow[ddr, bend right] & V \otimes_{\Bbbk} W \arrow[dd, dashed, "\exists! \text{ LINEAR MAP } \gamma"] \arrow[ddl, phantom, "\circlearrowleft"] \\ (v,w) \arrow[r, mapsto] & v \otimes w \\ & Z \text{ VS} \end{tikzcd}$$

SUCH THAT

$\forall$ BILINEAR MAPS β

HERE —

$\alpha_v : W \longrightarrow V \otimes_{\Bbbk} W$
$\quad w \longmapsto \alpha(v,w)$

$\alpha_w : V \longrightarrow V \otimes_{\Bbbk} W$
$\quad v \longmapsto \alpha(v,w)$

ARE LINEAR MAPS

$\forall v \in V, w \in W$

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II. Universal Property, Tensor Product of VS Revisited

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

USE ↑ TO CHARACTERIZE →

TAKE $\mathbb{k}$-VECTOR SPACES V, W

$$V \otimes_{\mathbb{k}} W := V \otimes W \overset{\text{DEF}}{=} \frac{\mathrm{SPAN}_{\mathbb{k}} \langle (v,w) \mid v \in V, w \in W \rangle}{\mathrm{SPAN}_{\mathbb{k}} \begin{pmatrix} \lambda(v,w) - (\lambda v, w) \quad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v',w) \\ (v, w+w') - (v,w) - (v,w') \end{pmatrix}}$$

(QUOTIENT SPACE)

$$\begin{tikzcd} V \times W \arrow[r, "\alpha"] \arrow[ddr, bend right, "\beta"'] & V \otimes_{\mathbb{k}} W \arrow[dd, dashed, "\gamma"] \\ (v,w) \arrow[r, mapsto] & v \otimes w \\ & Z \text{ vs} \end{tikzcd}$$

BILINEAR

SUCH THAT

$\forall$ BILINEAR MAPS β

$\exists!$ LINEAR MAP γ

HERE—

$\alpha_v : W \longrightarrow V \otimes_{\mathbb{k}} W$
$w \longmapsto \alpha(v,w)$

$\alpha_w : V \longrightarrow V \otimes_{\mathbb{k}} W$
$v \longmapsto \alpha(v,w)$

ARE LINEAR MAPS

$\forall v \in V, w \in W$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

TAKE $\mathbb{R}$-VECTOR SPACES V, W

$$V \otimes_{\mathbb{R}} W := V \otimes W \overset{\text{DEF}}{=} \frac{\mathrm{SPAN}_{\mathbb{R}}\langle (v,w) \mid v \in V, w \in W \rangle}{\mathrm{SPAN}_{\mathbb{R}} \begin{pmatrix} \lambda(v,w) - (\lambda v, w) \qquad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v',w) \\ (v, w+w') - (v,w) - (v,w') \end{pmatrix}}$$

(QUOTIENT SPACE)

WHY THE SAME?

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha \ \text{BILINEAR}"', "(v,w) \mapsto v \otimes w"] \arrow[drr, "\forall \ \text{BILINEAR MAP } \beta"'] & & V \otimes_{\mathbb{R}} W \arrow[d, "\exists! \ \text{LINEAR MAP } \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & Z \ \text{VS}
\end{tikzcd}$$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

TAKE $\mathbb{R}$-VECTOR SPACES V, W

$$V \otimes_{\mathbb{R}} W := V \otimes W \overset{\text{DEF}}{=} \frac{\text{SPAN}_{\mathbb{R}} \langle (v,w) \mid v \in V, w \in W \rangle}{\text{SPAN}_{\mathbb{R}} \begin{pmatrix} \lambda(v,w) - (\lambda v, w) \quad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v', w) \\ (v, w+w') - (v,w) - (v,w') \end{pmatrix}}$$

(QUOTIENT SPACE)

TAKE ARBITRARY BILINEAR MAP

$$\beta : V \times W \longrightarrow Z$$

WHY THE SAME?

$$\begin{tikzcd}
V \times W \arrow[r, "\alpha\ \text{BILINEAR}", "{(v,w) \mapsto v \otimes w}"'] \arrow[dr, "\forall\ \text{BILINEAR MAP } \beta"'] & V \otimes_{\mathbb{R}} W \arrow[d, "\exists!\ \text{LINEAR MAP } \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
& Z \ \text{VS}
\end{tikzcd}$$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

TAKE $\mathbb{R}$-VECTOR SPACES V, W

$$V \otimes_{\mathbb{R}} W := V \otimes W \overset{\text{DEF}}{=} \frac{\text{SPAN}_{\mathbb{R}}\langle (v,w) \mid v \in V, w \in W \rangle}{\text{SPAN}_{\mathbb{R}}\left(\begin{matrix} \lambda(v,w) - (\lambda v, w) \quad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v', w) \\ (v, w+w') - (v,w) - (v,w') \end{matrix} \right)}$$

(QUOTIENT SPACE)

TAKE ARBITRARY BILINEAR MAP

$$\beta : V \times W \longrightarrow Z$$

GET LINEAR MAP

$$\hat{\beta} : \text{SPAN}_{\mathbb{R}}\langle (v,w) \mid v \in V, w \in W \rangle \longrightarrow Z$$

$$\textstyle\sum_i \lambda_i (v_i, w_i) \longmapsto \sum_i \lambda_i \beta(v_i, w_i)$$

GET RELATION SPACE OF $V \otimes_{\mathbb{R}} W$

$$\subseteq \ker(\hat{\beta})$$

WHY THE SAME?

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha\ \text{BILINEAR}", "(v,w) \mapsto v \otimes w"'] \arrow[drr, "\forall\ \text{BILINEAR MAP}\ \beta"'] & \arrow[dr, phantom, "\circlearrowleft"] & V \otimes_{\mathbb{R}} W \arrow[d, "\exists!\ \text{LINEAR MAP}\ \gamma"] \\
& & Z\ \text{VS}
\end{tikzcd}$$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

TAKE $\mathbb{k}$-VECTOR SPACES V, W

$$\boxed{V \otimes_{\mathbb{k}} W := V \otimes W \overset{\text{DEF}}{=} \frac{\mathrm{SPAN}_{\mathbb{k}}\langle (v,w) \mid v \in V, w \in W \rangle}{\mathrm{SPAN}_{\mathbb{k}}\begin{pmatrix} \lambda(v,w)-(\lambda v,w) \qquad \lambda(v,w)-(v,\lambda w) \\ (v+v',w)-(v,w)-(v',w) \\ (v,w+w')-(v,w)-(v,w') \end{pmatrix}}}$$

(QUOTIENT SPACE)

TAKE ARBITRARY BILINEAR MAP

$$\beta : V \times W \longrightarrow Z$$

GET LINEAR MAP

$$\hat{\beta} : \mathrm{SPAN}_{\mathbb{k}}\langle (v,w) \mid v \in V, w \in W \rangle \longrightarrow Z$$

$$\textstyle\sum_i \lambda_i (v_i, w_i) \longmapsto \sum \lambda_i \beta(v_i, w_i)$$

GET RELATION SPACE OF $V \otimes_{\mathbb{k}} W$

$$\subseteq \ker(\hat{\beta})$$

YIELDS LINEAR MAP

$$\gamma : V \otimes_{\mathbb{k}} W \longrightarrow Z$$

$$v \otimes w \longmapsto \beta(v,w)$$

ARGUE UNIQUENESS OF γ...

WHY THE SAME?

$$\boxed{\begin{tikzcd}
V \times W \arrow[r, "\alpha\ \text{BILINEAR}", "(v,w) \mapsto v \otimes w"'] \arrow[dr, "\forall\ \text{BILINEAR MAP } \beta"'] & V \otimes_{\mathbb{k}} W \arrow[d, "\exists!\ \text{LINEAR MAP } \gamma"] \\
 & Z \ \text{VS}
\end{tikzcd}}$$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

TAKE $\Bbbk$-VECTOR SPACES V, W

$$V \otimes_{\Bbbk} W := V \otimes W \overset{\text{DEF}}{=} \frac{\text{SPAN}_{\Bbbk}\langle (v,w) \mid v \in V,\ w \in W \rangle}{\text{SPAN}_{\Bbbk}\begin{pmatrix} \lambda(v,w) - (\lambda v, w) \quad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v', w) \\ (v, w+w') - (v,w) - (v,w') \end{pmatrix}}$$

(QUOTIENT SPACE)

Ex. $V \otimes_{\Bbbk} W \cong W \otimes_{\Bbbk} V$

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha \ \text{BILINEAR}"] \arrow[rrd, "\forall \text{ BILINEAR MAP } \beta"'] & & V \otimes_{\Bbbk} W \arrow[d, "\exists! \text{ LINEAR MAP } \gamma"] \\
& \circlearrowleft & Z \text{ VS}
\end{tikzcd}$$

$(v,w) \longmapsto v \otimes w$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

TAKE $\Bbbk$-VECTOR SPACES V, W

COULD ESTABLISH $\cong$ DIRECTLY VIA $\rightarrow$

(TRICKY WORKING WITH QUOTIENTS ... SHOW MAPS ARE WELL-DEFINED)

$$\boxed{V \otimes_{\Bbbk} W := V \otimes W \overset{\text{DEF}}{=} \frac{\text{SPAN}_{\Bbbk}\langle (v,w) \mid v \in V, w \in W \rangle}{\text{SPAN}_{\Bbbk}\begin{pmatrix} \lambda(v,w) - (\lambda v, w) \quad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v',w) \\ (v, w+w') - (v,w) - (v,w') \end{pmatrix}}}$$

(QUOTIENT SPACE)

$$\boxed{\text{Ex.} \quad V \otimes_{\Bbbk} W \cong W \otimes_{\Bbbk} V}$$

$$\begin{tikzcd}
V \times W \arrow[r, "\alpha\ \text{BILINEAR}"] \arrow[dr, "\forall\ \text{BILINEAR MAP}\ \beta"'] & V \otimes_{\Bbbk} W \arrow[d, "\exists!\ \text{LINEAR MAP}\ \gamma"] \arrow[dl, phantom, "\circlearrowright"] \\
& Z\ \text{VS}
\end{tikzcd}$$

$(v, w) \mapsto v \otimes w$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

TAKE $\mathbb{R}$-VECTOR SPACES V, W

COULD ESTABLISH $\cong$ DIRECTLY VIA

$$V \otimes_{\mathbb{R}} W := V \otimes W \overset{\text{DEF}}{=} \frac{\mathrm{SPAN}_{\mathbb{R}}\langle (v,w) \mid v \in V, w \in W \rangle}{\mathrm{SPAN}_{\mathbb{R}}\begin{pmatrix} \lambda(v,w) - (\lambda v, w) \qquad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v',w) \\ (v, w+w') - (v,w) - (v,w') \end{pmatrix}}$$

(QUOTIENT SPACE)

(TRICKY WORKING WITH QUOTIENTS ... SHOW MAPS ARE WELL-DEFINED)

BETTER USING UNIV. PROPERTY:

Ex. $V \otimes_{\mathbb{R}} W \cong W \otimes_{\mathbb{R}} V$

HAVE BILINEAR MAP

$$\beta : V \times W \longrightarrow W \otimes_{\mathbb{R}} V$$
$$(v,w) \longmapsto w \otimes v$$

GET LINEAR MAP

$$\gamma : V \otimes_{\mathbb{R}} W \longrightarrow W \otimes_{\mathbb{R}} V$$
$$v \otimes w \longmapsto w \otimes v$$

$$\begin{tikzcd} V \times W \arrow[r, "\alpha", "\text{BILINEAR}"'] \arrow[dr, "\forall \text{ BILINEAR MAP } \beta"'] & V \otimes_{\mathbb{R}} W \arrow[d, "\exists! \text{ LINEAR MAP } \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\ & Z \text{ VS} \end{tikzcd}$$

$(v,w) \longmapsto v \otimes w$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

TAKE $\mathbb{R}$-VECTOR SPACES V, W

USING UNIV. PROPERTY:

$$V \otimes_{\mathbb{R}} W := V \otimes W \overset{\text{DEF}}{=} \frac{\mathrm{SPAN}_{\mathbb{R}}\langle (v,w) \mid v \in V,\, w \in W \rangle}{\mathrm{SPAN}_{\mathbb{R}}\begin{pmatrix} \lambda(v,w) - (\lambda v, w) \quad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v', w) \\ (v, w+w') - (v,w) - (v,w') \end{pmatrix}}$$

(QUOTIENT SPACE)

HAVE BILINEAR MAP

$$\beta : V \times W \longrightarrow W \otimes_{\mathbb{R}} V$$
$$(v, w) \longmapsto w \otimes v$$

GET

$$\gamma : V \otimes_{\mathbb{R}} W \longrightarrow W \otimes_{\mathbb{R}} V$$
$$v \otimes w \longmapsto w \otimes v$$

LINEAR MAP

Ex. $V \otimes_{\mathbb{R}} W \cong W \otimes_{\mathbb{R}} V$

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha", "\text{BILINEAR}"'] \arrow[drr, "\forall \text{ BILINEAR MAP } \beta"'] & & V \otimes_{\mathbb{R}} W \arrow[d, "\exists! \text{ LINEAR MAP } \gamma"] \\
& \arrow[ur, phantom, "\circlearrowleft"] & Z \text{ VS}
\end{tikzcd}$$

$(v, w) \longmapsto v \otimes w$

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II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED

TAKE $\mathbb{R}$-VECTOR SPACES V, W

$$V \otimes_{\mathbb{R}} W := V \otimes W \overset{\text{DEF}}{=} \frac{\operatorname{SPAN}_{\mathbb{R}}\langle (v,w) \mid v \in V, w \in W \rangle}{\operatorname{SPAN}_{\mathbb{R}}\begin{pmatrix} \lambda(v,w) - (\lambda v, w) \quad \lambda(v,w) - (v, \lambda w) \\ (v+v', w) - (v,w) - (v',w) \\ (v, w+w') - (v,w) - (v,w') \end{pmatrix}}$$

(QUOTIENT SPACE)

USING UNIV. PROPERTY:

HAVE BILINEAR MAP

$$\beta : V \times W \longrightarrow W \otimes_{\mathbb{R}} V$$
$$(v, w) \longmapsto w \otimes v$$

GET $\gamma : V \otimes_{\mathbb{R}} W \longrightarrow W \otimes_{\mathbb{R}} V$ LINEAR MAP

$$v \otimes w \longmapsto w \otimes v$$

Ex. $V \otimes_{\mathbb{R}} W \cong W \otimes_{\mathbb{R}} V$

$$\begin{tikzcd}
V \times W \arrow[r, "\alpha", "\text{BILINEAR}"'] \arrow[dr, "\forall \text{ BILINEAR MAP } \beta"'] & V \otimes_{\mathbb{R}} W \arrow[d, "\exists! \text{ LINEAR MAP } \gamma"] \\
 & Z \text{ VS}
\end{tikzcd}$$

$$(v, w) \longmapsto v \otimes w$$

LIKEWISE UNIV. PROP OF $W \otimes_{\mathbb{R}} V$ YIELDS LINEAR MAP

$$\bar{\gamma} : W \otimes_{\mathbb{R}} V \longrightarrow V \otimes_{\mathbb{R}} W$$
$$w \otimes v \longmapsto v \otimes w$$

γ & $\bar{\gamma}$ ARE MUTUALLY INVERSE

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III. MORE ON TENSOR PRODUCT & HOM OF VSPACES

OPERATIONS ON LINEAR MAPS

TAKE LINEAR MAPS $f: V \longrightarrow W$ AND $f': V' \longrightarrow W'$

GET LINEAR MAPS
$$\begin{cases} f \times f' : V \times V' \longrightarrow W \times W' \\ f + f' : V + V' \longrightarrow W + W' \\ f \oplus f' : V \oplus V' \longrightarrow W \oplus W' \end{cases}$$
AS EXPECTED

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III. MORE ON TENSOR PRODUCT & HOM OF VSPACES

OPERATIONS ON LINEAR MAPS

TAKE LINEAR MAPS $f: V \to W$ AND $f': V' \to W'$

GET LINEAR MAPS
$$\left\{ \begin{array}{l} f \times f' : V \times V' \longrightarrow W \times W' \\ f + f' : V + V' \longrightarrow W + W' \\ f \oplus f' : V \oplus V' \longrightarrow W \oplus W' \end{array} \right.$$
AS EXPECTED

ALSO HAVE LINEAR MAPS
$$f \otimes f' : V \otimes V' \longrightarrow W \otimes W'$$
$$v \otimes v' \longmapsto f(v) \otimes f'(v')$$

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III. MORE ON TENSOR PRODUCT & HOM OF VSPACES

OPERATIONS ON LINEAR MAPS

TAKE LINEAR MAPS $f : V \longrightarrow W$ AND $f' : V' \longrightarrow W'$

GET LINEAR MAPS
$$\begin{cases} f \times f' : V \times V' \longrightarrow W \times W' \\ f + f' : V + V' \longrightarrow W + W' \\ f \oplus f' : V \oplus V' \longrightarrow W \oplus W' \end{cases}$$
AS EXPECTED

ALSO HAVE LINEAR MAPS
$$f \otimes f' : V \otimes V' \longrightarrow W \otimes W'$$
$$v \otimes v' \longmapsto f(v) \otimes f'(v')$$

$$\mathrm{Hom}_{\mathbb{k}}(f, U) : \mathrm{Hom}_{\mathbb{k}}(W, U) \longrightarrow \mathrm{Hom}_{\mathbb{k}}(V, U)$$
$$g \longmapsto \left[V \xrightarrow{f} W \xrightarrow{g} U \right] =: gf$$

FOR VECTOR SPACE U

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III. MORE ON TENSOR PRODUCT & HOM OF VSPACES

OPERATIONS ON LINEAR MAPS

TAKE LINEAR MAPS $f: V \to W$ AND $f': V' \to W'$

GET LINEAR MAPS
$$\left\{ \begin{array}{l} f \times f' : V \times V' \longrightarrow W \times W' \\ f + f' : V + V' \longrightarrow W + W' \\ f \oplus f' : V \oplus V' \longrightarrow W \oplus W' \end{array} \right.$$
AS EXPECTED

ALSO HAVE LINEAR MAPS
$$\begin{aligned} f \otimes f' : V \otimes V' &\longrightarrow W \otimes W' \\ v \otimes v' &\longmapsto f(v) \otimes f'(v') \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_{\Bbbk}(f, U) : \mathrm{Hom}_{\Bbbk}(W, U) &\longrightarrow \mathrm{Hom}_{\Bbbk}(V, U) \\ g &\longmapsto [V \xrightarrow{f} W \xrightarrow{g} U] =: gf \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_{\Bbbk}(U, f') : \mathrm{Hom}_{\Bbbk}(U, V') &\longrightarrow \mathrm{Hom}_{\Bbbk}(U, W') \\ g &\longmapsto [U \xrightarrow{g} V' \xrightarrow{f'} W] =: f'g \end{aligned}$$

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III. MORE ON TENSOR PRODUCT & HOM OF VSPACES

OPERATIONS ON LINEAR MAPS

TAKE LINEAR MAPS $f: V \to W$ AND $f': V' \to W'$

GET LINEAR MAPS
$$\begin{cases} f \times f' : V \times V' \longrightarrow W \times W' \\ f + f' : V + V' \longrightarrow W + W' \\ f \oplus f' : V \oplus V' \longrightarrow W \oplus W' \end{cases}$$

$\boxed{V \otimes -}$ $\boxed{- \otimes V'}$
$\boxed{\mathrm{Hom}_{\mathbb{k}}(-, U)}$ $\boxed{\mathrm{Hom}_{\mathbb{k}}(U, -)}$
TAKES VS TO VS & LINEAR MAP TO LINEAR MAP.

ALSO HAVE LINEAR MAPS
$$f \otimes f' : V \otimes V' \longrightarrow W \otimes W'$$
$$v \otimes v' \longmapsto f(v) \otimes f'(v')$$

$$\mathrm{Hom}_{\mathbb{k}}(f, U) : \mathrm{Hom}_{\mathbb{k}}(W, U) \longrightarrow \mathrm{Hom}_{\mathbb{k}}(V, U)$$
$$g \longmapsto [V \xrightarrow{f} W \xrightarrow{g} U] =: gf$$

$$\mathrm{Hom}_{\mathbb{k}}(U, f') : \mathrm{Hom}_{\mathbb{k}}(U, V') \longrightarrow \mathrm{Hom}_{\mathbb{k}}(U, W')$$
$$g \longmapsto [U \xrightarrow{g} V' \xrightarrow{f'} W] =: f'g$$

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OPERATIONS ON LINEAR MAPS

TAKE LINEAR MAPS $f: V \longrightarrow W$ AND $f': V' \longrightarrow W'$

GET LINEAR MAPS
$$\begin{cases} f \times f' : V \times V' \longrightarrow W \times W' \\ f + f' : V + V' \longrightarrow W + W' \\ f \oplus f' : V \oplus V' \longrightarrow W \oplus W' \end{cases}$$

$\boxed{V \otimes -}$ $\boxed{- \otimes V'}$ $\boxed{\mathrm{Hom}_{\mathbb{k}}(-, U)}$ $\boxed{\mathrm{Hom}_{\mathbb{k}}(U, -)}$

TAKES VS TO VS & LINEAR MAP TO LINEAR MAP.

ALSO HAVE LINEAR MAPS

$$f \otimes f' : V \otimes V' \longrightarrow W \otimes W'$$
$$v \otimes v' \longmapsto f(v) \otimes f'(v')$$

$V \otimes -$ $- \otimes V'$

"COVARIANT"

DIRECTION OF MAP STAYS SAME

$$\mathrm{Hom}_{\mathbb{k}}(f, U) : \mathrm{Hom}_{\mathbb{k}}(W, U) \longrightarrow \mathrm{Hom}_{\mathbb{k}}(V, U)$$

CONTRAVARIANT

$$g \longmapsto [V \xrightarrow{f} W \xrightarrow{g} U] =: gf$$

$$\mathrm{Hom}_{\mathbb{k}}(U, f') : \mathrm{Hom}_{\mathbb{k}}(U, V') \longrightarrow \mathrm{Hom}_{\mathbb{k}}(U, W')$$

COVARIANT

$$g \longmapsto [U \xrightarrow{g} V' \xrightarrow{f'} W] =: f'g$$

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III. MORE ON TENSOR PRODUCT & HOM OF VSPACES

TENSOR-HOM ADJUNCTION

GIVES HOW $\otimes$ & HOM ASSIGNMENTS ARE RELATED

$\boxed{V \otimes -}$ $\boxed{- \otimes V'}$

$\boxed{\mathrm{Hom}_{\mathbb{k}}(U, -)}$ ALL <u>COVARIANT</u>

TAKES VS TO VS

& LINEAR MAP TO LINEAR MAP.

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III. MORE ON TENSOR PRODUCT & HOM OF VSPACES

TENSOR-HOM ADJUNCTION

GIVES HOW $\otimes$ & HOM ASSIGNMENTS ARE RELATED

$$\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(U, \mathrm{Hom}_{\mathbb{k}}(V, W))$$

$\boxed{V \otimes -}$ $\boxed{- \otimes V'}$

$\boxed{\mathrm{Hom}_{\mathbb{k}}(U,-)}$ ALL <u>COVARIANT</u>

TAKES VS TO VS & LINEAR MAP TO LINEAR MAP.

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III. MORE ON TENSOR PRODUCT & HOM OF VSPACES

TENSOR-HOM ADJUNCTION

GIVES HOW $\otimes$ & HOM ASSIGNMENTS ARE RELATED

$$\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(U, \mathrm{Hom}_{\mathbb{k}}(V, W))$$

$$\phi \mapsto \left[U \longrightarrow \mathrm{Hom}_{\mathbb{k}}(V, W) \right]$$

$\boxed{V \otimes -}$ $\boxed{- \otimes V'}$

$\boxed{\mathrm{Hom}_{\mathbb{k}}(U, -)}$ ALL COVARIANT

TAKES VS TO VS & LINEAR MAP TO LINEAR MAP.

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$$\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(U, \mathrm{Hom}_{\mathbb{k}}(V, W))$$

$$\phi \longmapsto \left[\begin{array}{l} U \longrightarrow \mathrm{Hom}_{\mathbb{k}}(V,W) \\ u \longmapsto [v \longmapsto \phi(u \otimes v)] \end{array}\right]$$

$\boxed{V \otimes -}$ $\boxed{- \otimes V'}$ $\boxed{\mathrm{Hom}_{\mathbb{k}}(U, -)}$ ALL COVARIANT

TAKES VS TO VS & LINEAR MAP TO LINEAR MAP.

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TENSOR-HOM ADJUNCTION

GIVES HOW $\otimes$ & HOM ASSIGNMENTS ARE RELATED

$$\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(U, \mathrm{Hom}_{\mathbb{k}}(V, W))$$

$$\phi \longmapsto \begin{bmatrix} U \longrightarrow \mathrm{Hom}_{\mathbb{k}}(V,W) \\ u \longmapsto [v \longmapsto \phi(u \otimes v)] \end{bmatrix}$$

$\boxed{V \otimes -}$ $\boxed{- \otimes V'}$

$\boxed{\mathrm{Hom}_{\mathbb{k}}(U, -)}$ ALL COVARIANT

TAKES VS TO VS & LINEAR MAP TO LINEAR MAP.

$$\begin{bmatrix} U \otimes V \longrightarrow W \\ \end{bmatrix} \longleftarrow\!\!\!\mid \ \psi$$

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TENSOR-HOM ADJUNCTION

GIVES HOW $\otimes$ & HOM ASSIGNMENTS ARE RELATED

$$\boxed{\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(U, \mathrm{Hom}_{\mathbb{k}}(V, W))}$$

$$\phi \longmapsto \begin{bmatrix} U \longrightarrow \mathrm{Hom}_{\mathbb{k}}(V,W) \\ u \longmapsto [v \longmapsto \phi(u \otimes v)] \end{bmatrix}$$

$$\begin{bmatrix} U \otimes V \longrightarrow W \\ u \otimes v \longmapsto \psi(u)(v) \end{bmatrix} \longleftarrow\!\!\mapsfrom \psi$$

$\boxed{V \otimes -}$ $\boxed{- \otimes V'}$ $\boxed{\mathrm{Hom}_{\mathbb{k}}(U, -)}$ ALL COVARIANT

TAKES VS TO VS & LINEAR MAP TO LINEAR MAP.

CHECK THESE MAPS ARE MUTUALLY INVERSE

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TENSOR-HOM ADJUNCTION

GIVES HOW $\otimes$ & HOM ASSIGNMENTS ARE RELATED

$$\boxed{\mathrm{Hom}_{\mathbb{k}}(U\otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(U, \mathrm{Hom}_{\mathbb{k}}(V,W))}$$

$$\phi \longmapsto \left[\begin{array}{ccc} U & \longrightarrow & \mathrm{Hom}_{\mathbb{k}}(V,W) \\ u & \longmapsto & [v \longmapsto \phi(u\otimes v)] \end{array}\right]$$

$$\left[\begin{array}{ccc} U\otimes V & \longrightarrow & W \\ u\otimes v & \longmapsto & \psi(u)(v) \end{array}\right] \longleftarrow\!\!\!\shortmid \; \psi$$

$\boxed{V\otimes -}$ $\boxed{-\otimes V'}$ $\boxed{\mathrm{Hom}_{\mathbb{k}}(U,-)}$ ALL <u>COVARIANT</u>

TAKES VS TO VS & LINEAR MAP TO LINEAR MAP.

ALSO GET:

$$\mathrm{Hom}_{\mathbb{k}}(U\otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(V, \mathrm{Hom}_{\mathbb{k}}(U,W))$$

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TENSOR-HOM ADJUNCTION

GIVES HOW $\otimes$ & HOM ASSIGNMENTS ARE RELATED

$$\boxed{\mathrm{Hom}_{\mathbb{k}}(U\otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(U, \mathrm{Hom}_{\mathbb{k}}(V,W))}$$

$\boxed{V\otimes -}$ $\boxed{-\otimes V'}$ $\boxed{\mathrm{Hom}_{\mathbb{k}}(U,-)}$ ALL COVARIANT

TAKES VS TO VS & LINEAR MAP TO LINEAR MAP.

$$\phi \longmapsto \left[\begin{array}{l} U \longrightarrow \mathrm{Hom}_{\mathbb{k}}(V,W) \\ u \longmapsto [v \longmapsto \phi(u\otimes v)] \end{array}\right]$$

$$\left[\begin{array}{l} U\otimes V \longrightarrow W \\ u\otimes v \longmapsto \psi(u)(v) \end{array}\right] \longleftarrow\!\!\dashv \psi$$

ALSO GET:

$$\boxed{\mathrm{Hom}_{\mathbb{k}}(U\otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(V, \mathrm{Hom}_{\mathbb{k}}(U,W))}$$

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$$\phi \longmapsto \begin{bmatrix} U \longrightarrow \mathrm{Hom}_{\mathbb{k}}(V,W) \\ u \longmapsto [v \longmapsto \phi(u \otimes v)] \end{bmatrix}$$

$\boxed{V \otimes -}$ $\boxed{- \otimes V'}$ $\boxed{\mathrm{Hom}_{\mathbb{k}}(U, -)}$ ALL COVARIANT

TAKES VS TO VS & LINEAR MAP TO LINEAR MAP.

$$\begin{bmatrix} U \otimes V \longrightarrow W \\ u \otimes v \longmapsto \psi(u)(v) \end{bmatrix} \longleftarrow\!\!\!\mid \psi$$

LATER (CHAPTER 2)

- VSPACES & LINEAR MAPS $\equiv$ "CATEGORY"
- $\otimes$, HOM ASSIGNMENTS $\equiv$ "FUNCTORS"
- RELATION BETWEEN ASSIGNMENTS $\equiv$ "ADJUNCTION"

ALSO GET:

$$\boxed{\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(V, \mathrm{Hom}_{\mathbb{k}}(U, W))}$$

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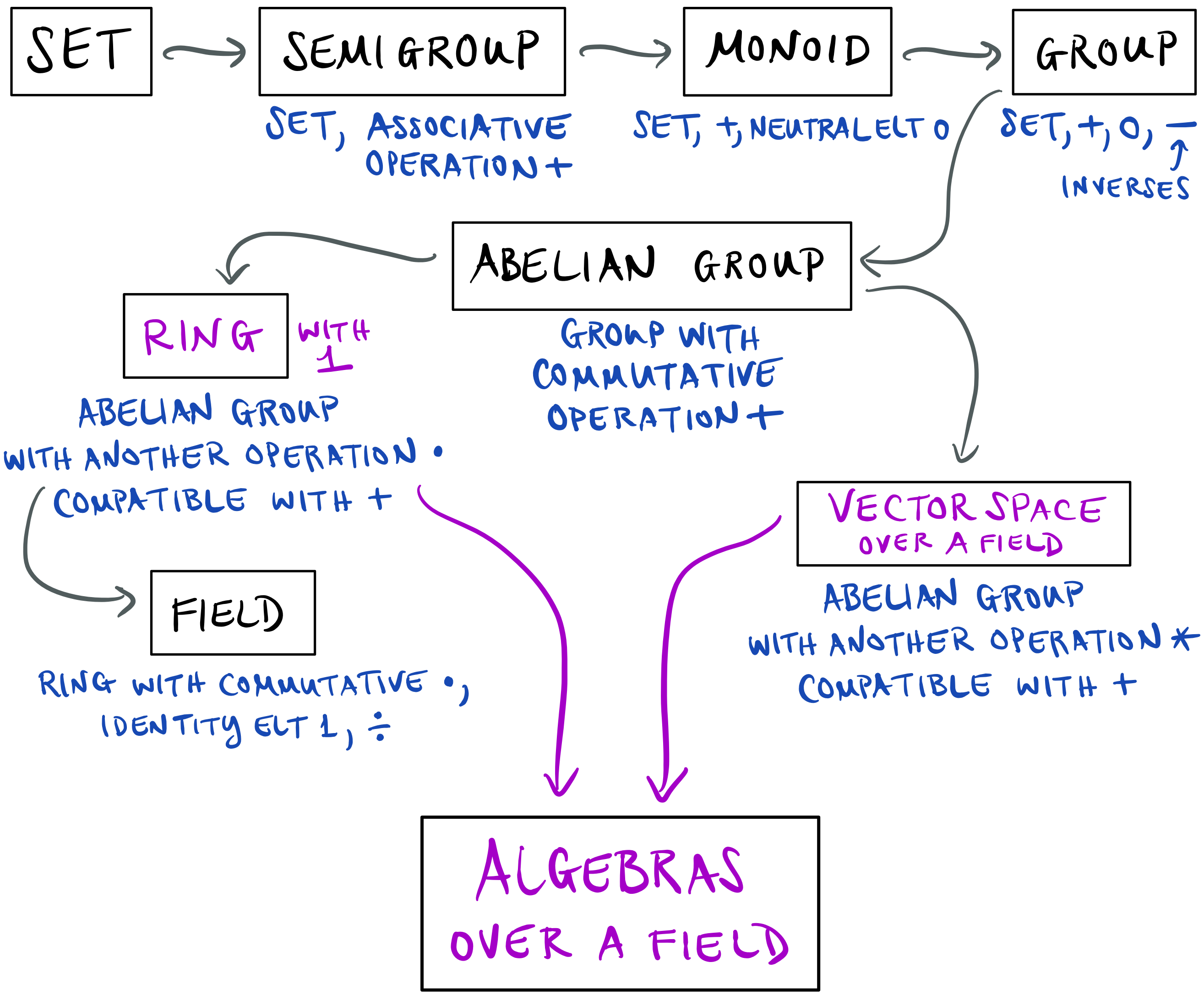
IV. ALGEBRAS OVER A FIELD

ALGEBRAIC STRUCTURES —



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$\boxed{\text{ALGEBRAS OVER A FIELD}}$

ABELIAN GROUP
WITH OPERATIONS • AND *
COMPATIBLE WITH EACH OTHER
& WITH +

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IV. ALGEBRAS OVER A FIELD

$\boxed{\text{ALGEBRAS OVER A FIELD}}$ $\equiv$ UNITAL RING MADE INTO A $\Bbbk$-VECTOR SPACE

ABELIAN GROUP
WITH OPERATIONS $\cdot$ AND $*$
COMPATIBLE WITH EACH OTHER
& WITH +

A UNITAL RING $(A, +, 0, \cdot, 1)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH A UNITAL RING MAP $\phi : \Bbbk \longrightarrow A$ SUCH THAT $\mathrm{im}(\phi) \subseteq Z(A)$

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ABELIAN GROUP
WITH OPERATIONS $\bullet$ AND $*$
COMPATIBLE WITH EACH OTHER
& WITH $+$

> A UNITAL RING $(A, +, 0, \bullet, 1)$ IS A **$\Bbbk$-ALGEBRA**
> IF IT COMES WITH A UNITAL RING MAP
> $\phi : \Bbbk \longrightarrow A$ SUCH THAT $\operatorname{im}(\phi) \subseteq Z(A)$

GET $* : \Bbbk \times A \longrightarrow A$ SCALAR MULTIP.

$$(\lambda, a) \longmapsto \lambda * a := \phi(\lambda) a$$

USING MULTIP IN A

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COMPATIBLE WITH EACH OTHER
& WITH $+$

$\equiv$

A $\Bbbk$-VSPACE MADE
INTO A UNITAL RING

$$\boxed{\begin{array}{c}\text{A UNITAL RING } (A,+,0,\cdot,1) \text{ IS A } \Bbbk\text{-ALGEBRA}\\ \text{IF IT COMES WITH A UNITAL RING MAP}\\ \phi : \Bbbk \longrightarrow A \quad \text{SUCH THAT } \operatorname{im}(\phi) \subseteq Z(A)\end{array}}$$

GET $* : \Bbbk \times A \longrightarrow A$ SCALAR MULTIP.

$(\lambda, a) \longmapsto \lambda * a := \phi(\lambda)a$ USING MULTIP IN A

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$\equiv$

A $\Bbbk$-VSPACE MADE
INTO A UNITAL RING

> A UNITAL RING $(A, +, 0, \cdot, 1)$ IS A **$\Bbbk$-ALGEBRA**
> IF IT COMES WITH A UNITAL RING MAP
> $\phi : \Bbbk \longrightarrow A$ SUCH THAT $\operatorname{im}(\phi) \subseteq Z(A)$

> A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH
> LINEAR MAPS $m : A \otimes A \longrightarrow A$ (MULTIPLICATION) & $u : \Bbbk \longrightarrow A$ (UNIT)
> $a \otimes b \longmapsto ab$ $\quad$ $1_\Bbbk \longmapsto 1_A$
>
> SUCH THAT THE FOLLOWING DIAGRAMS COMMUTE:

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IV. Algebras over a field

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ABELIAN GROUP
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COMPATIBLE WITH EACH OTHER
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$\equiv$

A $\Bbbk$-VSPACE MADE INTO A UNITAL RING

A UNITAL RING $(A, +, 0, \cdot, 1)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH A UNITAL RING MAP
$$\phi : \Bbbk \longrightarrow A \quad \text{SUCH THAT } \operatorname{im}(\phi) \subseteq Z(A)$$

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH LINEAR MAPS
$$m : A \otimes A \longrightarrow A \ (\text{MULTIPLICATION}) \quad \& \quad u : \Bbbk \longrightarrow A \ (\text{UNIT})$$
$$a \otimes b \longmapsto ab \qquad\qquad 1_{\Bbbk} \longmapsto 1_A$$

SUCH THAT THE FOLLOWING DIAGRAMS COMMUTE:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \Bbbk \cong A \cong \Bbbk \otimes A \arrow[r, "u \otimes id"] \arrow[d, "id \otimes u"'] \arrow[dr, "id"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

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ABELIAN GROUP
WITH OPERATIONS $\cdot$ AND $*$
COMPATIBLE WITH EACH OTHER
& WITH $+$

$\equiv$

A $\Bbbk$-VSPACE MADE
INTO A UNITAL RING

A UNITAL RING $(A, +, 0, \cdot, 1)$ IS A **$\Bbbk$-ALGEBRA**
IF IT COMES WITH A UNITAL RING MAP
$\phi : \Bbbk \longrightarrow A$ SUCH THAT $\operatorname{im}(\phi) \subseteq Z(A)$

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH
LINEAR MAPS $m : A \otimes A \longrightarrow A$ (MULTIPLICATION) & $u : \Bbbk \longrightarrow A$ (UNIT)

$$a \otimes b \longmapsto ab \qquad\qquad 1_{\Bbbk} \longmapsto 1_A$$

SUCH THAT THE FOLLOWING DIAGRAMS COMMUTE:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \Bbbk \cong A \cong \Bbbk \otimes A \arrow[r, "u \otimes id"] \arrow[d, "id \otimes u"'] \arrow[dr, "id"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

EXERCISE 1.3: $A \otimes \Bbbk \cong A \cong \Bbbk \otimes A$

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$\boxed{\text{ALGEBRAS OVER A FIELD}}$ $\equiv$ UNITAL RING MADE INTO A $\Bbbk$-VECTOR SPACE

A UNITAL RING $(A, +, 0, \bullet, 1)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH A UNITAL RING MAP $\phi : \Bbbk \longrightarrow A$ SUCH THAT $\operatorname{im}(\phi) \subseteq Z(A)$

ABELIAN GROUP WITH OPERATIONS $\bullet$ AND $*$ COMPATIBLE WITH EACH OTHER & WITH $+$

$\equiv$

A $\Bbbk$-VSPACE MADE INTO A UNITAL RING

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH

LINEAR MAPS $m : A \otimes A \longrightarrow A$, $a \otimes b \longmapsto ab$ (MULTIPLICATION) & $u : \Bbbk \longrightarrow A$, $1_{\Bbbk} \longmapsto 1_A$ (UNIT)

SUCH THAT THE FOLLOWING DIAGRAMS COMMUTE:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \Bbbk \cong A \cong \Bbbk \otimes A \arrow[r, "u \otimes id"] \arrow[d, "id \otimes u"'] \arrow[dr, "id"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

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$\boxed{\text{ALGEBRAS OVER A FIELD}}$ $\equiv$ UNITAL RING MADE INTO A $\Bbbk$-VECTOR SPACE

> A UNITAL RING $(A,+,0,\bullet,1)$ IS A $\Bbbk$-ALGEBRA IF IT COMES WITH A UNITAL RING MAP $\phi : \Bbbk \longrightarrow A$ SUCH THAT $\operatorname{im}(\phi) \subseteq Z(A)$

ABELIAN GROUP
WITH OPERATIONS $\bullet$ AND $*$
COMPATIBLE WITH EACH OTHER
& WITH $+$

$\equiv$

A $\Bbbk$-VSPACE MADE
INTO A UNITAL RING

> A $\Bbbk$-VSPACE $(A,+,0,*)$ IS A $\Bbbk$-ALGEBRA IF IT COMES WITH LINEAR MAPS
>
> $$m : A\otimes A \longrightarrow A \ \text{(MULTIPLICATION)}, \quad a\otimes b \longmapsto ab \qquad \& \qquad u : \Bbbk \longrightarrow A \ \text{(UNIT)}, \quad 1_{\Bbbk} \longmapsto 1_A$$
>
> SUCH THAT THE FOLLOWING DIAGRAMS COMMUTE:
>
> $$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "m\otimes id"] \arrow[d, "id\otimes m"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \arrow[d, "m"] \\ A\otimes A \arrow[r, "m"] & A \end{tikzcd} \qquad \begin{tikzcd} a\otimes b\otimes c \arrow[r, mapsto] \arrow[d, mapsto] & (ab)\otimes c \arrow[d, mapsto] \\ a\otimes(bc) \arrow[r, mapsto] & a(bc) = (ab)c \end{tikzcd}$$

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IV. Algebras over a field

$\boxed{\text{ALGEBRAS OVER A FIELD}}$ $\equiv$ UNITAL RING MADE INTO A $\Bbbk$-VECTOR SPACE

ABELIAN GROUP
WITH OPERATIONS $\cdot$ AND $*$
COMPATIBLE WITH EACH OTHER
& WITH $+$

$\equiv$

A $\Bbbk$-VSPACE MADE
INTO A UNITAL RING

> A UNITAL RING $(A, +, 0, \cdot, 1)$ IS A $\Bbbk$-ALGEBRA
> IF IT COMES WITH A UNITAL RING MAP
> $\phi : \Bbbk \longrightarrow A$ SUCH THAT $\operatorname{im}(\phi) \subseteq Z(A)$

> A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A $\Bbbk$-ALGEBRA IF IT COMES WITH
> LINEAR MAPS $m : A \otimes A \longrightarrow A$ (MULTIPLICATION) & $u : \Bbbk \longrightarrow A$ (UNIT)
> $a \otimes b \longmapsto ab$ $\qquad 1_{\Bbbk} \longmapsto 1_A$
>
> SUCH THAT THE FOLLOWING DIAGRAMS COMMUTE:
>
> $$\begin{tikzcd} a \otimes 1_{\Bbbk} = a = 1_{\Bbbk} \otimes a \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, mapsto] & 1_A \otimes a \arrow[d, mapsto] \\ a \otimes 1_A \arrow[r, mapsto] & a 1_A = a = 1_A a \end{tikzcd}$$
>
> $$\begin{tikzcd} A \otimes \Bbbk \cong A \cong \Bbbk \otimes A \arrow[r, "u \otimes id"] \arrow[d, "id \otimes u"'] \arrow[dr, "id"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

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ABELIAN GROUP
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$\equiv$

A $\Bbbk$-VSPACE MADE
INTO A UNITAL RING

> A UNITAL RING $(A, +, 0, \cdot, 1)$ IS A $\Bbbk$-ALGEBRA
> IF IT COMES WITH A UNITAL RING MAP
> $\phi : \Bbbk \longrightarrow A$ SUCH THAT $\mathrm{im}(\phi) \subseteq Z(A)$

> A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A $\Bbbk$-ALGEBRA IF IT COMES WITH
> LINEAR MAPS $m : A \otimes A \longrightarrow A$ (MULTIPLICATION) & $u : \Bbbk \longrightarrow A$ (UNIT)
> $a \otimes b \longmapsto ab$ $\qquad 1_{\Bbbk} \longmapsto 1_A$
>
> SUCH THAT THE FOLLOWING DIAGRAMS COMMUTE:
>
> $$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd} \qquad \begin{tikzcd} A \otimes \Bbbk \cong A \cong \Bbbk \otimes A \arrow[r, "u \otimes id"] \arrow[d, "id \otimes u"'] \arrow[dr, "id"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

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ABELIAN GROUP
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COMPATIBLE WITH EACH OTHER
& WITH +

$\equiv$

A $\mathbb{k}$-VSPACE MADE INTO A UNITAL RING

> A UNITAL RING $(A, +, 0, \bullet, 1)$ IS A $\mathbb{k}$-ALGEBRA IF IT COMES WITH A UNITAL RING MAP
> $$\phi : \mathbb{k} \longrightarrow A \quad \text{SUCH THAT } \operatorname{im}(\phi) \subseteq Z(A)$$

> A $\mathbb{k}$-VSPACE $(A, +, 0, *)$ IS A $\mathbb{k}$-ALGEBRA IF IT COMES WITH LINEAR MAPS
> $$m : A \otimes A \longrightarrow A \ (\text{MULTIPLICATION}) \quad \& \quad u : \mathbb{k} \longrightarrow A \ (\text{UNIT})$$
> $$a \otimes b \longmapsto ab \qquad\qquad 1_{\mathbb{k}} \longmapsto 1_A$$
>
> SUCH THAT THE FOLLOWING DIAGRAMS COMMUTE:
>
> ASSOCIATIVITY
> $$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd}$$
>
> UNITALITY
> $$\begin{tikzcd} A \otimes \mathbb{k} \arrow[r, phantom, "\cong"] \arrow[d, "id \otimes u"'] & A \arrow[r, phantom, "\cong"] \arrow[drr, "id"] \arrow[d, phantom, "\circlearrowright"] & \mathbb{k} \otimes A \arrow[r, "u \otimes id"] & A \otimes A \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowright"] \\ A \otimes A \arrow[rrr, "m"'] & & & A \end{tikzcd}$$

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ABELIAN GROUP
WITH OPERATIONS $\bullet$ AND $*$
COMPATIBLE WITH EACH OTHER
& WITH $+$

≡

A $\Bbbk$-VSPACE MADE
INTO A UNITAL RING

> A UNITAL RING $(A, +, 0, \bullet, 1)$ IS A **$\Bbbk$-ALGEBRA**
> IF IT COMES WITH A UNITAL RING MAP
> $\phi : \Bbbk \longrightarrow A$ SUCH THAT $\operatorname{im}(\phi) \subseteq Z(A)$

> EXERCISE 1.5 SHOW EQUIV. OF DEFNS $\Updownarrow$

> WILL USE
>
> A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH
> LINEAR MAPS $m : A \otimes A \longrightarrow A$ (MULTIPLICATION) & $u : \Bbbk \longrightarrow A$ (UNIT)
> $a \otimes b \longmapsto ab$ $\qquad$ $1_{\Bbbk} \longmapsto 1_A$
>
> SUCH THAT THE FOLLOWING DIAGRAMS COMMUTE:
>
> $$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$
>
> $$\begin{tikzcd} A \otimes \Bbbk \cong A \cong \Bbbk \otimes A \arrow[r, "u \otimes id"] \arrow[d, "id \otimes u"'] \arrow[dr, "id"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

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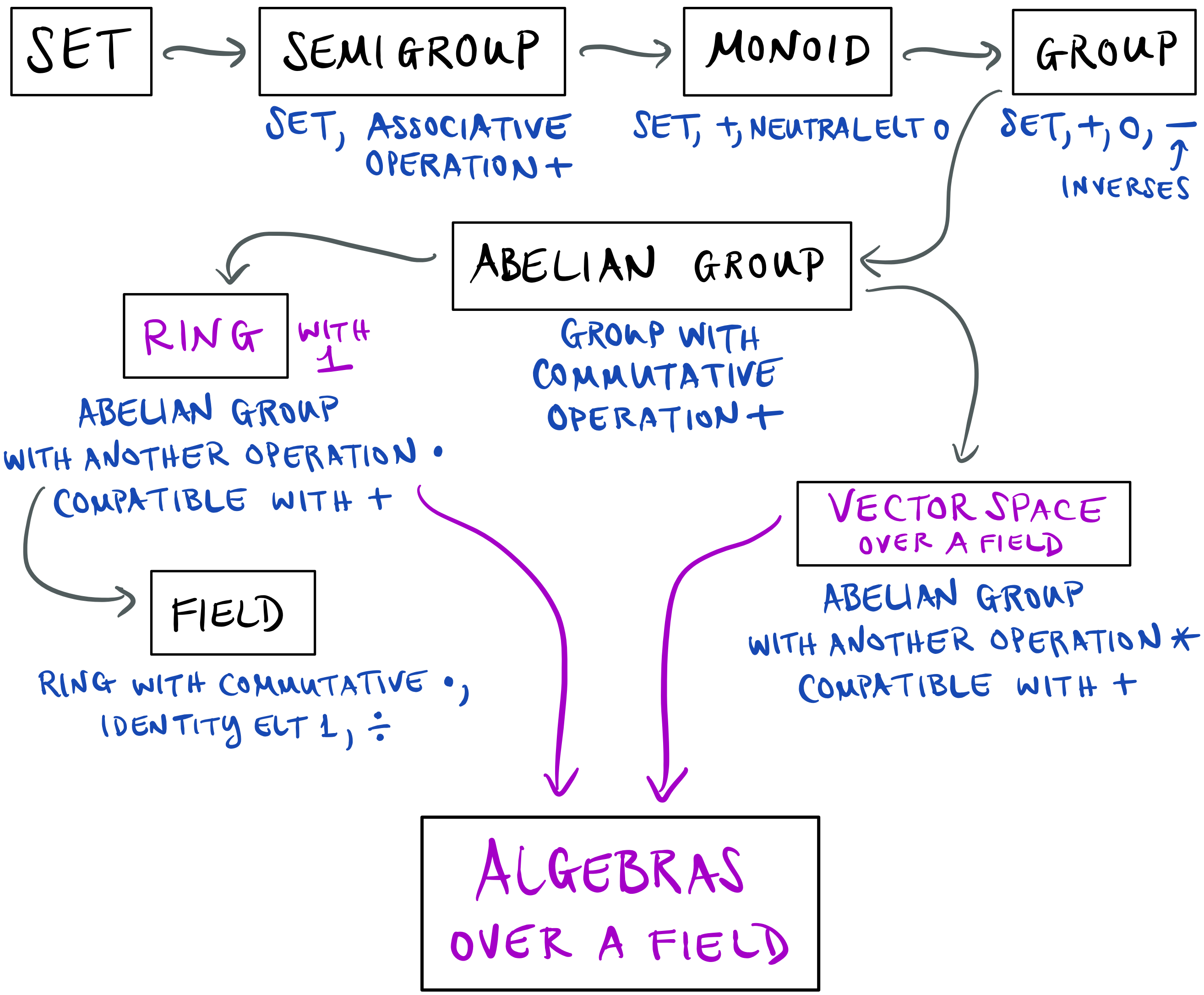
IV. ALGEBRAS OVER A FIELD

ALGEBRAIC STRUCTURES —



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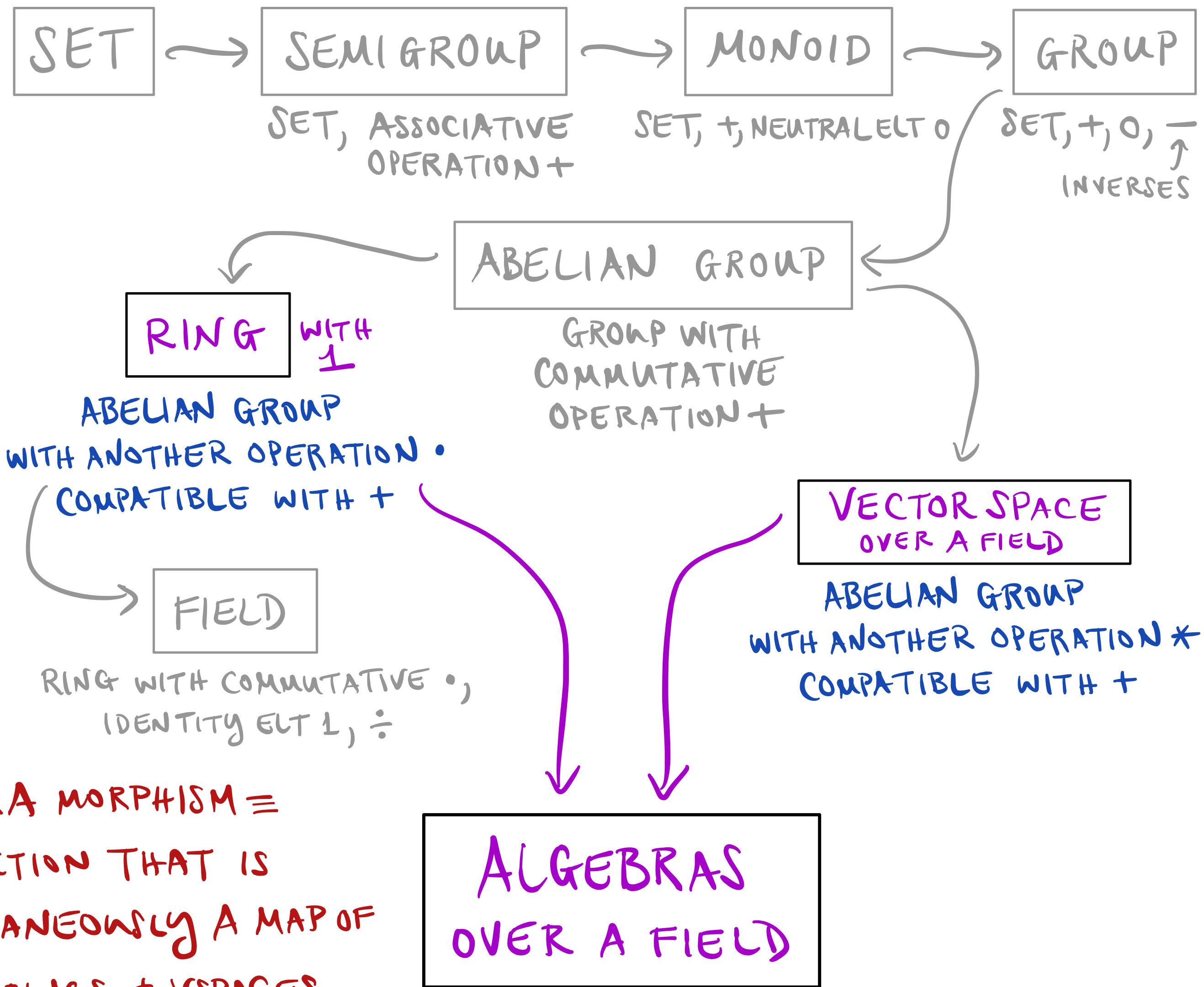
IV. ALGEBRAS OVER A FIELD

ALGEBRAIC STRUCTURES —



ALGEBRA MORPHISM ≡ A FUNCTION THAT IS SIMULTANEOUSLY A MAP OF UNITAL RINGS & VSPACES

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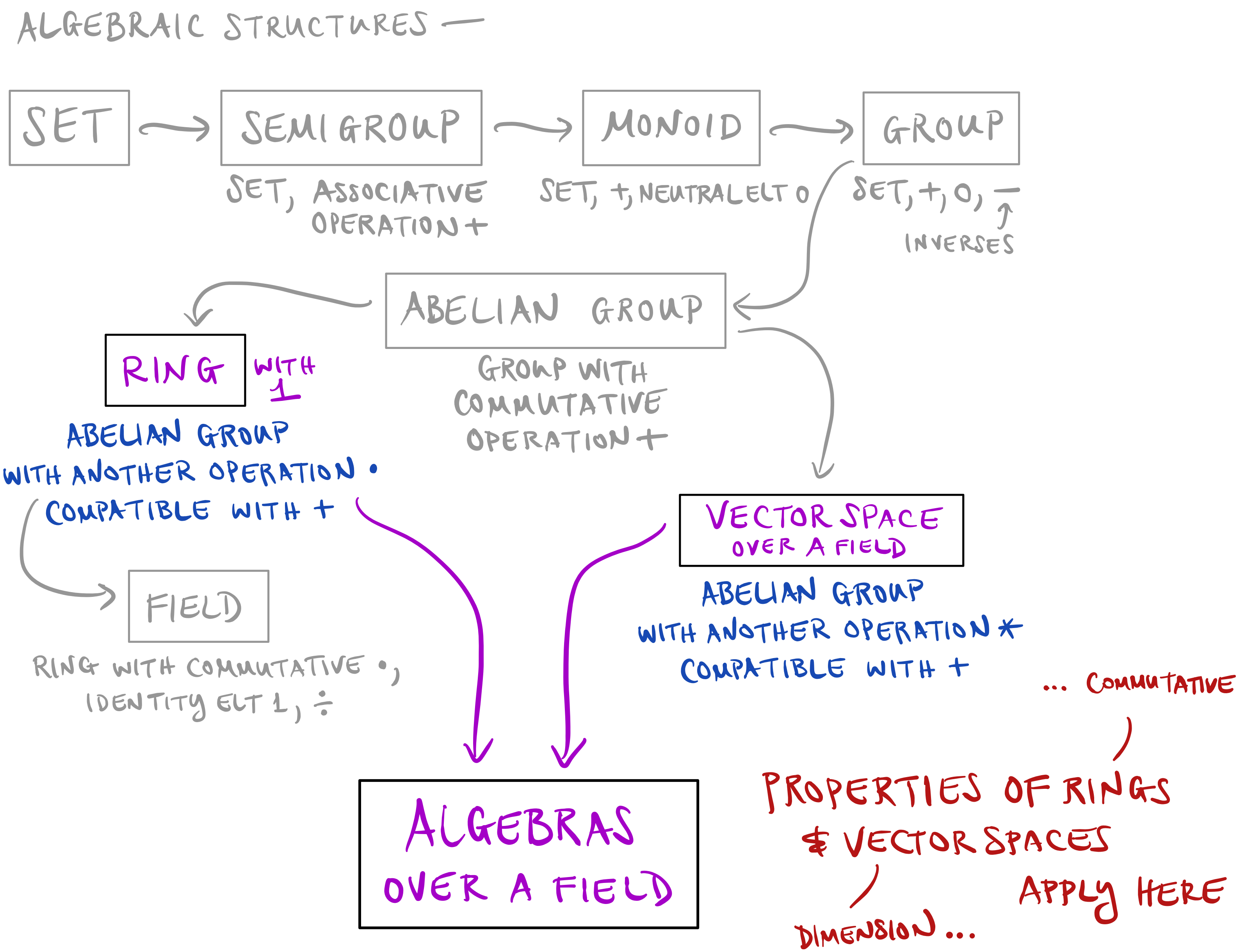
IV. ALGEBRAS OVER A FIELD



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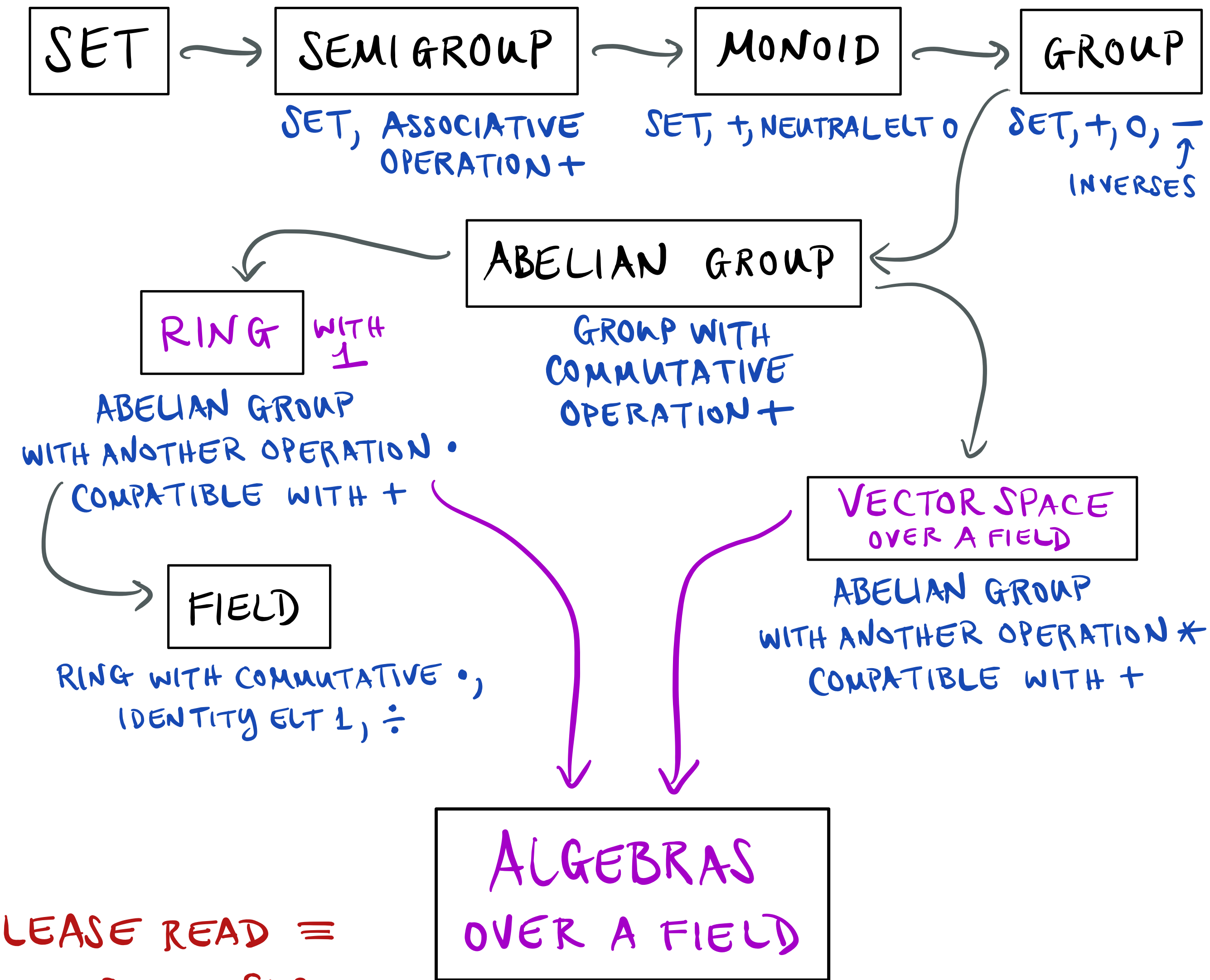
IV. ALGEBRAS OVER A FIELD

ALGEBRAIC STRUCTURES —



≡ PLEASE READ ≡
THE REST OF §1.2.1

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V. Examples/Types of Algebras over a Field

A $\Bbbk$-vspace $(A, +, 0, *)$ is a **$\Bbbk$-algebra** if it comes with linear maps $m: A \otimes A \to A$ (multiplication) & $u: \Bbbk \to A$ (unit)

s.t. $m(m \otimes id_A) = m(id_A \otimes m)$ (associativity) & $m(u \otimes id_A) = id_A = m(id_A \otimes u)$ (unitality)

You do!

Cook up some examples of

$\mathbb{R}$-algebras

$\Bbbk$-algebras

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V. EXAMPLES/TYPES OF ALGEBRAS OVER A FIELD

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH
LINEAR MAPS $m : A \otimes A \to A$ (MULTIPLICATION) & $u : \Bbbk \to A$ (UNIT)
s.t. $m(m \otimes id_A) = m(id_A \otimes m)$ (ASSOCIATIVITY) & $m(u \otimes id_A) = id_A = m(id_A \otimes u)$ (UNITALITY)

You do!

COOK UP SOME EXAMPLES OF

$\mathbb{R}$-ALGEBRAS
- $\mathbb{R}$
- $\mathbb{C}$
- $\mathbb{H}$ QUATERNIONS

$\Bbbk$-ALGEBRAS
- 0 ZERO VSPACE WITH ZERO m, u
- $\Bbbk$ ITSELF

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V. EXAMPLES / TYPES OF ALGEBRAS OVER A FIELD

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A $\Bbbk$-ALGEBRA IF IT COMES WITH LINEAR MAPS $m : A \otimes A \to A$ (MULTIPLICATION) & $u : \Bbbk \to A$ (UNIT)

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MATRIX ALGEBRAS

$\mathrm{Mat}_n(\Bbbk)$

- VS BASIS ≡ ELEMENTARY MATRICES $\{E_{k,l}\}_{k,l=1}^{n}$

  HAS 1 IN (k,l) SLOT
  0s ELSEWHERE

- MULTIP ≡ MATRIX MULTIPLICATION

- UNIT ≡ IDENTITY MATRIX $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

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V. EXAMPLES / TYPES OF ALGEBRAS OVER A FIELD

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH LINEAR MAPS $m: A \otimes A \longrightarrow A$ (MULTIPLICATION) & $u: \Bbbk \longrightarrow A$ (UNIT)

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MATRIX ALGEBRAS

$Mat_n(\Bbbk)$

- VS BASIS $\equiv$ ELEMENTARY MATRICES

$$\{E_{k,l}\}_{k,l=1}^{n}$$

HAS 1 IN (k,l) SLOT
0s ELSEWHERE

- MULTIP $\equiv$ MATRIX MULTIPLICATION
- UNIT $\equiv$ IDENTITY MATRIX $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

ENDOMORPHISM ALGEBRAS

$End_{\Bbbk}(V)$

GIVEN A $\Bbbk$-VECTOR SPACE V

- AS A VS $=$ $\underbrace{Hom_{\Bbbk}(V,V)}_{\Bbbk\text{-VS OF }\Bbbk\text{-LINEAR MAPS}}$

- MULTIP $\equiv$ FUNCTION COMPOSITION
- UNIT $\equiv$ IDENTITY MAP id_V

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V. EXAMPLES / TYPES OF ALGEBRAS OVER A FIELD

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH LINEAR MAPS $m: A \otimes A \to A$ (MULTIPLICATION) & $u: \Bbbk \to A$ (UNIT)

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MATRIX ALGEBRAS $\mathrm{Mat}_n(\Bbbk)$ $\cong$ ENDOMORPHISM ALGEBRAS $\mathrm{End}_{\Bbbk}(V)$

AS $\Bbbk$-ALGS

MATRIX ALGEBRAS $\mathrm{Mat}_n(\Bbbk)$

- VS BASIS $\equiv$ ELEMENTARY MATRICES $\{E_{k,l}\}_{k,l=1}^{n}$

HAS 1 IN (k,l) SLOT

0s ELSEWHERE

ENDOMORPHISM ALGEBRAS $\mathrm{End}_{\Bbbk}(V)$

GIVEN A $\Bbbk$-VECTOR SPACE V WITH $\dim_{\Bbbk} V = n$

- AS A VS $=$ $\mathrm{Hom}_{\Bbbk}(V, V)$

$\Bbbk$-VS OF $\Bbbk$-LINEAR MAPS

$\mathrm{Mat}(\phi) \longleftarrow \phi$

MATRIX OF ϕ WRT FIXED BASES

- MULTIP $\equiv$ MATRIX MULTIPLICATION
- UNIT $\equiv$ IDENTITY MATRIX $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

- MULTIP $\equiv$ FUNCTION COMPOSITION
- UNIT $\equiv$ IDENTITY MAP id_V

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V. Examples/Types of Algebras over a Field

A $\Bbbk$-vspace $(A, +, 0, *)$ is a **$\Bbbk$-algebra** if it comes with linear maps $m: A \otimes A \to A$ (multiplication) & $u: \Bbbk \to A$ (unit)

s.t. $m(m \otimes id_A) = m(id_A \otimes m)$ (associativity) & $m(u \otimes id_A) = id_A = m(id_A \otimes u)$ (unitality)

Free Algebras

$\Bbbk\langle v_i \rangle_{i \in I}$

Given variables $\{v_i\}_{i \in I}$

- VS basis $\equiv$ words in $\{v_i\}_{i\in I}$

  $(v_{i_1} v_{i_2} \cdots v_{i_r})$

- multip $\equiv$ word concatenation

- unit $\equiv$ empty word

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V. Examples / Types of Algebras over a Field

A $\Bbbk$-vspace $(A, +, 0, *)$ is a **$\Bbbk$-algebra** if it comes with linear maps $m: A \otimes A \longrightarrow A$ (multiplication) & $u: \Bbbk \longrightarrow A$ (unit)

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- unit $\equiv$ empty word

Ex.

$\Bbbk\langle v, w \rangle$

Basis elements include

$v,\ w,$

$v^2,\ vw,\ wv,\ w^2,\ \ldots$

$vw \cdot wv = vw^2v$

$1_{\Bbbk\langle v,w\rangle}$ thought of as $v^0 = w^0$

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V. Examples / Types of Algebras over a Field

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  $(v_{i_1} v_{i_2} \cdots v_{i_r})$
- multip $\equiv$ word concatenation
- unit $\equiv$ empty word

Tensor algebras

$T(V)$

Given a $\Bbbk$-vector space V

- As a VS $= \Bbbk \oplus V \oplus (V \otimes V) \oplus V^{\otimes 3} \oplus \ldots$
- Multip. induced by
  $V^{\otimes r} \otimes V^{\otimes s} \cong V^{\otimes (r+s)}$
- unit $\equiv$ inclusion $\Bbbk \hookrightarrow T(V) = \Bbbk \oplus V \oplus (V \otimes V) \oplus \ldots$

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V. EXAMPLES / TYPES OF ALGEBRAS OVER A FIELD

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A $\Bbbk$-ALGEBRA IF IT COMES WITH LINEAR MAPS $m : A \otimes A \to A$ (MULTIPLICATION) & $u : \Bbbk \to A$ (UNIT)

s.t. $m(m \otimes id_A) = m(id_A \otimes m)$ (ASSOCIATIVITY) & $m(u \otimes id_A) = id_A = m(id_A \otimes u)$ (UNITALITY)

FREE ALGEBRAS

$\Bbbk\langle v_i \rangle_{i \in I}$

GIVEN VARIABLES $\{v_i\}_{i \in I}$

- VS BASIS $\equiv$ WORDS IN $\{v_i\}_{i \in I}$ $(v_{i_1} v_{i_2} \cdots v_{i_r})$
- MULTIP $\equiv$ WORD CONCATENATION
- UNIT $\equiv$ EMPTY WORD

TENSOR ALGEBRAS

$T(V)$

GIVEN A $\Bbbk$-VECTOR SPACE V

- AS A VS = $\Bbbk \oplus V \oplus (V \otimes V) \oplus V^{\otimes 3} \oplus \dots$
- MULTIP. INDUCED BY $V^{\otimes r} \otimes V^{\otimes s} \cong V^{\otimes (r+s)}$

EXER. 1.3
$\oplus$ COMMUTES WITH $\otimes$

- UNIT $\equiv$ INCLUSION $\Bbbk \hookrightarrow T(V) = \Bbbk \oplus V \oplus (V \otimes V) \oplus \dots$

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V. EXAMPLES / TYPES OF ALGEBRAS OVER A FIELD

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FREE ALGEBRAS

$\Bbbk\langle v_i \rangle_{i \in I}$

TENSOR ALGEBRAS

$T(V)$

$\Bbbk\langle v_i \rangle_{i \in I} \cong T(V)$ AS $\Bbbk$-ALGS

GIVEN VARIABLES $\{v_i\}_{i \in I}$

- VS BASIS $\equiv$ WORDS IN $\{v_i\}_{i \in I}$

  $(v_{i_1} v_{i_2} \cdots v_{i_r}) \longmapsto v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \in V^{\otimes r}$

- MULTIP $\equiv$ WORD CONCATENATION
- UNIT $\equiv$ EMPTY WORD

GIVEN THE VSPACE $V = \bigoplus_{i=1}^{r} \Bbbk v_i$

- AS A VS $= \Bbbk \oplus V \oplus (V \otimes V) \oplus V^{\otimes 3} \oplus \cdots$

  $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \in V^{\otimes r}$

- MULTIP. INDUCED BY

  $V^{\otimes r} \otimes V^{\otimes s} \cong V^{\otimes (r+s)}$

- UNIT $\equiv$ INCLUSION $\Bbbk \hookrightarrow T(V) = \Bbbk \oplus V \oplus (V \otimes V) \oplus \cdots$

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- UNIT $\equiv$ INCLUSION $\Bbbk \hookrightarrow \underset{\Bbbk \oplus V \oplus (V \otimes V) \oplus \dots}{\underset{\parallel}{T(V)}}$

UNIVERSAL PROPERTY

$$V \xrightarrow[\text{LINEAR EMBEDDING}]{\alpha} T(V)$$

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V. EXAMPLES / TYPES OF ALGEBRAS OVER A FIELD

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH LINEAR MAPS $m: A \otimes A \longrightarrow A$ (MULTIPLICATION) & $u: \Bbbk \longrightarrow A$ (UNIT)

$\ni$ $m(m \otimes id_A) = m(id_A \otimes m)$ (ASSOCIATIVITY) & $m(u \otimes id_A) = id_A = m(id_A \otimes u)$ (UNITALITY)

UNIVERSAL PROPERTY

$$
\begin{tikzcd}
V \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & \arrow[d, phantom, "\circlearrowright"] & T(V) \arrow[dl, dashed, "\gamma"] \\
& Z_{vs} \sqsubset Z_{ALG} &
\end{tikzcd}
$$

LINEAR EMBEDDING α

$\forall$ LINEAR MAPS β

$\exists!$ ALGEBRA MAP γ

$\uparrow$ TAKE UNDERLYING VSPACE STRUCTURE

TENSOR ALGEBRAS $T(V)$

GIVEN A $\Bbbk$-VECTOR SPACE V

- AS A VS $= \Bbbk \oplus V \oplus (V \otimes V) \oplus V^{\otimes 3} \oplus \ldots$

- MULTIP. INDUCED BY
  $$V^{\otimes r} \otimes V^{\otimes s} \cong V^{\otimes (r+s)}$$

- UNIT $\equiv$ INCLUSION $\Bbbk \hookrightarrow \underset{\Bbbk \oplus V \oplus (V \otimes V) \oplus \ldots}{\underset{\parallel}{T(V)}}$

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V. EXAMPLES / TYPES OF ALGEBRAS OVER A FIELD

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH LINEAR MAPS $m : A \otimes A \to A$ (MULTIPLICATION) & $u : \Bbbk \to A$ (UNIT)

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MORE ALGEBRAS DEFINED BY UNIVERSAL PROPERTY

$$
\begin{tikzcd}
V \arrow[rr, "\substack{\text{LINEAR}\\ \text{EMBEDDING}\\ \alpha}"] \arrow[dr, "\substack{\forall\,\text{LINEAR}\\ \text{MAPS } \beta}"'] & \circlearrowright & T(V) \arrow[dl, dashed, "\substack{\exists!\ \text{ALGEBRA}\\ \text{MAP } \gamma}"] \\
& Z_{vs} \sqsubset Z_{alg} &
\end{tikzcd}
$$

↑ TAKE UNDERLYING VSPACE STRUCTURE

TENSOR ALGEBRAS

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V. Examples / Types of Algebras over a Field

A $\Bbbk$-vspace $(A, +, 0, *)$ is a **$\Bbbk$-algebra** if it comes with linear maps $m : A \otimes A \longrightarrow A$ (multiplication) & $u : \Bbbk \longrightarrow A$ (unit)

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More algebras defined by universal property

$$\begin{tikzcd} V \arrow[rr, "\text{linear embedding } \alpha"] \arrow[dr, "\beta"'] & \arrow[d, phantom, "\circlearrowleft"] & T(V) \arrow[dl, dashed, "\gamma"] \\ & Z_{vs} \subset Z_{ALG} & \end{tikzcd}$$

$\forall$ linear maps β

$\exists!$ algebra map γ

Z_{vs}: take underlying vspace structure

Tensor algebras

$$\begin{tikzcd} V \arrow[rr, "\text{linear } \alpha"] \arrow[dr, "\beta"'] & \arrow[d, phantom, "\circlearrowleft"] & S(V) \arrow[dl, dashed, "\gamma"] \\ & Z_{vs} \subset Z & \end{tikzcd}$$

$\forall$ linear maps β

$\exists!$ algebra map γ

Z comm alg.

Symmetric algebras

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V. Examples/Types of Algebras over a Field

A $\Bbbk$-vspace $(A, +, 0, *)$ is a **$\Bbbk$-algebra** if it comes with linear maps $m: A \otimes A \longrightarrow A$ (multiplication) & $u: \Bbbk \longrightarrow A$ (unit)

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More algebras defined by universal property

$$\begin{tikzcd} V \arrow[rr, "\substack{\text{linear}\\ \text{embedding}\\ \alpha}"] \arrow[dr, "\forall \text{ linear maps } \beta"'] & \arrow[d, phantom, "\circlearrowright"] & T(V) \arrow[dl, dashed, "\exists! \text{ algebra map } \gamma"] \\ & Z_{vs} \subset Z_{alg} & \end{tikzcd}$$

$\uparrow$ take underlying vspace structure

Tensor Algebras

$$\begin{tikzcd} V \arrow[rr, "\text{linear } \alpha"] \arrow[dr, "\forall \text{ linear maps } \beta"'] & \arrow[d, phantom, "\circlearrowright"] & S(V) \cong \Bbbk[v_1, \dots, v_n] \arrow[dl, dashed, "\exists! \text{ algebra map } \gamma"] \\ & Z_{vs} \subset Z_{\text{comm alg.}} & \end{tikzcd}$$

when $V = \bigoplus_{i=1}^{n} \Bbbk v_i$

Symmetric Algebras

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V. EXAMPLES / TYPES OF ALGEBRAS OVER A FIELD

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MORE ALGEBRAS DEFINED BY UNIVERSAL PROPERTY

$$\begin{tikzcd}
V \arrow[rr, "\alpha"', "\text{LINEAR EMBEDDING}"] \arrow[dr, "\beta"', "\forall \text{ LINEAR MAPS}"] & \circlearrowleft & T(V) \arrow[dl, dashed, "\exists! \text{ ALGEBRA MAP } \gamma"] \\
& Z_{vs} \subset Z_{ALG} &
\end{tikzcd}$$

$\uparrow$ TAKE UNDERLYING VSPACE STRUCTURE

TENSOR ALGEBRAS

$$\begin{tikzcd}
V \arrow[rr, "\text{LINEAR } \alpha"] \arrow[dr, "\beta"', "\forall \text{ LINEAR MAPS}"] & \circlearrowleft & S(V) \arrow[dl, dashed, "\exists! \text{ ALGEBRA MAP } \gamma"] \\
& Z_{vs} \subset Z_{\text{COMM ALG.}} &
\end{tikzcd}$$

SYMMETRIC ALGEBRAS

$$\begin{tikzcd}
V \arrow[rr, "\text{LINEAR } \alpha"] \arrow[dr, "\beta"', "\forall \text{ LINEAR MAPS}"] & \circlearrowleft & \Lambda(V) \arrow[dl, dashed, "\exists! \text{ ALGEBRA MAP } \gamma"] \\
& Z_{vs} \subset Z_{\text{ANTI-COMM. ALG}} &
\end{tikzcd}$$

EXTERIOR ALGEBRAS

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V. Examples/Types of Algebras over a Field

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More algebras defined by universal property

$$\begin{tikzcd}[column sep=huge, row sep=huge]
V \arrow[r, "\substack{\text{linear}\\ \text{embedding}\\ \alpha}"] \arrow[dr, "\substack{\forall \text{ linear}\\ \text{maps } \beta}"'] \arrow[dr, phantom, shift left=6ex, "\circlearrowright"] & T(V) \arrow[d, dashed, "\substack{\exists! \text{ algebra}\\ \text{map } \gamma}"] \\
& Z_{vs} \sqsubset Z_{ALG}
\end{tikzcd}$$

$\uparrow$ take underlying vspace structure

Tensor algebras

$$\begin{tikzcd}[column sep=huge, row sep=huge]
V \arrow[r, "\text{linear } \alpha"] \arrow[dr, "\substack{\forall \text{ linear}\\ \text{maps } \beta}"'] \arrow[dr, phantom, shift left=6ex, "\circlearrowright"] & S(V) \arrow[d, dashed, "\substack{\exists! \text{ algebra}\\ \text{map } \gamma}"] \\
& Z_{vs} \sqsubset Z_{\text{comm alg.}}
\end{tikzcd}$$

Symmetric algebras

$$\begin{tikzcd}[column sep=huge, row sep=huge]
V \arrow[r, "\text{linear } \alpha"] \arrow[dr, "\substack{\forall \text{ linear}\\ \text{maps } \beta}"'] \arrow[dr, phantom, shift left=6ex, "\circlearrowright"] & \Lambda(V) \arrow[d, dashed, "\substack{\exists! \text{ algebra}\\ \text{map } \gamma}"] \\
& Z_{vs} \sqsubset Z_{\text{anti-comm. alg}}
\end{tikzcd}$$

Exterior algebras

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CHELSEA WALTON
RICE U.

LECTURE #2

TOPICS:

✓ I. TENSOR PRODUCT OF VECTOR SPACES (VIA QUOTIENT) (§1.1.4)

✓ II. UNIVERSAL PROPERTY, TENSOR PRODUCT OF VS REVISITED (§1.1.4)

✓ III. MORE ON TENSOR PRODUCT & HOM OF VSPACES (§1.1.4)

✓ IV. ALGEBRAS OVER A FIELD (§1.1.5)

V. EXAMPLES/TYPES OF ALGEBRAS OVER A FIELD (§1.2)

$\mathrm{Mat}_n(\Bbbk)$, $\mathrm{End}_\Bbbk(V)$, $\Bbbk\langle v_i \rangle_{i \in I}$, $T(V)$, $S(V)$, $\Lambda(V)$

↳ NEXT: ALGS BUILT FROM GROUPS & GRAPHS.
ALSO MATRIX AVATARS OF ALGEBRAS

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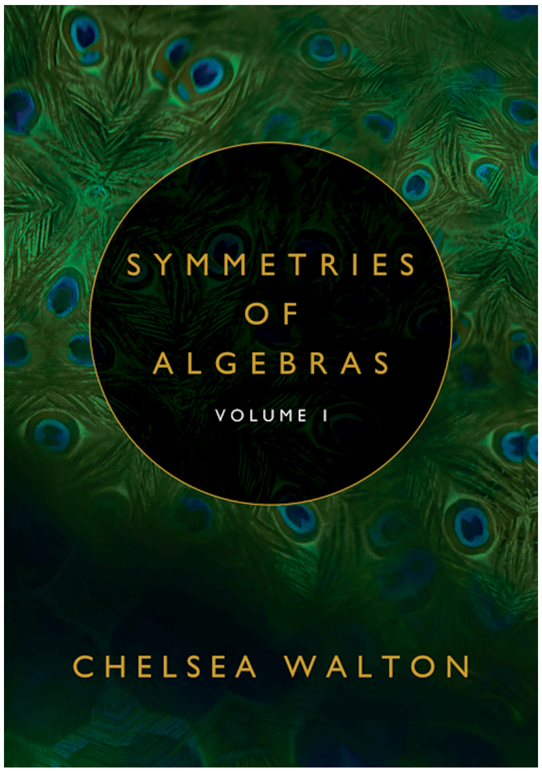
Enjoy this lecture?
You'll enjoy the textbook!

C. Walton's "Symmetries of Algebras, Volume 1" (2024)



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Lecture #2 keywords: algebra over a field, exterior algebra, free algebra, matrix algebra, symmetric algebra, Tensor-Hom adjunction, tensor algebra, tensor product, universal property, universal structure